Vysoké učení technické v Brně
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Aproximace a faktorizace ve fuzzy strukturách

Habilitační práce

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Obsah

1 Úvod

Při práci s fuzzy strukturami často narážíme na problém velkého rozsahu zpracovávaných dat způsobený velikostí (počtem prvků) používané množiny pravdivostních hodnot (stupňů pravdivosti). Tato práce popisuje způsob řešení tohoto problému, založený na eliminaci malých rozdílů mezi zkoumanými objekty.

Princip metody nastíníme na modelovém příkladě. Představme si, že se snažíme zjistit potenciální oblibu pěti různých druhů zboží. Průzkum trhu přinesl na otázku „Koupil/a byste si dané zboží?“ kladenou potenciálním zákazníkům různých věkových kategorií následující odpovědi:

zboží	$<$ 30 let	30–60 let	$>$ 60 let
A	0.25	0.45	0.82
B	0.13	0.38	0.73
C	0.82	0.50	0.03
D	0.31	0.27	0.38
E	0.25	0.63	0.30

Data v tabulce interpretujeme takto (např. pro řádek D a druhou věkovou kategorii): *Tvrzení „Průměrný zákazník ve věkové kategorii* 30–60 *let si chce koupit zboží D“ má pravdivostní hodnotu* 0.27.

Je zřejmé, že tabulka nese pro manažera rozhodujícího o zahájení výroby zboží zbytečně mnoho informací. Pokud budeme data v tabulce dále zpracovávat např. některou z metod data miningu (příkladem takové metody je fuzzy konceptuální analýza, kterou se z velké části zabývá tento text), narazíme také na zásadní problém, že množství informací získaných z tabulky prudce narůstá v závislosti na počtu použitých pravdivostních hodnot (tedy použité logické přesnosti).

Před další analýzou je tedy vhodné data upravit, a to i za cenu způsobení malé chyby, ke které dojde jak v samotných datech, tak v z nich získaném výsledku.

V tomto textu se věnujeme dvěma způsobům takové úpravy dat: *aproximaci* a *faktorizaci*. Oba jsou založeny na tom, že si uživatel zvolí míru (tzv. *práh*), po kterou je ochoten tolerovat chybu ve výstupních datech. V uvedeném případě by to mohlo například znamenat, že zanedbáme rozdíly v datech, které jsou menší než jedna pětina, tedy že zadáme hodnotu prahu rovnu $\frac{4}{5}$. Po zadání této hodnoty lze postupovat dvojím způsobem: u metody aproximace vyhledáme shluky podobných dat a každý nahradíme vhodnou střední hodnotou, u metody faktorizace budeme pracovat se samotnými shluky jako s jednotlivými (nedělitelnými) hodnotami.

Metoda aproximace tedy pracuje s malou (pro uživatele akceptovatelnou) úpravou vstupních dat, metoda faktorizace spočívá v abstrahování od malých rozdílů mezi jednotlivými hodnotami („simplification by abstraction").

Idea zmenšování velikosti fuzzy struktur a dat jimi generovaných pomocí aproximace a faktorizace je nová. Jejím hlavním autorem je Radim Bělohlávek (první práce je [3], v [8] se pak hovoří o logické přesnosti a jejím snižování), většina prací s touto tematikou (včetně prací tohoto textu) vznikla v jeho výzkumném týmu.

Výzkum vedl ke studiu nových nebo málo zkoumaných teoretických problémů: faktorizace struktur podle relace tolerance, středové body v reziduovaných svazech a systémech fuzzy množin a s nimi související problémy aproximace, dále faktorizace reziduovaných svazů a systémů fuzzy množin. Těmito problémy se zabývá velká část prací tohoto souboru [Práce 3, 7, 4, 5]. Mnohé z nich mají ovšem přímý dopad na problémy praktického charakteru (zejména faktorizaci konceptuálních svazů), nebo se jimi přímo zabývají [Práce 1].

Úvodní část toho textu shrnuje teoretické základy nutné pro pochopení přiložených prací (část 2), definuje základní fuzzy struktury, na něž v těchto pracích metodu aproximace a faktorizace aplikujeme (část 3), a shrnuje výsledky v pracích obsažené (část 4). Snažíme se postihovat souvislosti mezi jednotlivými pracemi. Na závěr úvodu nastiňujeme možnosti dalšího výzkumu (část 5).

Tuto úvodní část se snažíme podávat srozumitelným a přehledným způsobem. V případě nejasností najde čtenář exaktní výklad v přiložených pracích. Nabízíme také řadu ilustrativních příkladů, na které v přiložených pracích nezbýval prostor.

2 Fuzzy logika a fuzzy množiny

V této části podáváme základní poznatky o reziduovaných svazech, používaných jako struktury pravdivostních hodnot fuzzy logiky, a o fuzzy množinách a relacích. Uvádíme také několik příkladů. Podrobnosti lze nalézt v [8, 27, 28, 30].

2.1 Reziduované svazy

Úplným reziduovaným svazem rozumíme algebru $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ typu $(2, 2, 2, 2, 0, 0)$ takovou, že

1. $\langle L, \wedge, \vee, 0, 1\rangle$ je úplný svaz s nejmenším prvkem 0 a největším prvkem 1,

2. $\otimes$ je komutativní a asociativní operace splňující $a \otimes 1 = 1 \otimes a = a$ pro každé $a \in L$ (tedy $\langle L, \otimes, 1\rangle$ je komutativní monoid),

3. pro každé tři prvky $a, b, c \in L$ platí

$$a \otimes b \leq c \quad \text{právě když} \quad a \leq b \rightarrow c \tag{1}$$

(podmínka adjunkce).

Prvky reziduovaného svazu se nazývají *stupně pravdivosti*. Operace $\otimes$ a $\rightarrow$ se nazývají *součin* a *reziduum*.

Pokud není řečeno jinak, označujeme v celém tomto textu symbolem $\mathbf{L}$ pevně zvolený reziduovaný svaz. Uspořádání na množině L indukované operacemi $\wedge$ a $\vee$ označujeme $\leq$.

Poznámka 1. Reziduované svazy byly zavedeny v práci [42]. Použití reziduovaného svazu jako struktury pravdivostních hodnot ve fuzzy logice navrhl J. A. Goguen [24, 25].

Operace $\wedge$, $\vee$, $\otimes$, $\rightarrow$ reziduovaného svazu jsou interpretacemi logických spojek, resp. kvantifikátorů predikátové fuzzy logiky (po řadě: velký kvantifikátor, malý kvantifikátor, konjunkce, implikace). Podrobnosti lze najít například v [8].

Pomocí základních operací reziduovaných svazů definujeme operace $\neg$ (*negace*), a $\leftrightarrow$ (*bireziduum*), které jsou interpretacemi logických spojek fuzzy negace a fuzzy ekvivalence, takto:

$$\neg a = a \rightarrow 0, \tag{2}$$
$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a). \tag{3}$$

V literatuře se se strukturou úplného reziduovaného svazu setkáváme nejčastěji v následující podobě. Nosná množina L je rovna jednotkovému intervalu reálných čísel $[0, 1]$, operace $\otimes$ je libovolná *zleva spojitá* t*-norma*, tj. binární operace, která je zleva spojitá v prvním argumentu (jako reálná funkce dvou proměnných), komutativní, asociativní, monotonní a číslo 1 je jejím neutrálním prvkem [28], a pro reziduum $\rightarrow$ platí $a \rightarrow b = \bigvee\{c \in L \mid a \otimes c \leq b\}$. Pak $\langle [0, 1], \min, \max, \otimes, \rightarrow, 0, 1\rangle$ je úplným reziduovaným svazem.

Základní tři příklady adjungovaných dvojic operací $\otimes$ a $\to$ na intervalu $[0,1]$ jsou následující:

$$\text{Łukasiewicz:} \quad \begin{aligned} a \otimes b &= \max(a+b-1, 0), \\ a \to b &= \min(1-a+b, 1), \end{aligned} \tag{4}$$

$$\text{Gödel:} \quad \begin{aligned} a \otimes b &= \min(a, b), \\ a \to b &= \begin{cases} 1 & \text{jestliže } a \leq b, \\ b & \text{jinak,} \end{cases} \end{aligned} \tag{5}$$

$$\text{Goguen (součin):} \quad \begin{aligned} a \otimes b &= a \cdot b, \\ a \to b &= \begin{cases} 1 & \text{jestliže } a \leq b, \\ \frac{b}{a} & \text{jinak.} \end{cases} \end{aligned} \tag{6}$$

Každá z uvedených tří dvojic operací definuje strukturu úplného reziduovaného svazu na intervalu $[0,1]$. Vzniklé reziduované svazy se nazývají (po řadě) *standardní Łukasiewiczova, Gödelova, Goguenova (součinová) algebra.*

Třída úplných reziduovaných svazů obsahuje také konečné struktury. Položíme-li $L = \{a_0 = 0, a_1, \ldots, a_n = 1\} \subseteq [0,1]$, kde body $a_0 < \cdots < a_n$ jsou ekvidistantně rozloženy (tedy $a_{i+1} - a_i = a_{j+1} - a_j = \frac{1}{n}$ pro každé $i, j \in \{0, 1, \ldots, n-1\}$) a definujeme-li operace $\otimes$ a $\to$ jako zúžení Łukasiewiczových operací (4) na množinu L, pak dostaneme úplný reziduovaný svaz $\mathbf{L} = \langle L, \min, \max, \otimes, \to, 0, 1\rangle$, který se nazývá *ekvidistantní Łukasiewiczův řetězec.*

Na libovolné množině L, na které je dáno lineární uspořádání takové, že L s tímto uspořádáním tvoří úplný svaz, lze zavést strukturu úplného reziduovaného svazu s operacemi součinu a rezidua definovanými předpisem (5) (tento předpis, narozdíl od (4) a (6), využívá pouze uspořádání na množině L a faktu, že se jedná o lineární uspořádání). V případě, že množina L je konečná, dostáváme druhou skupinu příkladů konečného úplného reziduovaného svazu.

Příklad 1. Zvláštním příkladem úplného reziduovaného svazu je dvouprvková Booleova algebra $\langle\{0,1\}, \wedge, \vee, \otimes, \to, 0, 1\rangle$, označovaná symbolem **2**, která představuje strukturu pravdivostních hodnot klasické logiky. Operace $\wedge$, $\vee$, $\otimes$, $\to$ Booleovy algebry **2** jsou tedy interpretacemi odpovídajících spojek resp. kvantifikátorů klasické predikátové logiky.

2.2 Fuzzy množiny

$\mathbf{L}$-*množinou* (také *fuzzy množinou*) A *v univerzu* X rozumíme zobrazení $A\colon X \to L$. Pro $x \in X$ hodnotu $A(x) \in L$ interpretujeme jako „stupeň, ve kterém x je prvkem A." Množinu všech **L**-množin v univerzu X označujeme L^X (tedy standardním označením pro množinu všech zobrazení z X do L).

Pro **L**-množiny zavádíme následující označení. Pro libovolné $A \in L^X$ píšeme $A = \{^{A(x)}/x \mid x \in X\}$. Jestliže $A(x) \geq 0$ pouze pro $x \in \{x_1, \dots, x_n\}$, píšeme také $A = \{^{A(x_1)}/x_1, \dots, ^{A(x_n)}/x_n\}$ s případným zjednodušeným zápisem x místo $^1/x$. Pro **L**-množiny tvaru $\{^a/x\}$ tedy platí

$$\{^a/x\}(x') = \begin{cases} a & \text{pro } x' = x, \\ 0 & \text{jinak.} \end{cases} \tag{7}$$

Pro klasickou podmnožinu $A \subseteq X$ píšeme

$$A(x) = \begin{cases} 1 & \text{jestliže } x \in A, \\ 0 & \text{jestliže } x \notin A, \end{cases}$$

čímž ztotožňujeme množinu A s jistou **L**-množinou v X (charakteristickou funkcí). Díky tomuto ztotožnění můžeme klasické podmnožiny považovat za **L**-množiny.

Příklad 2. Množinu $L^{\{x\}}$ všech **L**-množin v jednoprvkové množině lze přirozeně ztotožnit s množinou L tak, že **L**-množinu $\{^a/x\}$ ztotožníme s prvkem $a \in L$. V tomto ztotožnění odpovídá průnik (resp. sjednocení) **L**-množin infimu (resp. supremu), S reziduu (relace $\subseteq$ na $L^{\{x\}}$ tedy odpovídá relaci $\leq$ na L) a $\approx$ bireziduu.

Klasické množinové operace a relace, jako jsou operace průniku a sjednocení a relace „býti podmnožinou", lze zobecnit i na **L**-množiny. Že se jedná skutečně o zobecnění, plyne z následujícího:

- Položíme-li $\mathbf{L} = \mathbf{2}$ (tedy přejdeme-li ke klasické logice), přejdou všechny nové pojmy v klasické množinové pojmy,
- aplikace zobecněných operací na klasické množiny vede ke stejnému výsledku jako aplikace příslušných klasických operací.

Operace s **L**-množinami jsou definovány po komponentách. *Průnik* **L**-množin $A, B \in L^X$ je **L**-množina $A \cap B$ v X taková, že $(A \cap B)(x) = A(x) \wedge B(x)$ pro každé $x \in X$, jejich *sjednocení* je **L**-množina $A \cup B$ v X taková, že $(A \cup B)(x) = A(x) \vee B(x)$ pro každé $x \in X$.

Průnik a sjednocení dvou **L**-množin lze zobecnit na libovolný počet **L**-množin, případně i na libovolnou **L**-množinu **L**-množin. Pro libovolnou **L**-množinu $U : L^X \to L$ (tedy $U \in L^{L^X}$; U je **L**-množina **L**-množin v X), *průnik* $\bigcap U$ a *sjednocení* $\bigcup U$ *systému* U jsou **L**-množiny v X takové, že

$$\bigcap U(x) = \bigwedge_{A \in L^X} U(A) \to A(x), \tag{8}$$

$$\bigcup U(x) = \bigvee_{A \in L^X} U(A) \otimes A(x), \tag{9}$$

pro každé $x \in X$.

Pro dvě **L**-množiny $A, B \in L^X$ klademe

$$S(A, B) = \bigwedge_{x \in X} A(x) \to B(x), \tag{10}$$

$$A \approx B = \bigwedge_{x \in X} A(x) \leftrightarrow B(x). \tag{11}$$

$S(A, B)$ a $A \approx B$ se nazývají (po řadě) *stupeň, ve kterém* A *je podmnožinou* B a *stupeň, ve kterém se* A *rovná* (případně *je podobné*) B. Pomocí pravidel sémantiky fuzzy logiky (odstavec 2.1) lze $S(A, B)$ interpretovat jako stupeň pravdivosti formule „Pro každé $x \in X$ platí: je-li x prvkem A, pak je x prvkem B." Podobně, $A \approx B$ je stupeň pravdivosti formule „Pro každé $x \in X$ platí: x je prvkem A právě když x je prvkem B." $\approx$ se nazývá *Leibnizova podobnost*.

Z definice operace birezidua (3) ihned plyne následující vztah:

$$A \approx B = S(A, B) \wedge S(B, A). \tag{12}$$

Z podmínky adjunkce (1) lze snadno odvodit, že $S(A, B) = 1$ právě když $A(x) \leq B(x)$ pro každé $x \in X$ (A je zcela obsaženo v B). Tuto skutečnost zapisujeme takto:

$$A \subseteq B,$$

což v případě klasických podmnožin vede přesně na klasický pojem podmnožiny.

Na **L**-množinách lze také zavést operace, které nemají obdobu v klasické teorii množin, v níž se redukují na triviální operace. Základním příkladem je tzv. posun a násobek (v literatuře se také setkáme s pojmem kotenzor a tenzor). Pro libovolný prvek $a \in L$ rozumíme *a-posunem* **L**-*množiny* $A \in L^X$ **L**-množinu $a \to A \in L^X$, definovanou předpisem

$$(a \to A)(x) = a \to A(x), \tag{13}$$

a *a-násobkem* A rozumíme **L**-množinu $a \otimes A \in L^X$, definovanou předpisem

$$(a \otimes A)(x) = a \otimes A(x), \tag{14}$$

pro každé $x \in X$.

Jednoduchým způsobem převedení $\mathbf{L}$-množiny na klasickou množinu je použití tzv. řezů. To spočívá ve volbě stupně pravdivosti $a \in L$ (nazývaného také *prahem*), který představuje hranici, po kterou jsme ochotni pravdivostní stupně akceptovat jako „dostatečně pravdivé“. Pro libovolnou $\mathbf{L}$-množinu $A \in L^X$ pak položíme

$$ {}^aA = \{x \in X \mid a \leq A(x)\}. \tag{15} $$

${}^aA$ je klasická podmnožina množiny X, která se nazývá *a-řezem* množiny A.

V pracích tohoto souboru je zvolený práh označován také písmenem e nebo ε.

2.3 Fuzzy relace, fuzzy ekvivalence

n-ární $\mathbf{L}$-relací mezi množinami $X_1, X_2, \ldots, X_n$ rozumíme libovolné zobrazení $R\colon X_1 \times X_2 \times \cdots \times X_n \to L$. $\mathbf{L}$-relace R je tedy $\mathbf{L}$-množina v kartézském součinu $X_1 \times X_2 \times \cdots \times X_n$, což odpovídá definici relace z klasické teorie množin. Je-li $X_1 = X_2 = \cdots = X_n = X$, hovoříme o *$n$-ární $\mathbf{L}$-relaci na množině X*.

Pro $x_1 \in X_1, x_2 \in X_2, \ldots, x_n \in X_n$ interpretujeme hodnotu $R(x_1, x_2, \ldots, x_n)$ jako stupeň, v němž jsou prvky $x_1, x_2, \ldots, x_n$ v relaci R.

V této práci se zabýváme především *binárními relacemi*, tedy případem $n = 2$. V tomto případě používáme pro hodnotu $R(x_1, x_2)$ také označení $x_1 R\, x_2$. Pojem *unární relace* na množině X (pro $n = 1$) splývá s pojmem $\mathbf{L}$-množiny v množině X.

Základní pojmy známé z klasických relací lze zobecnit i na $\mathbf{L}$-relace. Binární $\mathbf{L}$-relace R na množině X se nazývá

- *reflexivní*, jestliže $R(x, x) = 1$,
- *symetrická*, jestliže $R(x, y) = R(y, x)$,
- *tranzitivní*, jestliže $R(x, y) \otimes R(y, z) \leq R(x, z)$

pro každé $x, y, z \in X$. Binární $\mathbf{L}$-relace R se nazývá $\mathbf{L}$-*ekvivalence*, je-li reflexivní, symetrická a tranzitivní. Pokud navíc pro každé $x, y \in X$ z $R(x, y) = 1$ plyne $x = y$, nazývá se R $\mathbf{L}$-*rovností* na X.

Leibnizova podobnost $\mathbf{L}$-množin (11), je příkladem $\mathbf{L}$-rovnosti na množině L^X.

Řekneme, že n-ární $\mathbf{L}$-relace R na X je *kompatibilní* s $\mathbf{L}$-ekvivalencí $\approx$ na X, jestliže pro každé $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$ platí

$$ (x_1 \approx y_1) \otimes \cdots \otimes (x_n \approx y_n) \otimes R(x_1, \ldots, x_n) \leq R(y_1, \ldots, y_n). \tag{16} $$

Jelikož n-ární **L**-relace jsou speciální **L**-množiny, můžeme k nim také tvořit řezy. a-řezem ${}^aR$ **L**-relace R mezi množinami $X_1, \ldots, X_n$ je pak klasická n-ární relace mezi těmito množinami.

2.4 Tolerance

Je-li $\approx$ **L**-ekvivalence na množině X a $a \in L$ zvolený práh, je zjevně a-řez ${}^a\!\approx$ této **L**-ekvivalence (klasická) reflexivní a symetrická relace. Platí totiž

$$x \,{}^a\!\approx x, \text{ právě když } a \leq (x \approx x) = 1 \text{ a}$$
$$x \,{}^a\!\approx y, \text{ právě když } a \leq (x \approx y) = (y \approx x), \text{ právě když } y \,{}^a\!\approx x.$$

Je ale zřejmé, že a-řez **L**-ekvivalence nemusí být tranzitivní relace.

Ve speciálním případě $a = 1$ je 1-řez **L**-ekvivalence vždy (klasickou) ekvivalencí. Je-li $\approx$ navíc **L**-rovnost, je její 1-řez ${}^1\!\approx$ shodný s klasickou rovností $=$.

Binární (klasická) relace, která je reflexivní a symetrická, se nazývá *tolerance* [39, 35].

Poznámka 2. Rozdíl mezi relací tolerance a ekvivalence (která je proti toleranci navíc tranzitivní) lze charakterizovat tak, že zatímco ekvivalenci používáme, když sledujeme shodnost vybraných vlastností zkoumaných objektů, u tolerance nám jde pouze o jejich podobnost.

Mějme toleranci $\sim$ na množině X a prvek $x \in X$. *Třídou tolerance* $\sim$ *danou prvkem* x rozumíme množinu $[x]_\sim = \{y \in X \mid y \sim x\}$. Neprázdná množina $B \subseteq X$ se nazývá *blok tolerance* $\sim$, jestliže pro libovolné dva prvky $y_1, y_2 \in B$ platí $y_1 \sim y_2$.

Je zřejmé, že každý blok tolerance $\sim$ je podmnožinou třídy libovolného svého prvku. Třída tolerance ovšem nemusí být jejím blokem (jelikož relace $\sim$ nemusí být tranzitivní, neplatí obecně pro libovolné dva prvky $y_1, y_2 \in [x]_\sim$ vztah $y_1 \sim y_2$).

Blok B tolerance $\sim$ na množině X se nazývá *maximální blok*, jestliže pro každý jiný blok B' z $B \subseteq B'$ plyne $B = B'$. Množina všech maximálních bloků tolerance $\sim$ na množině X se nazývá *faktormnožina (factor set) množiny* X *podle tolerance* $\sim$ a označuje se $X/\!\sim$. Je zřejmé, že faktormnožina $X/\!\sim$ tvoří pokrytí množiny X (tj. $\bigcup X/\!\sim\, = X$).

Příklad 3. Jednoduchým příkladem relace tolerance je relace $\sim$ na množině R reálných čísel, daná vztahem

$$x \sim y, \quad \text{právě když} \quad 0.1 \leq |x - y|.$$

Maximálními bloky této tolerance jsou uzavřené intervaly délky 0.1, třída libovolného čísla $x \in R$ je rovna intervalu $[x - 0.1, x + 0.1]$.

Poznámka 3. Část prací tohoto souboru se zabývá problémem faktorizace množiny podle tolerance, vzniklé jako a-řez nějaké **L**-rovnosti. Problematika faktorizace podle tolerance je v matematice neobvyklá. Podle dostupných informací je zpracována pouze pro (úplné) svazy [21, 23].

2.5 Příklady

Příklad 4. Nechť $L = [0,1]$ a uspořádání na L je přirozené uspořádání reálných čísel (operace součinu a rezidua zatím přesněji nespecifikujeme). Dále nechť X je množina všech barev. Na množině X lze definovat několik **L**-množin, které mají přirozený význam. Položme například pro libovolnou barvu $x \in X$ hodnotu $A_R(x)$ rovnu hodnotě její červené složky. $A_R(x)$ lze interpretovat jako stupeň, v němž barva x má červenou složku (červená složka je obsažena v barvě x) a A_R jako **L**-množinu všech barev s červenou složkou.

Zkusme poněkud subjektivnější příklad a označme písmenem Q **L**-množinu v X barev, které se líbí nějaké testované osobě. **L**-množinu Q můžeme získat například dotazníkovým šetřením, kdy testovaná osoba bude požádána, aby u každé barvy určila, nakolik souhlasí s tvrzením „tato barva se mi líbí“.

Příklad 5. Pokračujme v předchozím příkladě a uvažme binární **L**-relaci $\approx$ na množině X, která charakterizuje podobnost barev, tedy takovou, že hodnotu $x \approx y$ lze interpretovat jako stupeň pravdivosti formule „Barvy x a y jsou podobné.“

L-relaci $\approx$ lze na množině X zavést mnoha způsoby, zřejmě ale bude v každém případě reflexivní a symetrická. Lze ji také přirozeným (tj. praxi dobře odpovídajícím) způsobem zavést tak, aby byla i tranzitivní.

Uvažme kromě **L**-množiny A_R z předchozího příkladu ještě analogické **L**-množiny A_G a A_B barev se zelenou, resp. modrou složkou. Jednoduchým a poměrně přirozeným způsobem, jak definovat podobnost dvou barev, je vycházet z podobnosti jejich jednotlivých složek a položit

$$x \approx y = \bigwedge_{c \in \{R,G,B\}} A_c(x) \leftrightarrow A_c(y) \tag{17}$$

(„Pro každou barevnou složku platí, že je obsažena v barvě x právě když je obsažena i v barvě y“).

Nyní lze snadno pomocí základních vlastností reziduovaných svazů ukázat, že **L**-relace $\approx$ je tranzitivní:

$$\begin{aligned}
&(x \approx y) \otimes (y \approx z) = \\
&= \left(\bigwedge_{c \in \{R,G,B\}} A_c(x) \leftrightarrow A_c(y) \right) \otimes \left(\bigwedge_{c \in \{R,G,B\}} A_c(y) \leftrightarrow A_c(z) \right) \leq \\
&\leq \bigwedge_{c \in \{R,G,B\}} (A_c(x) \leftrightarrow A_c(y)) \otimes (A_c(y) \leftrightarrow A_c(z)) \leq \\
&\leq \bigwedge_{c \in \{R,G,B\}} A_c(x) \leftrightarrow A_c(z) = \\
&= x \approx z.
\end{aligned}$$

L-relace $\approx$ je tedy **L**-ekvivalence na množině X.

Příklad 6. Jelikož každá barva je jednoznačně určena svými třemi komponentami, je **L**-relace $\approx$ z předchozího příkladu dokonce **L**-rovností na X. Pokud totiž $x \approx y = 1$, je podle definice této **L**-relace $A_c(x) \leftrightarrow A_c(y) = 1$ pro každé $c \in \{R, G, B\}$, což znamená, že pro každé $c \in \{R, G, B\}$ platí $A_c(x) = A_c(y)$.

V následujících příkladech obrátíme pozornost k příkladu s tyčemi, uvedeném v pracích [Práce 3, 6]. Uvedeme také úvahy, které mohou v praxi přispět k rozhodnutí o volbě vhodné strutury pravdivostních hodnot.

Příklad 7. Mějme sadu různě dlouhých železných tyčí. Úkolem je stanovit pro každé dvě tyče ze sady jejich podobnost, tj. číslo z intervalu $[0, 1]$ takové, že hodnota 1 znamená „tyče jsou úplně stejné“ a hodnota 0 „tyče se zcela liší.“ Jedná se o tenké tyče, jejich podobnost budeme stanovovat čistě porovnáním délek.

Předpokládejme nejprve, že máme podobnost dvou tyčí stanovit na základě fotografie, která zobrazuje pouze dvě porovnávané tyče. Neznáme přitom měřítko fotografie (tedy neznáme absolutní délku zobrazených tyčí), stejně tak nevíme, jak dlouhá je nejdelší tyč ze sady. Nejjednodušší možností v tomto případě je porovnávat tyče poměrem jejich délek. Stupeň podobnosti $p_1 \approx p_2$ tyčí p_1 a p_2 o délkách $l(p_1)$ a $l(p_2)$ tedy stanovíme vzorcem

$$p_1 \approx p_2 = \min\left(\frac{l(p_1)}{l(p_2)}, \frac{l(p_2)}{l(p_1)}\right). \tag{18}$$

Poměr délek tyčí lze zjistit i z fotografie, jejíž měřítko neznáme. Je-li totiž toto měřítko rovno číslu $c > 0$ (koeficient zvětšení), platí

$$p_1 \approx p_2 = \min\left(\frac{c \cdot l(p_1)}{c \cdot l(p_2)}, \frac{c \cdot l(p_2)}{c \cdot l(p_1)}\right). \tag{19}$$

Uvažme interval $[0, 1]$ s Goguenovou (součinovou) strukturou $\mathbf{L}$. Snadno lze ukázat, že v tomto případě je $\mathbf{L}$-relace $\approx$ $\mathbf{L}$-ekvivalencí na množině všech tyčí. Jelikož podmínky reflexivity a symetrie jsou zřejmé (a jsou splněny bez ohledu na volbu struktury reziduovaného svazu na intervalu $[0, 1]$), věnujme se pouze podmínce tranzitivity. Pro libovolné tři tyče p_1, p_2, p_3 máme

$$(p_1 \approx p_2) \otimes (p_2 \approx p_3) = \min\left(\frac{l(p_1)}{l(p_2)}, \frac{l(p_2)}{l(p_1)}\right) \cdot \min\left(\frac{l(p_2)}{l(p_3)}, \frac{l(p_3)}{l(p_2)}\right) \leq$$
$$\leq \frac{l(p_1)}{l(p_2)} \cdot \frac{l(p_2)}{l(p_3)} = \frac{l(p_1)}{l(p_3)}$$

a podobně

$$(p_1 \approx p_2) \otimes (p_2 \approx p_3) \leq \frac{l(p_2)}{l(p_1)} \cdot \frac{l(p_3)}{l(p_2)} = \frac{l(p_3)}{l(p_1)},$$

což vede k

$$(p_1 \approx p_2) \otimes (p_2 \approx p_3) \leq \min\left(\frac{l(p_1)}{l(p_3)}, \frac{l(p_3)}{l(p_1)}\right) = p_1 \approx p_3$$

a dokazuje tranzitivitu $\mathbf{L}$-relace $\approx$.

Příklad 8. Předpokládejme, že je známa maximální možná délka tyče a že je normovaná na hodnotu 1. Dále předpokládejme, že u fotografie tyčí známe měřítko a jsme z ní tedy schopni zjistit absolutní délku zobrazovaných tyčí. V takovém případě máme jinou (zřejmě přirozenější) možnost stanovení stupně podobnosti tyčí, založenou na rozdílu jejich délek:

$$p_1 \approx p_2 = 1 - |l(p_1) - l(p_2)|. \tag{20}$$

Ukážeme, že zvolíme-li jako množinu pravdivostních hodnot $\mathbf{L}$ interval $[0, 1]$ s Łukasiewiczovou strukturou, bude $\mathbf{L}$-relace $\approx$ $\mathbf{L}$-ekvivalencí na množině všech tyčí.

Podobně jako v předchozím příkladě je podmínka reflexivity a symetrie $\mathbf{L}$-relace $\approx$ splněna triviálně. Ověřme podmínku tranzitivity. Pro libovolné tři tyče p_1, p_2, p_3 máme

$$(p_1 \approx p_2) \otimes (p_2 \approx p_3) = \max\{(p_1 \approx p_2) + (p_2 \approx p_3) - 1, 0\},$$

kde

$$(p_1 \approx p_2) + (p_2 \approx p_3) - 1 = 1 - |l(p_1) - l(p_2)| - |l(p_2) - l(p_3)| \leq$$
$$\leq 1 - |l(p_1) - l(p_3)| = p_1 \approx p_3.$$

Tím je ověřeno, že $\mathbf{L}$-relace $\approx$ je $\mathbf{L}$-ekvivalence na množině všech tyčí.

3 Základní struktury

V této části uvádíme definice základních fuzzy struktur, které používáme v přiložených pracích: **L**-uzávěrových operátorů, **L**-automatů a **L**-konceptuálních svazů. Hlavní důraz klademe na **L**-konceptuální svazy, pro které uvádíme i řadu praktických příkladů. Tyto příklady ukazují, že práce tohoto souboru jsou součástí aktuálního výzkumu, který přináší zajímavé aplikace.

3.1 L-uzávěrové operátory

Uvažme pro libovolnou množinu X zobrazení $C\colon L^X \to L^X$ takové, že

$$A \subseteq C(A), \tag{21}$$

$$S(A_1, A_2) \leq S(C(A_1), C(A_2)), \tag{22}$$

$$C(A) = C(C(A)), \tag{23}$$

pro každé $A, A_1, A_2 \in L^X$. Zobrazení C se nazývá **L**-*uzávěrový operátor (fuzzy uzávěrový operátor)* na množině X [4, 8, 38].

Pevným bodem **L**-uzávěrového operátoru C nazýváme takovou **L**-množinu $A \in L^X$, že

$$C(A) = A. \tag{24}$$

Množinu všech pevných bodů **L**-uzávěrového operátoru C označujeme symbolem $\mathrm{fix}(C)$. Tato množina společně s množinovou inkluzí $\subseteq$ tvoří úplný svaz.

Z (22) a (12) také plyne následující vztah pro podobnost (11) **L**-množin v X:

$$(A_1 \approx A_2) \leq (C(A_1) \approx C(A_2)). \tag{25}$$

Poznámka 4. Pojem **L**-uzávěrového operátoru je zobecněním pojmu uzávěrového operátoru, známého z různých matematických disciplin: pro $L = \{0, 1\}$ tyto dva pojmy splývají.

Příklad 9. Několik příkladů klasických uzávěrových operátorů: uzávěr množiny v topologických (a tedy i metrických) prostorech, lineární obal podmnožiny vektorového prostoru, afinní obal podmnožiny afinního prostoru. Pevnými body těchto uzávěrových operátorů jsou vždy množiny, které hrají v příslušné teorii klíčovou úlohu: uzavřené množiny, vektorové, resp. afinní podprostory.

S dalším příkladem klasického uzávěrového operátoru, důležitým v kontextu této práce, se setkáváme v tzv. formální konceptuální analýze. Podstatná část této práce se týká zobecnění této discipliny, formální konceptuální analýzy s fuzzy hodnotami atributů, kterou uvádíme v odstavci 3.3 a v níž **L**-uzávěrové operátory hrají důležitou úlohu.

Následující příklady přebíráme především z [Práce 7] a připojujeme několik poznámek:

Příklad 10. V Příkladu 2 jsme uvedli, že reziduovaný svaz **L** lze ztotožnit se systémem fuzzy množin v libovolném jednoprvkovém univerzu $X = \{x\}$ tak, že prvek $a \in L$ ztotožníme s **L**-množinou $\{{}^{a}/x\}$. Identické zobrazení $C : L \to L$, tedy $C(a) = a$ je zjevně **L**-uzávěrový operátor na X a platí $\mathrm{fix}(C) = L$.

Příklad 11. Pro libovolnou množinu X a prvek $a \in L$ je zobrazení $C_a : L^X \to L^X$ dané předpisem

$$C_a(A) = a \to a \otimes A \tag{26}$$

(na pravé straně je použit posun a násobek **L**-množiny (13), (14)), **L**-uzávěrovým operátorem na X.

Příklad 12. Pro **L**-ekvivalenci $\approx$ na množině X položme

$$[C_{\approx}(A)](x) = \bigvee_{y \in X} A(y) \otimes (x \approx y). \tag{27}$$

$C_{\approx}$ je **L**-uzávěrový operátor, dobře známý v teorii fuzzy množin. Jeho axiomatické vlastnosti byly studovány v [9]. Množina pevných bodů tohoto **L**-uzávěrového operátoru obsahuje právě **L**-množiny, které jsou tzv. extensionální vzhledem k $\approx$, tj. splňující $A(x) \otimes (x \approx y) \leq A(y)$ („Je-li x prvkem A a x je podobné y, pak je y prvkem A“).

3.2 L-automaty

Z prací tohoto souboru se **L**-automaty zabývá pouze [Práce 1]. Proto zde uvedeme pouze stručnou definici a připojíme několik poznámek.

Podle [8] nazýváme **L**-*automatem (fuzzy automatem)* $\mathcal{M}$ *nad konečnou abecedou* Σ čtveřici $\langle Q, \Sigma, Q_I, Q_F, \delta \rangle$, kde Q je konečná množina stavů, Σ abeceda, $Q_I \subseteq Q$ **L**-množina počátečních stavů, $Q_F \subseteq Q$ **L**-množina koncových stavů a δ ternární **L**-relace, $\delta\colon Q \times \Sigma \times Q \to L$, nazývaná přechodová relace.

Pro stavy $q, q' \in Q$ a symbol $s \in \Sigma$ interpretujeme hodnotu $\delta(q, s, q')$ jako stupeň pravdivosti tvrzení „**L**-automat $\mathcal{M}$ může přejít ze stavu q do stavu q', má-li na vstupu symbol s“.

Pro vstupní slovo $\alpha = s_1 \dots s_n \in \Sigma^*$ klademe

$$\begin{aligned}
\delta(q, \alpha, q') &= \bigvee_{\substack{q_0, \dots, q_n \in Q \\ q_0 = q, q_n = q'}} \delta(q_0, s_1, q_1) \wedge \cdots \wedge \delta(q_{n-1}, s_n, q_n), \\
(\mathcal{L}(\mathcal{M}))(\alpha) &= \bigvee_{q, q' \in Q} Q_I(q) \wedge \delta(q, \alpha, q') \wedge Q_F(q').
\end{aligned}$$

Hodnota $\delta(q, \alpha, q')$ je interpretována jako stupeň, v němž může **L**-automat $\mathcal{M}$ přejít ze stavu q do stavu q', má-li na vstupu slovo α. **L**-množina $\mathcal{L}(\mathcal{M})$ se nazývá **L**-jazyk *přijímaný (rozpoznávaný)* **L**-automatem $\mathcal{M}$.

Poznámka 5. Pojem **L**-automatu je přímočarým a přirozeným zobecněním klasického pojmu nedeterministického konečného automatu (v nějž **L**-automat přejde, položíme-li $L = \{0, 1\}$, tedy přejdeme-li ke klasické logice).

L-jazyky (a tedy i **L**-automaty) mají své opodstatnění v tom, že je v některých situacích přirozené uvažovat jazyky, u nichž stupeň příslušnosti slova nemusí být čistě jen 0 nebo 1. Fuzzy automaty našly aplikace v různých oblastech (různé druhy simulací, rozhodování, rozpoznávání vzoru, zpracování signálu, získávání informací, viz např. [34]).

Podobně jako u klasických automatů i **L**-automaty mají zjednodušenou deterministickou verzi [7]: **L**-automat $\mathcal{M}$ se nazývá *deterministický*, existuje-li stav $q_0 \in Q$ (tzv. *počáteční stav*) takový, že

$$Q_I(q) = \begin{cases} 1 & \text{jestliže } q = q_0, \\ 0 & \text{jinak} \end{cases} \tag{28}$$

a pro každý stav$q_1 \in Q$ a symbol $s \in \Sigma$ existuje stav $q_2 \in Q$ takový, že

$$\delta(q_1, s, q) = \begin{cases} 1 & \text{jestliže } q = q_2, \\ 0 & \text{jinak.} \end{cases} \tag{29}$$

Pro q_1 a q_2 z předchozího vzorce píšeme $q_2 = \delta(q_1, s)$ (vzhledem k jinému počtu argumentů nehrozí záměna s **L**-relací δ). Funkce $\delta : Q \times \Sigma \to Q$ se nazývá *přechodová funkce*. Deterministický **L**-automat $\mathcal{M}$ s počátečním stavem q_0 a přechodovou funkcí δ zapisujeme jako čtveřici $\langle Q, \Sigma, q_0, Q_F, \delta \rangle$.

3.3 Formální konceptuální analýza dat s fuzzy hodnotami atributů

Formální konceptuální analýze dat s fuzzy hodnotami atributů (stručně fuzzy formální konceptuální analýze) je věnována většina prací tohoto souboru.

V této části definujeme souhrnně základní pojmy a uvedeme základní výsledky fuzzy formální konceptuální analýzy, které jsou v jednotlivých pracích uvedeny ve stručnější podobě.

Klasická formální konceptuální analýza (FCA, první práce [43], dále [23]), je jednou z používaných metod dolování dat (data mining). Formální konceptuální analýza dat s fuzzy hodnotami atributů, zavedená v [17] (náš přístup vychází z [36] a [2], standardní reference je [8]), je přirozeným zobecněním této teorie do prostředí fuzzy logiky a fuzzy množin.

V tomto úvodu se nevěnujeme zvlášť klasické FCA, která je speciálním případem pro $\mathbf{L} = \mathbf{2}$.

Základní strukturou, se kterou fuzzy FCA pracuje, je tzv. *formální* **L**-*kontext*. Ten je definován jako trojice $\langle X, Y, I\rangle$ tvořená množinou X (*množina objektů*), množinou Y (*množina atributů*) a **L**-relací $I : X \times Y \to L$ mezi množinami X a Y. Formální **L**-kontext $\langle X, Y, I\rangle$ reprezentuje datovou tabulku, která každému objektu $x \in X$ a atributu $y \in Y$ přiřazuje hodnotu $I(x, y) \in L$. Tuto hodnotu interpretujeme jako stupeň, v němž má objekt x atribut y.

Poznámka 6. V klasické FCA tedy máme pro objekt x a atribut y pouze dvě možnosti: objekt x atribut y buď má nebo nemá.

Pro libovolnou **L**-množinu objektů $A \in L^X$ definujeme **L**-množinu atributů $A^{\uparrow_I} \in L^Y$ předpisem

$$A^{\uparrow_I}(y) = \bigwedge_{x\in X} A(x) \to I(x, y). \tag{30}$$

Podobně pro **L**-množinu $B \in L^Y$ atributů definujeme **L**-množinu $B^{\downarrow_I}$ objektů předpisem

$$B^{\downarrow_I}(x) = \bigwedge_{y\in Y} B(y) \to I(x, y). \tag{31}$$

V případě, že nejsou pochybnosti o volbě **L**-relace I, používáme místo symbolů ${}^{\uparrow_I}$, ${}^{\downarrow_I}$ jednoduše ${}^{\uparrow}$, ${}^{\downarrow}$.

Předpisy (30) a (31) jsou definována zobrazení ${}^{\uparrow} : L^X \to L^Y$ a ${}^{\downarrow} : L^Y \to L^X$. Podle pravidel sémantiky fuzzy logiky je hodnota $A^{\uparrow}(y)$ rovna pravdivostní hodnotě tvrzení „Pro každý objekt x platí: jestliže x je prvkem množiny A, pak má atribut y“. Hodnota $B^{\downarrow}(x)$ je pak rovna pravdivostní hodnotě tvrzení „Pro každý atribut y platí: jestliže y je prvkem množiny B, pak je atributem objektu x“. V případě klasické logiky ($\mathbf{L} = \mathbf{2}$) je tedy množina $A^{\uparrow}$ rovna (klasické) množině všech atributů sdílených všemi objekty z množiny A a, podobně, množina $B^{\downarrow}$ je rovna množině všech objektů majících všechny atributy z B.

Zobrazení $^{\uparrow}$ a $^{\downarrow}$ splňují následující podmínky:

$$S(A_1, A_2) \le S(A_2^{\uparrow}, A_1^{\uparrow}), \qquad S(B_1, B_2) \le S(B_2^{\downarrow}, B_1^{\downarrow}), \tag{32}$$
$$A \subseteq A^{\uparrow\downarrow}, \qquad B \subseteq B^{\downarrow\uparrow}, \tag{33}$$

pro každé $A_1, A_2, A \in L^X$, $B, B_1, B_2 \in L^Y$.

Poznámka 7. Podmínky (32) a (32) znamenají, že dvojice $\langle^{\uparrow}, ^{\downarrow}\rangle$ tvoří tzv. $\mathbf{L}$-*Galoisovu konexi mezi množinami* X, Y (viz [8]).

Pokud v podmínce (32) předpokládáme, že levá strana nerovnosti je rovna 1, dostaneme následující speciální případ:

$$\text{Jestliže } A_1 \subseteq A_2, \text{ pak } A_2^{\uparrow} \subseteq A_1^{\uparrow}. \text{ Jestliže } B_1 \subseteq B_2, \text{ pak } B_2^{\downarrow} \subseteq B_1^{\downarrow}. \tag{34}$$

Tento vztah má srozumitelný význam: Zvětšíme-li množinu objektů, zmenší se množina atributů těmto objektům společných. Zvětšíme-li množinu atributů, zmenší se množina objektů tyto atributy sdílejících.

Z podmínek (32) a (33) také snadno plyne

$$S(A_1, A_2) \le S(A_1^{\uparrow\downarrow}, A_2^{\uparrow\downarrow}), \qquad S(B_1, B_2) \le S(B_1^{\downarrow\uparrow}, B_2^{\downarrow\uparrow}), \tag{35}$$
$$A^{\uparrow\downarrow\uparrow} = A^{\uparrow}, \qquad B^{\downarrow\uparrow\downarrow} = B^{\downarrow}, \tag{36}$$

což společně s (33) znamená, že zobrazení $^{\uparrow\downarrow}$ a $^{\downarrow\uparrow}$ jsou uzávěrové operátory.

Poznámka 8. Tato skutečnost je důležitá pro studium vlastností níže definovaných formálních $\mathbf{L}$-konceptů a $\mathbf{L}$-konceptuálních svazů. Díky ní lze také obecné výsledky z [Práce 6, 7], které se týkají $\mathbf{L}$-uzávěrových operátorů, aplikovat na formální $\mathbf{L}$-konceptuální svazy.

Dvojice $\langle A, B\rangle \in L^X \times L^Y$ se nazývá *formální* $\mathbf{L}$-*koncept*, jestliže platí $A^{\uparrow} = B$ a $B^{\downarrow} = A$. $\mathbf{L}$-množina A ve formálním konceptu $\langle A, B\rangle$ se nazývá *extent*, $\mathbf{L}$-množina B *intent*.

Poznámka 9. Definice formálního konceptu vystihuje důležitý aspekt lidského myšlení, který byl zaznamenán v Port-Royalské logice [1] a který souvisí s tím, jak myšlení pracuje s pojmy. Každý pojem totiž znamená jednak souhrn předmětů, které popisuje (objekty), a jednak souhrn vlastností, kterými je popisuje (atributy).

Označme $\mathrm{Ext}(X, Y, I)$, resp. $\mathrm{Int}(X, Y, I)$ množinu všech extentů, resp. intentů formálního kontextu $\langle X, Y, I\rangle$.

Poznámka 10. Z definice formálního konceptu plyne, že $\mathrm{Ext}(X,Y,I)$ je množinou všech pevných bodů uzávěrového operátoru $^{\uparrow\downarrow}$ a $\mathrm{Int}(X,Y,I)$ je množinou všech pevných bodů uzávěrového operátoru $^{\downarrow\uparrow}$.

Z (36) také plyne, že pro libovolnou $\mathbf{L}$-množinu $A \in L^X$ platí, že $A^{\uparrow}$ je intent s extentem $A^{\uparrow\downarrow}$ a pro libovolné $B \in L^Y$ platí, že $B^{\downarrow}$ je extent s intentem $B^{\downarrow\uparrow}$.

Formální konceptuální analýza se zabývá vyhledáváním a popisem všech formálních konceptů daného formálního kontextu.

Označme $\mathcal{B}(X,Y,I)$ množinu všech formálních konceptů formálního $\mathbf{L}$-kontextu $\langle X,Y,I\rangle$. Platí tedy

$$\mathcal{B}(X,Y,I) = \left\{\langle A,B\rangle \in L^X \times L^Y \mid A^{\uparrow} = B, B^{\downarrow} = A\right\}. \tag{37}$$

Na množině $\mathcal{B}(X,Y,I)$ zavádíme částečné uspořádání pomocí množinového uspořádání extentů. Klademe

$$\langle A_1,B_1\rangle \leq \langle A_2,B_2\rangle \quad \text{právě když} \quad A_1 \subseteq A_2 \tag{38}$$

(z (34) plyne, že tato podmínka je ekvivalentní podmínce $B_2 \subseteq B_1$). Z této definice také plyne, že uspořádaná množina $\mathcal{B}(X,Y,I)$ je izomorfní množině $\mathrm{Ext}(X,Y,I)$ s uspořádáním daným inkluzí $\mathbf{L}$-množin $\subseteq$ a duálně izomorfní množině $\mathrm{Int}(X,Y,I)$ s uspořádáním rovněž daným inkluzí $\subseteq$.

První část tzv. Hlavní věty $\mathbf{L}$-konceptuálních svazů (v klasickém případě dokázána v [43], pro formální $\mathbf{L}$-koncepty v [36, 5, 6]) říká, že množina $\mathcal{B}(X,Y,I)$ s uspořádáním $\leq$ je úplný svaz. Tento úplný svaz se nazývá $\mathbf{L}$-*konceptuální svaz indukovaný formálním kontextem* $\langle X,Y,I\rangle$.

Poznámka 11. Podmínku $\langle A_1,B_1\rangle \leq \langle A_2,B_2\rangle$ lze chápat tak, že formální koncept $\langle A_2,B_2\rangle$ je zobecněním formálního konceptu $\langle A_1,B_1\rangle$ a naopak, formální koncept $\langle A_1,B_1\rangle$ je specializací formálního konceptu $\langle A_2,B_2\rangle$.

Hlavní věta konceptuálních svazů pak říká, že každá množina formálních konceptů má přímé zobecnění (supremum) a přímou specializaci (infimum).

Pomocí $\mathbf{L}$-rovnosti $\approx$ definované pro libovolné univerzum X na systému $\mathbf{L}$-množin L^X (11) lze definovat $\mathbf{L}$-rovnost na $\mathbf{L}$-konceptuálním svazu tak, že pro každé $\langle A_1,B_1\rangle, \langle A_2,B_2\rangle \in \mathcal{B}(X,Y,I)$ položíme

$$\langle A_1,B_1\rangle \approx \langle A_2,B_2\rangle = A_1 \approx A_2 \quad (= B_1 \approx B_2). \tag{39}$$

Poznámka 12. $\mathbf{L}$-rovnost $\approx$ na $\mathbf{L}$-konceptuálním svazu $\mathcal{B}(X,Y,I)$ (a speciálně její a-řez ${}^a\!\approx$) hraje zásadní roli při faktorizaci konceptuálních svazů, která je tématem několika prací tohoto souboru.

3.4 Aplikace formální konceptuální analýzy

K základním aplikacím formální konceptuální analýzy patří vizualizace dat. Pomocí FCA lze nepřehlednou datovou tabulku převést na úplný svaz, který můžeme uživateli zobrazit pomocí Hasseova diagramu. Podle zkušeností z praxe se uživatelé v Hasseových diagramech snadno orientují a jsou schopni v nich vyhledávat různé dosud neznámé souvislosti v datech. Vizualizaci dat konceptuálním svazem lze ovšem použít pouze na nepříliš velké kolekce dat.

Poznámka 13. Druhá část Hlavní věty o konceptuálních svazech říká, že pokud v konceptuálním svazu vyznačíme několik důležitých konceptů (tzv. objektové a atributové koncepty), získáme dostatek informací k tomu, abychom z něj mohli zpětně rekonstruovat původní formální kontext. V tomto smyslu tedy konceptuální svaz nese úplnou informaci o původních datech.

Nejprve se v několika příkladech zmíníme o aplikacích klasické formální konceptuální analýzy.

FCA umožňuje uživateli zkoumat tabulková data v podobě snadno pochopitelného a přehledného Hasseova diagramu. Uvedeme jeden příklad aplikace tohoto typu.

Příklad 13. V práci [31] se autoři zabývají analýzou zdrojových kódů starých programů (napsaných v jazycích FORTRAN a COBOL), které používají velké množství globálních proměnných. Cílem bylo pokusit se zjednodušit strukturu těchto programů pomocí FCA. Množinou objektů X stanovili autoři množinu modulů (tj. jednotlivých zdrojových souborů), množinou atributů Y množinu globálních proměnných. Relace I zachycuje, které moduly pracují se kterými proměnnými.

V práci je uvedeno několik metod (heuristických i automatických), jak modifikovat vzniklý konceptuální svaz tak, aby se to promítlo do zjednodušení provázanosti jednotlivých modulů programu (ideálním případem je dosáhnout tzv. *tree-like lattice*, tj. svazu, jehož část vzniklá odebráním nejmenšího prvku je strom).

Základem těchto metod je zobrazení konceptuálního svazu. Metody nemusí být účinné, pokud je původní program příliš veliký a obsahuje příliš mnoho závislostí svých jednotlivých částí, tedy v případě, kdy je konceptuální svaz příliš veliký a složitý.

V následujícím příkladě ukážeme, že FCA lze použít i v situaci, kdy není možné nebo vhodné zobrazit celý konceptuální svaz v jednom obrázku. Jedná se o aplikaci z oblasti získávání informací (Information Retrieval), v níž upřesňujeme vyhledávací dotaz na základě nabídky, která je zkonstruována z informace o sousedech aktuálního prvku konceptuálního svazu.

Příklad 14. Nechť X je množina webových stránek, Y množina klíčových slov, $I(x, y)$ je pravdivostní hodnota tvrzení „Stránka x obsahuje klíčové slovo y“. Uvažme formální koncept, jehož intent B je roven množině {`"jaguar"`}. Extent tohoto formálního konceptu odpovídá webovým stránkám, které obsahují dané slovo. Jelikož v našem případě je hledaný výraz víceznačný, je vhodné vyhledávací dotaz upřesnit. Inteligentní vyhledávací program může podle slov obsažených v nalezených stránkách uživateli nabídnout vhodná upřesňující klíčová slova tak, že najde nejvýznamnější dolní sousedy aktuálního formálního konceptu. V našem případě se mezi nabídnutými slovy objeví slova `"cars"`, `"dealers"`, `"models"`, ale i `"cat"` a `"panthera onca"`. Pokud si uživatel některé z nabídnutých klíčových slov vybere, vyhledávač výsledek hledání upřesní.

Uvedený systém byl popsán v práci [19]. Na webové stránce

`http://credo.fub.it/`

je k dispozici experimentální implementace vyhledávače.

Některé další příklady použití FCA lze nalézt v knize [18], nejnovější aplikace pak ve sbornících konferencí ICFCA a CLA ([12], [22] jsou sborníky z posledních ročníků těchto konferencí). Existuje také několik softwarových nástrojů (včetně komerčních), které využívají FCA.

V následujících příkladech uvádíme několik aplikací formální konceptuální analýzy dat s fuzzy hodnotami atributů, zejména těch, které vzešly z práce výzkumného týmu pracoviště autora tohoto textu.

Příklad 15. Faktorová analýza (viz např. [26]) je matematická disciplína, která se zabývá možnostmi řešení následující úlohy: k dané číselné matici I nalézt matice A a B tak, aby matice A měla co nejmenší počet sloupců a aby platilo $I \approx A \cdot B$. Matice A (jejíž sloupce se nazývají *faktory*) pak může přibližně nahradit původní matici I v tom smyslu, že sloupce matice I lze přibližně vyjádřit jako lineární kombinace sloupců matice A.

Původní motivace faktorové analýzy pochází z experimentální psychologie. První práce o faktorové analýze [40] z roku 1904, se zabývala hledáním obecných faktorů lidské inteligence na základě většího množství empiricky zjištěných dat.

Binární faktorová analýza řeší analogický problém pro binární matice, její zobecnění pak pro matice s hodnotami v obecném reziduovaném svazu. V pracích [16, 10] je dokázáno, že v obou příkladech lze matice A a B zkonstruovat pomocí vybraných formálních konceptů matice I.

Příklad 16. Fuzzy FCA byla použita k vyhodnocování dotazníků IPAQ (*International Physical Activity Questionnaire*), zaměřených na zjišťování pohybové aktivity populace [14]. V těchto dotaznících odpovídají respondenti na otázky týkající se nejen přímo pohybové aktivity, ale i dalších životních podmínek (věk, pohlaví, zda mají práci, jak často se dívají na televizi, zda kouří apod.). Data jsou předzpracována tzv. škálováním a agregací objektů. Výsledkem je pak formální **L**-kontext s šesti agregovanými objekty (muži/ženy s nízkou/střední/vysokou pohybovou aktivitou) a 42 atributy. Zkoumáním formálních konceptů tohoto formálního kontextu pak lze nalézt odpovědi na otázky o společných vlastnostech lidí ze stejné skupiny, jako například: z mužů a žen s vysokou pohybovou aktivitou 81% vlastní jízdní kolo, 65% má zaměstnání, 21% kouří, 52% hodně chodí, 4% mají vysoký BMI (*Body Mass Index*), 1% je obézních.

Výhodou použití FCA proti obvyklým statistickým metodám v tomto případě je, že není nutno dopředu formulovat hypotézy, které se pak budou testovat.

Příklad 17. V práci [32] autoři jako příklad použili fuzzy FCA (v zobecněné podobě) k výběru časopisu vhodného k publikaci článku. Množinou objektů byla množina časopisů, které připadaly v úvahu, jako atributy byly zvoleny základní atributy časopisu podle citačního indexu (impakt faktor, poločas citovanosti atd.). Hledaný časopis vyšel jako objekt s největším stupněm příslušnosti v množině $B^{\downarrow}$, kde B je fuzzy množina atributů, charakterizující požadavky na hledaný časopis.

Poznámka 14. Na Katedře informatiky PřF UP v Olomouci vzniklo v rámci bakalářských a diplomových prací několik softwarových produktů, pracujících na bázi fuzzy FCA. Software, který je součástí práce [37], byl několik let nasazen v praxi a využíván.

4 Hlavní výsledky

V této části popíšeme hlavní výsledky prací tohoto souboru.

4.1 Problémy řešené metodou aproximace a faktorizace

Cílem mnoha úloh řešených ve fuzzy logice je nalézt fuzzy množinu nebo systém fuzzy množin v daném univerzu, splňující nějaké dopředu zadané požadavky (omezení). Základní úlohy, které jsou spojeny s pojmy uvedenými v předchozí části a které jsou studovány v pracích tohoto souboru, jsou tohoto typu:

- nalézt **L**-jazyk přijímaný daným konečným **L**-automatem,

- nalézt konceptuální svaz daného formálního **L**-kontextu,
- nalézt všechny pevné body daného **L**-uzávěrového operátoru.

Podívejme se podrobněji na uvedené úlohy. V případě **L**-automatu $\mathcal{M}$ se vstupní abecedou Σ je jazyk přijímaný tímto **L**-automatem **L**-množinou v univerzu Σ^*. Cílem první úlohy je tedy nalézt jednu **L**-množinu, která má ovšem sama složitější strukturu: jejími prvky jsou množiny (slova nad abecedou Σ).

Konceptuální svaz $\mathcal{B}(X, Y, I)$ je množinou všech formálních konceptů daného formálního kontextu $\langle X, Y, I \rangle$. Každý formální koncept je ovšem uspořádanou dvojicí extentu a intentu. Všechny formální koncepty tedy najdeme tak, že najdeme všechny extenty (nebo intenty). Ve druhé úloze tedy hledáme množinu všech extentů, tedy množinu všech **L**-množin $A \in L^X$ splňujících omezení $A^{\uparrow\downarrow} = A$ (nebo, ekvivalentně, množinu všech intentů což je množina **L**-množin $B \in L^Y$ splňujících $B^{\downarrow\uparrow} = B$).

Jelikož zobrazení ${}^{\uparrow\downarrow}$ je **L**-uzávěrovým operátorem na množině X, je třetí úloha zobecněním úlohy druhé. Pevné body **L**-uzávěrového operátoru na množině X jsou **L**-množiny v X, cílem třetí úlohy je tedy opět nalézt množinu **L**-množin v X, splňujících jisté dané podmínky.

Jedním z důsledků použití fuzzy logiky ve druhé a třetí úloze je, že množiny, které jsou jejich řešením, mohou být značně rozsáhlé. Velikost (počet prvků) těchto množin obecně závisí na velikosti použitého reziduovaného svazu, případně na konkrétní volbě jeho prvků.

U **L**-automatu (první úloha) vzniklého z externích (naměřených) dat můžeme zase narazit na problém, že jeho velikost bude podstatně záviset na zvolené logické přesnosti těchto dat (tedy na zvoleném reziduovaném svazu, jehož hodnoty data používají), případně na počtu stupňů pravdivosti v automatu použitých.

Na všechny tyto úlohy je tedy možné pokusit se aplikovat metodu aproximace a faktorizace, jejíž princip je popsán v Úvodu. Vznikají následující konkrétní problémy, které jsou řešeny v pracích tohoto souboru:

- K zadanému **L**-automatu nalézt minimální **L**-automat, který přijímá podobný jazyk [Práce 1].
- Prokoumat možnost aproximace daného formálního **L**-kontextu formálním kontextem s daty ze zvoleného menšího reziduovaného svazu $\mathbf{L}' \subseteq \mathbf{L}$ a odhadnout vzniklou chybu ve výsledném konceptuálním svazu [Práce 2].
- K zadanému **L**-konceptuálnímu svazu nalézt svaz o menším počtu prvků, který bude původní svaz dobře aproximovat [Práce 4, 5].

- K zadanému formálnímu kontextu najít podobný formální kontext takový, že výsledný konceptuální svaz bude dobře aproximovat původní a bude mít menší počet prvků [Práce 4, 5].

- K zadanému uzávěrovému operátoru nalézt množinu, která bude dobře aproximovat množinu jeho pevných bodů a bude mít menší počet prvků [Práce 6, 7].

Práce tohoto souboru se také zabývají dalšími problémy, které v souvislosti s řešením uvedených problémů vznikají (zejména aproximace a faktorizace v reziduovaných svazech [Práce 3, 4, 5]).

4.2 Přibližná minimalizace fuzzy automatů

Problémem přibližné minimalizace **L**-automatů se zabývá [Práce 1]. Pokud je známo, jedná se o první příspěvek, v němž je podmínka rovnosti jazyků přijímaných původním a minimalizovaným automatem nahrazena podmínkou podobnosti těchto jazyků v uživatelem předepsaném stupni.

Formulace problému je následující: K danému **L**-automatu $\mathcal{M}$ a stupni $a \in L$ nalézt co nejmenší **L**-automat $\mathcal{M}'$ takový, aby pro jazyky přijímané těmito automaty platilo

$$a \leq \mathcal{L}(\mathcal{M}) \approx \mathcal{L}(\mathcal{M}') \tag{40}$$

(relace $\approx$ je relace podobnosti **L**-množin (11)).

Poznámka 15. Prvotní motivace k řešení tohoto problému pocházela z počítačové grafiky. V práci [20] autoři navrhli novou metodu komprese bitmapového obrazu pomocí tzv. vážených automatů. Princip metody spočívá v tom, že se komprimovaná bitmapa popíše jazykem, k němuž se sestrojí vážený automat, který se poté minimalizuje. Vážené automaty představují zvláštní druh fuzzy automatů. Jejich přibližná minimalizace by odpovídala ztrátové kompresi obrazu (viz též podobnost barev v Příkladu 5).

Metoda se zatím ukazuje jako neperspektivní kvůli vysoké výpočetní náročnosti. Problematika je přehledně shrnuta v bakalářské práci [41], která obsahuje i softwarovou implementaci metody.

Přibližná minimalizace fuzzy automatů ovšem může najít uplatnění v jiných oblastech, kde se fuzzy automaty využívají.

Hlavní výsledky z [Práce 1] jsou tyto: je definován tzv. *stupeň dostupnosti* stavů **L**-automatu $\mathcal{M}$ a je dokázáno, že pokud **L**-automat $\mathcal{M}'$ vznikne z **L**-automatu $\mathcal{M}$ vypuštěním všech stavů, jejichž stupeň dostupnosti je menší nebo roven $\neg a$, pak je podmínka (40) splněna. Dále je pro dvojici deterministických **L**-automatů $\mathcal{M}$ a $\mathcal{M}'$ formulována nutná a dostatečná podmínka pro to, aby $\mathcal{M}'$ byl minimálním automatem splňujícím (40).

Poznamenejme, že podobně jako u klasických automatů je ke konstrukci minimálního automatu použita relace nerozlišitelnosti stavů automatu. V případě **L**-automatů se ovšem jedná o **L**-ekvivalenci na množině stavů, jejíž a-řez je relace tolerance. Při minimalizaci deterministických **L**-automatů se v práci používá pokrytí množiny stavů automatu jistými bloky této tolerance. Je tedy řešen speciální případ problému faktorizace množiny podle relace tolerance.

4.3 Matematické základy: aproximace a faktorizace v reziduovaných svazech

Základní motivací pro [Práce 3, 4] je, že pro studium možnosti redukce velikosti fuzzy systémů, případně množství dat jimi generovaných, pomocí **L**-relace podobnosti bude užitečné zabývat se **L**-relací podobností na podkladové struktuře stupňů pravdivosti samotné, tedy na reziduovaném svazu **L**.

Roli **L**-relace podobnosti na reziduovaném svazu **L** hraje operace birezidua $\leftrightarrow$, která je interpretací logické spojky ekvivalence. Uvedené práce se zabývají jejím e-řezem (pro pevně zvolené $e \in L$), tedy relací $\approx_e$ definovanou předpisem

$$a \approx_e b \quad \text{právě když} \quad e \leq a \leftrightarrow b \tag{41}$$

(relace $\approx_e$ je v [Práce 4] označována $\sim_e$, setkáme se též s označením ${}^e\!\!\leftrightarrow$, které je v souladu se symbolikou z (15)).

Problém aproximace v reziduovaných svazech lze obecně zformulovat takto: *Nalezněte dostatečně malou množinu stupňů pravdivosti K, která dobře aproximuje zadanou množinu stupňů pravdivosti M.*

Poznámka 16. S uvedeným problémem se můžeme setkat v následující situaci: Je-li $A\colon U \to L$ fuzzy množina v univerzu U, pak obraz $M = \{A(u) \mid u \in U\}$ lze chápat jako množinu stupňů pravdivosti „použitých v A.“ Úkolem je nalézt fuzzy množinu $B : U \to L$, která je dobrou aproximací množiny A a pro kterou je množina $K = \{B(u) \mid u \in U\}$ stupňů pravdivosti „použitých v B“ dostatečně malá.

Obecně řečeno, výhodou B proti A je jednoduchost. Množinu B může být například snazší interpretovat uživateli. Podle známého Millerova fenoménu [33] je pro člověka obtížné odlišit a konzistentně interpretovat více než 7 ± 2 hodnot dané škály (proměnné). Jestliže tedy A představuje stupně, v nichž předměty (např. zboží) splňují daná kritéria, mohlo by být lepší zobrazit uživateli množinu B místo A.

Následující definice byla formulována v [Práce 2] a upřesněna v [Práce 3]: Mějme dvě množiny $M, K \subseteq L$. *Stupeň* $\mathrm{app}(M, K)$*, v němž množina* K *aproximuje množinu* M, je definován vztahem

$$\mathrm{app}(M, K) = \bigwedge_{a \in M} \bigvee_{b \in K} (a \leftrightarrow b). \tag{42}$$

Jde tedy o stupeň pravdivosti formule „pro každé $a \in M$ existuje $b \in K$ takové, že a a b jsou podobné (blízké)".

V práci [Práce 3] jsou formulovány následující dvě úlohy aproximace v reziduovaných svazech.

Úloha 1. Najděte k zadané (konečné) množině $M \subseteq L$ a prahu $e \in L$ množinu K takovou, že

1. K aproximuje M alespoň ve stupni e, tedy

$$\mathrm{app}(M, K) \geq e, \tag{43}$$

2. neexistuje K' s $|K'| \leq |K|$, pro které $\mathrm{app}(M, K') > e$, tj. K je nejmenší (vzhledem k počtu prvků) množina, která splňuje (43).

Úloha 2. Najděte k zadané (konečné) množině $M \subseteq L$ a prahu $e \in L$ množinu K, která je řešením Úlohy 1 a navíc

3. Pro každé K' s $|K'| = |K|$ platí

$$\mathrm{app}(M, K) \geq \mathrm{app}(M, K'), \tag{44}$$

tj. mezi množinami s $|K|$ prvky K aproximuje množinu M nejlépe.

V [Práce 3] jsou zavedeny některé teoretické pojmy potřebné pro řešení problému aproximace v reziduovaných svazech (středová množina, středový bod, optimální středový bod) a dokázány některé jejich vlastnosti. Na základě těchto teoretických poznatků jsou uvedeny dva algoritmy, které řeší Úlohu 1 v případě lineárně uspořádaných reziduovaných svazů.

Možnou výhradou vůči uvedenému přístupu je, že množina K obecně není podalgebrou reziduovaného svazu $\mathbf{L}$ a že tedy ve skutečnosti není možné se pro další práci s jejími prvky omezit pouze na tuto množinu. Jedním z řešení tohoto problému je hledat množinu K pouze mezi podalgebrami reziduovaného svazu $\mathbf{L}$. Tato možnost byla rozpracována v [Práce 2].

Jinou možností je použít místo aproximace metodu faktorizace. Uvedeme základní principy jejího použití na reziduované svazy.

Relace $\approx_e$ (41) je relace tolerance na množině L. Hlavním výsledkem [Práce 4] je, zavedení struktury reziduovaného svazu na faktormnožině $L/{\approx_e}$ (označované také L/e). Prvky tohoto nového reziduovaného svazu $\mathbf{L}/{\approx_e}$ (označovaného také jednoduše $\mathbf{L}/e$) jsou maximální bloky tolerance $\approx_e$.

Poznámka 17. Definici operací reziduovaného svazu na množině L/e zde neuvádíme, čtenář ji najde v [Práce 4].

Poznámka 18. Reziduovaný svaz $\mathbf{L}/e$ lze chápat takto: uživatel zvolí hodnotu $e \in L$, která indikuje práh takový, že stupně pravdivosti, jejichž stupeň podobnosti je větší nebo roven e bude považovat za nerozlišitelné. Práh e tedy určuje maximální přípustnou chybu při změně stupně pravdivosti z $\mathbf{L}$. Prvky reziduovaného svazu $\mathbf{L}/e$ jsou množiny, které sdružují stupně pravdivosti z $\mathbf{L}$, jež je uživatel ochoten považovat za nerozlišitelné.

Přechodem od reziduovaného svazu $\mathbf{L}$ k $\mathbf{L}/e$ tedy dochází k *redukci logické přesnosti* [8].

Příklad 18. Nechť $\mathbf{L}_{10}$ je 11-prvkový Łukasiewiczův řetězec $L = \{0, 0.1, \ldots, 1\}$, $e = 0.6$. Pak $L/e = \{[0, 0.4], [0.1, 0.5], \ldots, [0.6, 1]\}$. Reziduovaný svaz $\mathbf{L}/e$ je izomorfní se sedmiprvkovým Łukasiewiczovým řetězcem $\mathbf{L}_6$.

Poznámka 19. Z předchozího příkladu jsou zřejmé i obtíže, se kterými je nutno při faktorizaci reziduovaných svazů počítat: 1. Prvky původního reziduovaného svazu $\mathbf{L}$ mohou náležet více (obvykle mnoha) blokům tolerance $\approx_e$ současně, 2. pro podstatnou redukci velikosti reziduovaného svazu je nutné přikročit ke značně velkorysé hodnotě prahu e.

I přes tyto výhrady lze pomocí faktorizace dojít k výrazným výsledkům (viz např. [Práce 4, Odstavec 4.4]).

[Práce 4] obsahuje podrobnou analýzu struktury faktorizovaného reziduovaného svazu $\mathbf{L}/e$. V [Práce 5] je tato problematika rozšířena na reziduované svazy s tzv. *zesilovači pravdivosti (Truth Stresser, Hedge)* [29].

4.4 Faktorizace v konceptuálních svazech

Problémy, o nichž hovoří odstavec 4.1, lze velmi dobře demonstrovat na příkladě fuzzy konceptuálních svazů. Uvažme následující příklad.

Příklad 19. Nechť $\mathbf{L}_2$ je tříprvkový Łukasiewiczův řetězec (tedy $L_2 = \{0, 0.5, 1\}$) a položme $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$. Dále definujme $\mathbf{L}_2$-relaci I tabulkou

I	y_1	y_2
x_1	0.5	1.0
x_2	0.0	0.0

Množina $\mathcal{B}_{\mathbf{L}_2}$ všech formálních $\mathbf{L}_2$-konceptů formálního $\mathbf{L}_2$-kontextu $\langle X, Y, I\rangle$ je tvořena následujícími čtyřmi $\mathbf{L}_2$-koncepty:

$$\langle\{x_1, x_2\}, \emptyset\rangle, \qquad \langle\{x_1, {}^{0.5}/x_2\}, \{{}^{0.5}/y_1, {}^{0.5}/y_2\}\rangle,$$
$$\langle\{x_1\}, \{{}^{0.5}/y_1, y_2\}\rangle, \qquad \langle\{{}^{0.5}/x_1\}, \{y_1, y_2\}\rangle.$$

Zvolíme-li místo reziduovaného svazu $\mathbf{L}_2$ pětiprvkový Łukasiewiczův řetězec $\mathbf{L}_4$ (tedy $L_4 = \{0, 0.25, 0.5,\ 0.75, 1\}$) a ostatní parametry zadání necháme stejné (až na triviální změnu oboru hodnot relace I), přibydou k výše uvedeným čtyřem formálním konceptům další čtyři:

$$\langle\{x_1, {}^{0.75}/x_2\}, \{{}^{0.25}/y_1, {}^{0.25}/y_2\}\rangle, \qquad \langle\{x_1, {}^{0.25}/x_2\}, \{{}^{0.5}/y_1, {}^{0.75}/y_2\}\rangle,$$
$$\langle\{{}^{0.75}/x_1, {}^{0.25}/x_2\}, \{{}^{0.75}/y_1, {}^{0.75}/y_2\}\rangle, \qquad \langle\{{}^{0.75}/x_1\}, \{{}^{0.75}/y_1, y_2\}\rangle.$$

Množina $\mathcal{B}_{\mathbf{L}_4}$ všech formálních $\mathbf{L}_4$-konceptů formálního $\mathbf{L}_4$-kontextu $\langle X, Y, I\rangle$ má tedy osm prvků.

Pokud bychom zvolili za $\mathbf{L}$ Łukasiewiczův řetězec s ještě větším (lichým) počtem prvků, bude výsledný konceptuální svaz opět větší. Počet prvků konceptuálního svazu v tomto případě závisí na počtu prvků Łukasiewiczova řetězce $\mathbf{L}$ kvadraticky. Platí totiž, že extentem je libovolná $\mathbf{L}$-množina $A \in L^X$, splňující podmínku:

$$A(x_1) = 1 \quad \text{nebo} \quad A(x_2) \leq A(x_1) - 0.5$$

a intentem libovolná $\mathbf{L}$-množina $B \in L^Y$, která splňuje

$$B(y_1) = B(y_2) \quad \text{nebo} \quad 0.5 \leq B(y_1) \leq B(y_2).$$

V krajním případě, pokud zvolíme za L celý interval $[0, 1]$, bude výsledný konceptuální svaz nekonečný.

V práci [3] (viz též [8, 11]) je navržena metoda redukce počtu prvků konceptuálního svazu pomocí faktorizace. Tato metoda využívá $\mathbf{L}$-relaci $\approx$, definovanou na libovolném $\mathbf{L}$-konceptuálním svazu $\mathcal{B}(X, Y, I)$ vztahem (39). Relace $\approx$ je $\mathbf{L}$-rovností na $\mathcal{B}(X, Y, I)$, pro libovolné dva $\mathbf{L}$-koncepty

$$\langle A_1, B_1\rangle, \langle A_2, B_2\rangle \in \mathcal{B}(X, Y, I)$$

hodnota $\langle A_1, B_1\rangle \approx \langle A_2, B_2\rangle$ vyjadřuje jejich „podobnost“ („blízkost“). Pro libovolné $a \in L$ je nyní řez ${}^a\!\approx$ relací tolerance (viz odstavec 2.4).

V práci [3] je ukázáno, že tato relace tolerance je kompatibilní se strukturou úplného svazu na $\mathcal{B}(X, Y, I)$. Podle výsledků [23] (které navazují na [21]) tedy na faktormnožině $\mathcal{B}(X, Y, I)/{}^a\!\approx$ existuje struktura úplného svazu.

Nepotřebujeme-li tedy rozlišovat mezi $\mathbf{L}$-koncepty, jejichž stupeň podobnosti je alespoň a (neboli, jsme-li ochotni „zanedbat malé rozdíly mezi koncepty“), můžeme přejít od jednotlivých $\mathbf{L}$-konceptů k množinám takto podobných $\mathbf{L}$-konceptů, tj. přejít od konceptuálního svazu $\mathcal{B}(X, Y, I)$ ke svazu $\mathcal{B}(X, Y, I)/{}^a\!\approx$.

V [Práce 4, Theorem 14] je dokázáno, že svaz $\mathcal{B}(X,Y,I)/^a\!\approx$ je konceptuálním svazem formálního kontextu, který lze snadno odvodit z původního **L**-kontextu $\langle X,Y,I\rangle$. Přesněji řečeno, svaz $\mathcal{B}(X,Y,I)/^a\!\approx$ je izomorfní $\mathbf{L}/a$-konceptuálnímu svazu $\mathcal{B}(X,Y,[I]^a)$ (definici $\mathbf{L}/a$-relace $[I]^a\colon X\times Y\to\mathbf{L}/a$ čtenář nalezne v [Práce 4]).

Poznámka 20. Pomocí tohoto výsledku lze tedy nalézt faktorizovaný **L**-konceptuální svaz tím, že se nalezne konceptuální svaz modifikovaných vstupních dat.

Poznámka 21. Tento výsledek je zajímavý tím, že převádí faktorizaci jisté struktury nad množinou stupňů pravdivosti **L** (**L**-konceptuálního svazu) podle tolerance dané stupni podobnosti jejích prvků na faktorizaci samotné množiny stupňů pravdivosti **L**.

Poznámka 22. Podle [13] je svaz $\mathcal{B}(X,Y,I)/a$ také izomorfní **L**-konceptuálnímu svazu $\mathcal{B}(X,\,Y,a\to I)$ (kde $a\to I$ je posun **L**-množiny I (13)). Tyto dva výsledky jsou velmi příbuzné, malá přednost našeho výsledku spočívá v tom, že reziduovaný svaz $\mathbf{L}/a$ má méně prvků, než reziduovaný svaz **L**, což znamená menší nároky na uložení vstupních dat a konstrukci konceptuálního svazu.

[Práce 5] zobecňuje tyto výsledky na tzv. *konceptuální svazy se zesilovači pravdivosti (hedges)* [15].

4.5 Zobecnění na množiny pevných bodů uzávěrových operátorů

V [Práce 6, 7] je problematika prací [Práce 3, 4] zobecněna na množiny pevných bodů uzávěrových operátorů.

V těchto pracích se uvažuje množina (univerzum) X s uzávěrovým operátorem $C : L^X \to L^X$ a zkoumají se možnosti snížení velikosti množiny $\mathrm{fix}(C)$ pevných bodů tohoto uzávěrového operátoru pomocí aproximace a faktorizace.

Poznámka 23. Příklad 10 ukazuje, že se skutečně jedná o zobecnění.

V [Práce 6] jsou uvedeny zobecněné verze výsledků první části [Práce 3]. Jsou definovány pojmy středového bodu, středové množiny a optimálního středového bodu v množině $\mathrm{fix}(C)$, které lze použít k aproximaci prvků této množiny.

V práci [Práce 7] je zkoumána faktormnožina $\mathrm{fix}(C)/^a\!\approx$, kde $^a\!\approx$ je a-řez (pro zvolené $a\in L$) podobnosti **L**-množin, definované předpisem (11). Je ukázáno, že tato množina má strukturu úplného svazu, a je také formulován a dokázán efektivní způsob nalezení všech jejích prvků. Maximálními bloky tolerance ${}^{a}{\approx}$, které jsou prvky této faktormnožiny, lze nahradit jednotlivé prvky původní množiny $\mathrm{fix}(C)$.

Význam výsledků prací [Práce 6, 7] spočívá v jejich obecnosti. Jelikož zobrazení ${}^{\uparrow\downarrow}$ indukované formálním **L**-kontextem je uzávěrový operátor a příslušný konceptuální svaz je izomorfní množině jeho pevných bodů, lze tyto výsledky aplikovat i v oblasti **L**-konceptuálních svazů.

Poznámka 24. Časopisecká práce [Práce 6] se zabývá zobecněním problematiky řešené v části konferenčního příspěvku [Práce 3]. Proto obě práce mají na začátku uveden stejný příklad. V chystané rozšířené časopisecké verzi práce [Práce 3] bude úvod změněn.

5 Možnosti dalšího výzkumu

Práce tohoto souboru se zabývají aktuální a originální problematikou v oblasti fuzzy struktur.

Kombinace výsledků uvedených v pracích tohoto souboru přináší různé náměty pro další výzkum. Uvedeme zde některé z těch, které jsou připraveny k publikaci, rozpracovány, případně které se chystáme v blízké době otevřít.

U deterministických **L**-automatů [Práce 1] zůstalo jako nedořešený problém dopracování hlavních výsledků v konkrétní postup (algoritmus) jejich přibližné minimalizace. Nabízí se využít novějších výsledků o aproximaci v reziduovaných svazech [Práce 3] a pokusit se například jako první krok minimalizace aproximovat stupně koncovosti jednotlivých stavů (tj. jednotlivé hodnoty **L**-množiny Q_F) hodnotami z nějaké menší množiny, vypočítané algoritmy, které jsou v práci [Práce 3] uvedeny. Podstatné pro výzkum v této oblasti bude i studium existujících aplikací deterministických **L**-automatů a možnosti využití výsledků o přibližné minimalizaci v těchto aplikacích. To by mohlo být perspektivní vzhledem k tomu, že přibližná minimalizace fuzzy automatů je novým, v literatuře zatím nezpracovaným tématem.

V oblasti aproximace v reziduovaných svazech je již připravena k publikaci rozšířená verze práce [Práce 3], obsahující algoritmus na řešení Úlohy 2 z odst. 4.3.

Problém redukce velikosti **L**-konceptuálního svazu je stále aktuálním tématem. Další výzkum se bude zabývat možnostmi použití aproximace v konceptuálních svazech, které byly zatím částečně zkoumány v pracích [Práce 2, 6], a také možnostmi efektivnější faktorizace.

Pomocí [Práce 4, Theorem 12] lze hlavní výsledek [Práce 7] rozšířit následujícím způsobem. Je-li $C\colon L^X \to L^X$ uzávěrový operátor, lze s jeho pomocí zavést nový uzávěrový operátor $C^{a\approx}$ na systému $(\mathbf{L}/a)^X$ všech $\mathbf{L}/a$-množin v X tak, že položíme

$$C^{a\approx}([A,B]) = [C(A)^a, (C(A)^a)_a]$$

(využíváme ztotožnění množin $(L/a)^X$ a L^X/a z [Práce 4, Odst. 4.1] a symboliku z [Práce 7]). Úplný svaz $\mathrm{fix}(C)/^a\approx$ (tj. množina pevných bodů operátoru C faktorizovaná podle tolerance $^a\approx$), kterým se zabývá [Práce 7] je pak roven (opět po ztotožnění $(L/a)^X$ a L^X/a) množině všech pevných bodů uzávěrového operátoru $C^{a\approx}$.

Zkušenosti s faktorizací reziduovaných svazů, **L**-konceptuálních svazů a množin pevných bodů **L**-uzávěrových operátorů lze využít k vypracování teorie faktorizace obecnějších **L**-struktur. Konkrétněji, máme-li dánu množinu X s **L**-ekvivalencí (nebo **L**-rovností) $\approx$, lze na faktormnožině $X/^a\approx$ (kde $a \in L$ je zadaný práh) přirozeným způsobem definovat novou „zbytkovou" fuzzy rovnost. Je-li navíc na množině X definována binární **L**-relace, je možné zavést indukovanou binární fuzzy relaci i na faktormnožině $X/^a\approx$. Do tohoto rámce spadá například problém faktorizace tzv. **L**-uspořádaných množin i další zajímavé aplikace. První výsledky v tomto směru jsou již připraveny k publikaci.

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ON APPROXIMATE MINIMIZATION OF FUZZY AUTOMATA

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Abstract. The paper presents a contribution to minimization of fuzzy automata. Traditionally, the problem of minimization of fuzzy automata is described as follows. Given a fuzzy automaton, describe an automaton with the minimal number of states which recognizes the same language as the given one. In this paper, we formulate a different problem. Namely, the minimal fuzzy automaton we are looking for is required to recognize a language which is similar to the language of the given fuzzy automaton to a certain degree a prescribed by a user, such as $a = 0.9$. That is, we relax the condition for the languages of the automata from being equal to a weaker condition of being similar to degree a. The condition of being equal is a special case for $a = 1$. We present answers to several problems which arise in this setting including a description of a minimal fuzzy automaton satisfying the above condition. The question of how to efficiently construct the minimal automaton is left as an open problem.

1 Problem setting

The idea of extending ordinary automata by principles of fuzzy logic goes back to the early eage of fuzzy logic, see e.g.[12], [13] for some of the early papers. Since then, many papers on fuzzy automata and their applications appeared, see [11] for an overview. Recent approaches use general structures of truth degrees, instead of the commonly used unit interval $[0,1]$. As a main motivation for studying fuzzy automata serves the fact that many languages are fuzzy in that words belong to such languages to possibly intermediate degrees, rather than just 1 (belongs) and 0 (does not belong).

One of the classic problems of finite automata is that of a minimization. Given an automaton $\mathcal{M}$, one looks for an equivalent automaton $\mathcal{M}'$ with as small number of states as possible. By "equivalent", one means "recognizing the same language". In all the papers we found, the problem of minimization of fuzzy automata is formulated essentially the same way as for the ordinary automata. That is, given a fuzzy automaton $\mathcal{M}$, one is looking for a fuzzy automaton $\mathcal{M}'$ with as small number of states as possible such that $\mathcal{L}(\mathcal{M}) = \mathcal{L}(\mathcal{M}')$. Note that $\mathcal{L}(\mathcal{M})$ denotes the language of $\mathcal{M}$, i.e. the fuzzy set of words recognized by $\mathcal{M}$. Such requirement might be considered too strong. Namely, one might require instead that $\mathcal{L}(\mathcal{M})$ and $\mathcal{L}(\mathcal{M}')$ be highly similar but not necessarily equal. With an appropriate definition of similarity of fuzzy languages, one might require that $\mathcal{L}(\mathcal{M})$ and $\mathcal{L}(\mathcal{M}')$ be similar in degree at least 0.9, for instance. Relaxing the requirement of equality of languages by replacing it with a weaker requirement of similarity (approximate equality), presents a new problem. The rationale behind is that

1. from the point of view of user's needs, an automaton recognizing approximately the same language may be acceptable,
2. with the weaker requirement of approximately equal languages, the number of states of the resulting minimal automaton may decrease compared to when we require equality of languages.

The present paper presents several results regarding approximate minimization of fuzzy automata including a description of a minimal automaton which recognizes a language which is similar to the language of a given automaton in a given degree a or higher.

2 Preliminaries

In this section, we survey basic notions from fuzzy logic and refer to [1], [3] for further details. Our basic structure of truth degrees is a complete residuated lattice. A reader not familiar with the framework of residuated lattices can go directly to Section 3.1 and use the particular setting described in the second sentence of Section 3.1.

A complete residuated lattice is an algebra

$$\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$$

such that $\langle L, \wedge, \vee, 0, 1 \rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1 \rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy so-called adjointness property: $a \otimes b \leq c$ iff $a \leq b \rightarrow c$, for each $a, b, c \in L$. Elements a of L are called truth degrees. $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication". "Fuzzy negation" $\neg$ is defined by $\neg a = a \rightarrow 0$ for $a \in L$.

A common choice of $\mathbf{L}$ is a structure with $L = [0, 1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding residuum $\rightarrow$. Three most important pairs of adjoint operations on the unit interval are Łukasiewicz: $a \otimes b = \max(a + b - 1, 0)$, $a \rightarrow b = \min(1 - a + b, 1)$; Gödel: $a \otimes b = \min(a, b)$, $a \rightarrow b = 1$ if $a \leq b$, $a \rightarrow b = b$ otherwise; Goguen (product): $a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$, $a \rightarrow b = \frac{b}{a}$ otherwise. Examples of finite residuated lattices include those defined on finite subchains of $[0, 1]$. Taking $L = \{0, 1\}$ gives us a two-element Boolean algebra (structure of truth degrees of classical logic).

Given $\mathbf{L}$ which serves as a structure of truth degrees, we define the usual notions: an $\mathbf{L}$-set (fuzzy set) A in universe U is a mapping $A : U \rightarrow L$, $A(u)$ being interpreted as "the degree to which u belongs to A". Let $\mathbf{L}^U$ denote the collection of all $\mathbf{L}$-sets in U. Operations with $\mathbf{L}$-sets are defined componentwise. For instance, the intersection of $\mathbf{L}$-sets $A, B \in \mathbf{L}^U$ is an $\mathbf{L}$-set $A \cap B$ in U such that $(A \cap B)(u) = A(u) \wedge B(u)$ for each $u \in U$, etc. Given $A, B \in \mathbf{L}^U$, we define a degree $A \approx B$ to which A and B are equal by

$$A \approx B = \bigwedge_{u \in U} \big(A(u) \leftrightarrow B(u)\big), \tag{1}$$

where $a \leftrightarrow b = (a \to b) \wedge (b \to a)$. (1) generalizes ordinary equality since $A = B$ iff $A \approx B = 1$. Described verbally, $A \approx B$ is a degree to which for each $u \in U$, u belongs to A iff u belongs to B.

3 Approximate minimization

3.1 Fuzzy automata

We use a definition of fuzzy automata from [1]. A reader not familiar with the framework of residuated lattices can, without losing much, replace L by the unit interval $[0,1]$; $\wedge$ and $\vee$ by the operations of minimum and maximum on $[0,1]$; $\otimes$ and $\to$ by Łukasiewicz operations (see Section 2), and take $a \leftrightarrow b = 1 - |a-b|$. For a complete residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$, an $\mathbf{L}$-*automaton* $\mathcal{M}$ over a finite alphabet Σ is defined as a tuple $\langle Q, \Sigma, Q_I, Q_F, \delta\rangle$ of a finite set Q of states, an alphabet Σ, an $\mathbf{L}$-set Q_I in Q of initial states, an $\mathbf{L}$-set Q_F in Q of final states, and an $\mathbf{L}$-relation δ between Q, Σ, and Q. For $q, q' \in Q$, $s \in \Sigma$, $\delta(q, s, q')$ is the degree to which the $\mathbf{L}$-automaton $\mathcal{M}$ can transfer from q to q' if the actual input symbol is s.

For any input word $\alpha = s_1 \dots s_n$ we set

$$\delta(q,\alpha,q') = \bigvee_{\substack{q_0,\dots,q_n \in Q \\ q_0 = q, q_n = q'}} \delta(q_0, s_1, q_1) \wedge \dots \wedge \delta(q_{n-1}, s_n, q_n),$$

$$(\mathcal{L}(\mathcal{M}))(\alpha) = \bigvee_{q,q' \in Q} Q_I(q) \wedge \delta(q,\alpha,q') \wedge Q_F(q').$$

$\delta(q,\alpha,q')$ is the degree to which $\mathcal{M}$ can transfer from q to q' having α at the input. If α is the empty word ε, we get

$$\delta(q,\varepsilon,q') = \begin{cases} 1 \text{ if } q = q' \\ 0 \text{ otherwise.} \end{cases}$$

The $\mathbf{L}$-set $\mathcal{L}(\mathcal{M})$ is called the $\mathbf{L}$-language *recognized* by $\mathcal{M}$.

The degree $Q_I(\alpha, q_0)$ to which $\mathcal{M}$ will reach the state q_0 by the input word α, and the degree $Q_F(q_0, \alpha)$ to which $\mathcal{M}$ will accept the input word α when starting from q_0 are defined as follows:

$$Q_I(\alpha, q_0) = \bigvee_{q \in Q} Q_I(q) \wedge \delta(q, \alpha, q_0), \; Q_F(q_0, \alpha) = \bigvee_{q \in Q} \delta(q_0, \alpha, q) \wedge Q_F(q).$$

One can easily see that

$$Q_I(q_0) = Q_I(\varepsilon, q_0), \; Q_F(q_0) = Q_F(q_0, \varepsilon).$$

3.2 Deterministic fuzzy automata

We use the definition of a deterministic fuzzy automaton from [2]. An $\mathbf{L}$-automaton $\mathcal{M}$ is *deterministic* if there is a state $q_0 \in Q$, called the *initial state*, such that

$$Q_I(q) = \begin{cases} 1 \text{ if } q = q_0, \\ 0 \text{ otherwise,} \end{cases}$$

and for any state $q_1 \in Q$ and symbol $s \in \Sigma$ there is a state $q_2 \in Q$ such that

$$\delta(q_1, s, q) = \begin{cases} 1 \text{ if } q = q_2, \\ 0 \text{ otherwise.} \end{cases}$$

For q_1 and q_2 above we write $q_2 = \delta(q_1, s)$. Therfore, δ can be regarded as an ordinary function, called *transition function*. If $\mathcal{M}$ is deterministic with the initial state q_0 and transition function δ then we write $\mathcal{M} = \langle Q, \Sigma, q_0, Q_F, \delta \rangle$. For a fixed symbol $s \in \Sigma$ we also write $\delta(q, s) = \delta_s(q)$ and thus obtain a mapping $\delta_s : Q \to Q$.

Note that it was proved in [2] that if our complete residuated lattice $\mathbf{L}$ satisfies that every complete sublattice generated by a finite $L' \subseteq L$ is finite, then every $\mathbf{L}$-automaton can be replaced by an equivalent deterministic automaton. As observed in [7], one case when this condition is satisfied is when the underlying lattice is distributive.

3.3 The problem of approximate minimization

Our problem can be formulated as follows. Given an $\mathbf{L}$-automaton $\mathcal{M}$ and a similarity threshold $a \in L$, find an $\mathbf{L}$-automaton $\mathcal{M}'$ such that

$$(\mathcal{L}(\mathcal{M}) \approx \mathcal{L}(\mathcal{M}')) \geq a. \tag{2}$$

Note that (2) is the case iff for each input word α we have

$$([\mathcal{L}(\mathcal{M})](\alpha) \leftrightarrow [\mathcal{L}(\mathcal{M}')](\alpha)) \geq a,$$

cf. (1). Measuring similarity of languages using $\leftrightarrow$ is the technical reason why we consider logical connectives on the lattice of truth degrees.

In the ordinary setting, minimization of an automaton involves removal of inaccessible states followed by factorization using an equivalence relation which represents indistinguishability of states. In our setting, (in)accessibility comes in degrees, i.e. there are degrees to which a state is (in)accessible. In Section 3.4, we show an appropriate "graded version" of a well-known fact saying that removing inaccessible states does not change the language recognized by an automaton. As to minimization, the situation is more complex in the setting of an approximate equality. We restrict ourselves to the case of deterministic fuzzy automata and present a result describing a minimal fuzzy automaton $\mathcal{M}'$ for a given $\mathcal{M}$ satisfying (2). The following example shows that approximate minimization is non-trivial in the sense that it can lead to a smaller number of states.

Example 1. Let n be a positive integer and $L = [0, 1]$ be equipped with the Łukasiewicz structure. Consider a deterministic $\mathbf{L}$-automaton $\mathcal{M} = \langle Q, \Sigma, q_0, Q_F, \delta \rangle$ with the set of states $Q = \{q_0, q_1, \ldots, q_n\}$, $\Sigma = \{s\}$, $Q_F(q_i) = 2^{-i}$ for $i < n$, $Q_F(q_n) = 0$, and δ defined by $\delta(q_i, s) = q_{i+1}$ for $i < n$ and $\delta(q_n, s) = q_n$. The automaton $\mathcal{M}$ for $n = 3$ is depicted in Fig. 1. The values $Q_F(q_i)$ are indicated inside the circles representing states q_i.

It is easy to see that the language $\mathcal{L} = \mathcal{L}(\mathcal{M})$ of $\mathcal{M}$ is given by

$$\mathcal{L}(s^i) = \begin{cases} 2^{-i} & \text{for } i < n, \\ 0 & \text{for } i \geq n. \end{cases}$$

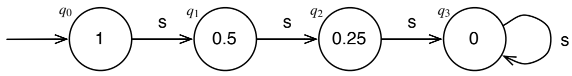

Fig. 1. Automaton $\mathcal{M}$.

Since for $i \neq j$ we have $Q_F(q_i) \neq Q_F(q_j)$ there does not exist a deterministic fuzzy automaton $\mathcal{M}'$ with the number of states less than $n+1$ recognizing the same language as $\mathcal{M}$. Therefore, the automaton is minimal in the ordinary sense. Consider now $a = \frac{3}{4}$. Then there exists and $\mathbf{L}$-automaton $\mathcal{M}'$ with just two states satisfying (2). Namely, one can put $Q' = \{q'_0, q'_1\}$, $\Sigma = \{s\}$, $Q'_F(q'_0) = 1$, $Q'_F(q'_1) = \frac{1}{4}$ and define δ' by $\delta'(q'_0, s) = q'_1$, $\delta(q'_1, s) = q'_1$ (see Fig. 2). Therefore, approximate minimization can, indeed, decrease

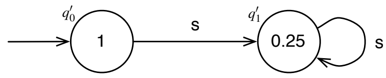

Fig. 2. Automaton $\mathcal{M}'$.

the number of states of a fuzzy automaton which is minimal in the ordinary sense.

3.4 Inaccessible states

For any subset $Y \subset Q$, we can construct a new $\mathbf{L}$-automaton $\mathcal{M}' = \langle Q', \Sigma, Q'_I, Q'_F, \delta' \rangle$ by *removing* the *states* belonging to Y from the automaton $\mathcal{M}$. The set of states Q' of $\mathcal{M}'$ is equal to the set $Q \setminus Y$, the $\mathbf{L}$-sets Q'_I and Q'_F are constructed by restriction of the $\mathbf{L}$-sets Q_I, and Q_F to the set Q', and the transition $\mathbf{L}$-relation δ' between Q', Σ, and Q' is constructed by restricting the $\mathbf{L}$-relation δ to the Cartesian product $Q' \times \Sigma \times Q'$.

For any state $q \in Q$ we define the *accessibility degree* $(\mathrm{Acc}(\mathcal{M}))(q)$ of q by

$$(\mathrm{Acc}(\mathcal{M}))(q) = \bigvee_{\alpha \in \Sigma^*} Q_I(\alpha, q). \tag{3}$$

This defines the $\mathbf{L}$-set $\mathrm{Acc}(\mathcal{M})$ of accessible states of $\mathcal{M}$.

Theorem 1. *If the* $\mathbf{L}$*-automaton* $\mathcal{M}'$ *results from an* $\mathbf{L}$*-automaton* $\mathcal{M}$ *by removing all states* q *such that* $(\mathrm{Acc}(\mathcal{M}))(q) \leq \neg a$ *then condition* (2) *is satisfied.*

Proof. For any input word $\alpha = s_1 \dots s_n$ and $(n+1)$-tuple of states $q_0, \dots, q_n$ set

$$\begin{aligned}(A(\alpha))(q_0 \dots q_n) = Q_I(q_0) \wedge \delta(q_0, s_1, q_1)\\ \wedge \dots \wedge \delta(q_{n-1}, s_n, q_n) \wedge Q_F(q_n)\end{aligned} \tag{4}$$

and

$$V(\alpha) = \bigvee_{\substack{q_0, \dots, q_n \in Q \\ \{q_0, \dots, q_n\} \not\subseteq Q'}} (A(\alpha))(q_0 \dots q_n). \tag{5}$$

The condition $\{q_0, \dots, q_n\} \not\subseteq Q'$ means that there is an index i such that for the state q_i, $(\text{Acc}(\mathcal{M}))(q_i) \leq \neg a$. Thus, $V(\alpha) \leq \neg a$. We have

$$(\mathcal{L}(\mathcal{M}))(\alpha) = (\mathcal{L}(\mathcal{M}'))(\alpha) \vee V(\alpha), \tag{6}$$

which implies that $(\mathcal{L}(\mathcal{M}'))(\alpha) \leq (\mathcal{L}(\mathcal{M}))(\alpha)$, which is equivalent to $(\mathcal{L}(\mathcal{M}'))(\alpha) \rightarrow (\mathcal{L}(\mathcal{M}))(\alpha) = 1$. It remains to be shown that $(\mathcal{L}(\mathcal{M}))(\alpha) \rightarrow (\mathcal{L}(\mathcal{M}'))(\alpha) \geq a$, i.e. $a \otimes (\mathcal{L}(\mathcal{M}))(\alpha) \leq (\mathcal{L}(\mathcal{M}'))(\alpha)$, which is true. Indeed,

$$\begin{aligned} a \otimes (\mathcal{L}(\mathcal{M}))(\alpha) &= a \otimes (V(\alpha) \vee (\mathcal{L}(\mathcal{M}'))(\alpha)) = \\ &= (a \otimes V(\alpha)) \vee (a \otimes (\mathcal{L}(\mathcal{M}'))(\alpha)) \leq \\ &\leq (a \otimes \neg a) \vee (a \otimes (\mathcal{L}(\mathcal{M}'))(\alpha)) = \\ &= a \otimes (\mathcal{L}(\mathcal{M}'))(\alpha) \leq (\mathcal{L}(\mathcal{M}'))(\alpha). \end{aligned}$$

□

Corollary 1. *If $\mathcal{M}'$ results from $\mathcal{M}$ by removing a state q, then*

$$\neg\text{Acc}(\mathcal{M})(q) \leq (\mathcal{L}(\mathcal{M}) \approx \mathcal{L}(\mathcal{M}')).$$

Proof. Immediately from Theorem 1 by observing that $\text{Acc}(\mathcal{M})(q) \leq \neg a$ is equivalent to $a \leq \neg\text{Acc}(\mathcal{M})(q)$, and that $b \leq c$ iff for each $a \in L$: $a \leq b$ implies $a \leq c$. □

Corollary 1 says that if we remove a state q, then the statement "if q is inaccessible then $\mathcal{L}(\mathcal{M})$ is equal to $\mathcal{L}(\mathcal{M}')$" is true (in degree 1) if interpreted in a fuzzy logic with $\mathbf{L}$ as the structure of truth degrees. To see this, just use basic rules of semantics of fuzzy logic [3]. Obviously, if $\mathbf{L}$ is the two-element Boolean algebra, this brings us to the realm of ordinary automata and Corollary 1 becomes the well-known statement saying that if $\mathcal{M}'$ results from $\mathcal{M}$ by removal of inaccessible states then $\mathcal{L}(\mathcal{M})$ equals $\mathcal{L}(\mathcal{M}')$. Note that if $\mathcal{M}$ is deterministic then the accessibility degree of any of its states is equal to 0 or 1 only. In this case, states with accessibility degree 0 and 1 are called *inaccessible* and *accessible*, respectively. If $\mathcal{M}$ contains accessible states only, it is called *accessible*, see e.g. [4].

3.5 Approximate minimization

Suppose that $\mathcal{M} = \langle Q, \Sigma, q_0, Q_F, \delta \rangle$ is an accessible deterministic $\mathbf{L}$-automaton. For $a \in L$ we call a set $P \subseteq Q$ a *set of a-similar states* if there is a $c \in L$ such that

$$\bigwedge_{q \in P} (Q_F(q) \leftrightarrow c) \geq a. \tag{7}$$

Let Q' be a covering of Q, i.e. a set of non-empty subsets of Q whose union is Q. Q' is called an *a-covering of $\mathcal{M}$* if it contains only sets of a-similar states. Q' is called *invariant* if for any $q'_1 \in Q'$, $s \in \Sigma$ there is a $q'_2 \in Q'$ such that $\delta_s(q'_1) \subseteq q'_2$.

Q' is called *minimal invariant a-covering* if it is an invariant a-covering with the minimal number of elements.

Since the partition $\{\{q\} \mid q \in Q\}$ is an invariant a-covering of $\mathcal{M}$, there always exists a minimal invariant a-covering of $\mathcal{M}$.

Let Q' be a minimal invariant a-covering of $\mathcal{M}$. We construct a new deterministic $\mathbf{L}$-automaton $\mathcal{M}' = \langle Q', \Sigma, q'_0, Q'_F, \delta' \rangle$ as follows. We choose q'_0 to be any element of Q' containing the initial state q_0 of $\mathcal{M}$ (such q'_0 exists since Q' is a covering), set $Q'_F(q') = c$, where $c \in L$ satisfies $\bigwedge_{q \in q'}(Q_F(q) \leftrightarrow c) \geq a$ (existence of c follows from the fact that q' is a set of a-similar states), and, finally, set $\delta'(q'_1, s) = q'_2$ where $q'_2 \in Q'$ is any element such that $\delta_s(q'_1) \subseteq q'_2$ (q'_2 exists since Q' is an invariant covering).

Example 2. Let $\mathbf{L}$ be the standard Łukasiewicz algebra on $[0,1]$. The deterministic $\mathbf{L}$-automaton $\mathcal{M} = \langle Q, \Sigma, q_0, Q_F, \delta \rangle$ in Fig. 3 over alphabet $\Sigma = \{s,t\}$ is minimal in the ordinary sense.

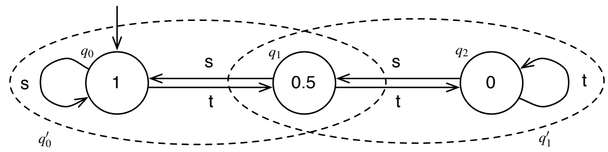

Fig. 3. 0.75-covering on $\mathbf{L}$-automaton.

Consider the covering $Q' = \{q'_0, q'_1\}$ of the set of states Q, where $q'_0 = \{q_0, q_1\}$ and $q'_1 = \{q_1, q_2\}$. The elements, i.e. sets of states, are indicated in Fig. 3 by dashed ovals. This covering is invariant:

$$\begin{aligned}
&\delta_s(q_0) = q_0,\ \delta_s(q_1) = q_0, \text{ hence } \delta_s(q'_0) \subseteq q'_0,\\
&\delta_s(q_1) = q_0,\ \delta_s(q_2) = q_1, \text{ hence } \delta_s(q'_1) \subseteq q'_0,\\
&\delta_t(q_0) = q_1,\ \delta_t(q_1) = q_2, \text{ hence } \delta_t(q'_0) \subseteq q'_1,\\
&\delta_t(q_1) = q_2,\ \delta_t(q_2) = q_2, \text{ hence } \delta_t(q'_1) \subseteq q'_1.
\end{aligned}$$

For $a = 0.75$, Q' is an a-covering of $\mathcal{M}$: for q'_0 we can set $c = 0.75$, and for q'_1, $c = 0.25$. This covering is a minimal invariant a-covering of $\mathcal{M}$. Indeed, the only covering with less number of elements is equal to $\{Q\}$, but the set Q is not a set of a-similar states. Thus, it is possible to reduce the number of states of the automaton to 2 and 2 is the minimal number of states of an $\mathbf{L}$-automaton $\mathcal{M}'$ satisfying (2). The automaton $\mathcal{M}'$ constructed from the covering Q' is shown in Fig. 4.

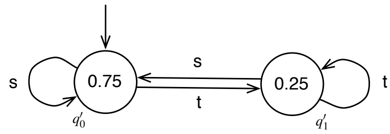

Fig. 4. Minimized $\mathbf{L}$-automaton.

Now we prove our main results concerning automaton $\mathcal{M}'$.

Theorem 2 (similarity of $\mathcal{M}$ and $\mathcal{M}'$). *Automata $\mathcal{M}$ and $\mathcal{M}'$ satisfy condition* (2).

Proof. By definition of $\mathcal{M}'$, $\delta(q_0, \varepsilon) = q_0 \in q'_0$ and if $q \in q'$ then for any symbol s, $\delta(q,s) \in \delta'(q',s)$. Put together, for any input word $\alpha \in \Sigma^*$, $\delta(q_0,\alpha) \in \delta'(q'_0,\alpha)$. Hence we get $Q_F(q_0,\alpha) \leftrightarrow Q'_F(q'_0,\alpha) \geq a$. □

Theorem 3 (minimality of $\mathcal{M}'$). *Let $\mathcal{M}''$ be a deterministic* **L**-*automaton such that $\mathcal{L}(\mathcal{M}'') \approx \mathcal{L}(\mathcal{M}) \geq a$. Then $|Q''| \geq |Q'|$.*

Proof. Let $\mathcal{M}'' = \langle Q'', \Sigma, q''_0, Q''_F, \delta'' \rangle$. For any state $q'' \in Q''$ we construct a set $P(q'') \subseteq Q$ by letting

$$P(q'') = \{\delta(q_0,\alpha) \mid \delta''(q''_0,\alpha) = q''\}.$$

Since $\mathcal{M}$ is accessible, for any $q \in Q$ there exists a word $\alpha \in \Sigma^*$ such that $\delta(q_0,\alpha) = q$. If we set $q'' = \delta''(q''_0,\alpha)$, we obtain $q \in P(q'')$. Hence, the system $\{P(q'') \mid q'' \in Q''\}$ is a covering of Q.

From the condition $\mathcal{L}(\mathcal{M}'') \approx \mathcal{L}(\mathcal{M}) \geq a$ we obtain that for any $q \in P(q'')$ it holds $Q_F(q,\alpha) \leftrightarrow Q''_F(q'',\alpha) \geq a$, thus the sets $P(q'')$ are sets of a-similar states.

Now we show that $\{P(q'') \mid q'' \in Q''\}$ is an invariant a-covering. Indeed, for any $q'' \in Q''$ and symbol $s \in \Sigma$ we have

$$\begin{aligned}
\delta_s(P(q'')) &= \{\delta(q,s) \mid q \in P(q'')\} = \\
&= \{\delta(q_0,\alpha s) \mid \alpha \in \Sigma \text{ such that } \delta''(q''_0,\alpha) = q''\} \subseteq \\
&\subseteq \{\delta(q_0,\alpha s) \mid \alpha \in \Sigma \text{ such that } \delta''(q''_0,\alpha s) = \delta''(q'',s)\} \subseteq \\
&\subseteq \{\delta(q_0,\beta) \mid \beta \in \Sigma \text{ such that } \delta''(q''_0,\beta) = \delta''(q'',s)\} = \\
&= P(\delta''_s(q'')).
\end{aligned}$$

Since the number of states of $\mathcal{M}''$ is greater than or equal to the number of elements of $\{P(q'') \mid q'' \in Q''\}$ (the mapping P need not to be injective), it is greater than or equal to the number of states of $\mathcal{M}'$. Indeed, $\{P(q'') \mid q'' \in Q''\}$ is an invariant a-covering while the set of states of $\mathcal{M}'$ forms a minimal invariant a-covering. This proves the theorem. □

4 Future research and open problem

In general, the future research should focus on various phenomena regarding finite fuzzy automata where the phenomenon of approximation plays a role. This paper presents one example of such phenomena. Related attempt can be found in [6] The following is an

> **Open Problem**: Find an efficient algorithm for a construction of a minimal approximately equivalent fuzzy automaton, as defined in this paper. Furthermore, study the related complexity issues.

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Approximating Infinite Solution Sets by Discretization of the Scales of Truth Degrees

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***Abstract*— The present paper discusses the problem of approximating possibly infinite sets of solutions by finite sets of solutions via discretization of scales of truth degrees. Infinite sets of solutions we have in mind in this paper typically appear in constraint-based problems such as "find all collections in a given finite universe satisfying constraint C". In crisp setting, i.e. when collections are conceived as crisp sets, the set of all such collections is finite and often computationally tractable. In fuzzy setting, i.e. when collections are conceived as fuzzy sets, the set of all such collections may be infinite and, *ipso facto*, computationally intractable when one uses the unit interval $[0,1]$ as the scale of membership degrees. A natural solution to this problem is to uses, instead of $[0,1]$, a finite subset K of $[0,1]$ which approximates $[0,1]$ to a satisfactory degree. This idea is pursued in the present paper. To be sufficiently specific, we illustrate the idea on a particular method, namely, on formal concept analysis. We present several results including estimation of degrees of similarity of the finitary approximation to the possibly infinite original case by means of the degree of approximation of K of $[0,1]$.**

I. Introduction and Problem Setting

Using the real unit interval $[0,1]$ as a scale of truth degrees is the most common choice in fuzzy logic applications. The aim of our paper is to bring to the attention one aspect of using $[0,1]$. Namely, given a universe set X, the set $[0,1]^X$ of all fuzzy sets in X is infinite even if X is finite. While this fact is advantageous from the point of view of representation capability of fuzzy sets in X, there are apparent disadvantages of using $[0,1]$ as well. Namely, using $[0,1]$ can lead to problems which are computationally not feasible even if the corresponding crisp problems, i.e., problems with $[0,1]$ replaced by $\{0,1\}$, are computationally tractable. Examples of these problems can be drawn from data mining tasks where one tries to extract all collections of elements of some set, say X, which satisfy certain constraint C. When "collection" is understood as a set, the search space is 2^X and the solution set $\{A \in 2^X \mid A \text{ satisfies } C\}$ can be efficiently computable. On the other hand, when "collection" is understood as a fuzzy set, both the search space $[0,1]^X$ and the solution set $\{A \in [0,1]^X \mid A \text{ satisfies } C\}$ can be infinite. In such a case, problem of computing the solution set is not feasible in principle. A natural idea in this case is to consider a finite subset K of $[0,1]$ which is a "good approximation" of $[0,1]$. Then both K^X and the solution set $\{A \in K^X \mid A \text{ satisfies } C\}$ are finite and the solution set can be a good approximation of the original infinite solution set $\{A \in [0,1]^X \mid A \text{ satisfies } C\}$. In our paper, we attempt to formalize the above considerations. We provide general results which address several issues of the idea of approximating an infinite solution set by a finite one. In particular, we obtain approximation formulas and a result which enables us to infer, given a required approximation level, how to select a K which guarantees the required approximation level.

We demonstrate our ability to approximate infinite solution sets by several examples. For illustration, we consider formal concept analysis (FCA, see [6]) which is a particular method of knowledge extraction. FCA deals with object-attribute data describing relationship between objects and attributes. In more detail: an input for FCA is a data table with rows corresponding to objects, columns corresponding to attributes, and table entries containing degrees to which objects have attributes. In its basic setting, FCA considers degrees 0 and 1 only, meaning that each object has/does not have an attribute. The output of FCA is a hierarchically ordered set of conceptual clusters extracted from the data. Since it is often the case that attributes are fuzzy rather than bivalent (attributes apply to objects to various degrees), one can consider an extension of FCA using arbitrary truth degrees from the real unit interval $[0,1]$ as table entries, see [2], [3]. The output of FCA in this case is again a hierarchy of clusters. It can happen, however, that the hierarchy is infinite due to the fact that we have shifted from a finite scale $\{0,1\}$ to the interval $[0,1]$. This infinite scale is in fact an infinite solution to the clustering problem which we want to approximate by a finite one. Using general results proposed in this paper, we are able to replace $[0,1]$ by a suitable finite scale $K \subseteq [0,1]$ which, being used as a structure of truth degrees, produces a finite solution (hierarchy with finitely many clusters) which is computationally tractable and approximates well the infinite one.

Section II presents preliminaries. Section III present our approach and results. In Section IV, we outline some further issues connected to the present problem.

II. Preliminaries

The basic concept in fuzzy logic is that of a structure of truth degrees which represents a set of truth degrees we use

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to describe graded truth of propositions (e.g., graded relationship of objects, graded properties of objects, similarities of values, etc.) plus logical connectives (e.g., conjunction, implications, ...) which are used to calculate truth degrees from other truth degrees. In this paper we are going to use so-called complete residuated lattices as our structures of truth degrees. This choice is general enough because complete residuated lattices include the popular t-norm-based structures of truth degrees as well as finite structures of truth degrees which we use to approximate the infinite ones. The rest of this section presents an introduction to the complete residuated lattices and derived notions we will need in the sequel. Further details can be found e.g. in [2], [7], [8], a good introduction to fuzzy logic and fuzzy sets is presented in [9].

A complete residuated lattice [8] is an algebra

$$\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle \qquad (1)$$

such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy so-called adjointness property:

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \rightarrow c \qquad (2)$$

for each $a, b, c \in L$. Elements a of L are called truth degrees. $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication". For each complete residuated lattice (1) we consider a derived (truth function of) logical connective $\leftrightarrow$ ("fuzzy equivalence") defined by $a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$. By a^n we denote $a \otimes \cdots \otimes a$ (n-times).

A common choice of $\mathbf{L}$ is a structure with $L = [0, 1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding $\rightarrow$. Three most important pairs of adjoint operations on the unit interval are:

$$\text{Łukasiewicz:} \quad \begin{array}{rcl} a \otimes b &=& \max(a + b - 1, 0), \\ a \rightarrow b &=& \min(1 - a + b, 1), \end{array} \qquad (3)$$

$$\text{Gödel:} \quad \begin{array}{rcl} a \otimes b &=& \min(a, b), \\ a \rightarrow b &=& \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise,} \end{cases} \end{array} \qquad (4)$$

$$\text{Goguen (product):} \quad \begin{array}{rcl} a \otimes b &=& a \cdot b, \\ a \rightarrow b &=& \begin{cases} 1 & \text{if } a \leq b, \\ \frac{b}{a} & \text{otherwise.} \end{cases} \end{array} \qquad (5)$$

Complete residuated lattices on $[0, 1]$ given by (3), (4), and (5) are called standard Łukasiewicz, Gödel, Goguen (product) algebras, respectively. The class of complete residuated lattices include finite structures as well. For instance, one can put

$$L = \{a_0 = 0, a_1, \ldots, a_n = 1\} \subseteq [0, 1], \qquad (6)$$

where $a_0 < \cdots < a_n$ and with $\otimes$ and $\rightarrow$ given by

$$a_k \otimes a_l = a_{\max(k+l-n,0)}, \qquad (7)$$

$$a_k \rightarrow a_l = a_{\min(n-k+l,n)}. \qquad (8)$$

Such an $\mathbf{L}$ is called a finite Łukasiewicz chain. If in addition $\{a_0, \ldots, a_n\} \subseteq [0, 1]$ are equidistant, in which case (7) and (8) are restrictions of the operations from (3), then $\mathbf{L}$ is called an equidistant Łukasiewicz chain. For instance,

$$L_3 = \{0, 0.5, 1\},$$
$$L_4 = \{0, \tfrac{1}{3}, \tfrac{2}{3}, 1\},$$
$$L_5 = \{0, 0.25, 0.5, 0.75, 1\}, \ldots$$

equipped with operations defined by (7) and (8) are equidistant Łukasiewicz chains. Another class of complete residuated lattices defined on finite subsets of $[0, 1]$ is the class of finite Gödel chains, where subsets of $L \subseteq [0, 1]$ are equipped with restrictions of Gödel operations (4) on $[0, 1]$ to L. A special case of a complete residuated lattice is the two-element Boolean algebra $\langle\{0, 1\}, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$, denoted by $\mathbf{2}$, which is the structure of truth degrees of the classical logic. That is, the operations $\wedge, \vee, \otimes, \rightarrow$ of $\mathbf{2}$ are the truth functions (interpretations) of the corresponding logical connectives of the classical logic.

With $\mathbf{L}$ taken as a structure of truth degrees, we use the following notions: an $\mathbf{L}$-set (fuzzy set) A in universe U is a mapping $A \colon U \rightarrow L$, $A(u)$ being interpreted as "the degree to which u belongs to A". Let $\mathbf{L}^U$ denote the collection of all $\mathbf{L}$-sets in U. Operations with $\mathbf{L}$-sets are defined componentwise. For instance, the intersection of $\mathbf{L}$-sets $A, B \in \mathbf{L}^U$ is an $\mathbf{L}$-set $A \cap B$ in U such that $(A \cap B)(u) = A(u) \wedge B(u)$ for each $u \in U$, etc. Binary $\mathbf{L}$-relations (binary fuzzy relations) between X and Y can be thought of as $\mathbf{L}$-sets in the universe $X \times Y$. That is, a binary $\mathbf{L}$-relation $I \in \mathbf{L}^{X \times Y}$ between a set X and a set Y is a mapping assigning to each $x \in X$ and each $y \in Y$ a truth degree $I(x, y) \in L$ (a degree to which x and y are related by I). For any $A, B \in \mathbf{L}^U$, we define a similarity degree

$$A \approx B = \textstyle\bigwedge_{u \in U} \big(A(u) \leftrightarrow B(u)\big), \qquad (9)$$

which expresses a degree to which fuzzy sets A and B are similar. In particular, we write $A = B$ if $A \approx B = 1$. We have $A = B$ (i.e., $A \approx B = 1$) iff $A(u) = B(u)$ ($u \in U$).

In the following we use well-known properties of residuated lattices and fuzzy structures which can be found in monographs [2], [8]. Throughout the rest of the paper, $\mathbf{L}$ denotes an arbitrary complete residuated lattice.

III. Discretization in Formal Concept Analysis

A. Introduction to FCA

We first present a brief account on formal concept analysis of data with fuzzy attributes, see e.g. [6] and [2], [3]. The input data to FCA consists of a data table describing a relationship between objects and attributes. The output of FCA consists of a hierarchically ordered collection of clusters. The clusters are called formal concepts and can be seen as natural concepts well-understandable and interpretable by humans. A data table with fuzzy attributes can be represented by a triplet $\langle X, Y, I\rangle$, called an $\mathbf{L}$-context, where X and Y are a non-empty sets of objects (table rows) and attributes (table columns), and I :

$X \times Y \to L$ is an $\mathbf{L}$-relation with $I(x,y)$ representing the degree to which object $x \in X$ has attribute $y \in Y$ (table entry corresponding to row x and column y). For $A \in \mathbf{L}^X$, $B \in \mathbf{L}^Y$ (i.e. A is a fuzzy set of objects, B is a fuzzy set of attributes), we define fuzzy sets $A^{\uparrow_I} \in \mathbf{L}^Y$ (fuzzy set of attributes), $B^{\downarrow_I} \in \mathbf{L}^X$ (fuzzy set of objects) by

$$A^{\uparrow_I}(y) = \bigwedge_{x\in X}\big(A(x) \to I(x,y)\big),$$
$$B^{\downarrow_I}(x) = \bigwedge_{y\in Y}\big(B(y) \to I(x,y)\big).$$

We put

$$\mathcal{B}(X,Y,I) = \{\langle A,B\rangle \in \mathbf{L}^X \times \mathbf{L}^Y \mid A^{\uparrow} = B,\ B^{\downarrow} = A\}$$

and define for $\langle A_1,B_1\rangle, \langle A_2,B_2\rangle \in \mathcal{B}(X,Y,I)$ a partial order $\leq$ by $\langle A_1,B_1\rangle \leq \langle A_2,B_2\rangle$ iff $A_1 \subseteq A_2$ (or, iff $B_2 \subseteq B_1$; both ways are equivalent). $\langle \mathcal{B}(X,Y,I),\leq\rangle$ is called a fuzzy concept lattice associated to $\langle X,Y,I\rangle$. Elements $\langle A,B\rangle$ of $\mathcal{B}(X,Y,I)$ are naturally interpreted as concepts (clusters) hidden in the input data represented by I. Namely, $A^{\uparrow_I} = B$ and $B^{\downarrow_I} = A$ say that B is the collection of all attributes shared by all objects from A, and A is the collection of all objects sharing all attributes from B. Note that these conditions represent exactly the definition of a concept as developed in the so-called Port-Royal logic; A and B are called the extent and the intent of the concept $\langle A,B\rangle$, respectively, and represent the collection of all objects and all attributes covered by the particular concept. Furthermore, $\leq$ models the natural subconcept-superconcept hierarchy—concept $\langle A_1,B_1\rangle$ is a subconcept of $\langle A_2,B_2\rangle$ iff each object from A_1 belongs to A_2 (dually for attributes).

We can see that formal concepts are just pairs $\langle A,B\rangle$ of fuzzy sets satisfying constraint $A^{\uparrow_I} = B$ and $B^{\downarrow_I} = A$. While in crisp case, i.e. $L = \{0,1\}$, $\mathcal{B}(X,Y,I)$ is finite and can be efficiently computed, it can be infinite in fuzzy setting, e.g. with $L = [0,1]$ with Łukasiewicz operations.

For two fuzzy concept lattices $\mathcal{B}_1 = \mathcal{B}(X,Y,I_1)$ and $\mathcal{B}_2 = \mathcal{B}(X,Y,I_2)$, we define a degree $\mathcal{B}_1 \approx_{\text{Ext}} \mathcal{B}_2$ to which $\mathcal{B}(X,Y,I_1)$ and $\mathcal{B}(X,Y,I_2)$ are similar via their extents by

$$\mathcal{B}_1 \approx_{\text{Ext}} \mathcal{B}_2 = \Big(\bigwedge_{\langle A_1,B_1\rangle\in\mathcal{B}_1}\bigvee_{\langle A_2,B_2\rangle\in\mathcal{B}_2}(A_1 \approx A_2)\Big) \wedge \wedge \Big(\bigwedge_{\langle A_2,B_2\rangle\in\mathcal{B}_2}\bigvee_{\langle A_1,B_1\rangle\in\mathcal{B}_1}(A_1 \approx A_2)\Big).$$

In an analogous way, one can define a degree $\mathcal{B}_1 \approx_{\text{Int}} \mathcal{B}_2$ to which $\mathcal{B}(X,Y,I_1)$ and $\mathcal{B}(X,Y,I_2)$ are similar via their intents.

B. Replacing infinite structures of truth degrees by finite scales

Suppose $\mathbf{L}_1 = \langle L_1,\wedge,\vee,\otimes,\to,0,1\rangle$ is a complete residuated lattice and $L_2 \subseteq L_1$. If L_2 is a nonempty subuniverse of L_2, i.e., if L_2 is nonempty and it is closed under all operations from $\mathbf{L}_1$, then L_2 equipped with restriction of the operations from $\mathbf{L}_1$ is a complete residuated lattice which is a subalgebra of $\mathbf{L}_1$. In such a case, we denote the subalgebra by $\mathbf{L}_2$. The new structure $\mathbf{L}_2$ can be seen as an approximation of the original structure of truth degrees $\mathbf{L}_1$. For example, an equidistant five-element Łukasiewicz chain (see Section II) can be seen as an approximation of the standard Łukasiewicz algebra defined on the real unit interval. Finite substructures seem to be good candidates for replacing infinite structures of truth degrees. One issue connected with moving from infinite structures to finite ones is our ability to estimate quality of approximation of the infinite structure by the finite one. Therefore, for a complete residuated lattice $\mathbf{L}_1$ and its subalgebra $\mathbf{L}_2$, we introduce the following degree of approximation:

$$\text{appr}(\mathbf{L}_1,\mathbf{L}_2) = \bigwedge_{a\in L_1}\bigvee_{b\in L_2}\big(a \leftrightarrow b\big). \qquad (10)$$

Using standard rules of fuzzy logic, one can see that formula (10) represents a degree to which it is true that "for each truth degree a from L_1 there is a truth degree b in L_2 which is equivalent to a". Thus, $\text{appr}(\mathbf{L}_1,\mathbf{L}_2)$ can be understood as a degree to which $\mathbf{L}_2$ is a faithful approximation of $\mathbf{L}_1$.

Example 1: Consider the standard Łukasiewicz algebra (denote it by $\mathbf{L}_1$) and its equidistant substructure with $L_2 = \{0, 0.25, 0.5, 0.75, 1\}$. Then $\text{appr}(\mathbf{L}_1,\mathbf{L}_2) = 0.75$. More generally, for

$$L_2 = \{0 = \tfrac{0}{n}, \tfrac{1}{n}, \dots, \tfrac{n-1}{n}, \tfrac{n}{n} = 1\}, \qquad (11)$$

one can see that $\text{appr}(\mathbf{L}_1,\mathbf{L}_2) = 1 - \frac{1}{n}$.

Consider now a complete residuated lattice $\mathbf{L}_1$ and an $\mathbf{L}_1$-context $\langle X,Y,I_1\rangle$. If we choose a finite substructure $\mathbf{L}_2$ of $\mathbf{L}_1$, we might consider transforming $\langle X,Y,I_1\rangle$ into an $\mathbf{L}_2$-context $\langle X,Y,I_2\rangle$ so that the formal concepts present in $\langle X,Y,I_2\rangle$ are good approximations of the formal concepts presented in $\langle X,Y,I_1\rangle$. We can think of $\mathcal{B}(X,Y,I_1)$ as the original solution set which may be infinite and $\mathcal{B}(X,Y,I_2)$ as its finite approximation. The first step in the process is to find a suitable $\mathbf{L}_2$-context $\langle X,Y,I_2\rangle$. To achieve this goal, we define a mapping assigning to each truth degree from L_1 its approximation in L_2. For simplicity, we restrict ourselves to cases where both $\mathbf{L}_1$ and $\mathbf{L}_2$ are linearly ordered and $\mathbf{L}_2$ is finite only. This is sufficient for the goal of approximating solution sets over $[0,1]$ by solutions sets over finite subsets of $[0,1]$.

Define $\text{disc}(L_1,L_2)\colon L_1 \to L_2$ as a mapping satisfying

$$(\text{disc}(L_1,L_2))(a) = b \quad\text{iff}\quad a \leftrightarrow b = \bigvee_{c\in L_2}\big(a \leftrightarrow c\big). \qquad (12)$$

Since L_2 is a finite chain, there is always at least one mapping satisfying (12). Note that $\text{disc}(L_1,L_2)$ is not given uniquely in general. The mapping satisfying (12) will be called a *discretization function.* For a fuzzy set $A \in L_1^U$ we define a fuzzy set $(\text{disc}(L_1,L_2))(A) \in L_2^U$ by a componentwise application of $\text{disc}(L_1,L_2)$ as follows:

$$\big((\text{disc}(L_1,L_2))(A)\big)(u) = (\text{disc}(L_1,L_2))(A(u)). \qquad (13)$$

The definition (13) can also be introduced for binary fuzzy relations $A \in L_1^{U\times V}$.

Now, given $\langle X,Y,I_1\rangle$ where $I_1 : X \times Y \to L_1$ we can consider $I_2\colon X \times Y \to L_2$ which equals to $(\text{disc}(L_1,L_2))(I_1)$. Hence, I_2 is a discretization of I_1 which is induced by mapping $\text{disc}(L_1,L_2)$, see (12) and (13).

Example 2: Let $\mathbf{L}_1$ be the finite Łukasiewicz chain and consider an $\mathbf{L}_1$-context which is given by the table

I_1	y_1	y_2	y_3
x_1	0.00	0.27	0.52
x_2	0.56	1.00	0.68
x_3	0.34	0.73	1.00

Furthermore, consider $L_2 = \{0, 0.25, 0.5, 0.75, 1\}$ and a discretization function $\text{disc}(L_1, L_2)$ defined by

$$(\text{disc}(L_1, L_2))(a) = \begin{cases} 0 & \text{if } a \in [0, 0.125), \\ 0.25 & \text{if } a \in [0.125, 0.375), \\ 0.5 & \text{if } a \in [0.375, 0.625), \\ 0.75 & \text{if } a \in [0.625, 0.875), \\ 1 & \text{if } a \in [0.875, 1]. \end{cases}$$

Then, the induced $\mathbf{L}_2$-context $I_2 = (\text{disc}(L_1, L_2))(I_1)$ will be the following:

I_2	y_1	y_2	y_3
x_1	0.00	0.25	0.50
x_2	0.50	1.00	0.75
x_3	0.25	0.75	1.00

Intuitively, if the discretization of the context looks similar to the original context (i.e., the truth degrees contained in the new context are similar to the corresponding degrees from the original table), then the two contexts should induce similar concepts. This is indeed so, as we are going to show in the sequel.

The following assertion says that every fuzzy set is similar to its discretization at least to a degree given by (10).

Lemma 1: Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be linear complete residuated lattices such that $\mathbf{L}_2$ is a finite substructure of $\mathbf{L}_1$. Then, for each $A_1 \in L_1^U$, we have:

$$\text{appr}(\mathbf{L}_1, \mathbf{L}_2) \leq A_1 \approx (\text{disc}(L_1, L_2))(A_1). \qquad (14)$$

Proof: We show that for each $u \in U$,

$$\text{appr}(\mathbf{L}_1, \mathbf{L}_2) \leq A_1(u) \leftrightarrow ((\text{disc}(L_1, L_2))(A_1))(u).$$

Using (10), (12) and (13), we get

$$\begin{aligned} \text{appr}(\mathbf{L}_1, \mathbf{L}_2) &\leq \textstyle\bigvee_{c \in L_2} \big(A_1(u) \leftrightarrow c\big) = \\ &= A_1(u) \leftrightarrow ((\text{disc}(L_1, L_2))(A_1(u)) = \\ &= A_1(u) \leftrightarrow ((\text{disc}(L_1, L_2))(A_1))(u). \end{aligned}$$

which finishes the proof. ■

The following assertion shows that the derivation operators $^{\downarrow}, ^{\uparrow}$ used in discretized contexts with discretized fuzzy sets of objects or attributes yield similar results as the original operators in the original contexts. The similarity of results is bounded from below by the degree of the approximation of the original structure of truth degrees by the finite one.

Lemma 2: Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be linear complete residuated lattices such that $\mathbf{L}_2$ is a finite substructure of $\mathbf{L}_1$. Moreover, let $\langle X, Y, I_1 \rangle$ be an $\mathbf{L}_1$-context and $\langle X, Y, I_2 \rangle$ be an $\mathbf{L}_2$-context such that $I_2 = (\text{disc}(L_1, L_2))(I_1)$. Then, for each $A_1 \in L_1^X$, $B_1 \in L_1^Y$, we have

$$\text{appr}(\mathbf{L}_1, \mathbf{L}_2)^2 \leq A_1^{\uparrow_{I_1}} \approx (\text{disc}(L_1, L_2))(A_1)^{\uparrow_{I_2}}, \qquad (15)$$

$$\text{appr}(\mathbf{L}_1, \mathbf{L}_2)^2 \leq B_1^{\downarrow_{I_1}} \approx (\text{disc}(L_1, L_2))(B_1)^{\downarrow_{I_2}}, \qquad (16)$$

where $\text{appr}(\mathbf{L}_1, \mathbf{L}_2)^2$ denotes $\text{appr}(\mathbf{L}_1, \mathbf{L}_2) \otimes \text{appr}(\mathbf{L}_1, \mathbf{L}_2)$.

Proof: Due to the limited scope of this paper, we present only a sketch of the proof. The full version of the proof is postponed to the full version of the paper.

We prove only (15) since the proof of (16) is symmetrical. Denote $\big(\text{disc}(L_1, L_2))(A_1)$ by A_2. Using adjointness and properties of residuated lattices, one can show that (15) is true iff for each $x \in X$ and $y \in Y$, the following inequalities are satisfied:

$$A_2(x) \otimes A_1^{\uparrow_{I_1}}(y) \otimes \text{appr}(\mathbf{L}_1, \mathbf{L}_2)^2 \leq I_2(x, y), \qquad (17)$$

$$A_1(x) \otimes A_2^{\uparrow_{I_2}}(y) \otimes \text{appr}(\mathbf{L}_1, \mathbf{L}_2)^2 \leq I_1(x, y). \qquad (18)$$

In order to prove (17), observe that

$$\begin{aligned} &A_2(x) \otimes A_1^{\uparrow_{I_1}}(y) \otimes \text{appr}(\mathbf{L}_1, \mathbf{L}_2)^2 \leq \\ &\leq A_2(x) \otimes (A_1(x) \to I_1(x, y)) \otimes \text{appr}(\mathbf{L}_1, \mathbf{L}_2)^2. \end{aligned}$$

Using Lemma 1, we get

$$\begin{aligned} \text{appr}(\mathbf{L}_1, \mathbf{L}_2) &\leq A_2(x) \to A_1(x), \\ \text{appr}(\mathbf{L}_1, \mathbf{L}_2) &\leq I_1(x, y) \to I_2(x, y), \end{aligned}$$

i.e. by adjointness,

$$\begin{aligned} A_2(x) \otimes \text{appr}(\mathbf{L}_1, \mathbf{L}_2) &\leq A_1(x), \\ I_1(x, y) \otimes \text{appr}(\mathbf{L}_1, \mathbf{L}_2) &\leq I_2(x, y), \end{aligned}$$

from which we get

$$\begin{aligned} &A_2(x) \otimes (A_1(x) \to I_1(x, y)) \otimes \text{appr}(\mathbf{L}_1, \mathbf{L}_2)^2 \leq \\ &\leq A_1(x) \otimes (A_1(x) \to I_1(x, y)) \otimes \text{appr}(\mathbf{L}_1, \mathbf{L}_2) \leq \\ &\leq I_1(x, y) \otimes \text{appr}(\mathbf{L}_1, \mathbf{L}_2) \leq I_2(x, y), \end{aligned}$$

which proves (17); the proof of (18) is symmetrical. ■

The following theorem shows to what degree the concepts present in the discretized data are similar to the concepts present in the original data.

Theorem 1: Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be linear complete residuated lattices such that $\mathbf{L}_2$ is a finite substructure of $\mathbf{L}_1$. Moreover, let $\langle X, Y, I_1 \rangle$ be an $\mathbf{L}_1$-context and $\langle X, Y, I_2 \rangle$ be an $\mathbf{L}_2$-context such that $I_2 = (\text{disc}(L_1, L_2))(I_1)$. Then,

$$\text{appr}(\mathbf{L}_1, \mathbf{L}_2)^2 \leq \mathcal{B}(X, Y, I_1) \approx_{\text{Ext}} \mathcal{B}(X, Y, I_2), \qquad (19)$$

$$\text{appr}(\mathbf{L}_1, \mathbf{L}_2)^2 \leq \mathcal{B}(X, Y, I_1) \approx_{\text{Int}} \mathcal{B}(X, Y, I_2). \qquad (20)$$

Proof: We present a sketch of the proof only. We focus on proving (19) because (20) will then be a consequence of results from [1]. Denote by E_1 and E_2 the sets of all extents of $\mathcal{B}(X, Y, I_1)$ and $\mathcal{B}(X, Y, I_2)$, respectively. It suffices to check

$$\text{appr}(\mathbf{L}_1, \mathbf{L}_2)^2 \leq \textstyle\bigwedge_{A_1 \in E_1} \bigvee_{A_2 \in E_2} A_1 \approx A_2, \qquad (21)$$

and

$$\text{appr}(\mathbf{L}_1, \mathbf{L}_2)^2 \leq \bigwedge_{A_2 \in E_2} \bigvee_{A_1 \in E_1} A_1 \approx A_2. \quad (22)$$

The inequality (21) is true iff, for each $A_1 \in E_1$,

$$\text{appr}(\mathbf{L}_1, \mathbf{L}_2)^2 \leq \bigvee_{A_2 \in E_2} A_1 \approx A_2. \quad (23)$$

Using Lemma 2, for $A_2 = \big((\text{disc}(L_1, L_2))(A_1^{\uparrow_{I_1}})\big)^{\downarrow_{I_2}}$, we get

$$\text{appr}(\mathbf{L}_1, \mathbf{L}_2)^2 \leq A_1^{\uparrow_{I_1}\downarrow_{I_1}} \approx \big((\text{disc}(L_1, L_2))(A_1^{\uparrow_{I_1}})\big)^{\downarrow_{I_2}},$$

which proves (23) because $A_1 = A_1^{\uparrow_{I_1}\downarrow_{I_1}}$ and $A_2 \in E_2$, i.e. the inequality (21) is true.

We now prove (22) by showing that for each $A_2 \in E_2$ there is $A_1 \in E_1$ such that $\text{appr}(\mathbf{L}_1, \mathbf{L}_2)^2 \leq A_1 \approx A_2$. Take $A_2 \in E_2$ and put $A_1 = A_2^{\uparrow_{I_2}\downarrow_{I_1}}$. Lemma 2 yields

$$\text{appr}(\mathbf{L}_1, \mathbf{L}_2)^2 \leq A_1 \approx \big((\text{disc}(L_1, L_2))(A_2^{\uparrow_{I_2}})\big)^{\downarrow_{I_2}}.$$

Observe that for each $a \in L_2$, we have $(\text{disc}(L_1, L_2))(a) = a$. Therefore, $(\text{disc}(L_1, L_2))(A_2^{\uparrow_{I_2}}) = A_2^{\uparrow_{I_2}}$. Since $A_2 = A_2^{\uparrow_{I_2}\downarrow_{I_2}}$, we get $\text{appr}(\mathbf{L}_1, \mathbf{L}_2)^2 \leq A_1 \approx A_2^{\uparrow_{I_2}\downarrow_{I_2}} = A_1 \approx A_2$. In addition to that, $A_1 \in E_1$, showing that (22) is true. ■

Remark 1: Let us see what the foregoing results say. Technically, they provide estimations of similarity degrees on the right hand side in terms of the degree $\text{appr}(\mathbf{L}_1, \mathbf{L}_2)$ of approximation of $\mathbf{L}_1$ by $\mathbf{L}_2$. For instance, Theorem 1 says that when we use $\mathbf{L}_2$ instead of $\mathbf{L}_1$ and transform $\langle X, Y, I_1\rangle$ to $\langle X, Y, I_2\rangle$, then the degree

$$\mathcal{B}(X, Y, I_1) \approx_{\text{Ext}} \mathcal{B}(X, Y, I_2)$$

to which $\mathcal{B}(X, Y, I_1)$ is similar to $\mathcal{B}(X, Y, I_2)$ is at least $\text{appr}(\mathbf{L}_1, \mathbf{L}_2)^2$. Two aspects of a result of this type need to be mentioned. First, computing the estimation $\text{appr}(\mathbf{L}_1, \mathbf{L}_2)^2$ is easy. When devising $\mathbf{L}_2$, computing the estimation $\text{appr}(\mathbf{L}_1, \mathbf{L}_2)^2$ enables us to see how well $\mathcal{B}(X, Y, I_1)$ is approximated by $\mathcal{B}(X, Y, I_2)$. Second, suppose we want to see what kind of approximation we need to use in order to have $\mathcal{B}(X, Y, I_1) \approx_{\text{Ext}} \mathcal{B}(X, Y, I_2)$ at least as high as a prescribed level a of similarity. Then the result tells us that we need to choose $\mathbf{L}_2$ such that $\text{appr}(\mathbf{L}_1, \mathbf{L}_2)^2 \geq a$, i.e. we know how fine the discretization $\mathbf{L}_2$ of $\mathbf{L}_1$ needs to be.

Example 3: If $\mathbf{L}_1$ is the standard Łukasiewicz algebra and $\mathbf{L}_2$ is its equidistant substructure with L_2 being (11) then in order to approximate the original concepts (computed using $\mathbf{L}_1$) at least to degree $0 \leq a < 1$, we need to take $\mathbf{L}_2$ so that

$$n \geq -\frac{2}{a-1},$$

e.g., for $a = 0.9$, we must take $n \geq 20$, i.e. $\mathbf{L}_2$ must contain 21 truth degrees to achieve the desired logical precision.

Remark 2: Using a finite scale instead of the infinite one is beneficial not only from the computational point of view. The discretization of structures of truth degrees can also be seen as a way of reducing the size of concept lattices. Concept lattices generated from contexts using t-norm based structures of truth degrees are usually too large or even infinite. Therefore, there is an effort to reduce the generated concepts lattices in that they contain just some (interesting) concepts (see, e.g., [4], [5]). Concept lattices generated using large scales of truth degrees often contain a large number of concepts which are similar to high degrees so that they are virtually indistinguishable for users. A reasonable choice of a finite scale of truth degrees can reduce the vast amount of concepts to a few representatives only. This is illustrated by the next example.

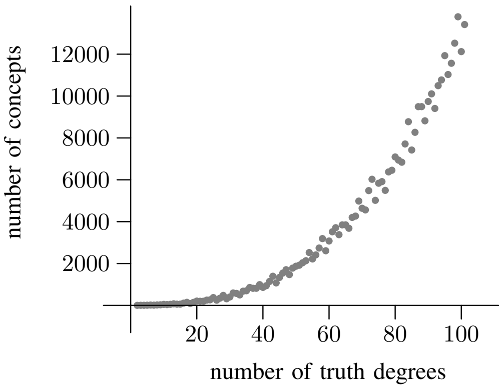

Fig. 1. Dependency between the size of the discrete scale of truth degrees and the number of extracted concepts

Example 4: Let $\mathbf{L}_1$ be the finite Łukasiewicz chain and consider again the $\mathbf{L}_1$-context

I_1	y_1	y_2	y_3
x_1	0.00	0.27	0.52
x_2	0.56	1.00	0.68
x_3	0.34	0.73	1.00

If we take $L_2 = \{\frac{n}{100} \,|\, 0 \leq n \leq 100\}$ equipped with Łukasiewicz operations, then $I_1 = (\text{disc}(L_1, L_2))(I_1)$, i.e. the discretized version of I_1, will coincide with I_1 itself. The concept lattice generated using $\mathbf{L}_2$ contains 13415 concepts which can be seen as not natural because the data table contains just three objects and three attributes. With smaller equidistant scales, we obtain smaller concept lattices generated from the data. The situation for I_1 and its discretizations using 2 up to 101 truth degrees is depicted in Fig. 1.

IV. General Approach and Further Issues

The present approach, which we illustrated on the case of formal concept analysis of data with fuzzy attributes, can be obviously generalized to other situations. A general framework, to which we will generalize our results in future work, is that of a predicate fuzzy logic with the assumption that our constraints are expressed by first-order formulas. Our present example and other examples then become a particular case of this general framework.

Moreover, the present idea of approximating a large (possibly infinite) scale L of truth degrees by a smaller set K

leads to related problems such as the one we are now going to outline. Suppose $A : X \to L$ is a fuzzy set taking values in L such that $M = \{A(x) \mid x \in X\}$ is the set of all degrees "used by A". One might wish to replace A by a different fuzzy set B which approximates A well and "uses" as small a set K of truth degrees as possible. Denote $K = \{B(x) \mid x \in X\}$. Then, given a similarity threshold e, our problem is to find B with

$$\text{appr}(M, K) \geq e. \tag{24}$$

such that K is minimal in terms of the number of its elements. Here, appr(M, K) is defined as earlier in our paper, see (10). A feasible approach to solve this problem is the following. Denote by $\sim_e$ the tolerance on L defined by

$$a \sim_e b \quad \text{iff} \quad a \leftrightarrow b \geq e \otimes e$$

and for any maximal block $B \subseteq L$ of $\sim_e$ set

$$c(B) = e \to \bigwedge B.$$

One can see that $\bigwedge B \in B$. Finally, for any $c \in L$ set

$$B(c) = \{a \in L \mid e \leq a \leftrightarrow c\}.$$

Then, one can show that for any maximal block $B \subseteq L$ of $\sim_e$ it holds $B = B(c(B))$, i.e. elements of maximal block B can be approximated by $c(B)$ with precision given by the threshold e. Moreover, one can prove the following theorem (its proof will be presented in the full version of our paper):

Theorem 2: Let $\Omega \subseteq L/\sim_e$ be a set of maximal blocks of the tolerance $\sim_e$ such that $M \subseteq \bigcup \Omega$ and $K = \{c(B) \mid B \in \Omega\}$. Then (24) is satisfied. Conversely, if $K \subseteq L$ satisfies (24) and for any $a \in K$, $B(a)$ is a maximal block of $\sim_e$ then there is a covering $\Omega \subseteq L/\sim_e$ of M such that $K = \{c(B) \mid B \in \Omega\}$.

The theorem tells how to find set K in terms of maximal blocks of of $\sim_e$ and shows universality of such approach. Moreover, an algorithm can be devised which finds a required K for a given M. Details of the just described method will be presented in our next paper.

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Central points and approximation in residuated lattices

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Abstract

Given a subset B of a complete residuated lattice, what are its points which are reasonably close to any point of B? What are the best such points? In this paper, we seek to answer these questions provided closeness is assessed by means of biresiduum, i.e. the truth function of equivalence in fuzzy logic. In addition, we present two algorithms which output, for a given input set M of points in a residuated lattice, another set K which approximates M to a desired degree. We prove that the algorithms are optimal in that the set K is minimal in terms of the number of its elements. Moreover, we show that the elements of any set K' with such property are bounded from below and from above by the elements produced by the two algorithms.

Keywords: Fuzzy Logic, Approximation, Residuated Lattice.

1 Motivation and preliminaries

Suppose there is a collection of metal poles of different lengths with the longest pole having (normalized) length 1. Suppose a person sees a picture of two poles from that collection and is asked to assess their similarity, i.e. the person is asked to tell a degree $p_1 \approx p_2$ to which the poles are similar. $p_1 \approx p_2 = 0$ and $p_1 \approx p_2 = 1$ indicate that the poles are not similar at all and that the poles are indistinguishable, respectively. Since the poles are narrow, the person assesses their similarity based solely on their lengths. The picture does not show a scale, i.e. the person does not know the actual lengths of the poles. An obvious way to asses the similarity s of poles p_1 and p_2 of lengths $l(p_1)$ and $l(p_2)$ is to put

$$p_1 \approx p_2 = \min\left(\frac{l(p_1)}{l(p_2)}, \frac{l(p_2)}{l(p_1)}\right), \qquad (1)$$

i.e. to make the similarity judgment based on the ratio of the lengths. Namely, the ratio does not depend on the actual lengths, i.e. $p_1 \approx p_2 = \min\left(\frac{c\cdot l(p_1)}{c\cdot l(p_2)}, \frac{c\cdot l(p_2)}{c\cdot l(p_1)}\right)$ for any $c > 0$, so it can be assessed even when the person does not know the actual magnification coefficient $c > 0$, i.e. does not know the scale for the picture. Given poles p_1 and p_2 with lengths $l(p_1)$ and $l(p_2)$, what is the length of the pole in the middle? That is, what is the length of the "central pole" p for which

$$p \approx p_1 = p \approx p_2,$$

i.e. for which the similarity to p_1 equals the similarity to p_2? An easy verification shows that the central pole p has length

$$l(p) = \sqrt{l(p_1)} \cdot \sqrt{l(p_2)}. \qquad (2)$$

Suppose now the person knows the scale, i.e. knows the lengths $l(p_1)$ and $l(p_2)$. Then there is another, perhaps more natural, way to assess the similarity. Namely, one can put

$$p_1 \approx p_2 = 1 - |l(p_1) - l(p_2)|, \qquad (3)$$

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i.e. the similarity is proportional to the distance of the normalized lengths of p_1 and p_2. If such measure of similarity is used, the length of the central pole p is

$$l(p) = \frac{l(p_1) + l(p_2)}{2}. \quad (4)$$

Obviously, given a set $B = \{p_1, \ldots, p_n\}$ of poles, the length of the optimal central pole for B is

$$l(p) = \sqrt{\min_i l(p_i)} \cdot \sqrt{\max_i l(p_i)}$$

for similarity given by (1) and

$$l(p) = \frac{\min_i l(p_i) + \max_i l(p_i)}{2}.$$

for similarity given by (3).

In this paper, we present theorems and algorithms motivated by the above types of problems. The first hint to a general framework for this kind of problems is the observation that in (1),

$$p_1 \approx p_2 = l(p_1) \leftrightarrow l(p_2) \quad (5)$$

with $\leftrightarrow$ being the biresiduum corresponding to product t-norm and that in (2),

$$l(p) = m \otimes \sqrt{l(p_1) \leftrightarrow l(p_2)} \quad (6)$$

with $m = \min\{l(p_1), l(p_2)\}$, $\otimes$ denoting the product t-norm and $\sqrt{}$ denoting its square root [4]. Likewise, (5) and (6) become (3) and (4) if $\leftrightarrow$ and $\otimes$ denote the Łukasiewicz biresiduum and t-norm. Henceforth, we consider the framework of left-continuous t-norms and their residua. In fact, we consider a more general framework of complete residuated lattices [6]. Recall that a complete residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice, $\langle L, \otimes, 1\rangle$ is a commutative monoid, and $\otimes$ and $\rightarrow$ satisfy so-called adjointness condition, i.e. $a \otimes b \leq c$ if and only if $a \leq b \rightarrow c$. Residuated lattices are the main structures of truth degrees used in fuzzy logic [2, 3]. We assume familiarity with examples and basic properties of residuated lattices.

Furthermore, we assume familiarity with basic concepts from tolerance and equivalence relations. Recall that a tolerance relation T on a set U is a reflexive and symmetric relation on U. An equivalence on U is a tolerance which is, moreover, transitive. A block of a tolerance T is a subset B of U for which $B \times B \subseteq T$, i.e. $u\,T\,v$ for every $u, v \in B$. A maximal block of T is a block B of T which is maximal with respect to set inclusion, i.e. such that if $B \subset B'$ then $B' \times B' \nsubseteq T$. A collection of maximal tolerance blocks of T is denoted by U/T. U/T forms a covering of U, i.e. every maximal block is nonempty and the union of all blocks yields U. A class of a tolerance T given by $u \in U$ is the set $[u]_T = \{v \in U \mid u\,T\,v\}$. If T is an equivalence relation, then maximal blocks of T as well as classes of T are just equivalence classes of T.

Given a complete residuated lattice $\mathbf{L}$, denote by $\approx_e$ the tolerance on L defined by

$$a \approx_e b \quad \text{iff} \quad a \leftrightarrow b \geq e$$

and put

$$\begin{aligned} a_e &= e \otimes a, \\ a^e &= e \rightarrow a, \\ [a]_e &= [a_e, (a_e)^e]. \end{aligned}$$

Note that $[p, q]$ denotes the interval $\{x \in L \mid p \leq x \leq q\} \subseteq L$. It can be easily verified that $\approx_e$ a compatible tolerance relation on the complete lattice $\langle L, \leq\rangle$. As a result, the following theorem follows directly from [7]:

Theorem 1 *The factor set $L/{\approx_e}$ is equal to the set $\{[a]_e \mid a \in L\}$.*

2 Maximal blocks and central sets

Let $B \subseteq L$ be a set. We set

$$\begin{aligned} &C_e(B) = \\ &\quad = \{c \in L \mid \text{for } b \in B,\, c \leftrightarrow b \geq e\}. \quad (7) \end{aligned}$$

$C_e(B)$ is called the *e-central set of* B (or simply a central set of B), its elements are called *e-central points of* B (or simply central points of B).

Lemma 1 *$c \in C_e(B)$ iff $(c \rightarrow \bigwedge B) \wedge (\bigvee B \rightarrow c) \geq e$.*

Proof. Follows easily from $c \to (\bigwedge_{b\in B} b) = \bigwedge_{b\in B}(c \to b)$ and $(\bigvee_{b\in B} b) \to c = \bigwedge_{b\in B}(b \to c)$. □

The following theorem shows how to compute the central set $C_e(B)$ of a subset $B \subseteq L$.

Theorem 2 *For any $B \subseteq L$, $C_e(B)$ is equal to $[e \otimes \bigvee B, e \to \bigwedge B]$.*

Proof. By adjointness, $e \le c \to \bigwedge B$ is equivalent to $c \le e \to \bigwedge B$ and $e \le \bigvee B \to c$ is equivalent to $e \otimes \bigvee B \le c$. Thus the assertion follows from Lemma 1. □

For $c \in L$ set

$$B_e(c) = \{b \in L \mid c \leftrightarrow b \ge e\}. \tag{8}$$

$B_e(c)$ is called the *closed ball with center c and radius e*. Since $c \in B_e(c)$, $B_e(c)$ is always nonempty. A closed ball $B_e(c)$ is called *maximal* if there is no $\bar{c} \ne c$ such that $B_e(c) \subset B_e(\bar{c})$.

Note that a closed ball $B_e(c)$ is exactly the class of tolerance $\approx_e$ determined by c.

Example 1 In the Łukasiewicz structure, $B_e(c)$ is just the interval $[c-(1-e), c+(1-e)] \cap [0,1]$. Hence the closed ball $B_{0.5}(0) = [0, 0.5]$ is not maximal: $B_{0.5}(0) \subset B_{0.5}(0.5) = [0,1]$.

The following result is a simple consequence of the above definitions. Note, however, that it does not say that the central set $C_e(B)$ is not empty.

Lemma 2 *For any subset $B \subseteq L$ and $c \in C_e(B)$ it holds $B \subseteq B_e(c)$.*

Proof. Let $b \in B$. Using (7), we get $c \leftrightarrow b \ge e$, and from (8), we get $b \in B_e(c)$. □

The following theorem provides an easy way to compute any closed ball with given center and radius.

Theorem 3 *For any $c \in L$, the closed ball $B_e(c)$ is equal to the interval $[e \otimes c, e \to c]$.*

Proof. The condition $b \leftrightarrow c \ge e$ from the definition of closed ball has two parts: $b \to c \ge e$ and $c \to b \ge e$. By adjointness, the first part is equivalent to $b \le e \to c$, the second to $b \ge e \otimes c$. □

Corollary 1 *For any $c \in L$, $c \in C_e(B_e(c))$.*

Proof. From Theorem 2 and Theorem 3 we obtain $C_e(B_e(c)) = [e \otimes (e \to c), e \to (e \otimes c)]$ and from adjointness, $e \otimes (e \to c) \le c \le e \to (e \otimes c)$. □

Now we turn our attention to the relationship between closed balls with radius e and blocks of the tolerance $\approx_{e^2}$ (where $e^2 = e \otimes e$) and show that maximal closed balls with radius e coincide with maximal blocks of this tolerance.

Lemma 3 *For any $c \in L$, $B_e(c)$ is a block of $\approx_{e^2}$.*

Proof. From Theorem 3, $e \otimes c$ and $e \to c$ are the least and the greatest elements of $B_e(c)$, respectively. From Theorem 1, the element $e \otimes e \otimes (e \to c)$ is the least element of a maximal block of $\approx_{e^2}$ containing $e \to c$. Since $e \otimes e \otimes (e \to c) \le e \otimes c$, $B_e(c)$ is contained in this maximal block, which proves the lemma. □

Lemma 4 *For any subset $B \subseteq L$, the central set $C_e(B)$ is nonempty if and only if B is a block of the tolerance $\approx_{e^2}$.*

Proof. According to Theorem 2, the nonemptiness of $C_e(B)$ is equivalent to the condition $e \otimes \bigvee B \le e \to \bigwedge B$, which is, according to adjointness, equivalent to $e \otimes e \le \bigvee B \to \bigwedge B$ which is equivalent to the fact that B is a block of $\approx_{e^2}$. □

Now we put together results of the previous lemmas.

Theorem 4 *Every maximal closed ball $B_e(c)$ is a maximal block of $\approx_{e^2}$. Conversely, if $B \subseteq L$ is a maximal block of $\approx_{e^2}$ then the central set $C_e(B)$ is nonempty and for any $c \in C_e(B)$ the closed ball $B_e(c)$ is maximal and equal to B.*

Proof. According to Lemma 3, a maximal ball $B_e(c)$ is a block of $\approx_{e^2}$. It is therefore contained in a maximal block B. From Lemma 4 it follows that this maximal block has at least one central point $c' \in B$. Now, Lemma 2 says that the closed ball $B_e(c')$ contains B. Put together, these considerations give $B_e(c) \subseteq B \subseteq B_e(c')$ and from the maximality of $B_e(c)$ we obtain $B_e(c) = B$.

To prove the converse, we first use Lemma 4 again to obtain $C_e(B) \neq \emptyset$. Now, any closed ball containing $B_e(c)$ should be equal to B, because it is a block of $\approx_{e^2}$ itself (Lemma 3). Any other closed ball containing $B_e(c)$ should be also equal to it by the same reason (Lemma 3). □

3 Optimal central points

So far, we investigated maximal sets which have nonempty sets of e-central points. Now we turn to another problem: find a maximal e such that the set of e-central points of a given set is nonempty.

We say that e is an *admissible radius* of set B if $C_e(B) \neq \emptyset$. From Lemma 2 it follows that if e is an admissible radius of B, then $B \subseteq B_e(c)$ for any $c \in C_e(B)$. Lemma 1 says that for any such c,

$$e \leq (c \to \bigwedge B) \wedge (\bigvee B \to c). \tag{9}$$

An *optimal central point* for $B \subseteq L$ is an element $c \in L$ such that for every m:

$$\bigwedge_{z \in B}(z \leftrightarrow m) \leq \bigwedge_{z \in B}(z \leftrightarrow c).$$

Since for any m we have $\bigwedge_{z \in B}(z \leftrightarrow m) = (m \to \bigwedge B) \wedge (\bigvee B \to m)$ (see, for example, the proof of Lemma 1), c is an optimal central point iff for every m:

$$(m \to \bigwedge B) \wedge (\bigvee B \to m) \leq (c \to \bigwedge B) \wedge (\bigvee B \to c) \tag{10}$$

Theorem 5 1. *For any optimal central point c of B, $e = (c \to \bigwedge B) \wedge (\bigvee B \to c)$ is the largest admissible radius of B.*

2. *If e is the largest admissible radius of B then the set of optimal central points of B is nonempty and is equal to $C_e(B)$.*

Proof. 1. By Lemma 1, $c \in C_e(B)$, which also means that e is an admissible radius of B. Now the assertion follows from (9) and (10).

2. The fact that $C_e(B)$ is nonempty follows directly from definition of admissible radius. Now, for any m, $(m \to \bigwedge B) \wedge (\bigvee B \to m)$ is an admissible radius, hence it is less than or equal to e. On the other hand, for any $c \in C_e(B)$ we have $B \subseteq B_e(c)$, which means $(c \to \bigwedge B) \wedge (\bigvee B \to c) \geq e$. Put together, (10) is satisfied for any $m \in L$, $c \in C_e(B)$. □

Lemma 5 *Let $d = \bigwedge B \to \bigvee B$. Then*

1. *e is an admissible radius of B iff $e \otimes e \leq d$.*

2. *For any $z \in L$, $e = z \wedge (z \to d)$ is an admissible radius of B.*

3. *e is an admissible radius of B iff $e = e \wedge (e \to d)$.*

Proof. 1. $e \otimes e \leq d$ if and only if B is a block of $\approx_{e^2}$, which is equivalent to the requirement that B is a subset of some closed ball $B_e(c)$ (Theorem 4). Hence, c is an e-central point of B.

2. We have $e \otimes e = (z \wedge (z \to d)) \otimes (z \wedge (z \to d)) \leq z \otimes (z \to d) \leq d$ and the result follows from 1.

3. From 1. and adjointness we obtain that e is an admissible radius of B iff $e \leq e \to d$, which is equivalent to $e = e \wedge (e \to d)$. □

Corollary 2 *The set*

$$\{z \wedge (z \to (\bigwedge B \to \bigvee B)) \mid z \in L\} \tag{11}$$

is the set of all admissible radii of B.

Proof. Follows from Lemma 5, parts 2. and 3. □

Theorem 6 (optimal central points) *Set B has optimal central points if and only if the set P from Corollary 2 has a largest element. This element is equal to the corresponding largest admissible radius e.*

Proof. Follows directly from Corollary 2 and Theorem 5. □

Now we derive a simple consequence of the previous results for the case of residuated lattices with square roots. We will use the concept of a square root introduced by Höhle [4]. A complete residuated lattice **L** has square roots if there is a function $\sqrt{\ }: L \to L$ satisfying

$$\sqrt{a} \otimes \sqrt{a} = a, \quad (12)$$
$$b \otimes b \leq a \quad \text{implies} \quad b \leq \sqrt{a}, \quad (13)$$

for every $a, b \in L$.

Remark 1 Łukasiewicz, product, and Gödel algebras on $[0, 1]$ have sqaure roots. They are given by

$$\sqrt{a} = \frac{a+1}{2} \quad \text{for Łukasiewicz},$$
$$\sqrt{a} = \text{ordinary number-theoretic square root of } a \text{ for product},$$
$$\sqrt{a} = a \quad \text{for Gödel}.$$

Theorem 7 *If* **L** *has square roots then any subset $B \subseteq L$ has optimal central points. For the corresponding largest admissible radius e it holds*

$$e = \sqrt{\bigwedge B \to \bigvee B}. \quad (14)$$

Proof. According to Lemma 5, part 1. and (12), e is the largest admissible radius of B. The rest follows from Theorem 5, part 1. □

4 Optimal algorithms for approximating sets of truth degrees

We now consider the following type of problems. Given a set M of truth degrees, find a reasonably small set K of truth degrees which approximates M well. We provide a precise statement below. Due to limited scope, we omit proofs in this section.

Note first that such problem naturally arises in the following scenario: Let $A : U \to L$ be a fuzzy set in universe U with $M = \{A(u) \,|\, u \in U\}$ being the set of truth degrees "used by A". Find a fuzzy set $B : U \to L$ which approximates A well and for which, in addition, the set $K = \{B(u) \,|\, u \in U\}$ of truth degrees "used by B" is small. In general terms, the advantage of B over A is its simplicity. As an example, B is easier to interpret. Due to the well-known Miller's 7 ± 2 phenomenon [5], people have difficulty to assign and interpret consistently more than 7 ± 2 values of a given variable. So if A represents degrees to which objects (such as products) meet certain criteria, it might be better to present B as an output instead of A.

We consider the following general definition.

A degree appr(M, K) *to which $M \subseteq L$ is approximated by $K \subseteq L$* is defined by

$$\text{appr}(M, K) = \bigwedge_{a \in M} \bigvee_{b \in K} (a \leftrightarrow b). \quad (15)$$

appr(M, K) can be seen as a truth degree of "for every $a \in M$ there is $b \in K$ such that a and b are similar (close)". Hence, appr(M, K) can be understood as a natural degree of approximation. Among the basic properties of appr$(\cdot, \cdot)$ are

1. appr$(M, K) = 1$ for $M \subseteq K$,
2. $K_1 \subseteq K_2$ implies appr$(M, K_1) \leq$ appr(M, K_2).

We now present two problems.

Problem 1 Given (finite) $M \subseteq L$ and a threshold $e \in L$, find (finite) K such that

1. K approximates M to degree at least e, i.e.
$$\text{appr}(M, K) \geq e, \quad (16)$$
2. there is no K' with $|K'| \leq |K|$ for which appr$(M, K') > e$, i.e. K is a least set in terms of the number of its elements which satisfies (16).

Problem 2 Given (finite) $M \subseteq L$ and a threshold $e \in L$, find (finite) K satisfying 1. and 2. of Problem 1, and

3. For any K' with $|K'| = |K|$,
$$\text{appr}(M, K) \geq \text{appr}(M, K'), \quad (17)$$
i.e. among the sets with $|K|$ elements, K provides the best approximation of M.

In the rest of this section, we assume that the complete residuated lattice $\mathbf{L}$ is linearly ordered, i.e. $a \leq b$ or $b \leq a$ for every $a, b \in L$. The following theorem provides a universal description of sets K satisfying (16).

Theorem 8 *Let* $\mathbf{L}$ *be linearly ordered.* 1. *Let* $\Omega \subseteq 2^L$ *be a covering of* M *and* $\varphi\colon \Omega \to L$ *a mapping such that for any* $B \in \Omega$, $\varphi(B) \in C_e(B)$. *Then* Ω *consists of blocks of the tolerance* $\approx_{e^2}$ *and for* $K = \varphi(\Omega)$, *(16) is satisfied.*

2. *If* $K \subseteq L$ *satisfies (16) then* $\Omega = \{B_e(c) \mid c \in K\} \subseteq 2^L$ *is a set of blocks of the tolerance* $\approx_{e^2}$ *which forms a covering of* M. *Moreover* $\varphi\colon \Omega \to L$ *defined by* $\varphi(B_e(c)) = c$ *satisfies* $\varphi(B) \in C_e(B)$.

Proof. Omitted due to lack of space. □

According to Theorem 2, $C_e(B)$ is equal to $[e \otimes \bigvee B, e \to \bigwedge B]$. Hence, we can construct a mapping φ from the first part of the above theorem by setting $\varphi(B)$ to any element of this interval (which is nonempty according to Lemma 4). Obviously, in order for K to provide a good approximation degree $\text{appr}(M, K)$, the best choice is to let $\varphi(B)$ be an optimal central point of B.

Example 2 Let $L = [0, 1]^2$ with Łukasiewicz structure, $M = L$, $e = \langle 0.25, 0.25\rangle$. Then

$$K = \{\langle 0, 0\rangle, \langle 0.5, 0\rangle, \langle 1, 0\rangle, \langle 0, 0.5\rangle, \langle 0, 1\rangle\}$$

satisfies (16). However, for $a = \langle 1, 1\rangle$, we have $\bigvee_{b \in K} a \leftrightarrow b = \langle 1, 1\rangle$, but $B_e(a) \cap K = \emptyset$ for there is no $b \in K$ such that $a \leftrightarrow b \geq e$. Therefore, assertion 2. from Theorem 8 does not hold.

We now present two algorithms which provide solutions to Problem 1. The first algorithm constructs K by "going up" in the set L of truth degrees.

Algorithm 1

```
1:  INPUT: M, e
2:  OUTPUT: K satisfying 1.  and 2.
            of Problem 1
3:  K ← ∅
4:  while M is not empty do
5:    min ← min(M)
6:    add e → min to K
7:    remove from M every
         element ≤ (e ⊗ e) → min
8:  endwhile
9:  return K
```

The second algorithm constructs K by "going down".

Algorithm 2

```
1:  INPUT: M, e
2:  OUTPUT: K satisfying 1.  and 2.
            of Problem 1
3:  K ← ∅
4:  while M is not empty do
5:    max ← max(M)
6:    add e ⊗ max to K
7:    remove from M every
         element ≥ e ⊗ e ⊗ max
8:  endwhile
9:  return K
```

As the next theorem shows, the algorithms stop and are correct, i.e. they produce a set K of minimal size for which $\text{appr}(M, K) \geq e$.

Theorem 9 (termination, correctness) 1. *Algorithms 1 and 2 terminate after at most* $|M|$ *steps.* 2. *Algorithms 1 and 2 are correct.*

Proof. Omitted due to lack of space. □

Note that Algorithms 1 and 2 work conceptually even for infinite sets M when replacing $\inf(M)$ by $\min(M)$ and $\sup(M)$ by $\max(M)$ in line 5.

Futhermore, the algorithms provide upper and lower bound for every set K' with the minimal number of elements which approximate M to degree at least e.

Theorem 10 (bounds) *Let the sets* K^u *and* K^l *produced by Algorithm 1 and Algorithm 2 consist of elements* $k_1^u < \cdots < k_m^u$ *and* $k_1^l < \cdots < k_m^l$, *respectively. If* K' *consisting of* $k'_1 < \cdots < k'_m$ *satisfies* $\text{appr}(M, K') \geq e$, *then*

$$k_1^l \leq k'_1 \leq k_1^u, \ldots, k_m^l \leq k'_m \leq k_m^u.$$

Proof. Omitted due to lack of space. □

The following example shows that not every $K' = \{k'_1, \ldots, k'_m\}$ for which $k^l_i \leq k'_i \leq k^u_i$ satisfies $\mathrm{appr}(M, K') \geq e$.

Example 3 Consider standard Łukasiewicz structure on $L = [0,1]$, $M = \{0.5, 0.7, 0.8\}$, and $e = 0.9$. Then $K^l = \{0.4, 0.7\}$ and $K^u = \{0.6, 0.9\}$. Let $K' = \{0.4, 0.9\}$. Then $\mathrm{appr}(M, K') = 0.8 < 0.9 = e$.

Although the set K produced by Algorithm 1 or Algorithm 2 is optimal in that it is one of the smallest sets for which $\mathrm{appr}(M, K) \geq e$, there can be a set K' of the same size, i.e. $|K'| = |K|$, for which $\mathrm{appr}(M, K') > \mathrm{appr}(M, K)$, i.e. K' provides a better approximation of M than K. From this point of view, the output set K from Algorithm 1 and Algorithm 2 can be improved. Namely, it is easily seen from the description of Algorithm 1 and Algorithm 2 that the set

$$\{B_e(k) \cap M \mid k \in K\}$$

forms a partition of M, i.e. sets $B_e(k) \cap M$ for $k \in K$ are pairwise disjoint and their union is M. Now, in general, k is not an optimal central point of $B_e(k) \cap M$. Therefore, we can improve K by replacing every $k \in K$ by an optimal central point of $B_e(k) \cap M$ (or a point which provides a better apprximation of $B_e(k) \cap M$ than k). By Theorem 7, if $\mathbf{L}$ has square roots, then any element from

$$\Big[\sqrt{\bigwedge(B_e(k) \cap M)} \otimes \bigvee(B_e(k) \cap M), \sqrt{\bigwedge(B_e(k) \cap M)} \rightarrow \bigwedge(B_e(k) \cap M)\Big]$$

can be used to replace k. For instance, for $M = \{0.5, 0.7, 0.8\}$ and $K = K^l = \{0.4, 0.7\}$ from Example 3, $B_{0.9}(0.7) \cap M = \{0.7, 0.8\}$ and $B_{0.9}(0.4) \cap M = \{0.5\}$. Hence, 0.4 can be replaced by 0.5 (optimal central point of $\{0.5\}$) and 0.7 can be replaced by 0.75 (optimal central point of $\{0.7, 0.8\}$). As a result, we get a set $K' = \{0.5, 0.75\}$ for which $\mathrm{appr}(M, K') = 0.95 > 0.9 = \mathrm{appr}(M, K)$, i.e. K' provides a better approximation of M than K does.

Still, such improvement does not, in general, satisfy condition 3. of Problem 2. That is, replacement of points k in K by better points k' which cover the same part of M, i.e. for which $B_e(k) \cap M = B_e(k') \cap M$, does not result in the best approximating set with size $|K|$. An algorithm which provides such set, i.e. which provides a solution to Problem 3, is the subject of our future research.

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Factorization of residuated lattices

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Abstract

We discuss the problem of factorization of residuated lattices by similarity relations. As the main result, we introduce a natural structure of residuated lattice on factorized residuated lattice. Some consequences are also discussed: the problem of representatives and factor projections, sequential factorization, application to fuzzy sets, application to factorization of concept lattices of data with fuzzy attributes.

Keywords: Fuzzy logic, Factorization, Residuated lattice, Similarity, Concept lattice

1 Introduction

Factorization is an important procedure for simplification of systems. The basic idea is to reduce complexity of a system by putting together elements, considered as similar. In other words, by factorizing-out small or unimportant differences between elements it is possible to obtain smaller amount of data. This method is also called Simplification by abstraction.

Factorization of algebraic systems by congruence relations (i.e., compatible equivalence relations) is well understood. Sometimes, however, the used relation is not transitive, i.e., it is only reflexive and symmetric (a tolerance relation), but is compatible with the structure of a system. This is usually the case when a relation is used for expressing similarity or proximity of elements of a system. Tolerance relations have been studied in the context of algebraic systems in many papers, see e.g. [6, 9].

In [7] and [10], factorization of lattices has been studied. The main result is that there can be introduced a lattice structure on lattice factorized by compatible tolerance. Lattices can serve as models of systems with hierarchically ordered elements. In this paper, we extend these results to residuated lattices.

Residuated lattices are heavily used in fuzzy logic as algebraic structures of truth degrees [8]. Roughly speaking, residuated lattice consists of truth degrees and is endowed with additional algebraic operations, generalizing classical logical connectives such as conjunction or implication.

In this paper, we study the possibility of factorizing residuated lattices by compatible tolerances (more precisely, by e-cuts of biresiduum, see beginning of Sec. 2.3), which are used for measuring similarity of truth values in fuzzy logic. As the main result, we introduce a structure of residuated lattice on the factor set and study its basic properties. We show that our way is the only natural way of introducing a residuated lattice structure on the factorized residuated lattice.

In this paper, we use results of [7, 10], who studied factorization of ordinary (non-residuated) lattices (complete ones in [10]). Their main results, needed in this paper, are summarized in Sec. 2.2. In Sec. 2.3, we recapitulate known consequences of this general theory to residuated lattices, as shown for example in [2, 3]. Our main result is presented in

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Sec. 3.1 where we define a structure of residuated lattice on any residuated lattice factorized by a cut of biresiduum.

Some consequences of the main result are given in the following sections. Section 3.2 deals with the problem of representatives and factor projections and contains examples showing that the structure of residuated lattice on factorized residuated lattices, introduced in this paper, is the only possible satisfying some natural requirements. In Section 3.3 we prove a result on factorization of residuated lattices which themselves are constructed by factorization of other residuated lattices.

In Sec. 4.1, we give a slight generalization of our results concerning factorization of the set of all fuzzy subsets in a given universe. In Sec. 4.3 we show an application of our results in the field of Formal Concept Analysis of data with fuzzy values of attributes. In the last section we outline a possible approach to the problem of minimization of any fuzzy system over a residuated lattice by means of factorization of the underlying residuated lattice.

Preliminary version of some results of this paper were previously published in Krupka M. Factorization of residuated lattices with application to concept lattices, *In*: R.Trappl (ed.) Cybernetics and Systems 2008, Austrian Society for Cybernetic Studies, Vienna (2008).

2 Preliminaries

In the following sections, we summarize basic known facts. Section 2.1 gives an overview of the theory of complete residuated lattices. In Sec. 2.2 we outline basic results on factorization of complete lattices by compatible tolerance relations, in Sec. 2.3 we give some consequences of these results for the case of complete residuated lattices and cuts of biresiduum as compatible tolerances.

2.1 Residuated lattices

We start by recapitulation of basics of the theory of complete residuated lattices. For more detailed review, we refer the reader to [2, 8].

A complete residuated lattice is defined as an algebra

$$\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle \tag{1}$$

such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with the least element 0 and the greatest element 1; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\to$ satisfy so-called adjointness property:

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \to c \tag{2}$$

for each $a, b, c \in L$. Elements a of L are called truth degrees. $\otimes$ and $\to$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication".

A biresiduum of $\mathbf{L}$ is a binary operation $\leftrightarrow$ defined by

$$a \leftrightarrow b = (a \to b) \wedge (b \to a). \tag{3}$$

It is a truth function of logical connective "fuzzy equivalence".

Example 1 A common choice of **L** is a structure with $L = [0,1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding $\rightarrow$. Three most important pairs of adjoint operations on the unit interval are:

$$\text{Łukasiewicz:} \quad \begin{aligned} a \otimes b &= \max(a+b-1, 0), \\ a \rightarrow b &= \min(1-a+b, 1), \end{aligned} \tag{4}$$

$$\text{Gödel:} \quad \begin{aligned} a \otimes b &= \min(a, b), \\ a \rightarrow b &= \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise,} \end{cases} \end{aligned} \tag{5}$$

$$\text{Goguen (product):} \quad \begin{aligned} a \otimes b &= a \cdot b, \\ a \rightarrow b &= \begin{cases} 1 & \text{if } a \leq b, \\ \frac{b}{a} & \text{otherwise.} \end{cases} \end{aligned} \tag{6}$$

Complete residuated lattices on $[0,1]$ given by (4), (5), and (6) are called standard Łukasiewicz, Gödel, Goguen (product) algebras, respectively.

Example 2 The class of complete residuated lattices includes finite structures as well. For instance, we can put

$$L = \{a_0 = 0, a_1, \ldots, a_n = 1\} \subseteq [0,1], \tag{7}$$

where $a_0 < \cdots < a_n$ are equidistant and $\otimes$ and $\rightarrow$ are restrictions of the operations from (4). In this case, **L** is called an equidistant Łukasiewicz chain. For instance,

$$\begin{aligned} L_3 &= \{0, 0.5, 1\}, \\ L_4 &= \{0, \tfrac{1}{3}, \tfrac{2}{3}, 1\}, \\ L_5 &= \{0, 0.25, 0.5, 0.75, 1\}, \end{aligned}$$

equipped with operations $\otimes$ and $\rightarrow$ which are restrictions of Łukasiewicz operations (4) are equidistant Łukasiewicz chains.

Example 3 A special case of a complete residuated lattice is the two-element Boolean algebra $\langle \{0,1\}, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$, denoted by **2**, which is the structure of truth degrees of the classical logic. That is, the operations $\wedge, \vee, \otimes, \rightarrow$ of **2** are the truth functions (interpretations) of the corresponding logical connectives of the classical logic.

2.2 Factorization by compatible tolerances

This section contains results derived from [7, 10]. We present main theorems without proofs.

A tolerance relation T on a nonempty set X is a binary relation which is reflexive and symmetric. A block of T is any subset $B \subseteq X$ such that for any $x_1, x_2 \in B$ it holds $\langle x_1, x_2 \rangle \in T$. A maximal block of T is a block which is maximal with respect to set inclusion, i.e., $B \subseteq X$ is a maximal block if and only if B is a block and for any other block B' from $B \subseteq B'$ it follows $B = B'$. The set of all maximal blocks of T is called the factor set of X by T and denoted X/T. The system X/Y is a covering of X (it consists of nonempty sets and for any $x \in X$ there is $B \in X/Y$ containing x), but need not form a partition of X (elements $B_1, B_2 \in X/T$ can overlap; in general $B_1 \cap B_2 \neq \emptyset$).

A compatible tolerance relation on a complete lattice $\mathbf{L}$ is a tolerance which preserves suprema and infima, i.e., a tolerance $\approx$ on $\mathbf{L}$ is compatible if from $a_j \approx b_j$ for any $j \in J$ follows $\bigvee_{j\in J} a_j \approx \bigvee_{j\in J} b_j$ and $\bigwedge_{j\in J} a_j \approx \bigwedge_{j\in J} b_j$.

For $a \in L$ denote

$$a^{\approx} = \bigvee\{b \in L \mid a \approx b\}, \quad a_{\approx} = \bigwedge\{b \in L \mid a \approx b\}, \tag{8}$$

$$[a]_{\approx} = [a_{\approx}, (a_{\approx})^{\approx}], \quad [a]^{\approx} = [(a^{\approx})_{\approx}, a^{\approx}] \tag{9}$$

($[a_1, a_2]$ denotes the interval $\{b \in L \mid a_1 \leq b \leq a_2\}$). From the fact that $\approx$ is a compatible tolerance we immediately obtain

$$a \in [a]_{\approx}, \quad a \in [a]^{\approx} \tag{10}$$

and, if we set $b = \bigvee[a]_{\approx}$, $c = \bigwedge[a]^{\approx}$,

$$[b]_{\approx} = [a]_{\approx}, \quad [c]^{\approx} = [a]^{\approx}. \tag{11}$$

The following theorem shows that maximal blocks of $\approx$ are exactly sets $[a]_{\approx}$ and $[a]^{\approx}$.

Theorem 1 *For a complete lattice* $\mathbf{L}$ *and compatible tolerance* $\approx$ *on* $\mathbf{L}$ *it holds* $L/{\approx} = \{[a]_{\approx} \mid a \in L\} = \{[a]^{\approx} \mid a \in L\}$.

Ordering on the set $L/{\approx}$ is introduced using suprema of maximal blocks and can be equivalently introduced using infima. For blocks $B_1, B_2 \in L/{\approx}$ we set

$$B_1 \leq B_2 \quad \text{iff} \quad \bigvee B_1 \leq \bigvee B_2. \tag{12}$$

Theorem 2 *The set* $L/{\approx}$ *together with the ordering* $\leq$ *is a complete lattice.*

If $\mathbf{L}$ is a complete lattice with the support set L then the complete lattice with the support set $L/{\approx}$ and the ordering we have just introduced will be denoted by $\mathbf{L}/{\approx}$. More formally, using the operations $\wedge$ and $\vee$ of infimum and supremum induced by the ordering $\leq$, we can write $\mathbf{L}/{\approx} = \langle L/{\approx}, \wedge, \vee, 0, 1\rangle$.

Note that for any $a \in L$ it holds

$$[a]_{\approx} = \bigwedge\{B \in L/{\approx} \mid a \in B\}, \tag{13}$$

$$[a]^{\approx} = \bigvee\{B \in L/{\approx} \mid a \in B\}. \tag{14}$$

The following theorem shows that suprema and infima in $\mathbf{L}/{\approx}$ are closed with respect to choices of representatives of maximal blocks.

Theorem 3 *Let for any* $j \in J$ *it holds* $B_j \in L/{\approx}$ *and* $b_j \in B_j$. *Then* $\bigvee_{j\in J} b_j \in \bigvee_{j\in J} B_j$ *and* $\bigwedge_{j\in J} b_j \in \bigwedge_{j\in J} B_j$.

2.3 Factorization by cuts of biresiduum

Results in this section are derived from [2, 3], where a more general approach is presented, namely sets of fixpoints of **L**-closure operators are considered in place of residuated lattice **L**. The main theorem is given without proof.

Using the operation of biresiduum $\leftrightarrow$ and a fixed element $e \in L$ we can define a binary relation $\sim_e$ on L by

$$a_1 \sim_e a_2 \quad \text{iff} \quad a_1 \leftrightarrow a_2 \geq e. \tag{15}$$

This relation is called the e-cut of biresiduum in **L**, "$a_1 \sim_e a_2$" being interpreted as "a_1 and a_2 are similar to a degree greater than or equal to e".

We introduce the following simplified notations: $a_e = a_{\sim_e}$, $a^e = a^{\sim_e}$, $[a]_e = [a]_{\sim_e}$, $[a]^e = [a]^{\sim_e}$. The factor lattice $\mathbf{L}/\sim_e$ will be denoted by $\mathbf{L}/e$.

We have the following simple result:

Theorem 4 *For any $a \in L$ it holds*

$$a_e = e \otimes a, \tag{16}$$
$$a^e = e \to a. \tag{17}$$

Example 4 As it can be easily proved, in Łukasiewicz structure, maximal blocks of $\sim_e$ are closed intervals of length $1-e$, in Goguen structure closed intervals of the form $[ea, a]$ for $e \neq 0$ and the interval $[0,1]$ for $e = 0$. In Gödel structure, maximal blocks of $\sim_e$ are singletons $\{a\}$ for $a < e$ and the interval $[e, 1]$.

As a consequence, we obtain the following equalities, which hold for any maximal block $B \in L/\sim_e$:

$$\bigvee B = e \to \bigwedge B, \tag{18}$$
$$\bigwedge B = e \otimes \bigvee B. \tag{19}$$

We shall use these two equalities frequently throughout the text.

Note the following equalities for the minimal and maximal elements $0, 1 \in L/e$:

$$0 = [0, e \to 0], \tag{20}$$
$$1 = [e, 1]. \tag{21}$$

3 Factorization of residuated lattices

The following sections contain main results of this paper. In Sec. 3.1 we introduce a structure of complete residuated lattice on the factor lattice $\mathbf{L}/e$.

Section 3.2 deals with the problem of representatives and factor projections. This problem can be described briefly as follows. There are two standard requirements for algebraic (and other) structures factorized by equivalence relations: operations on factor sets to be independent on choices of representatives and factor projections to be morphisms of considered structures. In the case of algebras (algebraic systems with operations only), the two requirements are equivalent (i.e. one is a reformulation of the other).

Our case is more complicated in that our factor sets are not partitions of original sets but their coverings only. As a consequence, the two mentioned requirements are generally not fulfilled: operations of product and residuum on the factor lattice are not determined by operations on the original lattice independently on the choice of representatives and there are several (at least two) mappings which can be called factor projections. However, Theorem 7 shows a weaker but still acceptable connection between products and residui of blocks from L/e and their representatives, Theorem 8 shows a similar weaker property of one of the possible factor projections. We also show in some examples that the structure of residuated lattice on factorized residuated lattices, introduced here, is the only one, which satisfies the weakened requirements on representatives and factor projections.

Having a structure of residuated lattice on $\mathbf{L}/e$ leads to a possibility of applying another factorization. For example, we might want to compute several concept lattices from a given data table, each with a lower degree of similarity than previous. In Sec. 3.3, we show that double (and multiple) factorization can be achieved in a single step, by choosing an appropriate, easily computable threshold.

3.1 Structure of factor residuated lattice

In this section, we introduce operations of product and residuum on the complete lattice $\mathbf{L}/e$ and prove their basic properties. As the main result, in Theorem 6 we show that these operations define a structure of residuated lattice on the set L/e.

For $B_1, B_2 \in L/e$ set

$$B_1 \otimes B_2 = \left[\bigvee B_1 \otimes \bigvee B_2\right]_e, \tag{22}$$

$$B_1 \to B_2 = \left[\bigvee B_1 \to \bigvee B_2\right]_e. \tag{23}$$

In Lemma 1 and Theorem 5 we show basic properties of operations $\otimes$ and $\to$ on L/e. Theorem 5 shows alternative ways to define these operations.

Lemma 1 *For any $B_1, B_2 \in L/e$ we have*

$$\bigvee B_1 \otimes \bigwedge B_2 = \bigwedge (B_1 \otimes B_2), \tag{24}$$

$$\bigvee B_1 \to \bigvee B_2 = \bigvee (B_1 \to B_2), \tag{25}$$

$$\bigwedge B_1 \to \bigwedge B_2 = \bigvee (B_1 \to B_2). \tag{26}$$

Proof. From (22) we have $\bigwedge(B_1 \otimes B_2) = e \otimes \bigvee B_1 \otimes \bigvee B_2$, which is equal to $\bigvee B_1 \otimes \bigwedge B_2$. This proves (24).

In (25) we have $\bigvee B_1 \to \bigvee B_2 = \bigvee B_1 \to (e \to \bigwedge B_2) = (e \otimes \bigvee B_1) \to \bigwedge B_2 = e \to (\bigvee B_1 \to \bigwedge B_2)$, hence $\bigvee B_1 \to \bigvee B_2$ is a supremum of some block (explicitly, it is the block $[\bigvee B_1 \to \bigwedge B_2]^e$). From (23) it follows that the block is equal to $B_1 \to B_2$.

To prove (26) we have $\bigwedge B_1 \to \bigwedge B_2 = (e \otimes \bigvee B_1) \to \bigwedge B_2 = \bigvee B_1 \to (e \to \bigwedge B_2) = \bigvee B_1 \to \bigvee B_2$. ∎

Theorem 5 *For any $B_1, B_2 \in L/e$ it holds*

$$B_1 \otimes B_2 = \left[\bigvee B_1 \otimes \bigwedge B_2\right]^e, \tag{27}$$

$$B_1 \to B_2 = \left[\bigwedge B_1 \to \bigwedge B_2\right]_e, \tag{28}$$

$$B_1 \to B_2 = \left[\bigvee B_1 \to \bigwedge B_2\right]^e. \tag{29}$$

Proof. (27) follows directly from (11) and (24). (28) follows from (11), (26), and (23). To prove (29) we first observe that $e \to (\bigvee B_1 \to \bigwedge B_2) = \bigwedge B_1 \to \bigwedge B_2$ and then use (28). ■

Now we introduce our main result.

Theorem 6 *The tuple $\mathbf{L}/e = \langle L/e, \wedge, \vee, \otimes, \to, 0, 1\rangle$ is a complete residuated lattice.*

Proof. We have to show that L/e together with the operation $\otimes$ and element 1 is a commutative monoid and that the operations $\otimes$ and $\to$ form an adjoint couple.

The operation $\otimes$ is obviously commutative and $1 = [e, 1] \in L/e$ is its unit element. For associativity, we have from (24) that for any $B_1, B_2, B_3 \in L/e$, $\bigwedge((B_1 \otimes B_2) \otimes B_3) = \bigvee B_1 \otimes \bigwedge B_2 \otimes \bigvee B_3 = \bigwedge(B_1 \otimes (B_2 \otimes B_3))$ and we can use Theorem 1.

To prove that $\langle \otimes, \to \rangle$ is an adjoint pair we show that $B_1 \to B_2$ is the greatest element of the set $K = \{B_3 \mid B_1 \otimes B_3 \le B_2\}$. First, we have $\bigwedge(B_1 \otimes (B_1 \to B_2)) = \bigwedge B_1 \otimes \bigvee(B_1 \to B_2) = \bigwedge B_1 \otimes (\bigwedge B_1 \to \bigwedge B_2) \le \bigwedge B_2$, from which it follows $B_1 \to B_2 \in K$. Second, it remains to be proved that for any $B_3 \in K$ it holds $B_3 \le B_1 \to B_2$. We have $\bigvee B_2 \ge \bigvee(B_1 \otimes B_3) \ge \bigvee B_1 \otimes \bigvee B_3$ (the last inequality follows from (10)), from which it follows $\bigvee B_3 \le \bigvee B_1 \to \bigvee B_2 = \bigvee(B_1 \to B_2)$. ■

From (22) and (27) it follows that for any $B_1, B_2 \in L/e$ it holds $\bigvee B_1 \otimes \bigvee B_2 \in B_1 \otimes B_2$ and $\bigvee B_1 \otimes \bigwedge B_2 \in B_1 \otimes B_2$. The following example demonstrates that it is not possible to introduce a product on L/e satisfying $\bigwedge B_1 \otimes \bigwedge B_2 \in B_1 \otimes B_2$.

Example 5 Consider $L = \{0, 0.5, 1\}$ with the structure of equidistant Łukasiewicz chain, i.e., with $0.5 \otimes 0.5 = 0$, and set $e = 0.5$. We have $L/e = \{\mathbf{0}, \mathbf{1}\}$, where $\mathbf{0} = \{0, 0.5\}$ and $\mathbf{1} = \{0.5, 1\}$. Now suppose that for a binary operation $\boxtimes$ on L/e it holds $\bigwedge B_1 \otimes \bigwedge B_2 \in B_1 \boxtimes B_2$ for any $B_1, B_2 \in L/e$. Hence in the case of $B_1 = B_2 = \mathbf{1}$ we obtain $0 = 0.5 \otimes 0.5 = \bigwedge \mathbf{1} \otimes \bigwedge \mathbf{1} \in \mathbf{1} \boxtimes \mathbf{1}$, which leads to $\mathbf{1} \boxtimes \mathbf{1} = \mathbf{0}$ and to the conclusion that $\mathbf{1}$ is not a unit element of $\boxtimes$.

3.2 Representatives, factor projections and uniqueness

As the following example demonstrates, in general case there is no structure of residuated lattice (in contrast to ordinary lattice, [7]) on the factor set L/e where product and residuum are independent on the choice of representatives of maximal blocks from L/e.

Example 6 Suppose that for blocks $B_1, B_2, B_3 \in L/e$ from $b_1 \in B_1$ and $b_2 \in B_2$ it follows $b_1 \otimes b_2 \in B_3$. Then, by Theorem 1, $\bigvee B_1, \bigwedge B_1 \in B_1$ and $\bigvee B_2, \bigwedge B_2 \in B_2$. Hence for $a = \bigvee B_1 \otimes \bigvee B_2$ we obtain $a \in B_3$, and by (19), $\bigwedge B_1 \otimes \bigwedge B_2 = e \otimes e \otimes \bigvee B_1 \otimes \bigvee B_2 = e \otimes e \otimes a \in B_3$, or $\bigwedge B_3 \le e \otimes e \otimes a$. Since $e \otimes \bigvee B_3 = \bigwedge B_3$ and $a \le \bigvee B_3$ then $e \otimes a \le \bigwedge B_3$ which implies $e \otimes a \le e \otimes e \otimes a$. This is generally satisfied in so called Heyting algebras (where $e \otimes a = e \wedge a$) but not, for example, in Łukasiewicz and Goguen structure.

Lemma 2 *Let $B_1, B_2 \in L/e,\ b_1 \in B_1,\ b_2 \in B_2$.*

1. *$b_1 \otimes b_2 \in B_1 \otimes B_2$ if and only if $b_1 \otimes b_2 \geq \bigvee B_1 \otimes \bigwedge B_2$.*
2. *$b_1 \to b_2 \in B_1 \to B_2$ if and only if $b_1 \to b_2 \leq \bigvee B_1 \to \bigvee B_2$.*

Proof. 1. From (22) it follows $b_1 \otimes b_2 \leq \bigvee(B_1 \otimes B_2)$. By (24), the condition $b_1 \otimes b_2 \geq \bigvee B_1 \otimes \bigwedge B_2$ is equivalent to $b_1 \otimes b_2 \geq \bigwedge(B_1 \otimes B_2)$ and the result follows from the fact that $B_1 \otimes B_2$ is an interval (Theorem 1).

2. Similarly, by (29) it always holds $b_1 \to b_2 \geq \bigwedge(B_1 \to B_2)$ and from (25) it follows that the condition $b_1 \to b_2 \leq \bigvee B_1 \to \bigvee B_2$ is equivalent to $b_1 \to b_2 \leq \bigvee(B_1 \to B_2)$. ■

Theorem 7 *Let $B_1, B_2 \in L/e,\ b_1 \in B_1,\ b_2 \in B_2$. Then*

$$\bigvee B_1 \otimes b_2 \in B_1 \otimes B_2, \tag{30}$$

$$\bigvee B_1 \to b_2 \in B_1 \to B_2, \tag{31}$$

$$b_1 \to \bigwedge B_2 \in B_1 \to B_2. \tag{32}$$

Proof. All three assertions are simple consequences of Lemma 2. In (32) we use the equality $\bigvee B_1 \to \bigvee B_2 = \bigwedge B_1 \to \bigwedge B_2$, which follows from (25) and (26). ■

A mapping $P\colon L \to L/e$ is called a factor projection if for any $a \in L$ it holds $a \in P(a)$. Factor projections P_e, P^e are defined by $P_e(a) = [a]_e$, $P^e(a) = [a]^e$. We have the following result regarding projection P_e:

Theorem 8 *(1) P_e is a homomorphism of structures $\langle L, \otimes\rangle$ and $\langle L/e, \otimes\rangle$.*

(2) The structure introduced in Section 3.1 is a unique structure of residuated lattice on L/e such that P_e is a homomorphism of these structures.

Proof. (1) For $b_1, b_2 \in L$ we have $\bigwedge P_e(b_1 \otimes b_2) = \bigwedge [b_1 \otimes b_2]_e = e \otimes b_1 \otimes b_2$ while $\bigwedge(P_e(b_1) \otimes P_e(b_2)) = \bigwedge([b_1]_e \otimes [b_2]_e) = \bigvee [b_1]_e \otimes \bigwedge [b_2]_e = (e \to (e \otimes b_1)) \otimes e \otimes b_2 = e \otimes (e \to (e \otimes b_1)) \otimes b_2 = e \otimes b_1 \otimes b_2$. Hence $\bigwedge [b_1 \otimes b_2]_e = \bigwedge([b_1]_e \otimes [b_2]_e)$ which shows that $P_e(b_1 \otimes b_2) = P_e(b_1) \otimes P_e(b_2)$.

(2) follows trivially from surjectivity of P_e. ■

As for the projection P^e, the following example demonstrates that not only P^e is not a homomorphism of structures $\langle L, \otimes\rangle$ and $\langle L/e, \otimes\rangle$ but, in general, there cannot be introduced a structure of residuated lattice on L/e such that P^e is such a homomorphism. This shows the importance of the projection P_e.

Example 7 Consider $L = \{0, 0.5, 1\}$ with the structure of equidistant Łukasiewicz chain from Example 5. We have $P^e(1) = P^e(0.5) = \mathbf{1}$, $P^e(0) = \mathbf{0}$. If P^e was a homomorphism then $\mathbf{1} = \mathbf{1} \otimes \mathbf{1} = P^e(0.5) \otimes P^e(0.5) = P^e(0.5 \otimes 0.5) = P^e(0) = \mathbf{0}$.

Regarding residuum, the following example shows that, in general, there exists no factor projection $P\colon L \to L/e$ preserving residuum.

Example 8 Consider L with the structure of equidistant Łukasiewicz chain from Example 5 again and set $e = 0.5$. The mappings P_e, P^e are the only factor projections from L to L/e.

For mapping P_e we have $P_e(0.5) = P_e(0) = \mathbf{0}$ and $P_e(0.5 \to 0.5) = P_e(1) = \mathbf{1}$ while $P_e(0.5 \to 0) = P_e(0.5) = \mathbf{0}$. Hence $P_e(0.5) = P_e(0)$ but $P_e(0.5 \to 0.5) \neq P_e(0.5 \to 0)$. Similarly, we have $P^e(0.5) = P^e(1) = \mathbf{1}$ but $P^e(0.5 \to 0) = P^e(0.5) = \mathbf{1}$ while $P^e(1 \to 0) = P^e(0) = \mathbf{0}$.

Hence, neither P_e, nor P^e is a morphism with respect to $\to$.

To summarize our previous results, the projection P_e preserves the product but it does not preserve residuum. The projection P^e does not preserve neither product, nor residuum. As counterexamples show, it is not possible to make this situation better, i.e., it is not possible to introduce operations of product and residuum on $\mathbf{L}/e$ such that P_e would preserve residuum, or P^e would preserve product or residuum.

The following theorem shows a weaker connection between residuum and projection P_e.

Theorem 9 *For any $a_1, a_2 \in L$, $P_e(a_1 \to a_2) \leq P_e(a_1) \to P_e(a_2)$.*

Proof. We have $\bigvee P_e(a_1 \to a_2) = e \to (e \otimes (a_1 \to a_2)) \leq e \to (a_1 \to (e \otimes a_2)) = (e \otimes a_1) \to (e \otimes a_2)$ and $\bigvee (P_e(a_1) \to P_e(a_2)) = (\bigwedge P_e(a_1)) \to (\bigwedge P_e(a_2)) = (e \otimes a_1) \to (e \otimes a_2)$. Hence $\bigvee P_e(a_1 \to a_2) \leq \bigvee (P_e(a_1) \to P_e(a_2))$. ■

Note that there are some important cases when the desired equality $P_e(a_1 \to a_2) = P_e(a_1) \to P_e(a_2)$ is true. In the following examples, we show some of them.

Example 9 Consider the case when $a_2 = \bigvee P_e(a_2)$. We have $a_2 = e \to (e \otimes a_2)$ and $\bigvee (P_e(a_1) \to P_e(a_2)) = (e \otimes a_1) \to (e \otimes a_2) = a_1 \to (e \to (e \otimes a_2)) = a_1 \to a_2 \leq \bigvee P_e(a_1 \to a_2)$. As a consequence, we obtain $\bigvee P_e(a_1 \to a_2) = a_1 \to a_2$.

Example 10 Suppose that there is only one block $B \in L/e$ containing $a_1 \to a_2$. In this case, $e \to e \otimes (a_1 \to a_2) = e \to (a_1 \to a_2)$ and $\bigvee (P_e(a_1) \to P_e(a_2)) = (e \otimes a_1) \to (e \otimes a_2) \leq (e \otimes a_1) \to a_2 = e \to (a_1 \to a_2) = e \to e \otimes (a_1 \to a_2) = \bigvee P_e(a_1 \to a_2)$.

As the last remark, we show some basic result for right inverses of factor projections P_e and P^e. We consider two mappings $Q_e, Q^e : L/e \to L$:

$$Q_e : B \mapsto \bigvee B, \tag{33}$$

$$Q^e : B \mapsto \bigwedge B. \tag{34}$$

Clearly, Q_e is a right inverse of P_e and Q^e is a right inverse of P^e.

It can be shown by counter-examples as above that Q_e does not preserve product and Q^e does not preserve neither product, nor residuum. However, the mapping Q_e does preserve residuum:

Theorem 10 *The mapping Q_e is a homomorphism of structures $\langle L/e, \to \rangle$ and $\langle L, \to \rangle$.*

Proof. Follows directly from (25). ■

3.3 Sequential factorization

Let $e_1, e_2 \in L$. We shall factorize residuated lattice $\mathbf{L}$ by two ways. First, we shall factorize $\mathbf{L}$ by the tolerance $\sim_{e_1}$ induced by e_1 and then the resulting residuated lattice $\mathbf{L}/e_1$ by the tolerance $\sim_E$ on $\mathbf{L}/e_1$ induced by the block $E = [e_2]_{e_1} \in L/e_1$, obtaining the factor residuated lattice $(\mathbf{L}/e_1)/E$. Second, we shall factorize $\mathbf{L}$ directly using the element $e_1 \otimes e_2 \in L$ obtaining the factor residuated lattice $\mathbf{L}/(e_1 \otimes e_2)$.

In this section, we show that both ways lead to the same result; more precisely, we find a canonical isomorphism between residuated lattices $(\mathbf{L}/e_1)/E$ and $\mathbf{L}/(e_1 \otimes e_2)$.

For any $\bar{B} \in (\mathbf{L}/e_1)/E$ and $B \in \mathbf{L}/(e_1 \otimes e_2)$ set

$$U(\bar{B}) = \bigcup \bar{B}, \qquad (35)$$

$$V(B) = \{A \in L/e_1 \mid A \subseteq B\}. \qquad (36)$$

Since $\bar{B}$ is a system of subsets of L then $U(\bar{B}) \subseteq L$. We also have $V(B) \subseteq L/e_1$.

Lemma 3 *For any $\bar{B} \in (\mathbf{L}/e_1)/E$ it holds $U(\bar{B}) \in \mathbf{L}/(e_1 \otimes e_2)$. $U(\bar{B})$ is equal to the interval $[a_0, a_1]$, where $a_0 = \bigwedge A_0$ for $A_0 = \bigwedge \bar{B}$ and $a_1 = \bigvee A_1 \in L$ for $A_1 = \bigvee \bar{B}$.*

Proof. We shall show that $U(\bar{B}) = [a_0, a_1] = [a_1]_{(e_1 \otimes e_2)}$.

Since for any $A \in \bar{B}$ and $a \in A$ it holds $a_0 \leq \bigwedge A \leq a \leq \bigvee A \leq a_1$ then $U(\bar{B}) \subseteq [a_0, a_1]$. To prove the converse inclusion we need to find to any $a \in [a_0, a_1]$ a block $A \in L/e_1$ such that $a \in A \subseteq [a_0, a_1]$. Set $A = [a \vee (e_1 \to a_0)]_{e_1}$. We have $a \vee (e_1 \to a_0) \geq a$ and $e_1 \otimes (a \vee (e_1 \to a_0)) = (e_1 \otimes a) \vee a_0 \leq a$, which means $a \in A$. From $a \vee (e_1 \to a_0) \leq a_1$ and $e_1 \to (e_1 \otimes a_1) = a_1$ it follows $\bigvee A = e_1 \to (e_1 \otimes (a \vee (e_1 \to a_0))) \leq a_1$. Finally, we have $(e_1 \otimes a) \vee a_0 \geq a_0$. Thus, $A \subseteq [a_0, a_1]$.

To finish the proof we shall show that $[a_0, a_1] = [a_1]_{(e_1 \otimes e_2)}$. We have from Lemma 1, $e_1 \otimes e_2 \otimes a_1 = \bigwedge E \otimes \bigvee A_1 = \bigwedge (E \otimes A_1) = \bigwedge A_0 = a_0$ and $(e_1 \otimes e_2) \to a_0 = \bigwedge E \to \bigwedge A_0 = \bigvee (E \to A_0) = \bigvee A_1 = a_1$. ■

Lemma 4 *For any $B \in \mathbf{L}/(e_1 \otimes e_2)$ it holds $V(B) \in (\mathbf{L}/e_1)/E$. $V(B)$ is equal to the interval $[A_0, A_1]$, where $A_0 = [a_0]^{e_1}$ for $a_0 = \bigwedge B$ and $A_1 = [a_1]_{e_1}$ for $a_1 = \bigvee B$.*

Proof. First, we shall show that $A_0 \subseteq B$ and $A_1 \subseteq B$. We have by Theorem 4 and adjointness, $a_0 = e_1 \otimes e_2 \otimes ((e_1 \otimes e_2) \to a_0) = e_1 \otimes (e_2 \otimes (e_2 \to (e_1 \to a_0))) \leq e_1 \otimes (e_1 \to a_0) \leq a_0$. Hence, $e_1 \otimes (e_1 \to a_0) = a_0$, which is equivalent to $a_0 = \bigwedge A_0$. Similarly, $a_1 = (e_1 \otimes e_2) \to (e_1 \otimes e_2 \otimes a_1) = e_1 \to (e_2 \to (e_2 \otimes e_1 \otimes a_1)) \geq e_1 \to (e_1 \otimes a_1) \geq a_1$. Hence, $e_1 \to (e_1 \otimes a_1) = a_1$ and $a_1 = \bigvee A_1$.

For $A \in L/e_1$, the condition $A \in [A_0, A_1]$ is equivalent to $a_0 \leq \bigwedge A$ and $\bigvee A \leq a_1$, which is equivalent to $A \subseteq B$. Thus, $V(B) = [A_0, A_1]$.

It remains to be proved that $E \otimes A_1 = A_0$ and $E \to A_0 = A_1$. This can be done by a similar way to the proof of Lemma 3: we have $\bigwedge (E \otimes A_1) = \bigwedge E \otimes \bigvee A_1 = e_1 \otimes e_2 \otimes a_1 = a_0 = \bigwedge A_0$ and $\bigvee (E \to A_0) = \bigwedge E \to \bigwedge A_0 = (e_1 \otimes e_2) \to a_0 = a_1 = \bigvee A_1$. ■

According to the previous two lemmas, equations (35), (36) define mappings $U : (\mathbf{L}/e_1)/E \to \mathbf{L}/(e_1 \otimes e_2)$ and $V : \mathbf{L}/(e_1 \otimes e_2) \to (\mathbf{L}/e_1)/E$. The following theorem shows their main property.

Theorem 11 *U is an isomorphism of residuated lattices with $U^{-1} = V$.*

Proof. From the above lemmas we obtain $\bigwedge U(\bar{B}) = \bigwedge \bigwedge \bar{B}$ and $\bigwedge \bigwedge V(B) = \bigwedge B$. Since elements of $\mathbf{L}/e_1$, $(\mathbf{L}/e_1)/E$, and $\mathbf{L}/(e_1 \otimes e_2)$ are uniquely identified by their infimas then V is the inverse of U.

Now for any $a \in L$, $\bigwedge P_{e_1 \otimes e_2}(a) = e_1 \otimes e_2 \otimes a$ and

$$\begin{aligned}
\bigwedge \bigwedge P_E(P_{e_1}(a)) = &= \bigwedge (E \otimes P_{e_1}(a)) = \\
&= \bigvee E \otimes \bigwedge P_{e_1}(a) = \\
&= \bigvee E \otimes e_1 \otimes a = \\
&= e_1 \otimes \bigvee E \otimes a = \\
&= e_1 \otimes e_2 \otimes a.
\end{aligned}$$

This shows that $P_{e_1 \otimes e_2} = U \circ P_E \circ P_{e_1}$ (and $P_E \circ P_{e_1} = V \circ P_{e_1 \otimes e_2}$). Hence the result follows from Theorem 8, part (2). ■

4 Application to fuzzy concept lattices

In this part, we show how the problem of factorization of concept lattices of data with fuzzy attributes can be solved using factorization of residuated lattices. This result can serve the reader as an example of a more general approach to factorization of fuzzy systems, which is outlined in the last section of this paper.

We start with Sec. 4.1, where we give a slight generalization of our previous results. Next, in Sec. 4.2, we recall basic definitions from formal concept analysis of data with fuzzy attributes and then, in Sec. 4.3, we give an overview of the problem of factorization of fuzzy concept lattices and then prove our result. Section 4.4 contains an example.

4.1 Factorization of $\mathbf{L}^X$

Recall that a fuzzy set (or, more precisely, an **L**-set) in a universe X is a mapping $A\colon X \to L$. The set of all **L**-sets in the universe X is denoted L^X, the direct product algebra (with operations defined pointwise by means of operations on **L**) is denoted by $\mathbf{L}^X$. $\mathbf{L}^X$ is a residuated lattice.

For two **L**-sets $A, B \in L^X$ we set

$$A \leq B \quad \text{iff} \quad A(x) \leq B(x) \text{ for any } x \in X. \tag{37}$$

The degree of similarity of two fuzzy sets $A, B \in \mathbf{L}^X$ is defined by

$$A \approx^X B = \bigwedge_{x \in X} (A(x) \leftrightarrow B(x)). \tag{38}$$

$\approx^X$ is a fuzzy equivalence on L^X, hence its e-cut $\approx^X_e$ defined by

$$A \approx^X_e B \quad \text{iff} \quad A \approx^X B \geq e, \tag{39}$$

is a tolerance relation on $\mathbf{L}^X$ (note that when identifying **L** with $\mathbf{L}^{\{1\}}$, this relation is equal to the relation $\sim_e$ introduced before).

The relation $\approx^X_e$ coincides with the e-cut $\sim_e$ of biresiduum on the residuated lattice $\mathbf{L}^X$ (see beginning of Sec. 2.3). Hence, if we identify e with the constant mapping $x \mapsto e$, then the factor set $\mathbf{L}^X/{\approx^X_e}$ is equal to the factor set $\mathbf{L}^X/e$.

In this section, we show how this factor set can be identified with the set $(\mathbf{L}/e)^X$ of all $\mathbf{L}/e$-sets in the universe X. First we give some notations and prove an auxiliary lemma.

For $S \subseteq \mathbf{L}^X$ and $x \in X$ we set

$$S_x = \{A(x) \mid A \in S\}. \tag{40}$$

Lemma 5 *A subset $S \subseteq L^X$ is a maximal block of $\approx_e^X$ if and only if for any $x \in X$ it holds $S_x \in L/e$.*

Proof. Let $S \in L^X/e$, $A, B \in S$. Then $A(x) \leftrightarrow B(x) \geq \bigwedge_{y\in X}(A(y) \leftrightarrow B(y)) = (A \approx B) \geq e$ showing that S_x is a block of $\sim_e$. Now let $a \in L$ be any element such that $a \sim_e b \geq e$ for any $b \in S_x$. Choose any $B \in S$ and set

$$C(y) = \begin{cases} a & \text{if } y = x, \\ B(y) & \text{otherwise.} \end{cases} \tag{41}$$

Now for any $A \in S$ we have $A \approx B \geq e$ which means that $a \in S_x$. This proves that S_x is a maximal block.

To prove the converse, we suppose that for any $x \in X$, $S_x \subseteq L$ is a maximal block of $\sim_e$. For $A, B \in S$ and $x \in X$ we have $A(x) \leftrightarrow B(x) \geq e$, which means that $A \approx B = \bigwedge_{x\in X}(A(x) \leftrightarrow B(x)) \geq e$. This shows that S is a block of $\approx_e^X$. Now let $A \in L^X$ satisfies $A \approx B \geq e$ for any $B \in S$. Then for $x \in X$ we have $A(x) \leftrightarrow B(x) \geq A \approx B \geq e$. Since B is arbitrary we obtain $A(x) \leftrightarrow b \geq e$ for any $b \in S_x$. S_x is a maximal block and $A(x) \in S_x$. This holds for any $x \in X$ which means that $A \in S$ and S is a maximal block of $\approx_e^X$. ∎

Now we can define a mapping $I\colon L^X/e \to (L/e)^X$ by

$$I(S)(x) = S_x. \tag{42}$$

Theorem 12 *The mapping I is a bijection.*

Proof. For $T \in (L/e)^X$ set $J(T) = \{A \in L^X \mid \text{for any } x \in X, A(x) \in T(x)\}$. We have $J(T) \subseteq L^X$ and by Lemma 5, $J(T) \in L^X/e$. Hence $J : (L/e)^X \to L^X/e$. Obviously, J is the inverse mapping to I. ∎

For any $\mathbf{L}$-set $A \in \mathbf{L}^X$ we shall use the symbols A^e, A_e, $[A]^e$, $[A]_e$ as before, where e is identified with the constant mapping $x \mapsto e$ (see beginning of Sec. 2.3 for definitions of these symbols). We have $A^e, A_e \in \mathbf{L}^X$, $[A]^e, [A]_e \in (\mathbf{L}^X)/e$.

In what follows, we shall not use the mapping I explicitly and not distinguish between sets L^X/e and $(L/e)^X$ and their elements. For example, we can consider $[A]_e$ as an element of $(L/e)^X$, having $[A(x)]_e = [A]_e(x) \in L/e$, for any $x \in X$.

4.2 Fuzzy concept lattices

In this section, we recall briefly basic definitions from formal concept analysis of data with fuzzy attributes. We refer the reader to [2] for details.

Let X, Y be sets, $I\colon X \times Y \to L$ an $\mathbf{L}$-relation between X and Y. The triple $\langle X, Y, I\rangle$ is called a data table with fuzzy attributes, elements of X and Y are called objects and attributes, respectively. $\langle X, Y, I\rangle$ represents a table which assigns to each $x \in X$ and $y \in Y$ a truth degree $I(x, y) \in L$ to which object x has the attribute y.

For **L**-set $A \in L^X$ of objects we define an **L**-set $A^{\uparrow} \in L^Y$ of attributes by

$$A^{\uparrow}(y) = \bigwedge_{x \in X} (A(x) \to I(x, y)) . \tag{43}$$

Similarly, for any **L**-set B of attributes we define an **L**-set $B^{\downarrow}$ of objects by

$$B^{\downarrow}(x) = \bigwedge_{y \in Y} (B(y) \to I(x, y)) . \tag{44}$$

Further we set

$$\mathscr{B}(X, Y, I) = \{\langle A, B\rangle \in L^X \times L^Y \mid A^{\uparrow} = B, B^{\downarrow} = A\}. \tag{45}$$

We define a partial ordering on $\mathscr{B}(X, Y, I)$ by

$$\langle A_1, B_1\rangle \le \langle A_2, B_2\rangle \quad \text{iff} \quad A_1 \le A_2 \tag{46}$$

(or, equivalently, $B_2 \le B_1$). $\mathscr{B}(X, Y, I)$ with this ordering is a complete lattice, called a concept lattice induced by $\langle X, Y, I\rangle$.

Elements $\langle A, B\rangle$ of $\mathscr{B}(X, Y, I)$ are called formal concepts, for each formal concept $\langle A, B\rangle$, A is called its extent, B intent. Formal concepts are interpreted as concepts/clusters hidden in the data table. Namely, the conditions $A^{\uparrow} = B$ and $B^{\downarrow} = A$ say that B is the collection of all attributes shared by all objects from A, and A is the collection of all objects sharing all attributes from B.

4.3 Factorization of fuzzy concept lattices

In this section, we recall the parametrized concept lattice factorization method, as introduced in [1]. Then we show that the factorized lattice $\mathscr{B}(X, Y, I)/{\approx_e}$ obtained by this method is in fact isomorphic to the concept lattice $\mathscr{B}(X, Y, [I]^e)$, computed from a data table with values lying in the factorized residuated lattice $\mathbf{L}/e$.

We introduce a similarity relation $\approx$ on the set $\mathscr{B}(X, Y, I)$ of all formal concepts of $\langle X, Y, I\rangle$ by

$$\langle A_1, B_1\rangle \approx \langle A_2, B_2\rangle = A_1 \approx^X A_2 \tag{47}$$

(see (38)). As it can be shown, we also have

$$\langle A_1, B_1\rangle \approx \langle A_2, B_2\rangle = B_1 \approx^X B_2. \tag{48}$$

Therefore, measuring similarity of fuzzy concepts via extents corresponds to measuring their similarity via intents.

$\langle A_1, B_1\rangle \approx \langle A_2, B_2\rangle$ is called the degree of similarity of formal concepts $\langle A_1, B_1\rangle$ and $\langle A_2, B_2\rangle$. $\approx$ is known to be a fuzzy equivalence on $\mathscr{B}(X, Y, I)$. Therefore, for any user-chosen threshold $e \in \mathbf{L}$, the e-cut $\approx_e$ is a (crisp) tolerance relation ("being similar to degree at least e") on $\mathscr{B}(X, Y, I)$. This tolerance is compatible with the lattice structure on $\mathscr{B}(X, Y, I)$.

From results of Sec. 2.2, namely Theorem 1, we now obtain that maximal blocks of $\approx_e$ are exactly intervals $[\langle A, B\rangle]_{\approx_e}$ (or, equivalently, intervals $[\langle A, B\rangle]^{\approx_e}$), and the factor set $\mathscr{B}(X, Y, I)/{\approx_e}$ together with the ordering given by (12) is a complete lattice.

We start by a summary of known facts, which we shall use to prove our result. Reader can refer [3] for details.

Theorem 13 *For* $\langle A, B\rangle \in \mathscr{B}(X, Y, I)$ *we have*

1. $e \to A$ *is an extent,* $e \to B$ *is an intent,*
2. $\langle A, B\rangle^{\approx_e} = \langle e \to A, (e \otimes B)^{\downarrow\uparrow}\rangle$,
3. $\langle A, B\rangle_{\approx_e} = \langle (e \otimes A)^{\uparrow\downarrow}, e \to B\rangle$,
4. $\langle A, B\rangle^{\approx_e} = ((\langle A, B\rangle^{\approx_e})_{\approx_e})^{\approx_e}$.

For the data table $\langle X, Y, I\rangle$, the **L**-relation I is a mapping $I\colon X \times Y \to L$. Using results of Section 4.1, we define an $\mathbf{L}/e$-relation $[I]^e\colon X \times Y \to L/e$ by

$$[I]^e(x, y) = [I(x, y)]^e \tag{49}$$

(as mentioned at the end of Section 4.1, we do not distinguish between elements of $(L/e)^{X\times Y}$ and $L^{X\times Y}/e$).

For $\bar{A} \in L^X/e$, $\bar{B} \in L^Y/e$ we consider the mappings $^\uparrow$, and $^\downarrow$ with respect to the formal context $\langle X, Y, [I]^e\rangle$

The following lemma has been proved in [1].

Lemma 6 *For any* $A_1, A_2 \in L^X$, $A_1 \approx^X A_2 \le A_1^\uparrow \approx^Y A_2^\uparrow$. *For any* $B_1, B_2 \in L^X$, $B_1 \approx^Y B_2 \le B_1^\downarrow \approx^X B_2^\downarrow$.

Proof. We have $A_1^\uparrow \approx^X A_2^\uparrow = \bigwedge_{y\in Y} A_1^\uparrow(y) \leftrightarrow A_2^\uparrow(y)$ and

$$\begin{aligned}
& A_1^\uparrow(y) \leftrightarrow A_2^\uparrow(y) = \\
= \; & \left(\bigwedge_{x\in X} A_1(x) \to I(x, y)\right) \leftrightarrow \left(\bigwedge_{x\in X} A_2(x) \to I(x, y)\right) \ge \\
\ge \; & \bigwedge_{x\in X} ((A_1(x) \to I(x, y)) \leftrightarrow (A_2(x) \to I(x, y))) \ge \\
\ge \; & \bigwedge_{x\in X} A_1(x) \leftrightarrow A_2(x) = \\
= \; & A_1 \approx^X A_2.
\end{aligned}$$

The second statement follows by duality. ■

Lemma 7 *For any* $\bar{A} \in L^X/e$ *with* $A = \bigvee \bar{A}$ *it holds* $\bar{A}^\uparrow = [A^\uparrow]^e$. *For any* $\bar{B} \in L^Y/e$ *with* $B = \bigvee \bar{B}$ *it holds* $\bar{B}^\downarrow = [B^\downarrow]^e$.

Proof. We have

$$\begin{aligned}
\bar{A}^{\uparrow}(y) &= \bigwedge_{x\in X} \bar{A}(x) \to [I]^e(x,y) = \\
&= \bigwedge_{x\in X} \bar{A}(x) \to [e \to I(x,y)]_e = \\
&= \bigwedge_{x\in X} [A(x)]_e \to [e \to I(x,y)]_e = \\
&= \bigwedge_{x\in X} [A(x) \to (e \to I(x,y))]_e = \\
&= \bigwedge_{x\in X} [e \to (A(x) \to I(x,y))]_e = \\
&= \bigwedge_{x\in X} [A(x) \to I(x,y)]^e = \\
&= \left[\bigwedge_{x\in X} (A(x) \to I(x,y))\right]^e = \\
&= [A^{\uparrow}(y)]^e.
\end{aligned}$$

The second statement follows by duality. ■

Lemma 8 *For any $\bar{A} \in L^X/e$, if $A \in \bar{A}$ then $A^{\uparrow} \in \bar{A}^{\uparrow}$. For any $\bar{B} \in L^Y/e$, if $B \in \bar{B}$ then $B^{\downarrow} \in \bar{B}^{\downarrow}$.*

Proof. This is a simple consequence of the previous two lemmas. If $A \in \bar{A}$ then $A \le \bigvee \bar{A}$ and $A \approx^X \bigvee \bar{A} \ge e$. Hence $A^{\uparrow} \ge (\bigvee \bar{A})^{\uparrow}$ and $A^{\uparrow} \approx^Y (\bigvee \bar{A})^{\uparrow} \ge e$ (Lemma 6). Thus, $A^{\uparrow} \in [(\bigvee \bar{A})^{\uparrow}]^e = \bar{A}^{\uparrow}$ (Lemma 7). The second statement can be proved similarly. ■

Lemma 9 *If $\bar{A} \in L^X/e$ is an extent of the formal context $\langle X, Y, [I]^e\rangle$ then $A = \bigvee \bar{A} \in L^X$ is an extent of the formal context $\langle X, Y, I\rangle$. Moreover, for the extent $A' = (e \otimes A)^{\uparrow\downarrow}$ it holds $\bigvee \bar{A} = e \to A'$.*

Proof. Denote $A = \bigvee \bar{A}$. We have from Lemma 8, $A^{\uparrow\downarrow} \in \bar{A}$ and, at the same time, $A^{\uparrow\downarrow} \ge \bigvee \bar{A}$. Hence the first assertion. Now, for the extent $A' = (e \otimes A)^{\uparrow\downarrow}$ we have $e \to A' \ge A$, but, according to Lemma 8 and because $e \to (e \otimes A) = A \in \bar{A}$, $e \to A' \in \bar{A}$. This shows that $e \to A' = A$ and completes the proof of the Lemma. ■

Lemma 10 *For any extent $A \in L^X$ of data table $\langle X, Y, I\rangle$, $[A]^e$ is an extent of the data table $\langle X, Y, [I]^e\rangle$ with the corresponding intent equal to $[(e \to A)^{\uparrow}]^e$.*

Proof. Set $B_1 = (e \to A)^{\uparrow}$. For $\bar{A} = [A]^e$ it holds $\bar{A} = [e \to A]_e$, $e \to A = \bigvee \bar{A}$. Applying first statement of Lemma 7 to $\bar{A}$ we obtain $[A]^{e\uparrow} = \bar{A}^{\uparrow} = [e \to A]^{\uparrow e} = [B_1]^e$.

By second statement of the same lemma, $[B_1]^{e\downarrow} = [(e \to B_1)^{\downarrow}]^e$. Hence, $\bigvee [B_1]^{e\downarrow} = e \to (e \to (e \to A)^{\uparrow})^{\downarrow}$. By Theorem 13, parts 2., 3, this is equal to the extent of the formal concept $((\langle A, B\rangle^{\approx_e})_{\approx_e})^{\approx_e}$. By the same theorem, $\bigvee [A]^e$ is equal to the extent of the formal concept $\langle A, B\rangle^{\approx_e}$. Since these two formal concepts are equal (Theorem 13, 4.) we have $[B_1]^{e\downarrow} = [A]^e$. ■

Let $\langle \bar{A}, \bar{B} \rangle \in \mathscr{B}(X, Y, [I]^e)$ be a formal context of data table $\langle X, Y, [I]^e \rangle$. From Lemma 9 it follows that $A_1 = \bigvee \bar{A}$ is an extent of the data table $\langle X, Y, I \rangle$ and there is another extent A of the same data table such that $A_1 = e \to A$ (recall that $\bar{A}$ is an element of $(L/e)^X$ which can be identified with $(L^X)/e$; see end of Sec. 4.1).

On the other hand, from Lemma 10 it follows that for any extent A of the data table $\langle X, Y, I \rangle$, $[A]^e$ is an extent of $\langle X, Y, [I]^e \rangle$. But $\bigvee [A]^e = e \to A$. We have therefore established a bijection F_1 from the concept lattice $\mathscr{B}(X, Y, [I]^e)$ to the set of **L**-sets of the form $e \to A$, where A is an extent of $\langle X, Y, I \rangle$. The bijection F_1 is formally given by

$$F_1(\langle \bar{A}, \bar{B} \rangle) = \bigvee \bar{A}, \tag{50}$$

$$F_1^{-1}(e \to A) = \langle [A]^e, [(e \to A)^{\uparrow}]^e \rangle. \tag{51}$$

Now, Theorem 13, part 2, together with Theorem 1 says that there is also a bijection F_2 from the set of **L**-sets of the form $e \to A$, where A is an extent of $\langle X, Y, I \rangle$ to the factor lattice $\mathscr{B}(X, Y, I)/\approx_e$. This bijection satisfies

$$F_2(e \to A) = [\langle A, A^{\uparrow} \rangle]^{\approx_e} = [\langle e \to A, (e \to A)^{\uparrow} \rangle]_{\approx_e}, \tag{52}$$

$$F_2^{-1}([\langle A, B \rangle]^{\approx_e}) = e \to A. \tag{53}$$

Hence we have constructed a bijection $F \colon \mathscr{B}(X, Y, [I]^e) \to \mathscr{B}(X, Y, I)/\approx_e$, $F = F_2 \circ F_1$. This bijection satisfies

$$F\left(\langle \bar{A}, \bar{B} \rangle\right) = \left[\left\langle \bigvee \bar{A}, \left(\bigvee \bar{A}\right)^{\uparrow} \right\rangle\right]_{\approx_e}, \tag{54}$$

$$F^{-1}([\langle A, B \rangle]^{\approx_e}) = \langle [A]^e, [(e \to A)^{\uparrow}]^e \rangle. \tag{55}$$

Using mapping F, we state the main result of this section:

Theorem 14 *Mapping F is an isomorphism of lattices.*

Proof. It remains to be shown that F and F^{-1} are morphisms of ordered sets. For $\langle \bar{A}, \bar{B} \rangle, \langle \bar{C}, \bar{D} \rangle \in \mathscr{B}(X, Y, [I]^e)$ denote $A_1 = \bigvee \bar{A}$, $C_1 = \bigvee \bar{C}$. We have $\langle \bar{A}, \bar{B} \rangle \leq \langle \bar{C}, \bar{D} \rangle$ iff $\bar{A} \leq \bar{C}$ (46), which is equivalent to $A_1 \leq C_1$ (12).

From Lemma 9, A_1 is an extent of $\langle X, Y, I \rangle$ and for extent $A = (e \otimes A_1)^{\uparrow\downarrow}$ of $\langle X, Y, I \rangle$ it holds $A_1 = e \to A$. Hence from Theorem 13, parts 2 and 3, $\bigvee [\langle A_1, A_1^{\uparrow} \rangle]_{\approx_e} = \langle A_1, A_1^{\uparrow} \rangle$ and, by the same argument, $\bigvee [\langle C_1, C_1^{\uparrow} \rangle]_{\approx_e} = \langle C_1, C_1^{\uparrow} \rangle$. Hence, $A_1 \leq C_1$ is equivalent to $[\langle A_1, A_1^{\uparrow} \rangle]_{\approx_e} \leq [\langle C_1, C_1^{\uparrow} \rangle]_{\approx_e}$. ■

4.4 Example

Consider $L = \{0, 0.5, 1\}$ with the structure of equidistant Łukasiewicz chain and a formal **L**-context $\langle X, Y, I \rangle$, where X is the set of nine planets and Y the set of their attributes related to their size and distance from the Sun [2]. The context is given by Fig. 1, the corresponding fuzzy concept lattice is depicted in Fig. 2.

Now set $e = 0.5$. We have $L/e = \{\mathbf{0}, \mathbf{1}\}$ where $\mathbf{0} = \{0, 0.5\}$ and $\mathbf{1} = \{0.5, 1\}$. Moreover, $[0]^e = \mathbf{0}$, $[0.5]^e = [1]^e = \mathbf{1}$. Now since for any $\langle x, y \rangle \in X \times Y$ it holds $[I]^e(x, y) = [I(x, y)]^e$ then we

	Size		From sun	
	small	**large**	**far**	**near**
Mercury	1	0	0	1
Venus	1	0	0	1
Earth	1	0	0	1
Mars	1	0	0.5	1
Jupiter	0	1	1	0.5
Saturn	0	1	1	0.5
Uranus	0.5	0.5	1	0
Neptune	0.5	0.5	1	0
Pluto	1	0	1	0

FIG. 1. **L**-context of planets and their attributes

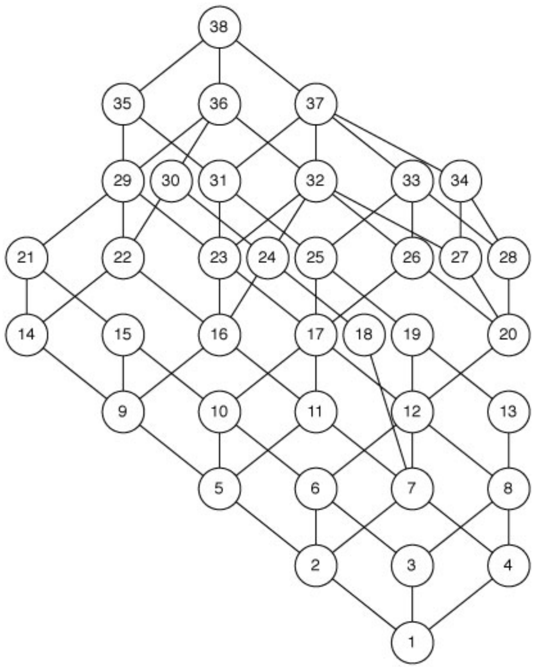

FIG. 2. **L**-concept lattice of the context given by Fig. 1

can easily compute all values of the $\mathbf{L}/e$-relation $[I]^e$. The resulting formal $\mathbf{L}/e$-context $\langle X, Y, [I]^e\rangle$ is depicted in Fig. 3.

Now the $\mathbf{L}/e$-concept lattice of this context can be computed. The resulting lattice, shown in Fig. 4, is isomorphic to the factor lattice $\mathscr{B}(X, Y, I)/e$, which is computed in [2].

Note that the same result can also be achieved using the formal context $\langle X, Y, e \to I\rangle$ with shifted incidence relation I. This follows from Theorem 13 and has been also discussed

	Size		From sun	
	small	large	far	near
Mercury	1	0	0	1
Venus	1	0	0	1
Earth	1	0	0	1
Mars	1	0	1	1
Jupiter	0	1	1	1
Saturn	0	1	1	1
Uranus	1	1	1	0
Neptune	1	1	1	0
Pluto	1	0	1	0

FIG. 3. $\mathbf{L}/e$-context $\langle X, Y, [I]^e \rangle$

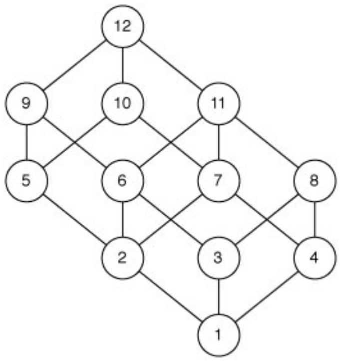

FIG. 4. $\mathbf{L}/e$-concept lattice of the context given by Fig. 3

in [4]. Our approach, however, admits a generalization, as suggested in the next section.

5 Conclusion and future research

The main result of this paper is that for any residuated lattice $\mathbf{L}$ and a threshold $e \in L$ there is a unique structure of residuated lattice on the factor set L/e (the set of maximal blocks of some tolerance relation induced by e) with some natural properties. We have shown an application of this result in Sec. 4, illustrated by the example in Sec. 4.4.

This application suggests the following generalized approach to complexity reduction of fuzzy systems. Consider a fuzzy system $\mathscr{S}$ over a residuated lattice $\mathbf{L}$ and choose an element $e \in L$. Since there is a well-defined structure of residuated lattice on the factor set L/e then it makes sense to consider transferring the system $\mathscr{S}$ to another fuzzy system, say $\mathscr{S}/e$, over the residuated lattice $\mathbf{L}/e$ by some meaningful way.

One measure of "meaningfulness" of this transfer could be the following (if applicable): if the system $\mathscr{S}$ outputs a result $a \in L$ in some situation, then the system $\mathscr{S}/e$ must output a result $B \in L/e$ such that $a \in B$ in the same situation.

The transfer $\langle X, Y, I\rangle \mapsto \langle X, Y, [I]^e\rangle$ from the previous section can be regarded as a meaningful transfer in this sense, since for any $\langle x, y\rangle \in X \times Y$ we have $I(x, y) \in [I(x, y)]^e$.

Since the number of elements in L/e is usually less than the number of elements in L (provided L is finite) then there is a possibility that $\mathscr{S}$, or any data associated with it, can be now reduced in size without losing any information.

The possibility of reducing the complexity of fuzzy systems by means of factorization of residuated lattices will be a subject of our future research. Some partial results have been already obtained in the field of minimization of fuzzy automata [5].

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Factorization of Concept Lattices with Hedges by Means of Factorization of Residuated Lattices

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Abstract. In the first part, we extend our results from a previous paper on factorization of residuated lattices to residuated lattices with hedges. In the second part, we show how this result can be applied to the problem of factorization of fuzzy concept lattices with hedges. Our approach is that instead of factorizing the original concept lattice with hedges we construct a new data table with fuzzy values of attributes in a factorized residuated lattice with hedges and show that the induced concept lattice is isomorphic to the factor concept lattice.

1 Introduction

Formal concept analysis (FCA) is a popular method for analysis of object-attribute data [11], [9]. Its aim is to process data in a tabular form (describing objects and their attributes) and extract interesting clusters, called formal concepts, which correspond to maximal rectangles in the processed data table. These formal concepts form a concept lattice, which represents the main output of the method.

In the case of formal concept analysis of data with fuzzy values of attributes the domain for data can consist of more than two elements (representing degrees to which particular objects can have particular attributes). Since the number of formal concepts can be large in this case, several methods of reducing the size of resulting concept lattice have been proposed. In this paper, we consider two of them: factorization and hedges.

The idea behind factorization of fuzzy concept lattices is that instead of considering the original concept lattice, which can be very large, we accept not to distinguish between formal concepts which are sufficiently similar. This can be done by choosing a degree of similarity of formal concepts and factorizing the concept lattice by the tolerance relation induced by this degree. As the result, we obtain a smaller lattice, whose size depends on the prescribed degree. This parametrized size reduction method has been introduced in [1] and further improved in [3], see also [2].

In [8], the notion of fuzzy concept lattice with hedges was introduced (see also [4], [5]). It can be viewed as another tool for reducing size of concept lattices. It introduces two additional parameters, called (truth-stressing) hedges, which are unary functions on the scale of truth degrees and can be seen as truth functions of connectives “very

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true". Hedges can be used as parameters selecting "important attributes" and "important objects". Stronger hedges lead to smaller number of extracted concepts.

In [6], these two approaches (factorization and hedges) were combined and a method of factorizing fuzzy concept lattices with hedges was introduced.

In [17], we dealt with residuated lattices, which are frequently used as structures of truth values in fuzzy logic, and as such are also used in the above papers. We showed (using results of [10] and [18]) that residuated lattices can be factorized by means of a prescribed degree of similarity of truth values. We also stated a general idea of approximate size reduction of fuzzy systems by factorizing the underlying structure of truth values (i.e., a residuated lattice) by a tolerance relation, induced by the user-prescribed degree to which we allow different truth values to be non-distinguishable. We also showed that this general idea is applicable to fuzzy concept lattices: factorized fuzzy concept lattice is in fact isomorphic to another concept lattice, constructed from a data table with values from factor residuated lattice.

In this paper, we first generalize our results from [17] to residuated lattices with hedges. We show that any hedge on a residuated lattice induces a hedge on the factorized residuated lattice. The only limitation is that the prescribed similarity degree must be a fixpoint of the used hedge (similar condition appears also in [6]).

In the next part we show that factor fuzzy concept lattices with hedges can be again described by means of factor residuated lattices with hedges. More precisely, we show that each factor fuzzy concept lattice with hedges is isomorphic to a fuzzy concept lattice with hedges built on a data table with values from the factorized residuated lattice.

This paper is organized as follows. In Section 2 we summarize basic known facts on residuated lattices, fuzzy sets, factorization of residuated lattices and factorization of concept lattices with hedges. In Section 3 we give our two main results on factorization of residuated lattices with hedges and factorization of concept lattices with hedges.

2 Preliminaries

2.1 Residuated lattices and fuzzy sets

We use complete residuated lattices as structures of truth values. We recall only basic facts here, for more detailed review, we refer the reader to [2], [12].

A complete residuated lattice is defined as an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with the least element 0 and the greatest element 1; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ (product) and $\rightarrow$ (residuum) satisfy so-called adjointness property: $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ for each $a, b, c \in L$. Elements of L are called truth degrees. $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication".

For each complete residuated lattice we consider a derived (truth function of) logical connective $\leftrightarrow$ ("fuzzy equivalence") defined by $a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$. $\leftrightarrow$ is called biresiduum and is used for measuring similarity of truth degrees.

A common choice of $\mathbf{L}$ is a structure with $L = [0, 1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding $\rightarrow$.

Three most important pairs of adjoint operations on the unit interval are:

$$\text{Łukasiewicz:}\quad \begin{aligned} a\otimes b &= \max(a+b-1,0),\\ a\to b &= \min(1-a+b,1),\end{aligned} \tag{1}$$

$$\text{Gödel:}\quad \begin{aligned} a\otimes b &= \min(a,b),\\ a\to b &= \begin{cases} 1 & \text{if } a\le b,\\ b & \text{otherwise,}\end{cases}\end{aligned} \tag{2}$$

$$\text{Goguen (product):}\quad \begin{aligned} a\otimes b &= a\cdot b,\\ a\to b &= \begin{cases} 1 & \text{if } a\le b,\\ \frac{b}{a} & \text{otherwise.}\end{cases}\end{aligned} \tag{3}$$

Complete residuated lattices on $[0,1]$ given by (1), (2), and (3) are called standard Łukasiewicz, Gödel, Goguen (product) algebras, respectively.

The class of complete residuated lattices include finite structures as well. For instance, we can put $L_{n+1} = \{a_0 = 0, a_1, \ldots, a_n = 1\} \subseteq [0,1]$, where $a_0 < \cdots < a_n$ are equidistant and $\otimes$ and $\to$ are restrictions of the operations from (1). In this case, the residuated lattice $\mathbf{L_{n+1}} = \langle L_{n+1}, \min, \max, \otimes, \to, 0, 1\rangle$ is called an equidistant Łukasiewicz chain.

A special case of a complete residuated lattice is the two-element Boolean algebra $\langle\{0,1\}, \wedge, \vee, \otimes, \to, 0, 1\rangle$, denoted by **2**, which is the structure of truth degrees of the classical logic. That is, the operations $\wedge, \vee, \otimes, \to$ of **2** are the truth functions (interpretations) of the corresponding logical connectives of the classical logic.

A hedge (or truth stresser) on residuated lattice **L** is a unary operation $*$ satisfying (i) $1^* = 1$, (ii) $a^* \le a$, (iii) $(a \to b)^* \le a^* \to b^*$, (iv) $a^{**} = a^*$, for $a, b \in L$. A hedge $*$ is a (truth function of) logical connective "very true" [13].

Among all hedges on any residuated lattice, the greatest one is given by $a^* = a$ and is called (obviously) identity. The smallest hedge is called globalization and is given by $1^* = 1$ and $a^* = 0$ for $a < 1$. In Fig. 1 there are depicted all possible hedges on $\mathbf{L_5}$.

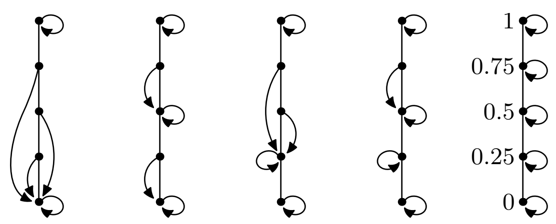

Fig. 1. All hedges on $\mathbf{L_5}$

Element $a \in L$ is said to be a fixpoint of hedge $*$ if $a^* = a$. For two fixpoints a_1, a_2 of $*$, the product $a \otimes b$ is also a fixpoint of $*$.

Recall that an **L**-set (or fuzzy set) A in universe X is a mapping $A\colon X \to L$. For any $x \in X$, $A(x)$ is interpreted as the degree to which x belongs to A. For two such **L**-sets

A_1, A_2, the degree of their similarity $A_1 \approx^X A_2 \in L$ is defined by

$$A_1 \approx^X A_2 = \bigwedge_{x \in X} A_1(x) \leftrightarrow A_2(x). \tag{4}$$

2.2 Factorization of residuated lattices

We use factorization of residuated lattices by compatible tolerances as the main tool in this paper. Regarding factorization of (complete) ordinary lattices we use results of Czédli [10] and Wille [18].

Recall that tolerance on a set X is a relation $\sim$ which is reflexive and symmetric. Each tolerance induces a covering of its underlying set, called the factor set. This set consists of all maximal blocks of the tolerance, i.e., the maximal subsets whose any two elements are in $\sim$. In the case of tolerance $\sim$ on the set X, the factor set is denoted $X/\sim$.

Compatible tolerance relation on a complete lattice $\mathbf{L}$ is a tolerance which preserves suprema and infima, i.e., a tolerance $\sim$ on $\mathbf{L}$ is compatible if from $a_j \sim b_j$ for any $j \in J$ follows $\bigvee_{j \in J} a_j \sim \bigvee_{j \in J} b_j$ and $\bigwedge_{j \in J} a_j \sim \bigwedge_{j \in J} b_j$.

For $a \in L$ we denote

$$a^{\sim} = \bigvee\{b \in L \mid a \sim b\}, \quad a_{\sim} = \bigwedge\{b \in L \mid a \sim b\}, \tag{5}$$

$$[a]_{\sim} = [a_{\sim}, (a_{\sim})^{\sim}], \quad [a]^{\sim} = [(a^{\sim})_{\sim}, a^{\sim}] \tag{6}$$

($[a_1, a_2]$ denotes the interval $\{b \in L \mid a_1 \leq b \leq a_2\}$).

Maximal blocks of $\sim$ are exactly sets $[a]_{\sim}$ and $[a]^{\sim}$, i.e., it holds $L/\sim = \{[a]_{\sim} \mid a \in L\} = \{[a]^{\sim} \mid a \in L\}$.

Ordering on the set $L/\sim$ is introduced using suprema of maximal blocks and can be equivalently introduced using infima. For blocks $B_1, B_2 \in L/\sim$ we set

$$B_1 \leq B_2 \quad \text{iff} \quad \bigvee B_1 \leq \bigvee B_2. \tag{7}$$

The set $L/\sim$ together with this ordering is a complete lattice, which is denoted by $\mathbf{L}/\sim$.

Now suppose that $\mathbf{L}$ is a residuated lattice. The following results can be found in [2], [3], where a more general approach is presented, namely sets of fixpoints of $\mathbf{L}$-closure operators are considered in place of residuated lattice $\mathbf{L}$.

For $e \in L$ we denote the e-cut of biresiduum in $\mathbf{L}$ by $\sim_e^{\mathbf{L}}$ or simply $\sim_e$. By definition of e-cuts of fuzzy relations, for any $a_1, a_2 \in L$, $a_1 \sim_e a_2$ if and only if $a_1 \leftrightarrow a_2 \geq e$. $\sim_e$ is a compatible tolerance on $\mathbf{L}$.

We introduce the following simplified notations: $a_e = a_{\sim_e}$, $a^e = a^{\sim_e}$, $[a]_e = [a]_{\sim_e}$, $[a]^e = [a]^{\sim_e}$. The factor lattice $\mathbf{L}/\sim_e$ will be denoted by $\mathbf{L}/e$.

It holds for any $a \in L$, $a_e = e \otimes a$, $a^e = e \to a$. As a consequence, we obtain the following equalities, which hold for any maximal block $B \in L/\sim_e$: $\bigvee B = e \to \bigwedge B$, $\bigwedge B = e \otimes \bigvee B$.

In [17] we introduced a structure of residuated lattice on the factor set L/e as follows. For $B_1, B_2 \in L/e$ we set

$$B_1 \otimes B_2 = \left[\bigvee B_1 \otimes \bigvee B_2\right]_e, \tag{8}$$

$$B_1 \to B_2 = \left[\bigvee B_1 \to \bigvee B_2\right]_e. \tag{9}$$

Now the set L/e together with elements $0,1 \in L/e$ and operations $\wedge, \vee$ given by the factor lattice structure and together with operations $\otimes, \to$ introduced in (8) and (9) is a complete residuated lattice, which is denoted by $\mathbf{L}/e$. More formally, $\mathbf{L}/e$ is equal to the tuple $\langle L/e, \wedge, \vee, \otimes, \to, 0, 1\rangle$.

In the following lemma, we introduce some basic properties of factor residuated lattices which will be needed later. For more systematic approach, the reader can refer to [17].

Lemma 1. *For any $a_1, a_2 \in L$, $B_1, B_2 \in L/e$ it holds*

$$[a_1 \to a_2]_e \;\leq\; [a_1]_e \to [a_2]_e, \tag{10}$$

$$[a_1 \to (e \to a_2)]_e \;=\; [a_1]_e \to [e \to a_2]_e, \tag{11}$$

$$\bigvee (B_1 \to B_2) \;=\; \bigvee B_1 \to \bigvee B_2. \tag{12}$$

2.3 Fuzzy concept lattices with hedges

In this section, we recall some basic notions and notations and state some basic results on fuzzy concept lattices with hedges and their factorization. We refer the reader to [2], [6], [8] for details.

Let X, Y be nonempty sets, $I\colon X \times Y \to L$ an $\mathbf{L}$-relation between X and Y. The triple $\langle X, Y, I\rangle$ is called a formal $\mathbf{L}$-context, elements of X and Y are called objects and attributes, respectively. $\langle X, Y, I\rangle$ represents a data table which assigns to each $x \in X$ and $y \in Y$ a truth degree $I(x,y) \in L$ to which object x has the attribute y.

For a hedge $*_X$ on $\mathbf{L}$ and $\mathbf{L}$-set $A \in L^X$ of objects we define an $\mathbf{L}$-set $A^{\uparrow} \in L^Y$ of attributes by

$$A^{\uparrow}(y) = \bigwedge_{x \in X} \left(A(x)^{*_X} \to I(x,y)\right). \tag{13}$$

Similarly, for any hedge $*_Y$ and $\mathbf{L}$-set B of attributes we define an $\mathbf{L}$-set $B^{\downarrow}$ of objects by

$$B^{\downarrow}(x) = \bigwedge_{y \in Y} \left(B(y)^{*_Y} \to I(x,y)\right). \tag{14}$$

The following lemma summarizes basic properties of mappings $\uparrow$ and $\downarrow$ [4]:

Lemma 2. *Mappings $\uparrow$ and $\downarrow$ defined by* (13) *and* (14) *satisfy the following properties:*

1. *$A^{*_X} \leq A^{\uparrow\downarrow}$ and $B^{*_Y} \leq B^{\downarrow\uparrow}$;*
2. *$A_1 \leq A_2$ implies $A_2^{\uparrow} \leq A_1^{\uparrow}$, and $B_1 \leq B_2$ implies $B_2^{\downarrow} \leq B_1^{\downarrow}$ (antitony);*
3. *$A^{\uparrow} = A^{*_X\uparrow}$ and $B^{\downarrow} = B^{*_Y\downarrow}$;*
4. *$A^{\uparrow *_Y} \leq A^{\uparrow\downarrow\uparrow} \leq A^{*_X\uparrow}$ and $B^{\downarrow *_X} \leq B^{\downarrow\uparrow\downarrow} \leq B^{*_Y\downarrow}$;*
5. *$A^{\uparrow\downarrow} = A^{\uparrow\downarrow\uparrow\downarrow}$ and $B^{\downarrow\uparrow} = B^{\downarrow\uparrow\downarrow\uparrow}$.*

Next we set

$$\mathscr{B}(X^{*_X}, Y^{*_Y}, I) = \{\langle A, B\rangle \in L^X \times L^Y \mid A^{\uparrow} = B, B^{\downarrow} = A\}. \tag{15}$$

We define a partial ordering on $\mathscr{B}(X, Y, I)$ by

$$\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle \quad \text{iff} \quad A_1 \leq A_2 \tag{16}$$

(or, equivalently, $B_2 \leq B_1$). $\mathscr{B}(X^{*_X}, Y^{*_Y}, I)$ with this ordering is a complete lattice, called an **L**-concept lattice induced by $\langle X, Y, I\rangle$ and hedges $*_X$, $*_Y$.

Elements $\langle A, B\rangle$ of $\mathscr{B}(X^{*_X}, Y^{*_Y}, I)$ are called formal concepts, for each formal concept $\langle A, B\rangle$, A is called its extent, B intent. Formal concepts are interpreted as concepts/clusters hidden in the data table. Namely, the conditions $A^{\uparrow} = B$ and $B^{\downarrow} = A$ say that B is the collection of all attributes shared by all objects (for which it is very true that they are) from A, and A is the collection of all objects sharing all attributes (for which it is very true that they are) from B.

The main idea of adding hedges to fuzzy concept lattices is that using hedges, one can affect the size of concept lattices. Namely, if we choose both $*_X$, $*_Y$ to be identities, we obtain an ordinary fuzzy concept lattice. Other choices lead to smaller concept lattices. For example, if both $*_X$, $*_Y$ are globalizations then the generated concept lattice consists of so called crisply generated formal concepts [7]. If $*_X$ and $*_Y$ are globalization and identity (respectively) then $\mathscr{B}(X^{*_X}, Y^{*_Y}, I)$ is isomorphic to so-called one-sided concept lattice [15].

Now we recall the parametrized concept lattice factorization method, as introduced in [1], and then mention its generalization to fuzzy concept lattices with hedges.

As we mentioned in Introduction, factorization represents another attempt to reduce the size of fuzzy concept lattice. In this method, user choses a degree $e \in L$ to which he/she considers two different concepts to be similar. Factorizing-out similar concepts by a tolerance relation induced by e a smaller lattice is obtained. This lattice do not preserve information on differences between similar concepts. Reader can refer [6], [8] for details on factorization of concept lattices and its generalization to concept lattices with hedges.

We introduce a similarity relation $\approx$ on the set $\mathscr{B}(X, Y, I)$ of all formal concepts of $\langle X, Y, I\rangle$ by

$$\langle A_1, B_1\rangle \approx \langle A_2, B_2\rangle = A_1 \approx^X A_2 \tag{17}$$

(see (4)).

$\langle A_1, B_1\rangle \approx \langle A_2, B_2\rangle$ is called the degree of similarity of formal concepts $\langle A_1, B_1\rangle$ and $\langle A_2, B_2\rangle$. $\approx$ is known to be a fuzzy equivalence on $\mathscr{B}(X, Y, I)$.

Since $\approx$ is a fuzzy equivalence on $\mathscr{B}(X, Y, I)$ then, for any user-chosen threshold $e \in \mathbf{L}$, the e-cut ${}^e{\approx}$ is a (crisp) tolerance relation ("being similar to degree at least e") on $\mathscr{B}(X, Y, I)$. This tolerance is compatible with the lattice structure on $\mathscr{B}(X, Y, I)$.

Maximal blocks of ${}^e{\approx}$ are exactly intervals $[\langle A, B\rangle]_{^e\approx}$ (or, equivalently, intervals $[\langle A, B\rangle]^{^e\approx}$, see (6)), and the factor set $\mathscr{B}(X, Y, I)/{}^e{\approx}$ together with the ordering given by (7) is a complete lattice.

This result can also be generalized to fuzzy concept lattices with hedges. First we show some properties of the fuzzy equivalence $\approx^X$ (resp. $\approx^Y$) on L^X (resp. L^Y) with connection to functions $\uparrow$ and $\downarrow$ [6]:

Lemma 3. *For $A_1, A_2 \in L^X$ and $B_1, B_2 \in L^Y$ we have $(A_1 \approx^X A_2)^{*_X} \leq A_1^{\uparrow} \approx^Y A_2^{\uparrow}$ and $(B_1 \approx^Y B_2)^{*_Y} \leq B_1^{\downarrow} \approx^X B_2^{\downarrow}$.*

For a concept lattice $\mathscr{B}(X^{*_X}, Y^{*_Y}, I)$, similarity of concepts is defined as above, as well as its e-cut, used for factorization. The factor set $\mathscr{B}(X^{*_X}, Y^{*_Y}, I)/{}^e{\approx}$ together with the ordering given by (7) is again a complete lattice. The structure of maximal blocks of ${}^{e}{\approx}$ on $\mathscr{B}(X^{*_X}, Y^{*_Y}, I)$ is given by the following lemma.

Lemma 4. *For* $\langle A,B\rangle \in \mathscr{B}(X,Y,I)$ *we have*

1. $\langle A,B\rangle^{e\approx} = \langle (e \to A)^{\uparrow\downarrow}, (e \otimes B)^{\downarrow\uparrow}\rangle$,
2. $\langle A,B\rangle_{e\approx} = \langle (e \otimes A)^{\uparrow\downarrow}, (e \to B)^{\downarrow\uparrow}\rangle$,
3. $\langle A,B\rangle^{e\approx} = ((\langle A,B\rangle^{e\approx})_{e\approx})^{e\approx}$,
4. $\langle A,B\rangle_{e\approx} = ((\langle A,B\rangle_{e\approx})^{e\approx})_{e\approx}$.

3 Results

3.1 Factorization of residuated lattices with hedges

The first main result of this paper concerns introducing a hedge on the factor residuated lattice $\mathbf{L}/e$ induced by a hedge on the original residuated lattice $\mathbf{L}$.

Suppose that $*$ is a hedge on residuated lattice $\mathbf{L}$ and $e \in L$ is its fixpoint, i.e., $e^* = e$. We define a new unary operation $*^e$ (or, simply, $*$ if e and underlying residuated lattice are obvious) on $\mathbf{L}/e$ by setting for any $B \in L/e$,

$$B^{*^e} = \left[\left(\bigvee B\right)^*\right]_e. \tag{18}$$

We have the following result for the new operation $*^e$:

Theorem 1. *If $e \in L$ is a fixpoint of the hedge $*$ then the operation $*^e$ on $\mathbf{L}/e$ is a hedge.*

Proof. Let $1 \in L$ and $\mathbf{1} \in L/e$ be unite elements. We have $\mathbf{1} = [1]_e$ and

$$\mathbf{1}^{*^e} = ([1]_e)^{*^e} = [1^*]_e = \mathbf{1},$$

which proves condition (i) for hedges.

Now let $B \in L/e$. Then

$$B^{*^e} = \left[\left(\bigvee B\right)^*\right]_e \leq \left[\bigvee B\right]_e = B,$$

which proves condition (ii).

To prove condition (iii) we use Lemma 1 and obtain for $B_1, B_2 \in L/e$,

$$\begin{aligned}(B_1 \to B_2)^{*^e} &= \left[\left(\bigvee (B_1 \to B_2)\right)^*\right]_e = \left[\left(\bigvee B_1 \to \bigvee B_2\right)^*\right]_e \leq \\ &\leq \left[\left(\bigvee B_1\right)^* \to \left(\bigvee B_2\right)^*\right]_e \leq \left[\left(\bigvee B_1\right)^*\right]_e \to \left[\left(\bigvee B_2\right)^*\right]_e = \\ &= B_1^{*^e} \to B_2^{*^e}.\end{aligned}$$

Let $B \in L/e$. To prove the equality $B^{*^e} = B^{*^e *^e}$ we show that infima of both sides are equal. Denote $\bigvee B = a$. We have $\bigwedge B^{*^e} = e \otimes a^*$ and $\bigwedge B^{*^e *^e} = e \otimes (e \to e \otimes a^*)^*$. Now, from condition (iii) for hedges and from the fact that $e \otimes a^*$ is a fixpoint of $*$ (both e and a^* are fixpoints) we obtain

$$\bigwedge B^{*^e *^e} \leq e \otimes (e^* \to (e \otimes a^*)^*) = e \otimes (e \to e \otimes a^*) = \bigwedge B^{*^e}.$$

The opposite inequality $\bigwedge B^{*^e} \leq \bigwedge B^{*^e *^e}$ follows from $(e \to e \otimes a^*)^* \leq e \to e \otimes a^*$ by multiplying both sides by e. This proves the remaining condition (iv) for hedges.

3.2 Factorization of fuzzy concept lattices with hedges

In this section, we present our second main result: the factorized **L**-concept lattice $\mathscr{B}(X^{*_X}, Y^{*_Y}, I)/^e\!\approx$ is isomorphic to an $\mathbf{L}/e$-concept lattice, constructed from a formal $\mathbf{L}/e$-context, which is easily computable from the original formal **L**-context $\langle X, Y, I\rangle$.

For any **L**-set $A \in \mathbf{L}^X$ we shall use the symbols A^e, A_e, $[A]^e$, $[A]_e$ as before, where e is identified with the constant mapping $x \mapsto e$. We have $A^e, A_e \in \mathbf{L}^X$, $[A]^e, [A]_e \in (\mathbf{L}^X)/e$.

In what follows, we shall not distinguish between sets L^X/e and $(L/e)^X$ and their elements. For example, we can consider $[A]_e$ as an element of $(L/e)^X$, having $[A(x)]_e = [A]_e(x) \in L/e$, for any $x \in X$ (see [17] for details).

For a formal context $\langle X, Y, I\rangle$, the **L**-relation I is a mapping $I : X \times Y \to L$. Using results from [17], we define an $\mathbf{L}/e$-relation $[I]^e : X \times Y \to L/e$ by

$$[I]^e(x,y) = [I(x,y)]^e \tag{19}$$

(like before, we do not distinguish between elements of $(L/e)^{X\times Y}$ and $L^{X\times Y}/e$).

Let $\langle X, Y, I\rangle$ be a formal context, $*_X$, $*_Y$ hedges, $e \in L$ a fixed threshold. We consider a new formal $\mathbf{L}/e$-context $\langle X, Y, [I]^e\rangle$. Using results of previous section, we introduce two thresholds $*_X^e$, $*_Y^e$ on the factor residuated lattice $\mathbf{L}/e$ such that e is their common fixpoint. Then we construct the concept lattice $\mathscr{B}(X^{*_X^e}, Y^{*_Y^e}, [I]^e)$.

When the underlying residuated lattice and e are obvious, we also denote the thresholds $*_X^e$, $*_Y^e$ simply by $*_X$, $*_Y$. Since there will be no possibility of confusion, we also denote the formal-context-defining operators with respect to the formal context $\langle X, Y, [I]^e\rangle$ and hedges $*_X^e$, $*_Y^e$ again by $^\uparrow$, and $^\downarrow$.

Lemma 5. *For any $\bar{A} \in L^X/e$ with $A = \bigvee \bar{A}$ it holds $\bar{A}^\uparrow = [A^\uparrow]^e$. For any $\bar{B} \in L^Y/e$ with $B = \bigvee \bar{B}$ it holds $\bar{B}^\downarrow = [B^\downarrow]^e$.*

Proof. From basic properties of blocks of compatible tolerances in residuated lattices and from (11) we obtain

$$\begin{aligned}
\bar{A}^\uparrow(y) &= \bigwedge_{x\in X} \bar{A}^{*_X^e}(x) \to [I]^e(x,y) = \\
&= \bigwedge_{x\in X} \bar{A}^{*_X^e}(x) \to [e \to I(x,y)]_e = \\
&= \bigwedge_{x\in X} [A^{*_X}(x)]_e \to [e \to I(x,y)]_e = \\
&= \bigwedge_{x\in X} [A^{*_X}(x) \to (e \to I(x,y))]_e = \\
&= \bigwedge_{x\in X} [e \to (A^{*_X}(x) \to I(x,y))]_e = \\
&= \bigwedge_{x\in X} [A^{*_X}(x) \to I(x,y)]^e = \\
&= \left[\bigwedge_{x\in X} (A^{*_X}(x) \to I(x,y))\right]^e = \\
&= [A^\uparrow(y)]^e.
\end{aligned}$$

The second statement follows by duality.

Lemma 6. *For any $\bar{A} \in L^X/e$, if $A \in \bar{A}$ then $A^\uparrow \in \bar{A}^\uparrow$. For any $\bar{B} \in L^Y/e$, if $B \in \bar{B}$ then $B^\downarrow \in \bar{B}^\downarrow$.*

Proof. This is a simple consequence of Lemma 5. If $A \in \bar{A}$ then $A \leq \bigvee \bar{A}$ and $A \approx^X \bigvee \bar{A} \geq e$. Hence $A^\uparrow \geq (\bigvee \bar{A})^\uparrow$ (Lemma 2, part 2) and $A^\uparrow \approx^Y (\bigvee \bar{A})^\uparrow \geq e^{*_X} = e$ (Lemma 3). Thus, $A^\uparrow \in [(\bigvee \bar{A})^\uparrow]^e = \bar{A}^\uparrow$ (Lemma 5). The second statement can be proved similarly.

Lemma 7. *For $\langle \bar{A}, \bar{B} \rangle \in \mathscr{B}(X^{*_X}, Y^{*_Y}, [I]^e)$, $(\bigvee \bar{B})^\downarrow$ is the least fixpoint of $\uparrow\downarrow$ in $\bar{A}$.*

Proof. Denote $B_0 = \bigvee \bar{B}$, $A_0 = B_0^\downarrow$. First we show that A_0 is a fixpoint of $\uparrow\downarrow$. The element $A_0^\uparrow$ is a fixpoint of $\downarrow\uparrow$ (Lemma 2, part 5). We have $B_0^{*_Y} \leq A_0^\uparrow$ (Lemma 2, part 1) and $A_0^\uparrow \leq B_0$ (Lemma 6, applied twice). Hence for fixpoint $A_0^{\uparrow\downarrow}$ of $\uparrow\downarrow$ we obtain (using Lemma 2, part 2), $B_0^\downarrow \leq A_0^{\uparrow\downarrow} \leq B_0^{*_Y\downarrow}$. But from Lemma 2, part 3, we have $B_0^\downarrow = B_0^{*_Y\downarrow}$, which shows that A_0 is a fixpoint of $\uparrow\downarrow$.

Now from antitony of $\uparrow$ and $\downarrow$ (Lemma 2, part 2) we have for any fixpoint $A \in \bar{A}$: $A \geq \bigwedge \bar{A}$, $A^\uparrow \leq (\bigwedge \bar{A})^\uparrow \leq B_0$ (Lemma 6), which leads to $A_0 \leq A^{\uparrow\downarrow} = A$.

Lemma 8. *For every $\langle \bar{A}, \bar{B} \rangle \in \mathscr{B}(X^{*_X}, Y^{*_Y}, [I]^e)$, the set $F(\langle \bar{A}, \bar{B} \rangle)$ of all $\langle A, B \rangle$ from $\mathscr{B}(X^{*_X}, Y^{*_Y}, I)$ such that $A \in \bar{A}$, is a maximal block of ${}^e\!\approx$ (i.e., $F(\langle \bar{A}, \bar{B} \rangle)$ belongs to $\mathscr{B}(X^{*_X}, Y^{*_Y}, I)/{}^e\!\approx$).*

Proof. According to Lemma 7, $A_0 = (\bigvee \bar{B})^\downarrow$ is the least fixpoint of $\uparrow\downarrow$ in $\bar{A}$. From Lemma 5 we have $e \to A_0 = \bigvee \bar{A}$ and $(e \to A_0)^{\uparrow\downarrow} = A_1$, where A_1 is the greatest fixpoint of $\uparrow\downarrow$ in $\bar{A}$. According to Lemma 6, $A_1 \in \bar{A}$.

It remains to be shown (Lemma 4) that $A_0 = (e \otimes A_1)^{\uparrow\downarrow} \in \bar{A}$. We have $(\bigvee \bar{A})^{*_X} \leq A_1 \leq \bigvee \bar{A}$ (Lemma 2, part 1) and from Lemma 2, parts 2, 3, the intent $B_1 = A_1^\uparrow$ is equal to $(\bigvee \bar{A})^\uparrow$. Hence, $\bigvee \bar{B} = e \to B_1$ (Lemma 5) and $(e \to B_1)^{\downarrow\uparrow}$ is the greatest intent of $\mathscr{B}(X^{*_X}, Y^{*_Y}, I)$ from $\bar{B}$. According to Lemma 4, the corresponding extent is equal to A_0. Applying Lemma 6 now completes the proof.

Lemma 9. *For any maximal block $K = [\langle A_0, B_0 \rangle, \langle A_1, B_1 \rangle] \in \mathscr{B}(X^{*_X}, Y^{*_Y}, I)/{}^e\!\approx$ there is exactly one formal concept $G(K) = \langle \bar{A}, \bar{B} \rangle \in \mathscr{B}(X^{*_X}, Y^{*_Y}, [I]^e)$ such that $\bigwedge \bar{A} \leq A_0$, $A_1 \leq \bigvee \bar{A}$. It holds $\bar{A} = [A_0]^e$.*

Proof. Since $A_0 \,{}^e\!\approx A_1$ then there exists a maximal block $A' \in L^X/e$ such that $A_0 \in A'$, $A_1 \in A'$. From Lemma 6 we have $A_0 \in A'^{\uparrow\downarrow}$, $A_1 \in A'^{\uparrow\downarrow}$. This gives existence of at least one $\langle \bar{A}, \bar{B} \rangle$ with desired properties.

Now suppose that $\langle \bar{A}, \bar{B} \rangle \in \mathscr{B}(X^{*_X}, Y^{*_Y}, [I]^e)$ is such that $\bigwedge \bar{A} \leq A_0$, $A_1 \leq \bigvee \bar{A}$. The element $(\bigvee \bar{B})^\downarrow$ is the least fixpoint of $\uparrow\downarrow$ in $\bar{A}$ (Lemma 7). Hence, $(\bigvee \bar{B})^\downarrow = A_0$ (K is a maximal block). From Lemma 5 we have $\bar{A} = [A_0]^e$ which proves the uniqueness of $\bar{A}$ as well as the desired equality.

Lemmas 8 and 9 give us mapping $F\colon \mathscr{B}(X^{*_X}, Y^{*_Y}, [I]^e) \to \mathscr{B}(X^{*_X}, Y^{*_Y}, I)/{}^e\!\approx$ and mapping $G\colon \mathscr{B}(X^{*_X}, Y^{*_Y}, I)/{}^e\!\approx \to \mathscr{B}(X^{*_X}, Y^{*_Y}, [I]^e)$ which are obviously mutually inverse. Using mapping F, we state our main result:

Theorem 2. *Mapping F is an isomorphism of lattices.*

Proof. It remains to be shown that F and G are morphisms of ordered sets. For two elements $\langle \bar{A}, \bar{B} \rangle, \langle \bar{C}, \bar{D} \rangle \in \mathscr{B}(X^{*X}, Y^{*Y}, [I]^e)$, denote $F(\langle \bar{A}, \bar{B} \rangle) = [\langle A_0, B_0 \rangle, \langle A_1, B_1 \rangle]$ and, similarly, $F(\langle \bar{C}, \bar{D} \rangle) = [\langle C_0, D_0 \rangle, \langle C_1, D_1 \rangle]$ (intervals taken in $\mathscr{B}(X^{*X}, Y^{*Y}, I)$).

If $\langle \bar{A}, \bar{B} \rangle \leq \langle \bar{C}, \bar{D} \rangle$ then $\bigvee \bar{A} \leq \bigvee \bar{C}$, from which and from Lemma 7 it follows $B_1 = (\bigvee \bar{A})^{\uparrow} \geq (\bigvee \bar{C})^{\uparrow} = D_1$. This means $[\langle A_0, B_0 \rangle, \langle A_1, B_1 \rangle] \leq [\langle C_0, D_0 \rangle, \langle C_1, D_1 \rangle]$.

To prove the opposite we start with $A_0 \leq C_0$. This and Lemma 5 give $\bigvee \bar{A} = e \to A_0 \leq e \to C_0 = \bigvee \bar{C}$, which finishes the proof.

4 Conclusion

The two main results of this paper can be interpreted as follows. If we are trying to reduce the complexity of some concept lattice with hedges by factorization, then we are, in fact, constructing another concept lattice with hedges, which is built over a data table with values in some factorized residuated lattice. Thus, the problem of factorization of concept lattice by similarity is replaced with the problem of factorization of the used set of truth degrees (residuated lattice) which indicate the similarity levels.

This paper extends our previous results from [17], where we considered residuated lattices and fuzzy concept lattices without hedges.

There is even more general approach ("Generalized concept lattice", [16]), which contains the notion of fuzzy concept lattice with hedges as a special case [14]. There arises a question whether the method of factorization of concept lattices can be generalized to this case. This question is open; the main obstacle seems to be that in this general framework there is no known natural notion of similarity of concepts.

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RESEARCH ARTICLE

Central Fuzzy Sets

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Let B be a collection of fuzzy sets. What are the fuzzy sets which are sufficiently similar to every fuzzy set from B, i.e. "central" fuzzy sets for B? Such question naturally arises if B is large and one wishes to replace B by a single fuzzy set—the representative of B. In this paper, we develop a framework which enables us to answer this question and related ones. We use complete residuated lattices as the structures of truth degrees, covering thus the real unit interval with left-continuous t-norm and its residuum as an important but particular case. We present results describing central fuzzy sets and optimal central fuzzy sets provided similarity of fuzzy sets is assessed by Leibniz rule.

Keywords: fuzzy set; similarity; tolerance; central point

1. Problem Setting

Suppose there is a collection of metal poles of different lengths. Suppose a person sees a picture of two poles from that collection and is asked to assess their similarity, i.e. the person is asked to tell a degree $p_1 \approx p_2$ to which the poles are similar. The degree has to be a value between 0 and 1, $p_1 \approx p_2 = 0$ and $p_1 \approx p_2 = 1$ indicate that the poles are not similar at all and that the poles are indistinguishable, respectively. Since the poles are narrow, the person assesses their similarity based solely on their lengths. The picture does not show a scale, i.e. the person does not know the actual lengths of the poles. An obvious way to asses the similarity s of poles p_1 and p_2 of lengths $l(p_1)$ and $l(p_2)$ is to put

$$p_1 \approx p_2 \,=\, \min\left(\frac{l(p_1)}{l(p_2)}, \frac{l(p_2)}{l(p_1)}\right), \tag{1}$$

i.e. to make the similarity judgment based on the ratio of the lengths. Namely, the ratio does not depend on the actual lengths, i.e. $p_1 \approx p_2 \,=\, \min\left(\frac{c\cdot l(p_1)}{c\cdot l(p_2)}, \frac{c\cdot l(p_2)}{c\cdot l(p_1)}\right)$ for any $c > 0$, so it can be assessed even when the person does not know the actual magnification coefficient $c > 0$, i.e. does not know the scale for the picture.

Given poles p_1 and p_2 with lengths $l(p_1)$ and $l(p_2)$, what is the length of the pole in the middle? That is, what is the length of the "central pole" p for which

$$p \approx p_1 = p \approx p_2,$$

i.e. for which the similarity to p_1 equals the similarity to p_2? An easy verification

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shows that the central pole p has length

$$l(p) = \sqrt{l(p_1)} \cdot \sqrt{l(p_2)}. \tag{2}$$

Suppose now that the longest possible pole has the length normalized to 1 and the person knows the scale, i.e. knows the lengths $l(p_1)$ and $l(p_2)$. Then there is another, perhaps more natural, way to assess the similarity. Namely, one can put

$$p_1 \approx p_2 \ = 1 - |l(p_1) - l(p_2)|, \tag{3}$$

i.e. the similarity is proportional to the distance of the normalized lengths of p_1 and p_2. If such measure of similarity is used, the length of the central pole p is

$$l(p) = \frac{l(p_1) + l(p_2)}{2}. \tag{4}$$

Obviously, given a set $B = \{p_1, \dots, p_n\}$ of poles, the length of the optimal central pole for B is

$$l(p) = \sqrt{\min_i l(p_i)} \cdot \sqrt{\max_i l(p_i)}$$

for similarity given by (1) and

$$l(p) = \frac{\min_i l(p_i) + \max_i l(p_i)}{2}.$$

for similarity given by (3).

In this paper, we present theorems and algorithms motivated by the above types of problems. The first hint to a general framework for this kind of problems is the observation that in (1),

$$p_1 \approx p_2 = l(p_1) \leftrightarrow l(p_2) \tag{5}$$

with $\leftrightarrow$ being the biresiduum corresponding to product t-norm and that in (2),

$$l(p) = m \otimes \sqrt{l(p_1) \leftrightarrow l(p_2)} \tag{6}$$

with $m = \min\{l(p_1), l(p_2)\}$, $\otimes$ denoting the product t-norm and $\sqrt{\ }$ denoting its square root, as introduced by Höhle (1995). Likewise, (5) and (6) become (3) and (4) if $\leftrightarrow$ and $\otimes$ denote the Łukasiewicz biresiduum and t-norm. Henceforth, we consider the framework of left-continuous t-norms and their residua. In fact, we consider a more general framework of complete residuated lattices (Ward and Dilworth 1939).

In general, we assume that B is a subset of a set $\mathcal{S}$ of fixpoints of some fuzzy closure operator C in a universe set X and study the "central fuzzy sets" of B, i.e. fuzzy sets from $\mathcal{S}$ which are sufficiently similar to any fuzzy set from B. If C is the identity, $\mathcal{S}$ is the set of all fuzzy sets in X, in which case no constraint is imposed, i.e. B as well as the central fuzzy sets may be arbitrary fuzzy sets in X. However, our setting with a general operator C allows us to consider only certain fuzzy sets (those which are the fixpoints of C) as the elements of B as well as the central fuzzy sets of B. Example 3.7 clarifies why we consider general operators C.

2. Preliminaries

2.1 *Tolerance Relations*

A tolerance relation, see e.g. (Pogonowski 1981, Schreider 1975), in a set X is a binary relation T in X which is reflexive and symmetric, i.e. for every $x, y \in X$, T satisfies

$$\langle x, x \rangle \in T,$$
$$\langle x, y \rangle \in T \text{ implies } \langle y, x \rangle \in T.$$

The concept of a tolerance relation generalizes the well-known concept of an equivalence relation. Namely, T is an equivalence relation if it is a tolerance relation which is, moreover, transitive, i.e. for every $x, y, z \in X$, if $\langle x, y \rangle \in T$ and $\langle y, z \rangle \in T$ then $\langle x, z \rangle \in T$.

Let T be a tolerance in X. A class of T given by $x \in X$ is the set $[x]_T = \{y \mid \langle x, y \rangle \in T\}$. A set $B \subseteq X$ is called a block of T if $B \times B \subseteq T$, i.e. if for every $x, y \in B$, $\langle x, y \rangle \in T$. A block B of T is called maximal if it is maximal with respect to set inclusion, i.e. if $B' \times B' \not\subseteq T$ for any $B' \supset B$. It is easy to see that if T is an equivalence relation, classes of T coincide with maximal blocks of T.

While equivalence relations serve as simple mathematical models of indistinguishability, tolerance relations serve as models of similarity. Namely, equivalence relations represent relationships defined by "have same features", while tolerance relations represent relationships defined by "have some features in common", see (Schreider 1975).

2.2 *Fuzzy Sets and Fuzzy Logic*

Residuated lattices as structures of truth degrees. In classical logic, the structure of truth degrees is the two-element Boolean algebra, i.e. a structure $\mathbf{L}$ which consists of a two-element set $L = \{0, 1\}$ of truth degrees and is equipped with truth functions of logical connectives. In fuzzy logic, there are more options, both for the set L of truth degrees and for the functions of logical connectives. As the structures of truth degrees, we use complete residuated lattices. Complete residuated lattices, introduced to fuzzy logic by Goguen (1968–69), and their variants are used in mathematical fuzzy logic (Gottwald 2008, Hájek 1998). Recall that a complete residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$ such that $\langle L, \wedge, \vee, 0, 1 \rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1 \rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy so-called adjointness property: $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ for each $a, b, c \in L$. That fact that $\langle L, \wedge, \vee, 0, 1 \rangle$ is a complete lattice means that the infimum $\bigwedge_{i \in I} a_i$ and supremum $\bigvee_{i \in I} a_i$ exist for any subset $\{a_i \mid i \in I\} \subseteq L$. Elements $a \in L$ are called truth degrees. Operations $\otimes$ and $\rightarrow$, called multiplication and residuum, are truth functions of logical connectives "fuzzy conjunction" and "fuzzy implication". A biresiduum of $\mathbf{L}$ is a binary operation $\leftrightarrow$ defined by

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a).$$

We denote by $\leq$ the lattice order induced by $\mathbf{L}$. Examples of residuated lattices are well known. A generic one is: Take a left-continuous t-norm $\otimes$. That is, $\otimes$ is binary operation on $[0, 1]$, which is left-continuous in its first argument (as a real function of two variables), commutative, associative, monotone, and has 1 as its neutral element (Hájek 1998). Put $a \to b = \bigvee\{c \in L \mid a \otimes c \leq b\}$. Then $\langle [0,1], \min, \max, \otimes, \to, 0, 1\rangle$ is a complete residuated lattice. Three most important pairs of adjoint operations on $[0,1]$ obtained this way are Łukasiewicz: $a \otimes b = \max(0, a+b-1)$, $a \to b = \min(1, 1-a+b)$; Gödel (minimum): $a \otimes b = a \wedge b$, $a \to b = b$ for $a > b$ and $a \to b = 1$ for $a \leq b$; Goguen (product): $a \otimes b = a \cdot b$, $a \to b = \frac{b}{a}$ for $a > b$ and $a \to b = 1$ for $a \leq b$. Throughout the rest of the paper, $\mathbf{L}$ denotes an arbitrary complete residuated lattice.

A special case of a complete residuated lattice is the two-element Boolean algebra $\langle \{0,1\}, \wedge, \vee, \otimes, \to, 0, 1\rangle$, denoted by $\mathbf{2}$. That is, the operations $\wedge, \vee, \otimes, \to$ of $\mathbf{2}$ are the truth functions (interpretations) of connectives of classical logic.

Fuzzy sets and fuzzy relations. Given $\mathbf{L}$, we define the usual notions regarding fuzzy sets and fuzzy relations: a fuzzy set (an $\mathbf{L}$-set) A in a universe X is a mapping $A\colon X \to L$, $A(x)$ being interpreted as "the degree to which x belongs to A". The set of all fuzzy sets in X is denoted by L^X. Operations with fuzzy sets are defined componentwise. For instance, the intersection of fuzzy sets $A, B \in L^X$ is a fuzzy set $A \cap B$ in X such that $(A \cap B)(x) = A(x) \wedge B(x)$ for each $x \in X$, etc. For fuzzy sets $A, B \in L^X$, put

$$S(A,B) = \bigwedge_{x \in X} (A(x) \to B(x)), \tag{7}$$

$$A \approx B = \bigwedge_{x \in X} (A(x) \leftrightarrow B(x)). \tag{8}$$

$S(A,B)$ and $A \approx B$ are called the degree of subsethood of A in B and the degree of equality of A and B, respectively. Note that $S(A,B)$ can be seen as a truth degree of "for each $x \in X$: if x belongs to A then x belongs to B". Similarly, $A \approx B$ can be seen as a truth degree of "for each $x \in X$: x belongs to A if and only if x belongs to B". $\approx$ is a fuzzy equivalence relation, i.e. $A \approx A = 1$ (reflexivity), $A \approx B = B \approx A$ (symmetry), and $(A \approx B) \otimes (B \approx C) \leq A \approx C$ (transitivity), which is called a Leibniz similarity. We denote the fact that $S(A,B) = 1$ by $A \subseteq B$ (A is fully contained in B). Hence, we have

$$A \subseteq B \text{ if and only if for each } x \in X:\ A(x) \leq B(x). \tag{9}$$

For more details we refer to (Belohlavek 2002, Hájek 1998).

3. Central Points

3.1 *Fuzzy Closure Operators*

Suppose $\mathcal{S}$ is a system of fuzzy sets in X, i.e. $\mathcal{S} \subseteq L^X$. We are going to consider the following type of problems. Given $B \subseteq \mathcal{S}$, what are the fuzzy sets $A \in \mathcal{S}$ which are similar to every $A' \in B$ to a degree at least ε? To asses similarity of A and A', we use $\approx$ defined by (8). That is, A being similar to A' to a degree at least ε means $A \approx A' \geq \varepsilon$. Furthermore, we assume that $\mathcal{S}$ is a system of fixpoints of an $\mathbf{L}$-closure operator (fuzzy closure operator) C in X, see Example 3.1, Example 3.6, and Example 3.7 for particular examples.

Recall (Belohlavek 2001, 2002, Rodríguez *et al.* 2003) that an $\mathbf{L}$-closure operator

C in X is a mapping $C : L^X \to L^X$ satisfying

$$A \subseteq C(A), \tag{10}$$

$$S(A_1, A_2) \leq S(C(A_1), C(A_2)), \tag{11}$$

$$C(A) = C(C(A)), \tag{12}$$

for every $A, A_1, A_2 \in L^X$. As a consequence, we also have

$$(A_1 \approx A_2) \leq (C(A_1) \approx C(A_2)). \tag{13}$$

The set $\mathrm{fix}(C)$ of all fixpoints of C is defined by

$$\mathrm{fix}(C) = \{A \in L^X \mid C(A) = A\}.$$

$\langle \mathrm{fix}(C), \subseteq \rangle$ is a complete lattice in which the infima $\bigwedge$ and suprema $\bigvee$ are given by

$$\bigwedge_{j \in J} A_j = \bigcap_{j \in J} A_j,$$

$$\bigvee_{j \in J} A_j = C\Big(\bigcup_{j \in J} A_j\Big),$$

for every $\{A_j \mid j \in J\} \subseteq \mathrm{fix}(C)$. In this paper, we often denote subsets of $\mathrm{fix}(C)$ by B. Correspondingly, we denote the infimum and the supremum of B by $\bigwedge B$ and $\bigvee B$, respectively.

Example 3.1 Clearly, the identity mapping $C : L^X \to L^X$, i.e. $C(A) = A$ for every $A \in L^X$, is an **L**-closure operator in X. In this case, $\mathrm{fix}(C) = L^X$. □

Remark 1: The concept of an **L**-closure operator generalizes the well-known concept of a closure operator. Namely, for $L = \{0, 1\}$, **L**-closure operators coincide with ordinary closure operators.

3.2 *Central Points, Closed Balls, and Blocks*

Definition 3.2: Let $B \subseteq \mathrm{fix}(C)$. Given a threshold $\varepsilon \in L$, let

$$C_\varepsilon(B) = \{A \in \mathrm{fix}(C) \mid \text{for every } A' \in B : A \approx A' \geq \varepsilon\}.$$

We call the elements of $C_\varepsilon(B)$ *ε-central points of B.*

That is, $C_\varepsilon(B)$ is the set of all fixpoints of C for which the degree of equality to every $A' \in B$ is at least ε. In a sense, $C_\varepsilon(B)$ contains all fixpoints which are ε-similar to every fixpoint from B.

Example 3.3 If B is empty or $\varepsilon = 0$ then $C_\varepsilon(B) = \mathrm{fix}(C)$.

Definition 3.4: Let $A \in \mathrm{fix}(C)$. Given a threshold $\varepsilon \in L$, let

$$B_\varepsilon(A) = \{A' \in \mathrm{fix}(C) \mid A \approx A' \geq \varepsilon\}.$$

We call the set $B_\varepsilon(A)$ a *closed ε-ball with center A.*

Example 3.5 If $\varepsilon = 0$ then $B_\varepsilon(A) = \mathrm{fix}(C)$.

Note that it follows immediately from the definitions that

$$B_\varepsilon(A) = C_\varepsilon(\{A\}). \tag{14}$$

Remark 2: The concept of similarity can be regarded as dual to the concept of a distance. A simple way to illustrate this correspondence is the following one. For any metric space M with a distance function d there can be introduced an $\mathbf{L}$-equivalence $\approx$ on M, with $\mathbf{L}$ being the unit real interval $[0,1]$ with Goguen (product) structure, by putting

$$(x \approx y) = e^{-d(x,y)},$$

where $d(x,y)$ is the distance of the points x and y. On the other hand, for any $\mathbf{L}$-equivalence $\approx$ on M satisfying

$$x \approx y = 1 \quad \text{iff} \quad x = y,$$

we can define a metric on M by

$$d(x,y) = -\lg(x \approx y).$$

Note that the above relationship is a special case of a general relationship between metric distances and fuzzy equivalences which are transitive w.r.t. a continuous Archimedean t-norm, such as the Goguen (product) t-norm, which is described in (De Baets and Mesiar 2002).

Now, any closed ε-ball with center A in $\text{fix}(C)$ coincides with the closed ball with center A and radius $-\lg\varepsilon$ in the metric space $\langle \text{fix}(C), d\rangle$. This illustrates the fact that the concept of a closed ball has its well-known counterpart in the theory of metric spaces. However, let us emphasize that such counterpart is available only for $L = [0,1]$ equipped with a continuous Archimedean t-norm $\otimes$.

The notion of ε-central point seems to have no counterpart in the theory of metric spaces.

Example 3.6 The notions of ε-central points and closed ε-balls generalize those studied by Belohlavek and Krupka (2008a). Namely, Belohlavek and Krupka (2008a) introduced the following concepts. Let $\mathbf{L}$ be a complete residuated lattice with a support set L. For $B \subseteq L$ and $\varepsilon \in L$, the set $C_\varepsilon(B)$ of central points and the closed ε-ball with center $c \in L$ were defined by

$$\begin{aligned} C_\varepsilon(B) &= \{a \in L \mid \text{ for each } b \in B : \ a \leftrightarrow b \geq \varepsilon\}, \\ B_\varepsilon(c) &= \{a \in L \mid a \leftrightarrow c \geq \varepsilon\}. \end{aligned}$$

Clearly, if we let $X = \{x\}$ and identify the $\mathbf{L}$-sets in X with truth degrees from L, i.e. identify $A \in \mathbf{L}^X$ s.t. $A(x) = a$ with a, then the notions of ε-central points and closed ε-balls are particular examples of the corresponding notions introduced in this paper. □

Example 3.7 Another example in which central points and closed balls naturally appear comes from concept analysis of data with fuzzy attributes (Belohlavek 2002), see also (Ganter and Wille 1999) for formal concept analysis of data with binary attributes. Let $\langle X, Y, I\rangle$ be a formal fuzzy context, i.e. X and Y are sets of objects and attributes, and $I : X \times Y \to L$ is a fuzzy relation between X and Y. For $x \in X$ and $y \in Y$, $I(x,y)$ is interpreted as the degree to which object x has attribute y. Let $^{\uparrow}: L^X \to L^Y$ and $^{\downarrow}: L^Y \to L^X$ denote the associated operators, i.e.

$$A^{\uparrow}(y) = \bigwedge_{x\in X} (A(x) \to I(x,y)),$$
$$B^{\downarrow}(x) = \bigwedge_{y\in Y} (B(y) \to I(x,y)).$$

Let $\mathcal{B}(X,Y,I) = \{\langle A,B\rangle \mid A^{\uparrow} = B, B^{\downarrow} = A\}$ denote the associated concept lattice. Elements $\langle A,B\rangle \in \mathcal{B}(X,Y,I)$ are called formal concepts and represent particular clusters in the data described by $\langle X,Y,I\rangle$. A and B are called the extent and the intent of $\langle A,B\rangle$ and represent the collection of all objects and attributes covered by the formal concept $\langle A,B\rangle$. Consider the set

$$\mathrm{Ext}(X,Y,I) = \{A \mid \langle A,B\rangle \in \mathcal{B}(X,Y,I) \text{ for some } B \in L^Y\}$$

of all extents of $\langle X,Y,I\rangle$. It can be easily shown that $\mathrm{Ext}(X,Y,I) = \mathrm{fix}(C)$ for the **L**-closure operator $C: L^X \to L^X$ defined by $C(A) = A^{\uparrow\downarrow}$.

Since $\mathcal{B}(X,Y,I) = \{\langle A, A^{\uparrow}\rangle \mid A \in \mathrm{Ext}(X,Y,I)\}$, $\mathcal{B}(X,Y,I)$ can be identified with $\mathrm{Ext}(X,Y,I)$. Given a threshold $\varepsilon \in L$ and a set $B \subseteq \mathcal{B}(X,Y,I)$ of formal concepts, $C_\varepsilon(B)$, i.e. the set of ε-central points, is the set of all formal concepts which are similar to every formal concept from B to a degree at least ε. Such set may be desirable particularly if B is large and we need just a representative formal concept(s) instead of B. In such case, it is particularly interesting to ask for the best such representative formal concept, i.e. such for which the similarity degree to every formal concept from B is the largest possible. We call such elements optimal central points and investigate them in Section 3.3. □

Remark 3 : (a) Recall that for a binary relation T between sets U and V, the Galois connection (Ore 1944) induced by T is a pair of mappings $^{\uparrow_T}: 2^U \to 2^V$ and $^{\downarrow_T}: 2^V \to 2^U$ defined for $M \in 2^U$ and $N \in 2^V$ by

$$M^{\uparrow_T} = \{v \in V \mid \text{ for each } u \in M: \langle u,v\rangle \in T\},$$
$$N^{\downarrow_T} = \{u \in U \mid \text{ for each } v \in N: \langle u,v\rangle \in T\}.$$

If $U = V$ and T is symmetric, then $^{\uparrow_T}$ coincides with $^{\downarrow_T}$ and we write just M^T instead of $M^{\uparrow_T}$ or $M^{\downarrow_T}$.

(b) Consider the Galois connection induced by the ε-cut ${}^{\varepsilon}\!\approx$ of $\approx$, i.e. by the symmetric binary relation ${}^{\varepsilon}\!\approx$ between $\mathrm{fix}(C)$ and $\mathrm{fix}(C)$ defined for $A, A' \in \mathrm{fix}(C)$ by

$$\langle A, A'\rangle \in {}^{\varepsilon}\!\approx \quad \text{if and only if} \quad A \approx A' \geq \varepsilon. \tag{15}$$

Clearly, ${}^{\varepsilon}\!\approx$ is a tolerance relation which need not be transitive. $\langle A, A'\rangle \in {}^{\varepsilon}\!\approx$ means that A and A' are similar to degree at least ε. As a result of the definitions, for $B \subseteq \mathrm{fix}(C)$ and $A \in \mathrm{fix}(C)$, we have

$$C_\varepsilon(B) = B^{{}^{\varepsilon}\!\approx} \quad \text{and} \quad B_\varepsilon(A) = \{A\}^{{}^{\varepsilon}\!\approx}.$$

Note also that $B_\varepsilon(A)$ is just the class of tolerance ${}^{\varepsilon}\!\approx$ given by A. □

From the basic properties of Galois connections, we get the following assertions.

Lemma 3.8: *For* $B, B_1, B_2 \subseteq \mathrm{fix}(C)$*,*

$$B_1 \subseteq B_2 \text{ implies } C_\varepsilon(B_1) \subseteq C_\varepsilon(B_2), \tag{16}$$

$$B \subseteq C_\varepsilon(C_\varepsilon(B)), \tag{17}$$

$$C_\varepsilon(B) = C_\varepsilon(C_\varepsilon(C_\varepsilon(B))), \tag{18}$$

$$C_\varepsilon(B) = \bigcap_{A \in B} B_\varepsilon(A). \tag{19}$$

□

Note that (19) says that ε-central points of B are just the points common to all closed ε-balls with centers $A \in B$.

As a consequence, we get the following lemma.

Lemma 3.9: *For* $B \subseteq \mathrm{fix}(C)$*,*

$$B \subseteq \bigcap_{A \in C_\varepsilon(B)} B_\varepsilon(A). \tag{20}$$

For $A \in \mathrm{fix}(C)$*,*

$$A \in C_\varepsilon(B_\varepsilon(A)). \tag{21}$$

Proof: (20) follows from (17) and (19). Due to (17), $\{A\} \subseteq C_\varepsilon(C_\varepsilon(\{A\})) = C_\varepsilon(B_\varepsilon(A))$, whence (20). □

The following lemma is another direct consequence of the observation made in Remark **3** and the well-known properties of Galois connections.

Lemma 3.10: *1. The mapping* $cl_\varepsilon : 2^{\mathrm{fix}(C)} \to 2^{\mathrm{fix}(C)}$ *defined for* $D \subseteq \mathrm{fix}(C)$ *by* $cl_\varepsilon(D) = C_\varepsilon(C_\varepsilon(D))$ *is an ordinary closure operator in* $\mathrm{fix}(C)$*.*

2. The set $\mathrm{fix}(cl_\varepsilon) = \{D \subseteq \mathrm{fix}(C) \mid D = cl_\varepsilon(D)\}$ *of all fixpoints of* cl_ε *equipped with* $\subseteq$ *is a complete lattice.*

3. $D \in \mathrm{fix}(cl_\varepsilon)$ *if and only if* $D = C_\varepsilon(B)$ *for some* $B \subseteq \mathrm{fix}(C)$*, i.e.* $\mathrm{fix}(cl_\varepsilon)$ *contains just sets of* ε*-central points.*

We now present a description of the set $C_\varepsilon(B)$ of central points in our general setting. First, we need the following lemma.

Lemma 3.11: $A \in C_\varepsilon(B)$ *iff* $S(A, \bigwedge B) \wedge S(\bigvee B, A) \geq \varepsilon$*.*

Proof: By definition, $A \in C_\varepsilon(B)$ means that for each $A' \in B$, $S(A, A') \geq \varepsilon$ and $S(A', A) \geq \varepsilon$. Hence, to prove the assertion, it suffices to check that (a) $S(A, A') \geq \varepsilon$ for each $A' \in B$ is equivalent to $S(A, \bigwedge B) \geq \varepsilon$, and (b) $S(A', A) \geq \varepsilon$ for each $A' \in B$ is equivalent to $S(\bigvee B, A) \geq \varepsilon$.

(a): By definition and using $(\bigwedge B)(x) = \bigwedge_{A' \in B} A'(x)$,

$$S\left(A, \bigwedge B\right) = \bigwedge_{x \in X} \left(A(x) \to \bigwedge_{A' \in B} A'(x)\right) = \bigwedge_{x \in X} \bigwedge_{A' \in B} (A(x) \to A'(x)).$$

Hence, $S(A, \bigwedge B) \geq a$ iff for every $A' \in B$, $S(A, A') \geq a$.

(b): Since $A \in \mathrm{fix}(C)$ we have

$$S\left(\bigvee B, A\right) = S\left(C\left(\bigcup B\right), C(A)\right) \geq S\left(\bigcup B, A\right)$$

by (11). On the other hand, (10) yields $\bigvee B \supseteq \bigcup B$, which implies $S(\bigvee B, A) \leq S(\bigcup B, A)$. Hence,

$$S(\bigvee B, A) = S(\bigcup B, A) = \bigwedge_{x \in X} \bigwedge_{A' \in B} (A'(x) \to A(x)) = \bigwedge_{A' \in B} S(A', A)$$

and thus $S(\bigvee B, A) \geq a$ iff for each $A' \in B$, $S(A', A) \geq a$. □

The next theorem shows that central points form particular intervals in the lattice $\langle \mathrm{fix}(C), \subseteq \rangle$.

Theorem 3.12: *For any* $B \subseteq \mathrm{fix}(C)$,

$$C_\varepsilon(B) = \left[C\left(\varepsilon \otimes \bigvee B\right), \varepsilon \to \bigwedge B\right].$$

Note that $[\_, \_]$ denotes an interval in $\langle \mathrm{fix}(C), \subseteq \rangle$, i.e.

$$\left[C\left(\varepsilon \otimes \bigvee B\right), \varepsilon \to \bigwedge B\right] = \left\{A \in \mathrm{fix}(C) \,|\, C\left(\varepsilon \otimes \bigvee B\right) \subseteq A \subseteq \varepsilon \to \bigwedge B\right\},$$

and that fuzzy sets $\varepsilon \otimes \bigvee B$ and $\varepsilon \to \bigwedge B$ are defined by

$$\left(\varepsilon \otimes \bigvee B\right)(x) = \varepsilon \otimes \left(\bigvee B\right)(x) \quad \text{and} \quad \left(\varepsilon \to \bigwedge B\right)(x) = \varepsilon \to \left(\bigwedge B\right)(x).$$

Proof: By Lemma 3.11, A is a central point iff $S(A, \bigwedge B) \geq \varepsilon$ and $S(\bigvee B, A) \geq \varepsilon$, which is equivalent to $A \subseteq \varepsilon \to \bigwedge B$ and $\varepsilon \otimes \bigvee B \subseteq A$. Since fixpoints of C are closed under $\to$-shifts, see (Belohlavek 2002), we get $\varepsilon \to \bigwedge B \in \mathrm{fix}(C)$. However, $\varepsilon \otimes \bigvee B$ need not be a fixpoint. The least fixpoint greater than or equal to $\varepsilon \otimes \bigvee B$ is $C(\varepsilon \otimes \bigvee B)$. This proves the theorem. □

The following theorem describes closed balls.

Theorem 3.13: *For any* $A \in \mathrm{fix}(C)$,

$$B_\varepsilon(A) = [C(\varepsilon \otimes A), \varepsilon \to A].$$

Proof: Directly from Theorem 3.12 using (14). □

Consider now, in addition to ${}^{\varepsilon}\!\approx$, cf. (15), the binary relation ${}^{\varepsilon^2}\!\approx$ on $\mathrm{fix}(C)$ defined by

$$\langle A, A' \rangle \in {}^{\varepsilon^2}\!\approx \quad \text{if and only if} \quad A \approx A' \geq \varepsilon^2 = \varepsilon \otimes \varepsilon. \tag{22}$$

Since $\varepsilon \otimes \varepsilon \leq \varepsilon$, $\langle A, A' \rangle \in {}^{\varepsilon}\!\approx$ implies $\langle A, A' \rangle \in {}^{\varepsilon^2}\!\approx$. Hence, classes (i.e., closed balls, cf. Remark **3** (b)) of ${}^{\varepsilon}\!\approx$ are contained in classes of ${}^{\varepsilon^2}\!\approx$, i.e. $B_\varepsilon(A) \subseteq B_{\varepsilon^2}(A)$.

Likewise, blocks of ${}^{\varepsilon}\!\approx$ are blocks of ${}^{\varepsilon^2}\!\approx$. However, there is an interesting relationship between the closed balls $B_\varepsilon(A)$ and maximal blocks of ${}^{\varepsilon^2}\!\approx$ which we now investigate.

Lemma 3.14: *For each $A \in \mathrm{fix}(C)$, $B_\varepsilon(A)$ is a block of ${}^{\varepsilon^2}\!\approx$.*

Proof: By Theorem 3.13, $B_\varepsilon(A) = [C(\varepsilon \otimes A), \varepsilon \to A]$. It follows from (Belohlavek and Krupka 2008b, Theorem 2) that

$$B = [C(\varepsilon^2 \otimes (\varepsilon \to A)), \varepsilon^2 \to C(\varepsilon^2 \otimes (\varepsilon \to A))]$$

is a maximal block of ${}^{\varepsilon^2}\!\approx$ which contains the fixpoint $\varepsilon \to A$. Now, since $\varepsilon^2 \otimes (\varepsilon \to A) \subseteq \varepsilon \otimes A$, we get $C(\varepsilon^2 \otimes (\varepsilon \to A)) \subseteq C(\varepsilon \otimes A)$. Similarly, since $\varepsilon^2 \otimes (\varepsilon \to A) \subseteq C(\varepsilon^2 \otimes (\varepsilon \to A))$, we get $\varepsilon \to A \subseteq \varepsilon^2 \to C(\varepsilon^2 \otimes (\varepsilon \to A))$. We proved $B_\varepsilon(A) \subseteq B$ which finishes the proof. □

Lemma 3.15: *For $B \subseteq \mathrm{fix}(C)$, $C_\varepsilon(B)$ is non-empty if an only if B is a block of ${}^{\varepsilon^2}\!\approx$.*

Proof: Due to Theorem 3.12, $C_\varepsilon(B)$ is non-empty iff $C(\varepsilon \otimes \bigvee B) \le \varepsilon \to \bigwedge B$. Furthermore, B is a block of ${}^{\varepsilon^2}\!\approx$ iff $\varepsilon^2 \le S(\bigvee B, \bigwedge B)$. Indeed, this follows by a slight modification of (Ganter and Wille 1999, Proposition 54) by observing that $S(\bigvee B, \bigwedge B) = \bigvee B \approx \bigwedge B$ and that, due to (Belohlavek and Krupka 2008b, Lemma 1), ${}^{\varepsilon^2}\!\approx$ is a complete tolerance relation on $\langle \mathrm{fix}(C), \subseteq \rangle$. To prove the lemma, we thus need to check that

$$C\left(\varepsilon \otimes \bigvee B\right) \subseteq \varepsilon \to \bigwedge B \quad \text{iff} \quad \varepsilon^2 \le S\left(\bigvee B, \bigwedge B\right). \tag{23}$$

Let $C(\varepsilon \otimes \bigvee B) \le \varepsilon \to \bigwedge B$. Since $\varepsilon \otimes \bigvee B \le C(\varepsilon \otimes \bigvee B)$, we get $\varepsilon \otimes \bigvee B \le \varepsilon \to \bigwedge B$ from which $\varepsilon^2 \le S(\bigvee B, \bigwedge B)$ readily follows.

Conversely, if $\varepsilon^2 \le S(\bigvee B, \bigwedge B)$ then $\varepsilon \otimes \bigvee B \le \varepsilon \to \bigwedge B$, from which we get $C(\varepsilon \otimes \bigvee B) \le C(\varepsilon \to \bigwedge B) = \varepsilon \to \bigwedge B$, because of monotony of C and the fact that $\varepsilon \to \bigwedge B$ is a fixpoint of C. The proof is finished. □

We say that a closed ε-ball $B_\varepsilon(A)$ is maximal if $B_\varepsilon(A) = B_\varepsilon(A')$ for every A' with $B_\varepsilon(A) \subseteq B_\varepsilon(A')$. The following theorem describes a relationship between closed balls and maximal blocks of ${}^{\varepsilon^2}\!\approx$.

Theorem 3.16: *For $B \subseteq \mathrm{fix}(C)$, B is a maximal closed ε-ball if and only if B is a maximal block of ${}^{\varepsilon^2}\!\approx$. In particular, if B is a maximal block of ${}^{\varepsilon^2}\!\approx$ then $C_\varepsilon(B) \ne 0$ and $B = B_\varepsilon(A)$ for every $A \in C_\varepsilon(B)$.*

Proof: Let $B_\varepsilon(A)$ be maximal. Due to Lemma 3.14, $B_\varepsilon(A)$ is a block of ${}^{\varepsilon^2}\!\approx$. There exists a maximal block B of ${}^{\varepsilon^2}\!\approx$ for which for which $B_\varepsilon(A) \subseteq B$ (the existence of B follows from Zorn lemma). Due to Lemma 3.15, $C_\varepsilon(B) \ne \emptyset$. Take an arbitrary $A' \in C_\varepsilon(B)$. Due to (20), $B \subseteq B_\varepsilon(A')$. Therefore, $B_\varepsilon(A) \subseteq B \subseteq B_\varepsilon(A')$. Maximality of $B_\varepsilon(A)$ as a closed ε-ball yields $B_\varepsilon(A) = B$, i.e. $B_\varepsilon(A)$ is a maximal block of ${}^{\varepsilon^2}\!\approx$

Conversely, let B be a maximal block of ${}^{\varepsilon^2}\!\approx$. Observe fist that if $B \subseteq B_\varepsilon(A)$ then $B = B_\varepsilon(A)$. Indeed, due to Lemma 3.15, $B_\varepsilon(A)$ is a block of ${}^{\varepsilon^2}\!\approx$ and hence $B = B_\varepsilon(A)$ follows from the fact that B is a maximal block of ${}^{\varepsilon^2}\!\approx$. Therefore, to prove the claim, it is sufficient to realize that $C_\varepsilon(B) \ne \emptyset$ (Lemma 3.15) and that for every $A \in C_\varepsilon(B)$ we have $B \subseteq B_\varepsilon(A)$ due to (20). □

3.3 *Optimal Central Points*

Consider now the following problem. Theorem 3.12 describes the set $C_\varepsilon(B)$ of ε-central points of B. Every $A \in C_\varepsilon(B)$ is good in the sense that the degree $A \approx A'$ of its similarity to any $A' \in B$ is at least ε. However, some central points from $C_\varepsilon(B)$ may be better than others. We call the best ones the optimal central points of B.

Definition 3.17: Let $B \subseteq \text{fix}(C)$. $A \in \text{fix}(C)$ is called an *optimal central point of* B if and only if

$$\bigwedge_{A' \in B} (D \approx A') \leq \bigwedge_{A' \in B} (A \approx A') \tag{24}$$

for every $D \in \text{fix}(C)$.

Remark 4: Note that according to the principles of fuzzy logic,

$$\bigwedge_{A' \in B} (D \approx A')$$

can be understood as the truth degree of "for every $A' \in B$: D is similar to A'". Therefore, for an optimal central point of B, such degree is the highest possible.

We now turn to a characterization of optimal central points and their existence in terms of radii. We need the following concepts.

Definition 3.18: We say that $\varepsilon \in L$ is an *admissible radius of* $B \subseteq \text{fix}(C)$ if $C_\varepsilon(B) \neq \emptyset$. We call ε the *radius of* B *for* A if ε is the largest radius for which $A \in C_\varepsilon(B)$.

Observe that for any B and A, the radius of B for A is $\bigwedge_{A' \in B}(A \approx A')$. This observation and (24) thus yield an alternative characterization of optimal central points:

Lemma 3.19: *A is an optimal central point of B if and only if for every $D \in \text{fix}(C)$, the radius of B for A is larger than or equal to the radius of B for D.*

The following theorem provides a characterization of optimal central points of B.

Theorem 3.20: *Conditions* 1., 2., *and* 3. *are equivalent.*

1. *The set of all optimal central points of B is non-empty and ε is the radius of B for some optimal central point A.*
2. *The set of all optimal central points of B is non-empty and ε is the radius of B for any of the optimal central points.*
3. *ε is the largest admissible radius of B.*

Any of conditions 1., 2., *and* 3. *implies condition* 4.

4. *The set of all optimal central points is equal to $C_\varepsilon(B)$.*

Proof: "1. $\Rightarrow$ 2.": (24) implies that the radii of B for any two optimal central points A_1 and A_2 are equal.

"2. ⇒ 3.": Assume 2. Clearly, ε is an admissible radius of B. If ε' is an admissible radius of B then for any $D \in C_{\varepsilon'}(B)$, we have $\varepsilon' \leq \bigwedge_{A' \in B}(D \approx A')$. Now, for any optimal central point A of B, (24) and the assumption $\bigwedge_{A' \in B}(A \approx A') = \varepsilon$ give $\bigwedge_{A' \in B}(D \approx A') \leq \varepsilon$, whence $\varepsilon' \leq \varepsilon$, proving 3.

"3. ⇒ 1.": For $A \in C_\varepsilon(B)$, $\varepsilon \leq \bigwedge_{A' \in B}(A \approx A')$. On the other hand, since $\bigwedge_{A' \in B}(A \approx A')$ is an admissible radius (the radius of B for A), we have $\bigwedge_{A' \in B}(A \approx A') \leq \varepsilon$, whence $\bigwedge_{A' \in B}(A \approx A') = \varepsilon$. Since for any D, $\bigwedge_{A' \in B}(D \approx A')$ is an admissible radius, we get $\bigwedge_{A' \in B}(D \approx A') \leq \varepsilon = \bigwedge_{A' \in B}(A \approx A')$, proving 1.

To finish the proof, we check "2. ⇒ 4.": Assume 2. Clearly, every optimal central point of B is in $C_\varepsilon(B)$. If A is not optimal then $\bigwedge_{A' \in B}(A \approx A') < \varepsilon$ and hence $A \notin C_\varepsilon(B)$. □

Remark 5 : Note that condition 4. of Theorem 3.20 nor the condition saying that the set of optimal central points of B is non-empty and is equal to $C_\varepsilon(B)$ implies conditions 1., 2, and 3. Consider the following example (cf. Example 3.6). Let $\mathbf{L}$ be the Gödel algebra on the real unit interval $L = [0, 1]$. Let $X = \{x\}$ (singleton). Then $\mathcal{S} = \{\{^0/x\}, \{^{0.5}/x\}, \{^1/x\}\}$ is a set of fixpoints of an $\mathbf{L}$-closure operator C. This claim follows from (Belohlavek 2001) by verification of the fact that $\mathcal{S}$ is closed under intersections and that $a \to A \in \mathcal{S}$ for every $a \in L$ and $A \in \mathcal{S}$. Consider $B = \{\{^{0.5}/x\}, \{^1/x\}\}$. A moment's reflection shows that the set of optimal points of B is B. Now, $B = C_{0.4}(B)$ but the largest admissible radius of B is 0.5.

We now turn to the existence of optimal central points of B. We need the following lemma.

Lemma 3.21:

1. *ε is an admissible radius of B if and only if $\varepsilon \otimes \varepsilon \leq S(\bigvee B, \bigwedge B)$.*
2. *For every $z \in L$, $z \wedge (z \to S(\bigvee B, \bigwedge B))$ is an admissible radius of B.*
3. *ε is an admissible radius of B if and only if $\varepsilon = \varepsilon \wedge (\varepsilon \to S(\bigvee B, \bigwedge B))$.*
4. *The set*

$$R = \left\{z \wedge \left(z \to S\left(\bigvee B, \bigwedge B\right)\right) \mid z \in L\right\} \tag{25}$$

is the set of all admissible radii of B.

Proof: Denote $d = S(\bigvee B, \bigwedge B)$.

1. Using Theorem 3.12, $C_\varepsilon(B) \neq \emptyset$ iff $[C(\varepsilon \otimes \bigvee B), \varepsilon \to \bigwedge B] \neq \emptyset$ iff $C(\varepsilon \otimes \bigvee B) \subseteq \varepsilon \to \bigwedge B$ iff $\varepsilon \otimes \varepsilon \leq S(\bigvee B, \bigwedge B)$ (the last two conditions are equivalent due to (23)).

2. $(z \wedge (z \to d)) \otimes (z \wedge (z \to d)) \leq z \otimes (z \to d) \leq d$, hence the claim follows from 1.

3. Using 1., ε is an admissible radius of B iff $\varepsilon \leq \varepsilon \to d$ which is equivalent to $\varepsilon = \varepsilon \wedge (\varepsilon \to d)$.

4. A consequence of 2. and 3. □

The following theorem presents a necessary and sufficient condition for the existence of optimal central points of B.

Theorem 3.22 : *A set $B \subseteq \mathrm{fix}(C)$ has optimal central points if and only if the set R from (25) has a largest element. This element is the largest admissible radius ε of B and $C_\varepsilon(B)$ is the set of optimal central points of B.*

Proof: Follows from 4. of Lemma 3.21 and from Theorem 3.20. □

For some of the well-known structures of truth degrees the description of optimal central points can be made more particular. As an example, consider the setting of Example 3.6 and assume that the complete residuated lattice $\mathbf{L}$ is the real unit interval $[0,1]$ equipped with Łukasiewicz t-norm and its residuum. Then if $B = [a,b]$, the largest admissible radius of B is $\frac{a-b+2}{2}$ and the set of optimal central points of B contains just one $c \in [0,1]$, namely $c = \frac{a+b}{2}$. In the rest of this paper, we show that such more particular descriptions are available if the complete residuated lattice $\mathbf{L}$ has square roots. According to Höhle (1995), a complete residuated lattice $\mathbf{L}$ has square roots if there is a function $\sqrt{\ } : L \to L$ satisfying

$$\sqrt{a} \otimes \sqrt{a} = a, \tag{26}$$

$$b \otimes b \leq a \quad \text{implies} \quad b \leq \sqrt{a}, \tag{27}$$

for every $a, b \in L$.

Example 3.23 (Höhle 1995) Łukasiewicz, product, and Gödel algebras on $[0,1]$ have square roots. They are given by

$$\sqrt{a} = \frac{a+1}{2} \quad \text{for Łukasiewicz},$$
$$\sqrt{a} = \text{ordinary number-theoretic square root of } a \text{ for product},$$
$$\sqrt{a} = a \quad \text{for Gödel}.$$

Theorem 3.24: *If $\mathbf{L}$ has square roots then any subset $B \subseteq L$ has optimal central points. For the corresponding largest admissible radius ε it holds*

$$\varepsilon = \sqrt{S\left(\bigwedge B, \bigvee B\right)}. \tag{28}$$

Proof: According to 1. of Lemma 3.21 and (26), ε is the largest admissible radius of B. The rest follows from Theorem 3.20. □

Corollary 3.25: *If If $\mathbf{L}$ has square roots then for any subset $B \subseteq L$, the set of optimal central points is equal to*

$$\left[\sqrt{S\left(\bigwedge B, \bigvee B\right)} \otimes \bigvee B, \sqrt{S\left(\bigwedge B, \bigvee B\right)} \to \bigwedge B\right].$$

Example 3.26 Consider the setting of Example 3.6 and let $L = [0,1]$. In this case, for $a, b \in L$ we have $S(a,b) = a \to b$. Let $B \subseteq [0,1]$ and denote $[a,b] = [\bigwedge B, \bigvee B]$. For Łukasiewicz, product, and Gödel algebras on $[0,1]$, Theorem 3.24 gives the following description of the set $\mathcal{O}$ of optimal central points of B.

$$\mathcal{O} = \left\{ \frac{a+b}{2} \right\} \quad \text{for Łukasiewicz,}$$

$$\mathcal{O} = \begin{cases} \{\sqrt{a} \cdot \sqrt{b}\} & \text{if } a > 0 \text{ or } a = b = 0, \\ [0,1] & \text{if } 0 = a < b, \end{cases} \quad \text{for product,}$$

$$\mathcal{O} = \begin{cases} [a,1] & \text{if } a < b, \\ \{a\} & \text{if } a = b, \end{cases} \quad \text{for Gödel.}$$

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Grouping fuzzy sets by similarity

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ABSTRACT

The paper presents results on factorization of systems of fuzzy sets. The factorization consists in grouping those fuzzy sets which are pairwise similar at least to a prescribed degree *a*. An obstacle to such factorization, well known in fuzzy set theory, is the fact that "being similar at least to degree *a*" is not an equivalence relation because, in general, it is not transitive. As a result, ordinary factorization using equivalence classes cannot be used. This obstacle can be overcome by considering maximal blocks of fuzzy sets which are pairwise similar at least to degree *a*. We show that one can introduce a natural complete lattice structure on the set of all such maximal blocks and study this lattice. This lattice plays the role of a factor structure for the original system of fuzzy sets. Particular examples of our approach include factorization of fuzzy concept lattices and factorization of residuated lattices.

1. Introduction

Factorization represents a fundamental construction in mathematics. Its main aim is to capture the process of simplification by abstraction. An input to a factorization is a mathematical structure, typically a system of elements equipped possibly with relations and functions. An output of a factorization consists of another structure, called a factor structure (or a quotient structure), which can be considered a simplified version of the input structure. Elements of the factor structure are groups of elements of the original structure, which are indistinguishable from a certain point of view. The indistinguishability is usually represented by an equivalence relation and the groups of elements are the corresponding equivalence classes. To be able to introduce a naturally inherited structure on the groups of indistinguishable elements, the equivalence relation needs to be compatible with functions and relations of the original structure.

In this paper, we present a general framework for factorization of systems of fuzzy sets by similarity. The input structure consists of a system of fuzzy sets equipped with a subsethood relation. The indistinguishability relation which we use for factorization is represented by the relation "being similar at least to degree a" where similarity degrees are assessed by means of a well-known Leibniz similarity relation, see Section 2, i.e. the indistinguishability is represented by an a-cut of a particular fuzzy equivalence relation. We assume that the fuzzy sets are fixpoints of some fuzzy closure operator. Examples of such systems are fuzzy concept lattices, fuzzy sets in a given universe, or complete residuated lattices.

Such assumptions are natural: We deal with a system $\mathscr{S}$ of fuzzy sets and if $\mathscr{S}$ is considered too large, we want to simplify it by putting together those fuzzy sets from $\mathscr{S}$ which are pairwise similar to a prescribed degree a (threshold, a parameter to the factorization). However, the ordinary factorization cannot be used. Namely, an obstacle consists in the fact, well known in fuzzy set theory, that "being similar at least to degree a" is not a transitive relation and hence not an equivalence relation. We overcome this obstacle by utilizing results from lattice theory on factorization of complete lattices by compatible reflex-

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ive and symmetric relations. We show that "being similar at least to degree a" is a compatible relation and thus, one can introduce the structure of a complete lattice on the set of all maximal blocks of this relation. We study this lattice and provide an efficient description of the blocks which can be used to compute the factor system. Namely, we show that the upper bounds of the maximal blocks are just fixpoints of a particular fuzzy closure operator for which we present an explicit description.

2. Preliminaries from fuzzy logic

In classical logic, the structure $\mathbf{L}$ of truth degrees consists of the two-element set $L = \{0, 1\}$ of truth degrees and the truth functions of logical connectives. In fuzzy logic, there are more options, both for the set L of truth degrees and for the functions of logical connectives. As the structures of truth degrees, we use complete residuated lattices in our approach. Complete residuated lattices are general structures of truth degrees and several variants of them are used in fuzzy logic. A complete residuated lattice [5,13] is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\to$ satisfy the adjointness property: $a \otimes b \leqslant c$ iff $a \leqslant b \to c$ for every $a, b, c \in L$. The fact that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice means that the infimum $\bigwedge_{i\in I} a_i$ and the supremum $\bigvee_{i\in I} a_i$ exist for every subset $\{a_i \mid i \in I\} \subseteq L$. Elements $a \in L$ are called truth degrees. Operations $\otimes$ and $\to$, called multiplication and residuum, are the truth functions of logical connectives "fuzzy conjunction" and "fuzzy implication". A biresiduum of $\mathbf{L}$ is a binary operation $\leftrightarrow$ defined by

$$a \leftrightarrow b = (a \to b) \wedge (b \to a).$$

We denote by $\leqslant$ the lattice order induced by $\mathbf{L}$. Examples of residuated lattices are well known. A generic one is: Take $L = [0, 1]$ and a left-continuous t-norm $\otimes$. That is, $\otimes$ is binary operation, which is left-continuous in its first argument (as a real function of two variables), commutative, associative, monotone, and has 1 as its neutral element [13]. Put $a \to b = \bigvee\{c \in L \mid a \otimes c \leqslant b\}$. Then $\langle [0, 1], \min, \max, \otimes, \to, 0, 1\rangle$ is a complete residuated lattice.

Particular well-known examples include the following t-norms and residua: Łukasiewicz ($a \otimes b = \max(0, a + b - 1), a \to b = \min(1, 1 - a + b)$); Gödel (minimum) ($a \otimes b = a \wedge b$, $a \to b = b$ for $a > b$ and $a \to b = 1$ for $a \leqslant b$); Goguen (product) ($a \otimes b = a \cdot b, a \to b = \frac{b}{a}$ for $a > b$ and $a \to b = 1$ for $a \leqslant b$). Another well-known class of examples includes residuated lattices which are finite chains, e.g. $L = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ equipped with restrictions of the abovementioned Łukasiewics or Gödel operations. Throughout the rest of the paper, $\mathbf{L}$ denotes an arbitrary complete residuated lattice.

Given $\mathbf{L}$, we define the usual notions regarding fuzzy sets and fuzzy relations: a fuzzy set (an $\mathbf{L}$-set) A in universe X is a mapping $A : X \to L, A(x)$ being interpreted as "the degree to which x belongs to A". The set of all fuzzy sets in X is denoted by L^X. Operations with fuzzy sets are defined componentwise. For instance, the intersection of fuzzy sets $A, B \in \mathbf{L}^X$ is a fuzzy set $A \cap B$ in X such that $(A \cap B)(x) = A(x) \wedge B(x)$ for each $x \in X$, etc. For fuzzy sets $A, B \in L^X$, put

$$S(A, B) = \bigwedge_{x\in X}(A(x) \to B(x)), \tag{1}$$

$$A \approx B = \bigwedge_{x\in X}(A(x) \leftrightarrow B(x)). \tag{2}$$

$S(A, B)$ and $A \approx B$ are called the degree of subsethood of A in B and the degree of equality of A and B, respectively. Note that $S(A, B)$ can be seen as the truth degree of "for each $x \in X$: if x belongs to A then x belongs to B". Similarly, $A \approx B$ can be seen as the truth degree of "for each $x \in X$: x belongs to A if and only if x belongs to B". $\approx$ is a fuzzy equivalence relation, i.e. $A \approx A = 1$ (reflexivity), $A \approx B = B \approx A$ (symmetry), and $(A \approx B) \otimes (B \approx C) \leqslant A \approx C$, which is called the Leibniz similarity. Furthermore, $S(A, B) = 1$ iff $A(x) \leqslant B(x)$ for each $x \in X$ (A is fully contained in B). This fact is denoted by

$$A \subseteq B.$$

For more details we refer to [5,12,13].

3. Factorization by similarity

Suppose $\mathscr{S}$ is a system of fuzzy sets in X, i.e. $\mathscr{S} \subseteq L^X$. Suppose furthermore that $\mathscr{S}$ is a system of fixpoints of an $\mathbf{L}$-closure operator (fuzzy closure operator) C in X. Recall [2,5,15] that an $\mathbf{L}$-closure operator C in X is a mapping $C : L^X \to L^X$ satisfying

$$A \subseteq C(A), \tag{3}$$

$$S(A_1, A_2) \leqslant S(C(A_1), C(A_2)), \tag{4}$$

$$C(A) = C(C(A)), \tag{5}$$

for every $A, A_1, A_2 \in L^X$. As a consequence, we also have

$$(A_1 \approx A_2) \leqslant (C(A_1) \approx C(A_2)). \tag{6}$$

The set fix(C) of all fixpoints of C is defined by

$$\mathrm{fix}(C) = \{A \in L^X \mid A = C(A)\}.$$

Remark 1. The concept of an **L**-closure operator generalizes the well-known concept of a closure operator. Namely, for $L = \{0,1\}$, **L**-closure operators coincide with ordinary closure operators.

We thus assume $\mathscr{S} = \mathrm{fix}(C)$ for an **L**-closure operator C. Next, we present three examples of such systems.

Example 1. The first example comes from formal concept analysis of data with fuzzy attributes [5–7,17]. Let $\langle X, Y, I\rangle$ be a formal fuzzy context, i.e. X and Y are sets of objects and attributes, and $I : X \times Y \to L$ is a fuzzy relation between X and Y. For $x \in X$ and $y \in Y$, $I(x,y)$ is interpreted as the degree to which object x has attribute y. Let $^{\uparrow} : L^X \to L^Y$ and $^{\downarrow} : L^Y \to L^X$ denote the associated operators, i.e.

$$A^{\uparrow}(y) = \bigwedge_{x\in X}(A(x) \to I(x,y)), \quad B^{\downarrow}(x) = \bigwedge_{y\in Y}(B(y) \to I(x,y)).$$

Let $\mathscr{B}(X,Y,I) = \{\langle A,B\rangle | A^{\uparrow} = B, B^{\downarrow} = A\}$ denote the associated concept lattice. $\mathscr{B}(X,Y,I)$ equipped with $\leqslant$ defined by $\langle A_1,B_1\rangle \leqslant \langle A_2,B_2\rangle$ iff $A_1 \subseteq A_2$ (iff $B_1 \supseteq B_2$) is indeed a complete lattice. Elements $\langle A,B\rangle \in \mathscr{B}(X,Y,I)$ are called formal concepts and represent particular clusters in the data described by $\langle X,Y,I\rangle$. A and B are called the extent and the intent of $\langle A,B\rangle$ and represent the collection of all objects and attributes covered by the formal concept $\langle A,B\rangle$. Consider the set

$$\mathrm{Ext}(X,Y,I) = \{A \mid \langle A,B\rangle \in \mathscr{B}(X,Y,I) \text{ for some } B \in L^Y\}$$

of all extents of $\langle X,Y,I\rangle$. It is well-known [2] that the mapping $C : L^X \to L^X$ defined by $C(A) = A^{\uparrow\downarrow}$ is an **L**-closure operator for which $\mathrm{fix}(C) = \mathrm{Ext}(X,Y,I)$, $\mathrm{Ext}(X,Y,I)$ is the set of all fixpoints of C. Note that $\langle \mathrm{Ext}(X,Y,I), \subseteq\rangle$ is isomorphic to $\langle \mathscr{B}(X,Y,I), \leqslant\rangle$.

Now, putting $\mathscr{S} = \mathrm{Ext}(X,Y,I)$, we have our first example of a system of fuzzy sets. Since $\mathscr{B}(X,Y,I) = \{\langle A, A^{\uparrow}\rangle \mid A \in \mathrm{Ext}(X,Y,I)\}$, $\mathscr{B}(X,Y,I)$ can be identified with $\mathrm{Ext}(X,Y,I)$. Therefore, loosely speaking, $\mathscr{S}$ is the concept lattice associated to the input data $\langle X,Y,I\rangle$. Note also that every **L**-closure operator in X can be obtained this way, i.e. from some $\langle X,Y,I\rangle$, see [2,5].

Example 2. Every complete residuated lattice **L** can be thought of as a system $\mathscr{S}$ of fuzzy sets. Namely, putting $X = \{x\}$, we can identify L with the set L^X of all fuzzy sets in X (just identify a with $\{a/x\}$). Consider the identity mapping $C : L \to L$, i.e. $C(a) = a$. Obviously, C is an **L**-closure operator on X and $\mathrm{fix}(C) = L$. $\mathscr{S} = L$ is our second example. Note that in this example, $S(a,b) = a \to b$, $a \approx b = a \leftrightarrow b$, see Section 2.

Example 3. Let $\equiv$ be a fuzzy equivalence on X and put $[C_{\equiv}(A)](x) = \bigvee_{y\in X} A(y) \otimes (x \equiv y)$. $C_{\equiv}$ is an **L**-closure operator which is well known in fuzzy set theory. Putting $\mathscr{S} = \mathrm{fix}(C)$, $\mathscr{S}$ contains just the fuzzy sets in X which are called extensional w.r.t. $\equiv$, i.e. those satisfying $A(x) \otimes (x \equiv y) \leqslant A(y)$ (reads: if x is in A and x is similar to y then y is in A).

It is easily seen that fix(C) equipped with inclusion $\subseteq$, see Section 2, is a complete lattice in which

$$\bigwedge_{j\in J} A_j = \bigcap_j A_j, \quad \bigvee_{j\in J} A_j = C\left(\bigcup_j A_j\right).$$

Define a binary relation ${}^a\!\approx$ on fix(C) by

$$A\,{}^a\!\approx B \quad \text{iff } (A \approx B) \geqslant a.$$

That is, ${}^a\!\approx$ is the a-cut of $\approx$. $A\,{}^a\!\approx B$ means that A and B are similar at least to degree a. The following lemma follows from [4]:

Lemma 1. *${}^a\!\approx$ is a complete tolerance on $\langle \mathrm{fix}(C), \subseteq\rangle$. That is, ${}^a\!\approx$ is a reflexive and symmetric relation on* fix(C) *which is compatible with infima and suprema in $\langle \mathrm{fix}(C), \subseteq\rangle$.*

Note that compatibility of ${}^a\!\approx$ with infima and suprema means that for any $A_j, B_j \in \mathrm{fix}(C)$ $(j \in J)$, if $A_j\,{}^a\!\approx B_j$ for all $j \in J$, then

$$\bigwedge_{j\in J} A_j\,{}^a\!\approx \bigwedge_{j\in J} B_j \quad \text{and} \quad \bigvee_{j\in J} A_j\,{}^a\!\approx \bigvee_{j\in J} B_j.$$

Because ${}^a\!\approx$ is a complete tolerance on the complete lattice $\langle \mathrm{fix}(C), \subseteq\rangle$ (Lemma 1), we can apply the construction of factorization of complete lattices by complete tolerances described in [10], see also [9,16], and define the factor lattice of fix(C) by ${}^a\!\approx$.

Denote by $\mathrm{fix}(C)/{}^a\!\approx$ the collection of all maximal blocks of ${}^a\!\approx$, i.e.

$$\mathrm{fix}(C)/{}^a\!\approx = \{B \subseteq \mathrm{fix}(C) \mid (B \times B) \subseteq {}^a\!\approx \quad \text{and} \quad (B' \times B') \nsubseteq {}^a\!\approx \text{ whenever } B' \supset B\}.$$

That is, blocks B of $\mathrm{fix}(C)/^a\!\approx$ are maximal sets of fixpoints of C which are pairwise similar at least to degree a. These blocks are particular intervals in the complete lattice $\langle \mathrm{fix}(C), \subseteq \rangle$. Namely, put

$$A_a = \bigwedge_{B \in \mathrm{fix}(C), A^a \approx B} B, \quad A^a = \bigvee_{B \in \mathrm{fix}(C), A^a \approx B} B.$$

for $A \in \mathrm{fix}(C)$. The following theorem follows from [10, Proposition 55, Theorem 14]. Namely, parts (1) and (2) are particular cases of [10, Proposition 55] and [10, Theorem 14] for a a complete tolerance $^a\!\approx$ on the complete lattice $\langle \mathrm{fix}(C), \subseteq \rangle$.

Theorem 2

(1) $\mathrm{fix}(C)/^a\!\approx = \{[A_a, (A_a)^a] \mid A \in \mathrm{fix}(C)\}$, *i.e. the blocks of* $\mathrm{fix}(C)/^a\!\approx$ *are certain intervals in* $\langle \mathrm{fix}(C), \subseteq \rangle$.
(2) With respect to a partial order $\leqslant$ *on* $\mathrm{fix}(C)/^a\!\approx$, *defined for* $[u_1, u_2], [v_1, v_2] \in \mathrm{fix}(C)/^a\!\approx$ *by*

$$[u_1, u_2] \leqslant [v_1, v_2] \quad \text{iff } u_1 \subseteq v_1 (\text{iff } u_2 \subseteq v_2),$$

$\langle \mathrm{fix}(C)/^a\!\approx, \leqslant \rangle$ *is a complete lattice, called the factor lattice of* $\langle \mathrm{fix}(C), \subseteq \rangle$ *by tolerance* $^a\!\approx$.

Note that for $A_1, A_2 \in \mathrm{fix}(C)$, the interval $[A_1, A_2]$ is defined by $[A_1, A_2] = \{A \in \mathrm{fix}(C) | A_1 \subseteq A \subseteq A_2\}$. Note also that it follows from [10, Theorem 14] that a mapping sending A to $[A_a, (A_a)^a]$ is a $\bigvee$-morphism of $\mathrm{fix}(C)$ to $\mathrm{fix}(C)/^a\!\approx$, i.e. preserves arbitrary suprema (but not arbitrary infima); dually, a mapping sending A to $[(A^a)_a, A^a]$ is a $\bigwedge$-morphism of $\mathrm{fix}(C)$ to $\mathrm{fix}(C)/^a\!\approx$, i.e. preserves arbitrary infima (but not arbitrary suprema).

Our particular setting enables us to describe maximal blocks in a simple way. Note that for a truth degree $a \in L$ and a fuzzy set $A \in L^X$, the fuzzy sets $a \otimes A \in L^X$ and $a \to A \in L^X$ are defined by $(a \otimes A)(x) = a \otimes A(x)$ and $(a \to A)(x) = a \to A(x)$. We start with following auxiliary result.

Lemma 3. *For* $A \in \mathrm{fix}(C), A_a = C(a \otimes A), A^a = a \to A$.

Proof. $A_a = C(a \otimes A)$: Since

$$a \leqslant (A \approx a \otimes A) \leqslant (C(A) \approx C(a \otimes A)) = (A \approx C(a \otimes A)),$$

$C(a \otimes A)$ is a fixpoint of C similar to A at least to degree a. Furthermore, if $B \in \mathrm{fix}(C)$ satisfies $a \leqslant (A \approx B)$ then $a \otimes A \subseteq B$ (due to adjointness), hence $C(a \otimes A) \subseteq C(B) = B$ by monotony of C. As a result, $C(a \otimes A)$ is the least fixpoint of C similar to A at least to degree a which immediately yields $A_a = C(a \otimes A)$.

$A^a = a \to A$: [2] yields that $\mathrm{fix}(C)$ is closed under $\to$-shifts, i.e. if $A \in \mathrm{fix}(C)$ then $a \to A \in \mathrm{fix}(C)$. Let $B \in \mathrm{fix}(C)$ be similar to A at least to degree a, i.e. $a \leqslant (A \approx B)$. Then $B \subseteq a \to A$ (due to adjointness), i.e. $a \to A$ is the largest fixpoint of C similar to A at least to degree a, whence $A^a = a \to A$. □

Example 4

(1) Consider Example 1. That is, $\mathrm{fix}(C) = \mathrm{Ext}(X, Y, I)$ is the set of all extents of formal concepts of $\mathscr{B}(X, Y, I)$. As mentioned above, $\langle \mathrm{fix}(C), \subseteq \rangle$ is isomorphic to $\langle \mathscr{B}(X, Y, I), \leqslant \rangle$. One can easily see that the factor lattice $\mathrm{fix}(C)/^a\!\approx$ is isomorphic to $\mathscr{B}(X, Y, I)/^a\!\approx$. That is, in this example, the factor lattice yields the factor concept lattice given by similarity threshold a, see [1,6].
(2) Consider Example 2. That is, $\mathrm{fix}(C) = L$ is a support set of a complete residuated lattice **L**. For $c \in L, c_a = a \otimes c$ and $c^a = a \to c$. The factor lattice $L/^a\!\approx$ coincides with the lattice part of a factor algebra of the complete residuated lattice **L** modulo $^a\!\approx$, see [14].

The following lemma describes the mappings sending A to A_a, and to A^a.

Lemma 4. *The mappings* $f : A \mapsto C(a \otimes A)$ *and* $g : B \mapsto a \to B$ *satisfy*

$$S(A_1, A_2) \leqslant S(f(A_1), f(A_2)), \tag{7}$$
$$S(B_1, B_2) \leqslant S(g(B_1), g(B_2)), \tag{8}$$
$$f(g(A)) \subseteq A, \tag{9}$$
$$B \subseteq g(f(B)). \tag{10}$$

Proof. By routine verification using standard properties of complete residuated lattices. □

Remark 2. Note that mappings satisfying (7)–(10). were studied in [11].

Lemma 4 provides us with useful properties. For example, as a direct consequence of Lemma 4, $f(A) = fgf(A)$ and $g(B) = gfg(B)$, i.e. $A_a = ((A_a)^a)_a$ and $A^a = ((A^a)_a)^a$. Because of this, $[A_a, (A_a)^a] = [((A_a)^a)_a, (A_a)^a]$, i.e. every $[A_a, (A_a)^a]$ is of the form $[(B^a)_a, B^a]$. By similar arguments, every $[(B^a)_a, B^a]$ is of the form $[A_a, (A_a)^a]$. This implies a possibility of a dual description of the blocks, namely, $\mathrm{fix}(C)/^a\!\approx = \{[(A^a)_a, A^a] \mid A \in L^X\}$, Let us now turn to a description of blocks of the factor lattice $\mathrm{fix}(C)/^a\!\approx$ which provides a way to compute the factor lattice efficiently.

Let us first note that if we can efficiently compute closures $C(A)$ for $A \in L^X$ (and if "everything is finite"), we can compute $\mathrm{fix}(C)/^a\!\approx$ by computing first $\mathrm{fix}(C)$ and then computing the blocks of $\mathrm{fix}(C)/^a\!\approx$. Namely, the set $\mathrm{fix}(C)$ of fixpoints of a fuzzy closure operator C can be computed by a modification [3] of Ganter's NEXTCLOSURE algorithm [10]. In addition, the blocks are blocks of a tolerance relation and they can be computed by available algorithms (these algorithms come from graph theory; namely, maximal blocks of a tolerance relation T are just maximal independent sets in the graph of the complement of T). However, there is a better way than this "naive" one. We present it below.

As we know from Theorem 2, the elements of $\mathrm{fix}(C)/^a\!\approx$ are $^a\!\approx$-blocks and every such a block is an interval of the form $[A_a, (A_a)^a]$ for $A \in \mathrm{fix}(C)$. By the previous results, each such block is determined by its upper bound $(A_a)^a$. To compute all elements of $\mathrm{fix}(C)/^a\!\approx$, it is therefore sufficient to compute the set

$$\mathrm{UB} = \{B \in L^X \mid [A, B] \in \mathrm{fix}(C)/^a\!\approx \text{ for some } A\}$$

of all upper bounds of blocks from $\mathrm{fix}(C)/^a\!\approx$. Taking into account Lemma 4, the following claim is easy to check.

Lemma 5. *The mapping*

$$C_a : A \mapsto a \to C(a \otimes A)$$

sending A to $(A_a)^a = a \to C(a \otimes A)$ is an **L***-closure operator in X.*

Now, C_a provides a useful description of UB.

Theorem 6. $\mathrm{UB} = \mathit{fix}(C_a)$.

Proof. Let $B \in \mathrm{UB}$. Then there exists an A such that $[A, B] \in \mathrm{fix}(C)/^a\!\approx$. By Theorem 2 (1) and Lemma 4 and its consequences, $B = (B_a)^a$, i.e. by Lemma 3, $B = a \to C(a \otimes B)$, i.e. $B \in \mathrm{fix}(C_a)$.

Conversely, let $B \in \mathrm{fix}(C_a)$, i.e. $B = a \to C(a \otimes B) = (B_a)^a$. Due to Theorem 2 (1), in order to see that $B \in \mathrm{UB}$, it suffices to verify that $B \in \mathrm{fix}(C)$. This is, indeed, true: Clearly, $C(a \otimes B) \in \mathrm{fix}(C)$. Therefore, $B = a \to C(a \otimes B) \in \mathrm{fix}(C)$ due to the fact that $\mathrm{fix}(C)$ is closed under a-shifts [2]. □

Therefore, the fixpoints of C_a are just the upper bounds UB. Since, as mentioned above, $\mathrm{fix}(C)/^a\!\approx$ can be restored from UB, we reduced the problem of computing the factor lattice $\mathrm{fix}(C)/^a\!\approx$ to the problem of computing a set of fixpoints of a fuzzy closure operator, namely, of C_a. This way is more efficient than the "naive" one because we need not compute all the fixpoints of C (note that $\mathrm{fix}(C_a) \subseteq \mathrm{fix}(C)$) and, moreover, we need not compute the maximal blocks of fixpoints which is a time-consuming step even when one employs specialized algorithms. Note that some of the formulas and results obtained in this section generalize those from [6] where a particular case of concept lattices was considered, see Example 1. In particular, it was demonstrated in [6] that the speed-up in computing $\mathrm{fix}(C)/^a\!\approx$ using C_a is high and depends in a natural way on the threshold a.

Alternatively, one can proceed in a dual way and use the lower bounds of blocks from $\mathrm{fix}(C)/^a\!\approx$. In the rest of this section, we briefly describe the approach. Recall [8], [11] that an **L**-interior operator I in X is a mapping $I : L^X \to L^X$ satisfying

$$I(A) \subseteq A, \tag{11}$$
$$S(A_1, A_2) \leqslant S(I(A_1), I(A_2)), \tag{12}$$
$$I(A) = I(I(A)), \tag{13}$$

for every $A, A_1, A_2 \in L^X$. The set $\mathrm{fix}(I)$ of all fixpoints of I is defined by

$$\mathrm{fix}(I) = \{A \in L^X | I(A) = A\}.$$

In addition, define

$$\mathrm{LB} = \{A \in L^X \mid [A, B] \in \mathrm{fix}(C)/^a\!\approx \text{ for some } A\}.$$

LB is the set of lower bounds of blocks from $\mathrm{fix}(C)/^a\!\approx$. The following lemma follows easily from Lemma 4.

Lemma 7. *The mapping*

$$I_a : A \mapsto C(a \otimes (a \to A))$$

sending A to $(A^a)_a = C(a \otimes (a \to A))$ is an **L***-interior operator in X.*

Theorem 8. $\mathrm{LB} = \mathit{fix}(I_a)$,

Proof. Let $A \in \mathrm{LB}$. Then there exists a B such that $[A, B] \in \mathrm{fix}(C)/^a\!\approx$. From Theorem 2 (1) and consequences of Lemma 4 we have $A = B_a = C(a \otimes B)$, $B = A^a = a \to A$. Put together, $A = C(a \otimes (a \to A)) = I_a(A)$ and $A \in \mathrm{fix}(I_a)$,

Conversely, let $A \in \mathrm{fix}(I_a)$, i.e. $A = C(a \otimes (a \to A)) = (A^a)_a$. This means that A is a fixpoint of C. Since a-shifts of fixpoints are also fixpoints [2], we get that $B = a \to A = A^a$ is a fixpoint of C. Now, $[A, B] = [B_a, (B_a)^a]$ (Lemma 4 and its consequences), which is an element of $\mathrm{fix}(C)/^a\!\approx$ (Theorem 2). This shows $A \in \mathrm{LB}$. $\square$

4. Conclusions

We have demonstrated that the straightforward idea of grouping fuzzy sets by putting together those which are sufficiently similar, i.e. similar at least to a prescribed degree a, leads to feasible structures in spite of the fact that "similar to degree at least a" is not an equivalence relation. Particular examples of the general procedure presented here include factorization of concept lattices and factorization of complete residuated lattices. Note also that our results are degenerate in case of ordinary sets. Namely, similarity to degree 1 means equality of ordinary sets, while similarity to degree 0 presents no constraint. Correspondingly, $\mathrm{fix}(C)/^1\!\approx$ is isomorphic to $\mathrm{fix}(C)$ and $\mathrm{fix}(C)/^0\!\approx$ consists of a single block containing all fixpoints from $\mathrm{fix}(C)$. From this point of view, this paper points out a phenomenon which is hidden in case of ordinary sets.

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