BRNO UNIVERSITY OF TECHNOLOGY

FACULTY OF MECHANICAL ENGINEERING

HYPERSTRUCTURES AND ORDERING: ONE POSSIBLE APPROACH

Habilitation thesis

RNDr. Michal Novák, Ph.D.
(Applied mathematics)

BRNO 2018

I hereby declare that the contents of this habilitation thesis is based solely on sources cited in References and on information given in “Origin of results” on page vii.

Brno, 26^{th} April 2018

Acknowledgement:

I am lucky to have met people providing me with ideas, time to implement them, and with support and critical revision of my suggestions. Thanks should be expressed to all of them because by being what they are they contributed to this book immensely. To name just a few, I would like to thank to Prof. RNDr. Jan Chvalina, DrSc., doc. RNDr. Zdeněk Šmarda, CSc., doc. dr. Irina Cristea, Ph.D. and Mgr. Štepán Křehlík, Ph.D. However, the greatest thanks should be expressed to Petra for her immeasurable patience.

To my children so that they could
– once, in some distant future –
learn why I had been so often
mentally away in moments
when they wanted me to be with them.

Contents

Introduction

Giving a name is a most difficult task as it influences the destiny of the named thing or person considerably. Giving a title to a scientific work is no easier. A too special title limits its audience as only true enthusiasts will read it. On the other hand, a general title is likely to attract attention of many yet arouse too great expectations. Moreover, it may put the writer in the position of an amateur writing superficially about things already known or ignoring the context or true relevance of his results. I have decided on a compromise: a general universally understandable title with a subtitle humbly admitting that this book is not meant to be the compendium of the topic. Thus the name I gave it is "Hyperstructures and ordering: one possible approach".

I was introduced to the (algebraic) hyperstructure theory in 2005 by Jan Chvalina. First, I helped him in preparing contributions to a few local conferences and meetings where he decided to present results on his hyperstructure generalizations in the theory of automata. Later on, when I noticed that the attention of the small group of his colleagues is broader and covers various kinds of operators (which, as I learned later, is motivated by some very classical results), I realized that many of their conclusions are based on one particular lemma. When asking about some more results in this direction, I got a surprising answer that none exist yet if they existed they would help. So I set myself on a path of exploring what had first been called *"Ends lemma"–based hyperstructures* and later, when Piergiulio Corsini did not like the name as he thought it was too long, *EL–hyperstructures*. I was encouraged when the *European Journal of Combinatorics* decided to publish some results on what has gradually become my small obsession and even more encouraged when they soon accepted another paper on the topic. In 2014, at the main meeting of the "pure hyperstructuralists", the International Congress on Algebraic Hyperstructures and Applications (AHA), Bijjan Davvaz and his colleagues and Ph.D. students presented a number of results on *ordered hyperstructures*, a concept that is younger yet – thanks to the intensity and prolific nature of Bijjan's work – more spread. At the same meeting I communicated two contributions of our Brno group, both of which were centered around ordering. When soon after this meeting some Iranian colleagues decided to continue in the research of EL–hyperstructures and applied the idea in the n–ary context and in the difficult direction of "hyperstructures constructed from hyperstructures", i.e. what they called EL^2*–hyperstructures*, I realized that the answer I had got a few years ago – "There are no results but they would help." – may have indeed been true.

In 2014, Štěpán Křehlík, a Ph.D. student of our department, asked me for help with a paper he was preparing for publication. I am not sure it will ever appear. Anyway, we have published a sequence of four papers, in which we have gradually shifted our attention from the pure theory around EL–hyperstructures to applying the idea, since. I am even more pleased that Kyriakos Ovaliadis, a colleague of mine from Kavala, Greece, believes that the research on hyperstructures and ordering will help to make advances in the important yet problematic and so far not standardised area of collecting data from underwater wireless sensor networks.

Since Štěpán is my junior and Kyriakos is an engineer, they have been asking naturally arrising simple questions. Answering them not only showed my limitations in knowledge but also made me go back to the roots and study works of the "old bards (and ladies)". I have had a great priviledge of having private discussions with some of these, who not only provided fundamental results of "our" theory but also remember the historical context and motivation of their own teachers and colleagues. I am also happy to have met Irina Cristea, a person of my age, whose professional experience, similar to mine, has always encouraged me to learn more about the topic, to which she herself had been introduced by another "old bard", Piergiulio Corsini, and his student, Violeta Leoreanu. Thus in the end it turned out that the topic I started to explore is broader and its roots much firmer and older than I thought.

To be specific, this book deals with a class of algebraic hyperstructures in the sense of Marty which are constructed from quasi-ordered semigroups. I discuss both semihypergroups and ring– or lattice–like hyperstructures of this type. Since I deal with one particular class of hyperstructures, I relate them to other similar concepts such as *quasi-order hypergroups*, *ordered hyperstructures* or BCK–algebras. Apart from the introductory chapter, the reader will find historical references, footnotes or links to numerous papers on the topic of algebraic hyperstructures. This is because I am explicitly discussing concepts which have already been implicitly noticed by a number of authors. This is the reason why the subtitle of this book reads "one possible approach".

Notation

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	number domains
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\mathcal{P}(S)$	power set of set S
$\mathcal{P}^*(S)$	$\mathcal{P}(S) \setminus \emptyset$
H^n	Cartesian product $\underbrace{H \times \ldots \times H}_{n}$
$\approx$	non-empty intersection
$\leq$	relation; generally speaking any preordering (i.e. quasi-ordering)
$\preceq$	partial ordering of *ordered hyperstructures*
$+, \cdot$	operations
$*, \oplus, \odot, \circ, \bullet, \star$	hyperoperations
$\circ$	composition (hyper)operation (Subsection 2.5.4 only)
$*$	BCI–algebra operation (Section 1.2, Subsection 2.6.4 only)
$\wedge, \vee$	lattice operations
$\bigwedge, \bigvee$	lattice-like hyperoperations
$\sim$	equivalence relation
$\mid, \subseteq, \equiv$, etc.	specific relations such as divisibility, set inclusion, congruence, etc.
$\min, \max, \inf, \sup$	minimum, maximum, infimum, supremum
$\gcd, \mathrm{lcm}$	greatest common divisor, least common multiple
$[a)_\leq$	$\{x \mid a \leq x\}$
$_\leq(a]$	$\{x \mid x \leq a\}$
a/b	$\{x \mid a \in x * b\}$
$b \setminus a$	$\{x \mid a \in b * x\}$
a^n	$\underbrace{a * \ldots * a}_{n}$ or $\underbrace{a \cdot \ldots \cdot a}_{n}$ (context dependent)
$i(a)$	set of (hyperstructure) inverses of a
$f(a_1^{i-1}, x_i, a_{i+1}^n)$	$f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots a_n)$
$\mathrm{perm}\{a_1, \ldots, a_n\}$	set of all permutations of $a_1, \ldots a_n$
$\mathbb{M}_{n,n}(S)$	set of square $n \times n$ matrices with entries from S
$\mathbf{A}$	matrix

For notation used in Chapter 4, see Subsection 4.1.1 and Section 4.2.

Origin of results

Results presented in this book have been collected from several papers published between 2005 and 2017. A majority of results included in Section 2.4 was published by *European Journal of Combinatorics* as Novák [244] and Novák [242]. Results of Subsection 2.4.4 were published as Novák [246], results of Subsection 2.4.10 were included in Novák, Cristea and Křehlík [248]. Subsection 2.5.2 and 2.5.3 is based on Novák [243] while results of Subsection 2.5.4 were taken from Novák and Cristea [247] accepted in *Hacettepe Journal of Mathematics and Statistics*. Results included in Subsection 2.5.5 were published by *Analele Ştiinţifice ale Universităţii "Ovidius" Constanţa* as Křehlík and Novák [187]. In Chapter 3, Section 3.1 is based on Novák [240] published by *Analele Ştiinţifice ale Universităţii "Ovidius" Constanţa* while Section 3.3 (together with Subsection 2.4.6) was published by *Soft Computing* as Novák and Křehlík [249]. Some of the examples throughout Chapter 2 and Chapter 3 as well as some results in Section 4.1 were collected from papers such as [23,50–54,58,60,69,70,76] (published often as conference proceedings) written by a group of students and colleagues of Jan Chvalina, to which I belong. Results motivated by Section 4.2 were published by *Analele Ştiinţifice ale Universităţii "Ovidius" Constanţa* as Chvalina, Křehlík and Novák [61]. The study of the mathematical model of Section 4.3 was initiated by Novák, Ovaliadis and Křehlík [251].

Some minor parts of Chapter 2 and Chapter 3 were adapted from papers written by authors other than myself. Subsection 2.4.9 includes results obtained in Ghazavi, Anvariyeh and Mirvakili [137] (published by *Journal of Algebraic Systems*). Subsection 3.1.3 includes results published by *Soft Computing* as Ghazavi and Anvariyeh [135] while in Subsection 3.2 some results published by Ghazavi, Anvariyeh and Mirvakili [136] in *Iranian Journal of Mathematical Sciences and Informatics* are included.

Of the above papers published in journals assigned with an impact factor, I am the sole author of [240, 242, 244]. For papers, where their authors are given in the alphabetical order, the co-authorship is equal. Where the alphabetical order is not followed, authors are listed based on their contribution.

Chapter 1

Preliminaries

1.1 Basic notions

The difference between algebraic *structures* and *hyperstructures* comes from the difference between the notion of an *operation* and a *hyperoperation*. While an operation on a set H is a mapping from H^n to H, a hyperoperation on H is a mapping from H^n to $\mathcal{P}^*(H)$, the set of all non-empty subsets of H. In other words, the result of an operation on H is an *element* of H, while the result of a hyperoperation (sometimes called a *hyperproduct*) on H is a *subset* of H. Typically, the operation of addition applied on two real numbers results in another real number. A hyperoperation applied on the same pair of numbers may result in e.g. the interval between these numbers.

The generalization towards hyperstructures was done in 1934 when Marty in [201] introduced the concept of a *hypergroup*. The study of multi-valued aspects enabled by the transition from operations to hyperoperations has attracted a lot of attention since. As a result, different branches of the hyperstructure theory exist. In this book we deal exclusively with *algebraic* hyperstructures in the sense of Marty as studied in the classical introductory books [92,95,108,111] written by Corsini, his former Ph.D. student Leoreanu (Leoreanu–Fotea after her marriage), Davvaz and Cristea. We do not study *topological hyperstructures* introduced by Dunkl, Jewett and Spector [131], which are used in harmonic analysis or probability theory.[1] For a basic introduction to this approach to the hyperstructure theory see e.g. Bloom and Heyer [28], for a short overview of its history see e.g. Ross et al. [273], p. 77.

[1]See the definition of a hypergroup used in Bloom and Heyer [28], p. 9, which has nothing in common with the definition used in the algebraic context as it is based on vector spaces and locally compact Hausdorff spaces and mentions continuous mappings. Given the names above, these hypergroups are often called *DJS–hypergroups*.

This introductory section includes definitions and some basic results on algebraic hyperstructures. Most of these are taken from [92, 95, 111] and to some extent also from Vougiouklis [300], which the reader is asked to confer for further reference.

1.1.1 Hyperstructures

In abstract algebra, a *groupoid* is a set G, which is closed with respect to a binary operation. A *hypergroupoid* is a set H closed with respect to a hyperoperation.[2] Notice that some authors (especially students of Mittas and Massouros) speak of *hypercomposition* and *hypercompositional structures* instead of *hyperoperation* and *hyperstructures*, respectively.[3]

Definition 1.1.1. Let H be a non-empty set. By an *n–hyperoperation* we mean a mapping $f : H^n \to \mathcal{P}^*(H)$. The number n is called the *arity* of f.

Binary hyperoperations, i.e. those where $n = 2$, are often denoted by symbols such as "$*, \star, \bullet, \circ, \odot, \oplus, \otimes$" so that they could be easily distinguished from the usual binary operations. However, especially for cases of hyperrings, they are often denoted by symbols "$+, \cdot$" which may be sometimes somewhat confusing. Hyperoperations of arity greater than 2 are denoted by $f(x_1, \ldots, x_n)$. A set H may be endowed with one or more hyperoperations.

Definition 1.1.2. A set H endowed with a family of hyperoperations is called a *hyperstructure* (or a *multivalued algebra*). A set H endowed with one binary hyperoperation is called a *hypergroupoid*. A set H endowed with one hyperoperation of arity $n > 2$ is called an *n–ary hypergroupoid*; in case of $n = 3$ the word *ternary* is used.

Hypergroupoids as well as hyperstructures are usually denoted as pairs (`set`,`hyperoperation(s)`); if the hyperoperation(s) is / are obvious or irrelevant, they are denoted by the set only.

[2] The following definition is taken over from Corsini and Leoreanu [95]. We could permit that the image of f is $\mathcal{P}(H) = \mathcal{P}^*(H) \cup \emptyset$. It is easy to prove that, in a hypergroup, $f(a, b)$ is always non-empty. However, if we permit the image of f to be $\mathcal{P}(H)$, then in weaker structures such as semihypergroups there can be $f(a, b) = \emptyset$. In such a case we speak of *partial hypergroupoids*. Further on, however, we will be interested in hypergroupoids only.

[3] This is partially motivated by the fact that in arithmetics, by a *hyperoperation* one means any of the members of the recursively defined *hyperoperation sequence*, i.e. a sequence of operations, the first members of which are usually named "successor", "addition", "multiplication", "exponentiation", etc. However – unfortunately – sometimes one may step upon other names such as *non-deterministic operator* (or *nd-operator*) used by Martínez et al. [200] in their study of hyperstructure generalizations of lattices.

Commutativity and *associativity* are defined in the same way as in the usual non-hyperstructure way. However, proving associativity is often not easy in the hyperstructure theory as we have to prove equality of sets instead of elements. A *semihypergroup* is defined by means of associativity.

Definition 1.1.3. A hypergroupoid $(H, *)$ is called *commutative* if for all $a, b \in H$ there holds $a * b = b * a$. An n–ary hypergroupoid (H, f) is called *associative* if

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}) \tag{1.1}$$

holds for every $i, j \in \{1, 2, \ldots, n\}$ and $x_1, x_2, \ldots, x_{2n-1} \in H$. In case of a hypergroupoid $(H, *)$ this means that $a*(b*c) = (a*b)*c$ for all $a, b, c \in H$. By a *semihypergroup* we mean an associative hypergroupoid; by an *n–ary semihypergroup* we mean an associative n–ary hypergroupoid.

Given an arbitrary hypergroupoid $(H, *)$, for all $A, B \subseteq H$ and $x \in H$ there is[4]

$$A * B = \bigcup_{a \in A, b \in B} a * b, \quad x * A = \{x\} * A, \quad B * x = B * \{x\}. \tag{1.2}$$

The concept of the *neutral element* is generalized by means of inclusion. In the hyperstructure theory there is no equivalent of monoids, i.e. semigroups with a neutral element. In the n–ary context, neutral elements were originally defined for hypergroups but this was a side effect of the fact that when starting these considerations in [115], Davvaz and Vougiouklis were not interested in the study of semihypergroups. Therefore, as suggested in e.g. Novák [240], the concept can be used without any modifications in n–ary semihypergroups as well.

Definition 1.1.4. An element $e \in H$, where $(H, *)$ is a hypergroupoid, is called an *identity* (or a *neutral element* or a *unit*) of H if there is

$$x \in x * e \cap e * x \tag{1.3}$$

for all $x \in H$. In case (H, f) is an n–ary semihypergroup, $e \in H$ is an *identity* (or a *neutral element* or a *unit*) of H if

$$f(\underbrace{e, \ldots, e}_{i-1}, x, \underbrace{e, \ldots, e}_{n-i}) \tag{1.4}$$

includes x for all $x \in H$ and all $1 \leq i \leq n$.

[4]These days, the *set union notation* given below is regarded standard. However, in the very first papers on the hyperstructure theory such as e.g. Wall [307], the *additive notation*, where for $A = a_1 + a_2 \ldots a_p$ and $B = b_1 + b_2 + \ldots b_q$, the (hyper)product is defined as $AB = \sum\limits_{i=1}^{p} \sum\limits_{j=1}^{q} a_i b_j$, is used.

The concept of a *group* is defined by means of associativity, existence of a neutral element and existence of inverse elements. This results in the fact that, in a group $(G, \cdot)$, equations $a \cdot x = b$ and $x \cdot a = b$ have unique solutions. A hypergroup is defined without explicitly mentioning inverse elements.[5]

Definition 1.1.5. By *reproductive law* we mean a condition

$$a * H = H * a = H \tag{1.5}$$

satisfied for all $a \in H$, where $(H, *)$ is a hypergroupoid, or a condition

$$f(H^{i-1}, x, H^{n-i}) = H \tag{1.6}$$

satisfied for all $x \in H$ and all $i = \{1, 2, \ldots, n\}$, where (H, f) is an n–ary hypergroupoid. A hypergroupoid $(H, *)$ in which the reproductive law holds is called a *quasi-hypergroup*. A semihypergroup in which the reproductive law holds is called a *hypergroup*. An n–ary hypergroupoid (H, f) in which the reproductive law holds is called an *n–ary hypergroup*.

Notice that the reproductive law might in the n–ary context be written in a more intuitive way (using "$*$" to denote the hyperoperation) as

$$\underbrace{H * \ldots * H}_{i-1} * x * \underbrace{H * \ldots * H}_{n-i} = H \tag{1.7}$$

for all $x \in H$ and all $i = \{1, 2, \ldots, n\}$. Also notice that an equivalent definition of n–ary hypergroups is such that they are n–ary semihypergroups (H, f) in which the equation

$$b \in f(a_1^{i-1}, x_i, a_{i+1}^n) \tag{1.8}$$

has the solution $x_i \in H$ for every $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n, b \in H$ and $1 \leq i \leq n$, which is a generalization of the group property and in fact the original definition given by Davvaz and Vougiouklis [115] in 2006. Notice that by (1.8) we mean that b is an element of $f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots a_n)$. Since the generalization of (semi)hypergroups to n–ary context is rather new, names *n–semihypergroup* and *n–hypergroup* can sometimes, e.g. in [193] published in 2008, be found.

In monoids, if an inverse exists, then it is unique. Since in the hyperstructure theory we have no monoids, and neutral elements (or rather, identities) are defined by means of inclusion, an element can have multiple hyperstructure inverses. Therefore, instead of "the" inverse we say "an" inverse just as instead of "the" identity (or neutral element) we say "an" identity.

[5] For some classical wordings and motivations of the definition see e.g. Marty [201], Dresher and Ore [128], Utumi [294] or Comer [82].

Definition 1.1.6. If $(H, *)$ is a hypergroup endowed with at least one identity, then an element $a' \in H$ is called an *inverse* of $a \in H$ if there is an identity $e \in H$ such that $a * a' \ni e \in a' * a$. The set of inverses of $a \in H$ is denoted by $i(a)$.

Inverse elements in n–ary hyperstructures were studied e.g. by Ameri and Norouzi in [11]. The property of *having a unique inverse element* required in [11] is taken over from the definition of a *canonical n–ary* hypergroup included in Leoreanu [190]. Notice that canonical n–ary hypergroups are a special class of *commutative* n–ary hyperstructures (moreover, with the unique identity e having a certain further property). Without this requirement of uniqueness (thus applicable to any n–ary hypergroup), the definition of inverses in the n–ary context would be as follows.[6]

Definition 1.1.7. Element x' of an n–ary hypergroup (H, f) is called an *inverse element* to $x \in H$ if there exists an identity $e \in H$ such that

$$e \in f(\mathrm{perm}\{x, x', \underbrace{e, \ldots, e}_{n-2}\}) \tag{1.9}$$

for every $1 \leq i \leq n$.

In this form the above definition is used in Novák [240] and will be used in Section 3.1.

The analogy of "the" identity is called *scalar identity* in the hyperstructure theory. Apart from scalar identities we often make use of other special elements where the hyperproduct is a one-element (or some special) set.[7]

Definition 1.1.8. An element e of a hypergroupoid $(H, *)$ is called

1. a *strong identity* if there is $x * e = e * x \subseteq \{x, e\}$
2. a *scalar identity* if there is $x * e = \{x\} = e * x$,
3. an *absorbing element* if there is $x * e = \{e\} = e * x$

for all $x \in H$.

While a strong identity need not be unique, a scalar identity, once it exists in H, is always unique. Also, some authors such as Jafarabadi et.

[6]Notice that in the following text the notation $\mathrm{perm}\{a_1, \ldots, a_n\}$ stands for the set of all permutations of elements $a_1, \ldots, a_n$.

[7]For some results concerning strong identities in various types of hyperstructures see e.g. Jantosciak, Massouros and Massouros [170, 210].

al [162, 163] denote absorbing elements by "0" and use names *zero scalar element* or simply *zero scalar* instead of *absorbing element*. Furthermore, the condition for absorbing elements is often given in an equivalent way as $e * H = e = H * e$. Moreover, scalar elements are often defined in hypergroups only as they are important for defining *canonical hypergroups* introduced by Mittas [225] or *polygroups* introduced by Comer [83] (for the history of this concept see Comer [81, 82]; polygroups are also called *quasi-canonical hypergroups* – see e.g. Massouros [208] or Corsini and Leoreanu [95]).

Definition 1.1.9. A hypergroup H is called *canonical* if

1. it is commutative,
2. it has a scalar identity,
3. every element has a unique inverse,
4. it is reversible.

A non-commutative canonical hypergroup is called *quasi-canonical hypergroup* (or *polygroup*). (The notion of reversibility will be defined later in Subsection 1.1.3.)

The idea of *idempotency*, important in e.g. making distinctions between semigroups and groups, is transferred to hyperstructures in the usual way of inclusion.

Definition 1.1.10. An element a of a semihypergroup $(S, *)$ is called an *idempotent* if $a \in a * a$.[8] The semihypergroup $(S, *)$ is called an *idempotent semihypergroup* if all its elements are idempotent. A nonempty subset A of a semihypergroup $(S, *)$ is called an *idempotent subset* if $A \subseteq A * A$.

Motivated to study classical geometries from the point of view of the hyperstructure theory, Prenowitz and Jantosciak [168, 264, 265] developped the concept of a *join space* and its weaker non-commutative version called *transposition hypergroup*, which make use of the following "fraction notation": in a hypergroupoid $(H, *)$ we denote

$$a/b = \{x \in H \mid a \in x * b\} \quad b \setminus a = \{x \in H \mid a \in b * x\} \tag{1.10}$$

for an arbitrary pair of $a, b \in H$. If the hyperoperation "$*$" is commutative, then obviously $a/b = a \setminus b$. Notice that – even though the fraction notation may seem a bit complicated – we are talking about half lines in (1.10) and the classical Pasch's axiom in (1.11).

[8]Notice that some authors, such as Polat [262], use a more strict definition of *idempotency* where $a * a = \{a\}$. This is motivated by the need to study geometry motivated join spaces.

Definition 1.1.11. A commutative hypergroup $(H,*)$ is called a *join space* if for all $a,b,c,d \in H$ the following implication holds:

$$a/b \cap c/d \neq \emptyset \Rightarrow a*d \cap b*c \neq \emptyset. \tag{1.11}$$

The above condition is often written as $a/b \approx c/d \Rightarrow a*d \approx b*c$, where "$\approx$" denotes non-empty intersection.

Weakening of the axioms of associativity and commutativity enabled Vougiouklis in 1990s to construct some equivalence relations such that quotients of these "weak" hyperstructures, or H_v*–structures* as he called them, with respect to these equivalence relations are always (single-valued) structures. Thus since [304] and the compendium [300] we speak of H_v*–structures* such as H_v*–groups*, H_v*–fields* or even H_v*–matrices*.

Definition 1.1.12. A hypergroupoid $(H,*)$ is *weak associative* (often abbreviated to WASS) if

$$a*(b*c) \cap (a*b)*c \neq \emptyset \tag{1.12}$$

for all $a,b,c \in H$, and *weak commutative* (often abbreviated to COW) if

$$a*b \cap b*a \neq \emptyset \tag{1.13}$$

for all $a,b,c \in H$. A weak associative hypergroupoid is called an H_v*–semigroup*. An H_v–semigroup is called H_v*–group* if it satisfies the reproductive law.

As en example of such equivalence relations let us mention relation β^* which one can define on every hypergroup or every H_v–group $(H,\cdot)$ by

$$a\beta b \Leftrightarrow \exists n \in \mathbb{N}, \exists x_1, x_2, \dots x_n \in H : a \in \prod_{i=1}^{n} x_i \ni b \tag{1.14}$$

The transitive closure of β, denoted by β^*, is called *fundamental equivalence relation* on H. Vougiouklis [300] proved that if H is an H_v–group, then β^* is the smallest equivalence relation on H such that the quotient H/β^* is a group. In this respect notice that, given a groupoid $(H,+)$, e.g. a commutative hyperoperation "$*$" can be defined on H by putting e.g. $a*b = a+b$ for such $a,b \in H$ that $a+b = b+a$ and $a*b = \{a+b, b+a\}$ for such $a,b \in H$ that $a+b \neq b+a$. For details regarding this procedure, called *uniting elements* *method* by Corsini and Vougiouklis, see [96, 304].[9] The history of equivalence relations in the hyperstructure theory has two origins: Koskas [183] and Vougiouklis. For a nice comparison of their approaches see Antampoufis and Hošková–Mayerová [8].

Algebraic *hyperstructures with two (hyper)operations* are defined in a similar way as the single-valued structures. However, the terminology is often ambiguous and not codified.[10] Just as for algebraic structures, we need two (hyper)structures (hypergroups, H_v–groups, semihypergroups, groups, semigroups, etc.) and distributivity of one (hyper)operation over the other. Historically, hyperrings were first defined by Krasner [186] as hyperstructures with one hyperoperation and one operation. For this reason, by "hyperring" *Krasner hyperrings* are often meant.[11]

Definition 1.1.13. Let "$\oplus$" be a hyperoperation and "$\cdot$" an operation defined on H such that "$\cdot$" is distributive over "$\oplus$" from both left and right. Denote "0" the two-sided absorbing element with respect to "$\cdot$".

1. If $(H, \oplus)$ is a canonical hypergroup and $(H, \cdot)$ is a semigroup with 0,

---

[9] H_v–structures were introduced in [304] in a very short Section 5. However, the motivation is explained very clearly there: "Examples of H_v–groups can be obtained in the following way. Let $(H, \cdot)$ be a group and let d denote a property that is desirable of a quotient. Put together, in the same class, every pair of elements that cause the non-validity of d and form the quotient by this partition. The resulting quotient structure is an H_v–group. The *uniting elements procedure*, introduced in [96], extends to this situation. By forming the $\beta^\star$ quotient of this H_v–group, a group that satisfies property d is obtained."

[10] The reason may be that some of the fundamental papers on the theory are not in English (e.g. of Mittas and Stratigopoulos written in French [223, 224, 290]) or are otherwise difficult to access. In short, one can say that Krasner [186] introduced the notion of the *hyperfield* and then *hyperring* in order to approximate a local field of positive characteristic by a system of local fields of characteristic zero. The additive part of this hyperring was a special hypergroup (later known as *canonical hypergroup*) while the multiplicative part was a semigroup. Constructions of these structures can be found in [185, 207, 230]. While studying polynomials over Krasner's hyperrings, Mittas [226], introduced *superrings*, in which both parts, additive and multiplicative, were hyperstructures. G. Massouros, approching the theory of languages and automata from the point of view of hyperstructure theory, was led to the introduction of the concepts of *hyperringoid* and *join hyperring* [211, 214]. Then, Vougiouklis [302] generalizing Mittas' superring introduced *hyperrings in the general sense*. On contrary, Rota [276] studied hyperstructures in which not the additive but the multiplicative part is a hypergroup while the additive part, i.e. not the multiplicative part, is a semigroup.

[11] Notice that in the process of evolution of the hyperring theory, the results obtained by Massouros [202, 207] are fundamental as he proved that there exist other Krasner hyperrings than the residual or quotient ones which can be obtained by the original Krasner's constructions [185, 186], i.e. in order to achieve them means of classical algebra are insufficient. For a detailed discussion of the topic written in an easy-to-follow style, see the expository and survey paper [230] written by Nakasis. Also see Massouros [205].

then $(H, \oplus, \cdot)$ is called *Krasner hyperring*.

2. If $(H, \oplus)$ is a canonical hypergroup and $(H \setminus \{0\}, \cdot)$ is a group, then $(H, \oplus, \cdot)$ is called *hyperfield*.

It is to be noted that since we work with hyperoperations, in distributive laws we compare sets instead of elements. Therefore, it is rather natural to weaken the distributive laws by permitting inclusions instead of requesting equality. Thus, Spartalis and Vougiouklis studied hyperrings from a more general perspective as hyperstructures with two hyperoperations; for a basic introduction see e.g. their papers [284, 302]. In their theory, Krasner hyperrings are a class of *additive* hyperrings. Vougiouklis also introduced the notion of *strong* (or *good*) hyperrings as a counterpart to hyperstructures in which only the inclusive version of distributivity holds. In his classification he also defined *semihyperrings*.

Definition 1.1.14. Let "$\oplus$" and "$\odot$" be hyperoperations and "$+$" and "$\cdot$" operations defined on H. By *inclusion distributivity* of "$\odot$" over "$\oplus$" (and likewise for other possible combinations of the above (hyper)operations) we mean validity of[12]

$$\begin{aligned} x \odot (y \oplus z) &\subseteq (x \odot y) \oplus (x \odot z) \\ (x \oplus y) \odot z &\subseteq (x \odot z) \oplus (y \odot z) \end{aligned} \tag{1.15}$$

for all $x, y, z \in H$.

1. If $(H, \oplus)$ is a hypergroup, $(H, \odot)$ is a semihypergroup and "$\odot$" is inclusively distributive over "$\oplus$", then $(H, \oplus, \odot)$ is called *hyperring in the general sense*.

2. If $(H, \oplus)$ is a hypergroup, $(H, \odot)$ is a semihypergroup and "$\odot$" is distributive over "$\oplus$", then $(H, \oplus, \odot)$ is called *good* (or *strong*) *hyperring in the general sense*.

3. If $(H, \oplus)$ is a hypergroup, $(H, \cdot)$ is a semigroup and "$\cdot$" is inclusively distributive over "$\oplus$", then $(H, \oplus, \cdot)$ is called *additive hyperring*.

4. If $(H, \oplus)$ is a hypergroup, $(H, \cdot)$ is a semigroup and "$\cdot$" is distributive over "$\oplus$", then $(H, \oplus, \cdot)$ is called *good* (or *strong*) *additive hyperring*.

5. If $(H, \oplus)$ and $(H, \odot)$ are semihypergroups and "$\odot$" is inclusively distributive over "$\oplus$", then $(H, \oplus, \odot)$ is called *semihyperring in the general sense*.

[12] In [302] Vougiouklis actually uses the symbol "$\subset$" yet he does so in the sense of "$\subseteq$". Also, he himself does not use the term "inclusion distributivity" – see Remark 1.1.16.

6. If $(H, \oplus)$ and $(H, \odot)$ are semihypergroups and "$\odot$" is distributive over "$\oplus$", then $(H, \oplus, \odot)$ is called *good* (or *strong*) *semihyperring.*

Just as is the case with semirings in classical algebra, some authors define semihyperrings as hyperstructures, where the scalar identity with respect to one hyperoperation is absorbing with respect to the other while others assume semigroups only and omit the axiom of absorption. This is e.g. the case of [255], where Omidi and Davvaz introduced (long after the classification of Vougiouklis) the term *ordered semihyperring*, which will be mentioned in Section 1.2.

In his study of H_v–structures summed up in [300], Vougiouklis also coined the term *H_v–ring.* Notice that the following definition uses the concept of *weak distributivity* which is different from *inclusion distributivity.* In fact, it is a parallel to *weak commutativity* and *weak associativity.*

Definition 1.1.15. If $(H, \oplus)$ is an H_v–group, $(H, \odot)$ is an H_v–semigroup and hyperoperation "$\odot$" is weakly distributive over "$\oplus$", i.e. for all $x, y, z \in H$ there is

$$\begin{aligned} x \odot (y \oplus z) \quad &\cap \quad (x \odot y) \oplus (x \odot z) \neq \emptyset \\ (x \oplus y) \odot z \quad &\cap \quad (x \odot z) \oplus (y \odot z) \neq \emptyset, \end{aligned} \tag{1.16}$$

then $(H, \oplus, \odot)$ is called *H_v–ring.*

Remark 1.1.16. As regards various types of hyperstructure distributivity, one must consider historical perspective. The classification of Vougiouklis, i.e. Definition 1.1.14, is the oldest one, and therefore its distributivity did not require a name as the distinction was provided by words "good" (or "strong"). However, after the concepts of *weak associativity* and *weak commutativity* emerged in [300], their parallel in hyperrings was called *weak distributivity* and, to make the distinction clear, the original "non-strong" distributivity of Definition 1.1.14 became known as *inclusion distributivity.* Unfortunately, some authors such as Hedayati, Ameri [144] use the term "weak" in the sense of "inclusive" – see Definition 2.5.19 on page 109.

When classifying hyperrings, Vougiouklis [302] mentions also *multiplicative hyperrings*, which are (in his sense) intuitive counterparts to additive hyperrings, i.e. hyperstructures $(H, +, \odot)$, where "$\odot$" is (weakly) distributive over the single-valued operation "$+$". However, when giving a definition of the term in 1982 (and studying the concept in [274, 277]), Rota [276] listed two more conditions – commutativity and existence of unique single-valued inverses. However, her definition, which follows below as Definition 1.1.17, must be read with the historical perspective in mind as Rota gave it in 1982, i.e. *before* the classification of Vougiouklis, when the term *hyperring* universally meant “Krasner hyperring”, i.e. a concept which is based on canonical hypergroups, i.e. hyperstructures which are commutative and in which every element has the unique inverse. In this way, multiplicative hyperrings are in fact Krasner hyperrings with swapped hyper– and single-valued operations.

Definition 1.1.17. If $(H,+)$ is a commutative group, $(H,\odot)$ is a semihypergroup, hyperoperation “$\odot$” is (inclusively)[13] distributive over “$+$”, i.e.

$$\begin{aligned} a\odot(b+c) &\subseteq (a\odot b)+(a\odot c)\\ (b+c)\odot a &\subseteq (b\odot a)+(c\odot a), \end{aligned}\tag{1.17}$$

for all $a,b,c\in H$ and for all $a,b\in H$ there is $a\odot(-b)=(-a)\odot b=-(a\odot b)$, then $(H,+,\odot)$ is called *multiplicative hyperring.*

If in the above definition hyperoperation “$\odot$” is distributive over “$+$” in the sense of equalities instead of inclusions, then Rota and later Davvaz and Leoreanu [111] call $(H,+,\odot)$ *strongly distributive multiplicative hyperring*. In the sense of classification done by Vougiouklis, names *multiplicative hyperring* and *good* (or *strong*) *multiplicative hyperring* would be used.

Obviously, we have to be very careful when interpreting this definition as $a\odot b$ and $a\odot c$ are sets on which the definition applies a single-valued operation “$+$”. In Section 2.5 we will discuss this topic in detail. We will also specify whether we mean multiplicative hyperrings in the sense of Vougiouklis or in the sense of Rota. For more details on multiplicative hyperrings see Davvaz and Leoreanu–Fotea [111], chapter 4. In Subsection 2.5.4 we develop results obtained by Cristea and Jančić-Rašović [98, 166]. Since they adopt the terminology of Spartalis [284], it is to be noted that by “hyperring” Spartalis means “good hyperring in the general sense” in the sense of Definition 1.1.14[14] and by “semihyperring” he means “good semihyperring” in the sense of Definition 1.1.14. Other hyperstructures such as e.g. *hyperringoids* will be defined in Subsection 2.5.3. H_v*–matrices* will be defined in Subsection 2.5.5.

Finally, we must mention that some authors have recently adopted the terminology advocated by Ameri, Hedayati and Davvaz in papers such as [10, 103], where by semihyperrings they mean structures $(S,\oplus,\cdot)$ such that $(S,\oplus)$ is a semihypergroup, $(S,\cdot)$ is a semigroup and “$\cdot$” distributes over “$\oplus$”. It

[13] Neither Rota uses the word “inclusive”. In [276] she calls the property “proprietà distributive” while strong distributivity is “fortemente distributive”.

[14] This usage is contrary to some more recent papers in which many authors use “hyperring” in the sense of “Krasner hyperring”. The paper of Spartalis dates back to 1989 and he worked with Vougiouklis.

is to be noted, though, that early papers of Davvaz such as [104] follow the classification of Vougiouklis and Definition 1.1.14.

In the algebra of single-valued structures, the notion of a lattice naturally combines the point of view of algebraic operations and the point of view of ordering. The notion of a *hyperlattice*, which transfers this double approach to the hyperstructure theory, was introduced in the hyperstructure theory twice: by Benado [21] in 1950s and Konstantinidou and Mittas [182] in 1977. Further on we will concentrate on the line stemming from the approach of Konstantinidou and Mittas, which received more attraction; for implications of Benado's work see Pickett [259] or Spanish authors such as Martínez et al. [200] or a PhD thesis by Golzio [142] (who all speak of *multilattices*).

Konstantinidou and Mittas defined hyperlattices as sets with one hyperoperation "$\bigvee$" and one single-valued operation "$\wedge$" under the following Definition 1.1.18.[15] In this way hyperlattices are also defined in Corsini and Leoreanu [95].

Definition 1.1.18. Let H be a set, "$\bigvee$" a hyperoperation on H and "$\wedge$" an operation. We say that $(H, \bigvee, \wedge)$ is a *hyperlattice* if the following conditions are satisfied, for all $a, b, c \in H$:

1. $a \in a \bigvee a$ and $a \wedge a = a$

2. $a \bigvee b = b \bigvee a$ and $a \wedge b = b \wedge a$

3. $(a \bigvee b) \bigvee c = a \bigvee (b \bigvee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$

4. $a \in [a \bigvee (a \wedge b)] \cap [a \wedge (a \bigvee b)]$

5. $a \in a \bigvee b \Rightarrow b = a \wedge b$

In the course of time other approaches have emerged and further concepts have been defined. The most prominent of these include *total hyperlattices*, also known as *superlattices*, which are structures with two hyperoperations. Also, notice that some authors such as Soltani Lashkenari, Rasouli and Davvaz [270, 283] make distinction between *join* and *meet hyperlattices*, which they consider to be hyperstructures with one hyperoperation and one single-valued operation in the sense of Definition 1.1.18 yet such that only axioms 1–4 hold. (In their terminology the above definition defines the *join hyperlattice*.) If moreover axiom 5 of Definition 1.1.18 holds, they call the

[15] In axiom 4, notice that for sets $X, Y \subseteq H$ there is $X \wedge Y = \{x \wedge y \mid x \in X, y \in Y\}$.

join hyperlattice a *strong join hyperlattice*.[16] Naturally, join and meet hyperlattices are mutual duals, i.e. dually, a *strong meet hyperlattice* may be defined.

The introduction of *ordered hyperstructures* [146] combined with numerous results obtained by Konstantinidou and Seramifidis (for references see the list in [95]; most papers written in French), enhanced the study of hyperstructure generalizations of lattice-like structures and enabled to view these from the two perspectives common in lattices. Xiao and Zhao in [309] introduced and studied the concept of a *hypersemilattice* as a multivalued analogy of a semilattice while Dehghan Nezhad and Davvaz in [116] presented numerous results on *hypersemilattices* and especially *H_v–semilattices*, which are the natural weakening of the strong hypersemilattice concept.

Definition 1.1.19. Let L be a nonempty set with a binary hyperoperation "$*$" on L such that, for all $a, b, c \in L$, the following conditions hold:

1. $a \in a * a$ (idempotency)

2. $a * b = b * a$ (commutativity)

3. $(a * b) * c \cap a * (b * c) \neq \emptyset$ (weak associativity)

Then $(L, *)$ is called an *H_v–semilattice*. When in the condition 3 we have equality, then $(L, *)$ is called a *hypersemilattice*.

1.1.2 Subhyperstructures

The notion of *subhyperstructure* is often a straightforward transfer of the notions of the single-valued classical concepts.

Definition 1.1.20. A pair $(K, *)$, where K is a non-empty subset of a hypergroupoid $(H, *)$ such that $K * K \subseteq K$, is called a *subhypergroupoid* of H. A subhypergroupoid $(K, *)$ of H is called a *subsemihypergroup* of $(H, *)$ if it is associative or a *sub-quasi-hypergroup* of $(H, *)$ if the reproductive law is valid for K. If $(K, *)$ is a hypergroup, it is called a *subhypergroup* of H. The definitions for *n–ary hypergroupoids* are analogous.

Terms such as *sub-join space* or *sub-canonical hypergroup* are not used so often, even though they can be defined in a similar way. We say e.g. "the subhypergroup K of the join space H is a join space" instead.

[16] I.e. in the original definition given by Konstantinidou and Mittas [182] a *hyperlattice* is a *strong join hyperlattice*.

Given certain specific properties we distinguish several special kinds of subhyperstructures. These will be mentioned in Subsection 1.1.3.

Subhyperstructures of hyperstructures with two (hyper)operations are defined mostly in a similar fashion as subsets which themselves are hyperstructures of the given type. However, sometimes (most often for Krasner hyperrings) equivalent definitions are used which may result in some level of confusion.

Definition 1.1.21. A non-empty subset S of a semihyperring $(R, \oplus, \odot)$ is called a *subsemihyperring* of R if $(S, \oplus, \odot)$ itself is a semihyperring. If $(R, \oplus, \odot)$ and $(S, \oplus, \odot)$ are Krasner hyperrings, then S is called a *subhyperring* of R.

Since in Krasner hyperrings we have unique neutral elements and unique inverses, some authors use an equivalent definition of a subhyperring defining it as a non-empty subset S of R such that $x \ominus y \subseteq S$ and $x \odot y \in S$ for all $x, y \in S$, where "$\ominus y$" is the unique inverse of $y \in S$. For reasons of convenience signs "$+, \cdot, -$" are often used instead of "$\oplus, \odot, \ominus$". Of the authors who study the issue of ordering in hyperstructure theory, e.g. Asokkumar [14] or Davvaz in some of his papers do so. Alternatively, it is possible to define subhyperrings as subsets S of a Krasner hyperring $(R, \oplus, \odot)$ such that $(S, \oplus)$ is a canonical subhypergroup of $(R, \oplus)$ and $S \odot S \subseteq S$, where $S \odot S \subseteq S$ has the same meaning as $x \odot y \subseteq S$ for all $x, y \in S$. The actual form of the definition influences the form of definition of a *hyperideal* of a hyperring. Below, we give the most natural definition corresponding to the definition of a subhyperring as included in Davvaz and Leoreanu–Fotea [111] at the beginning of a chapter discussing Krasner hyperrings. Naturally, the ideal of a hyperideal remains the same regardless of the hyperstructure it is applied on.

Definition 1.1.22. A subhyperring S of a Krasner hyperring $(R, \oplus, \odot)$ is called a *left hyperideal* of R if $r \odot s \in S$ (or a *right hyperideal* of R if $s \odot r \in S$) for all $s \in S, r \in R$. If S is both a left and a right hyperideal of R, i.e. if $R \odot S \subseteq S$ and $S \odot R \subseteq S$, it is called a *hyperideal* of R.

In semihypergroups, hyperideals are hyperstructure analogies of ideals too.

Definition 1.1.23. A non-empty subset I of a semihypergroup $(H, *)$ is called a *left hyperideal* of H if $h * i \in I$ (or a *right hyperideal* of H if $i * h \in I$) for all $i \in I, h \in H$. If I is both a left and a right hyperideal of H, it is called a *hyperideal* of H.

In Section 2.4 we will use some results concerning hyperideals such as those obtained by Chattopadhyay [42]; this paper may be also used as a source of reference for definitions and results concerning some basic types of hyperideals such as *prime*, *semiprime*, *principal* or *maximal* in the context of semihypergroups. Also, a number of authors such as Azizpour, Changphas, Dine, Davvaz, Hedayati, Hila [40,145,147] study the notion of *bi-hyperideals*. Further on, in Section 3.2, we will include some results obtained by Ghazavi, Anvariyeh and Mirvakili [137] concerning ideals in hyperstructures presented in Chapter 2.

1.1.3 Properties

Now we are going to mention a few definitions of some very basic properties of hyperstructures. Some more special properties will be defined at respective places further on in the text.

Definition 1.1.24. If A is a non-empty subset of a hypergroupoid $(H, *)$, then A is called:

1. *Reflexive* in H if for an arbitrary pair of elements $x, y \in H$ the fact that $x * y \cap A \neq \emptyset$ implies that $y * x \cap A \neq \emptyset$.

2. *Invariant (normal)* in H if for an arbitrary element $x \in H$ there is $x * A = A * x$.

3. *Invertible on the left* if for an arbitrary pair $x, y \in H$ the fact that $y \in A * x$ implies the fact that $x \in A * y$. It is called *invertible on the right* if the fact that $y \in x * A$ implies the fact that $x \in y * A$ and *invertible* if it is invertible both on the left and on the right.

Definition 1.1.25. If A is a non-empty subset of H, where $(H, *)$ is a semihypergroup, then A is called a *complete part* of H if the following implication holds:

$$\forall n \in \mathbb{N}, \forall (x_1, \ldots, x_n) \in H^n, \prod_{i=1}^{n} x_i \cap A \neq \emptyset \Rightarrow \prod_{i=1}^{n} x_i \subset A$$

The intersection of all complete parts of a semihypergroup S which contain A, is called *complete closure of A in S* and is denoted by $\mathcal{C}(A)$. A semihypergroup $(S, *)$ is called *complete* if, for all $x, y \in S$, there is $\mathcal{C}(x * y) = x * y$.

Definition 1.1.26. A semihypergroup $(H, *)$ is called *simplifiable on the left* if for all $x, a, b \in H$ the fact that $x * a \cap x * b \neq \emptyset$ implies that $a = b$.

The concepts of *simplifiable on the right* and *simplifiable* are defined in a way analogous to invertibility. It was already Marty [201] who proved that any hypergroup simplifiable on the left (or on the right) is a group.

Definition 1.1.27. If $(H, *)$ is a hypergroup and G its subhypergroup, then G is called:

1. *Closed on the left (right)* in H if for arbitrary $a \in H$ and $x, y \in G$ the fact that $x \in a * y$ $(x \in y * a)$ implies $a \in G$. The subhypergroup G of H is called *closed* in H if it is closed both from the left and from the right.

2. *Ultraclosed on the left* if for an arbitrary $x \in H$ there holds $G * x \cap (H \setminus G) * x \neq \emptyset$. The properties of being *ultraclosed on the right* and *ultraclosed* are defined in a way analogous to the concept of closed on the right and closed.

Definition 1.1.28. A hypergroup is called *regular* if it has at least one identity and each element has at least one inverse. A regular hypergroup is called *reversible* if for any triple $x, y, z \in H$ there holds (i) if $y \in a * x$, then there exists an inverse a' of a such that $x \in a' * y$ and (ii) if $y \in x * a$, then there exists an inverse a'' of a such that $x \in y * a''$. A commutative hypergroup $(H, *)$ is called *inner irreducible* if for any pair of its subhypergroups G_1, G_2 such that $G_1 * G_2 = H$ there holds $G_1 \cap G_2 \neq \emptyset$.

Finally, we mention *homomorphisms* of hypergroupoids, a topic which will be discussed only very briefly further on.[17]

Definition 1.1.29. Let $(H, \circ)$ and $(K, *)$ be hypergroupoids and $f : H \to K$ be a mapping. If, for all $a, b \in H$, there is $f(a \circ b) \subset f(a) * f(b)$, we say that f is a *homomorphism*. If, for all $a, b \in H$, there is $f(a \circ b) = f(a) * f(b)$, we say that f is a *good homomorphism*.

1.1.4 Ordering

The constructions, which we present further on, rely on the commonly-known notion of *quasi-* (or *partially*) *ordered semigroups*. The concepts we are going to use are rather basic and need not be recalled.[18]

[17] Names "good" and "strong" are often equivalents in the hyperstructure theory. However, this is not true with homomorphisms as *strong homomorphisms* are different from *good* ones. See Corsini and Leoreanu [95], p. 4.

[18] For a deeper study of the topic a great number of monographs, including easy to follow books [29, 141], can be referred to. We will often make use of results included in Chvalina [44].

Definition 1.1.30. By a *quasi-ordered* (or a *partially ordered*) *semigroup* we mean a semigroup $(S, \cdot, \leq)$ endowed with a reflexive and transitive (or a reflexive, antisymmetric and transitive) relation "$\leq$" such that

$$a \leq b \Rightarrow a \cdot c \leq b \cdot c \text{ and } c \cdot a \leq c \cdot b \tag{1.18}$$

for all $a, b, c \in S$.

The condition used in the above definition is often called *compatibility condition* and the relation "$\leq$" is said to be "compatible with the operation '$\cdot$'". Alternatively, we speak of *monotone condition*. In proofs of our results we will often make use the fact that in quasi-ordered semigroups $(S, \cdot, \leq)$ the fact that $a \leq b$ and $c \leq d$ implies $a \cdot c \leq b \cdot d$.

Remark 1.1.31. At this place a *remark on terminology* must be included. In accordance with Chvalina [44] and a number of papers stemming from it (including those with results which make a substantial part of this book) we use terms "quasi-order(ing)", "quasi-ordered semigroup" and "partial order(ing)" and "partially ordered semigroup". In English, terms such as "preorder" or "preordered set" are often used instead of the former. Also, terms "proset" and "poset" are often used to refer to a set endowed with a "preorder", i.e. "quasi-ordering", and "partial order(ing)", respectively. Furthermore, since in many of our results it will be important whether the relation in question is or is not antisymmetric (on top of being reflexive and transitive) the adjectives "quasi" or "partially" will always be used. This is especially important to notice as many authors who focus their study on partially ordered sets exclusively can afford to omit the adjective "partially" as in their context it is redundant.

Remark 1.1.32. Just as we have discussed terminology, we must make a *remark on notation*. Given a relation R one usually writes aRb to denote that elements a and b are in relation R and denotes by $R(x)$ the set $\{y \in X \mid xRy\}$, i.e. the set of all elements which are in relation R with x. If R is a partial ordering, most authors use the standard notation $a \leq b$; if R is a quasi-ordering, $a \preceq b$ is used. For an equivalence relation R, the notation $a \sim b$ is often used.

In our text we either write $a \leq b$ and specify the type of the relation or use standard notation such as "$|$" for divisibility relation, "$\subseteq$" for set inclusion or "$\equiv$" for congruence modulo m. To avoid confusion, we also – whenever possible – describe the relation in plain words.

When moving from *binary relations* to *n–ary relations*, some concepts can be transferred while others cannot. The following definition has been taken from Cristea and Ştefănescu [99] who make use of results obtained by Novák and Novotný [253] and continue their research presented in [252–254].

Definition 1.1.33. The relation ρ on a non-empty set H is called:

1. *Reflexive* if, for any $x \in H$, the n-tuple $(x, \dots, x) \in \rho$,

2. *n–transitive* if it has the following property:
   if $(x_1, \dots, x_n) \in \rho$, $(y_1, \dots, y_n) \in \rho$ and there exist natural numbers $i_0 > j_0$ such that $1 < i_0 \leq n$, $1 \leq j_0 < n$, $x_{i_0} = y_{j_0}$, then the n-tuple $(x_{i_1}, \dots x_{i_k}, y_{j_{k+1}}, \dots y_{j_n}) \in \rho$ for any natural number $1 \leq k < n$ and $i_1, \dots i_k, j_{k+1}, \dots j_n$ such that $1 \leq i_1 < \dots < i_k < i_0$, $j_0 < j_{k+1} < \dots < j_n \leq n$,

3. *Strongly symmetric* if $(x_1, \dots x_n) \in \rho$ implies $(x_{\sigma(1)}, \dots x_{\sigma(n)}) \in \rho$ for any permutation σ on the set $\{1, \dots, n\}$,

4. *n–ary preoredering* (or *n–ary quasi-ordering*[19]) on H if it is reflexive and n–transitive,

5. An *n–equivalence* on H if it is reflexive, strongly symmetric and n–transitive.

In Section 3.1 we will see that the impossibility to reasonably transfer the notion of antisymmetry to the context of n–ary relations enabled Ghazavi and Anvariyeh [135] to obtain some interesting results concerning the topic of this book. In [135] the following definition is given as a parallel to the binary context.

Definition 1.1.34. An algebraic structure $(H, \cdot, \rho)$ is called an *n–ary preordering groupoid* (or *n–ordered groupoid*) if $(H, \cdot)$ is a groupoid and ρ is an n–ary preordering on H such that for all $(x_1, x_2, \dots, x_n, y) \in H^{n+1}$ with the property $(x_1, x_2, \dots x_n) \in \rho$ there holds $(yx_1, yx_2, \dots, yx_n) \in \rho$ and $(x_1y, x_2y, \dots x_ny) \in \rho$.

The following idea (and corresponding notation) will be crucial in our considerations: for a set S endowed with a relation "$\leq$" we denote

$$[a)_{\leq} = \{a \in S \mid a \leq x\} \tag{1.19}$$

and

$$(a]_{\leq} = \{a \in S \mid x \leq a\}. \tag{1.20}$$

[19] This term is not used in the definition of [99] or in Ghazavi and Anvariyeh [135], which we mention below, but we include it because of Remark 1.1.31.

These sets are often called *upper* (or *lower*) *cone of S generated by $a \in S$*. Based on Chvalina [44] we call them *principal end* (or *principal beginning*) *of S generated by $a \in S$*.

Some of our proofs will make use of concepts of the *smallest* / the *greatest* or *minimal* / *maximal* elements. Strictly speaking, all of these concepts are defined for partially ordered sets only because if antisymmetry of the relation "$\leq$" is missing, we may run into difficulties (notice that the concept of a maximal element actually *can* be defined in the context of quasi–ordered sets; for an example of use see e.g. Kovar and Chernikava [184]). Therefore, if we use the greatest or the smallest elements in our proofs, we need to adjust assumptions of the theorems accordingly. However, in cases which regard maximal elements, we will in fact handle elements, principal ends of which are one-element sets. Since the relation "$\leq$" will always be reflexive, the fact that $[a)_{\leq} = \{a\}$ will simply mean that there exists no element $x \in S$ different from a such that $a \leq x$, which is an equivalent definition of a maximal element. In such a case it is in fact irrelevant whether "$\leq$" is a partial or quasi-order. Since we will be interested in *binary* relations, we ignore the case of *cyclically ordered sets*. However, even if we did not, $[a)_{\leq}$ would not be a one-element set (for $S \neq \{a\}$). Therefore, let us include the following definition with which we waive all possible future problems with different approaches to the concept of maximality in quasi-ordered sets.

Definition 1.1.35. Let $(S, \leq)$ be a quasi-ordered set. An element $a \in S$ such that $[a)_{\leq} = \{x \in S \mid a \leq x\} = \{a\}$, is called an *EL–maximal element*.

However, one must be aware of the following: if for some element $a \in S$ there holds $x \leq a$ for all $x \in S$, where "$\leq$" is a quasi-ordering only, then – unfortunately – we *cannot* conclude that $[a)_{\leq} = \{a\}$.

1.2 Various approaches to ordering in the hyperstructure theory

The connection between *ordering* and the idea of a *hyperoperation*, i.e. a mapping from H^n to the set of non-empty subsets of H (denoted as $\mathcal{P}^*(H)$), is a very natural one. It was in fact the idea of a line segment being generated by its endpoints, i.e. a hyperoperation

$$a * b = \{x \in \mathbb{R} \mid a \leq x \leq b\} \tag{1.21}$$

for all $a, b \in \mathbb{R}$, that was the motivation[20] with which Prenowitz and Jantosciak in 1972 introduced the concept of a *join space* [264]. Already in 1948, Iwasava [161] worked with a hyperoperation "$*$" defined on a lineraly ordered group G by

$$a * b = \{x \in G \mid \min\{a, b\} \leq x \leq \max\{a, b\}\}. \tag{1.22}$$

Corsini [87, 91] used a very much similar idea when constructing join spaces from fuzzy sets.[21]

Definition 1.2.1. Let X be a non-empty set. Any function $\mu : X \to \langle 0, 1\rangle$ is called a *membership function* of X. The pair (X, μ) is called a *fuzzy set*. For a given $x \in X$, $\mu(x)$ is called the *grade of membership* of x in (X, μ).

Suppose we have a hypergroupoid $(H, *)$. For a given element $u \in H$ denote by $Q(u)$ the set of all pairs $(a, b) \in H^2$ such that $u \in a * b$, i.e. $Q(u) = \{(a, b) \in H^2 \mid u \in a * b\}$ and by $q(u)$ denote the cardinality of $Q(u)$, i.e. $q(u) = |Q(u)|$. Now, for all $u \in H$, define

$$\tilde{\mu}(u) = \frac{\sum\limits_{(x,y)\in Q(u)} \frac{1}{|x*y|}}{q(u)}. \tag{1.23}$$

If $Q(u) = \emptyset$ or in case of infinite cardinality of $Q(u)$, we define $\tilde{\mu}(u) = 0$. Obviously, $\tilde{\mu}(u) \in \langle 0, 1\rangle$ for all $u \in H$, i.e. $\tilde{\mu}$ is a membership function of H. Corsini [87] showed that if we define on H a hyperoperation "$*_{\tilde{\mu}}$" by

$$x *_{\tilde{\mu}} y = \{z \in H \mid \min\{\tilde{\mu}(x), \tilde{\mu}(y)\} \leq \tilde{\mu}(z) \leq \max\{\tilde{\mu}(x), \tilde{\mu}(y)\}\} \tag{1.24}$$

for all $x, y \in H$, then $(H, *_{\tilde{\mu}})$ is a join space.

---

[20] Notice how simple and evident it is to show that $(H, *)$, where "$*$" is a line segment generated by its endpoints, is a hypergroup, or rather to justify the wording of hyperstructure axioms. Also recall a quote from the abstract of Nieminen [238] which reads: "A join space is an abstract model for partially ordered linear spherical and projective geometries." Cf. also [18], in which Bandelt and Mulder discuss the relation of pseudo-median graphs and join spaces, to be more precise, the text on p. 15. Recall that Prenowitz had worked on the idea of geometries seen from the perspective of the hyperstructure theory since early 1940s [263], which makes the topic – and the link to ordering – one of the most classical ones in the hyperstructure theory. Recently, a similar geometrical approach can be observed in works of Antampoufis, Dramalids and Vougiouklis [7, 9, 125, 127] on *geometrical hyperoperations*. However, by far the nicest link between geometry and hyperstructure theory can be found in [204], in which Massouros relates some of Euclid's postulates to hyperoperations. For more of geometrical motivation of some concepts of hyperstructure theory see also Mittas and Massouros [203, 227].

[21] This idea was later developped e.g. by Ştefănescu and Cristea [289].

Also, on page 29 we mention Pickett [259] who in 1967 gave an example of a hyperoperation constructed using lattice ordering. In 1975, Varlet [295] established a *connection between distributive lattices and join spaces* which had been introduced only 3 years before. We quote Varlet's result as Theorem 2.5.57 on page 129; notice that the hyperoperation he uses is defined on a lattice $(L, \wedge, \vee, \leq)$ by

$$a \diamond b = \{x \in L \mid a \wedge b \leq x \leq a \vee b\} \tag{1.25}$$

for all $a, b \in L$. The papers that we make reference to above are a tiny selection. However, it may not be so easy to identify elements of hyperstructure theory in many works before 1980s since the terminology had not been codified for long. For example, when *using join spaces for classification of median algebras*, Bandelt and Hedlíková [17] use the term "operation" when referring to a hyperoperation (see [17], p.7). Even though the paper in fact relies on the hyperstructure theory, it never uses the prefix "hyper–" to make this connection obvious. Also, hyperstructures have often been called *multistructures*, and some authors may have adopted this (now obsolete) terminology. For example, Mihail Benado, a Romanian, studied generalizations of lattices in 1950s, i.e. even before Konstantinidou and Mittas [182]. However, his works such as [21] (written in Romanian)[22] have not drawn enough attention in the algebraic hyperstructure theory (or remained unfinished). Recently, Martínez et. al [200], picked up his research – however, instead of using a standard name "hyperoperation" they speak of "non-deterministic operators" (or "*nd*–operators").

There are several areas in which the hyperstructure theory and ordering intersect substantially. First of all, it is the *lattice theory* and the results of Varlet on *distributive lattices* [295], Konstantinidou, Seramifidis and Kehagias [177, 179] on *lattice-ordered join spaces* and results of Comer, Mittas, Konstantinidou [80, 85, 182] and later of Călugăreanu and Leoreanu [37] on *hypergroupoids associated to semi-lattices and lattices* as well as on *characterization of modular lattices*. For a collection of these results see Corsini and Leoreanu [95], chapter 4.

Nieminen [237, 238] established a *connection between join spaces and connected simple graphs* by defining the hyperoperation on the set of vertices of a graph as certain paths, i.e. subsets of the set V of vertices of a given graph. One can see that this idea is similar to (1.25) or the idea of a line segment as a result of a hyperoperation applied on its endpoints.[23] Also

[22] For details on his research and impact of his ideas on contemporary theoretical informatics see an overview paper Rudeanu and Vaida [278].

[23] The use of join spaces for the study of graphs is by far not a closed topic. See e.g. recently published Polat [262].

Corsini [88, 89], Rosenberg [272] and V. Leoreanu and L. Leoreanu [197] studied *hyperoperations associated to graphs*. With a motivation similar to Nieminen, Kalampakas, Spartalis et al. [173–175] studied *path hyperoperations* in graphs and based on these *picture hyperstructures* which enable us to see the hyperoperation as a mapping from the set of pixels of a movie frame to the subset of pixels such as path or an image. In this way it can be used in image processing to e.g. detect motion.

Chvalina's book [44] is a collection of his results on functional graphs, quasi-ordered sets and commutative hypergroups. One layer of this book studies hyperoperations related to a general set H endowed with a quasi-ordering "$\leq$". In 1994, Chvalina [43] classified certain types of hyperstructures which in [44] (published in Czech) and later in [53] were named *quasi-order* hypergroups.[24] The following definition is adapted from [44], p.158.

Definition 1.2.2. By a *quasi-order hypergroup* we mean a commutative hypergroup $(H, *)$ such that, for all $a \in H$, there is $a * a * a = a * a$, i.e. $a^3 = a^2$.

Later on, Chvalina [44] approached the topic from a slightly different perspective than in [43] and defined, on the set of quasi-order hypergroups, a binary relation $R_* \subseteq H * H$ by

$$R_* = \{(a, b) \in H * H \mid a * b * a = a * a\} \tag{1.26}$$

and showed that R_* is a quasi-ordering on the hypergroup H which is moreover antisymmetric if and only if, for all $a, b \in H$, the fact that $a^2 = b^2$ implies $a = b$. Also, Chvalina showed that if (H, R) is a quasi-ordered set, then $(H, *_R)$, where

$$a *_R b = R(a) \cup R(b) \tag{1.27}$$

for all $a, b \in H$, is an extensive commutative hypergroup. Corsini and Leoreanu, inspired by [43], include the following definition and proposition.

Definition 1.2.3. Let $(H, *)$ be a hypergroupoid. We say that H is a *quasi-order hypergroup*, i.e. a *hypergroup determined by a quasi-order*, if, for all $a, b \in H$, $a \in a^3 \subseteq a^2$ and $a * b = a^2 \cup b^2$. Moreover, if the following implication holds,

$$a^2 = b^2 \Rightarrow a = b, \tag{1.28}$$

for all $a, b \in H$, then $(H, *)$ is called an *order hypergroup*.

[24] Or, alternatively – and originally, *quasi-ordering* hypergroups. Also notice that in some earlier papers such as [47, 148] Hort and Chvalina use also the name *qoset-hypergroup*.

Proposition 1.2.4. *A hypergroupoid is a (quasi–) order hypergroup if and only if there exists a (quasi–) order R on the set H such that, for all $a, b \in H$*

$$a * b = R(a) \cup R(b).$$

Notice that since in Remark 1.1.32 we chose not to follow the R notation of relations, the condition of Proposition 1.2.4 rewrites to

$$a * b = [a)_{\leq} \cup [b)_{\leq}, \tag{1.29}$$

which can be easily compared to (2.1) in Subsection 2.1.[25]

Thus we have seen that quasi-order hypergroups are hyperstructures constrcuted from a pair *set – relation*. This book discusses *EL–hyperstructures*, i.e. hyperstructures constructed from a triple *semigroup – single-valued operation – relation*. The study of hyperstructure generalization of lattices, initiated by Konstantinidou and Mittas [182] and developped especially by Konstantinidou [179–181], urged the need to study such hyperstructures not only from the algebraic point of view of hyperstructures with two (hyper)operations but also from the point of view of ordered sets. Starting with Heidari and Davvaz [146], *ordered hyperstructures*, in which the triple *set – hyperoperation – relation* is used, have been studied.[26] Notice that in the following definition we use the symbol "$\preceq$" not in the sense of a preorder, i.e. a quasi-ordering, but as a symbol for partial ordering on a hyperstructure. This will enable us, in Section 2.6, to easily distinguish between quasi-ordered semigroups $(H, \cdot, \leq)$ and ordered semihypergroups $(H, *, \preceq)$.

Definition 1.2.5. An *ordered semihypergroup* $(H, *, \preceq)$ is a semihypergroup $(H, *)$ together with a partial ordering "$\preceq$" which is compatible with the hyperoperation, i.e.

$$x \preceq y \Rightarrow a * x \preceq a * y \text{ and } x * a \preceq y * a \tag{1.30}$$

for all $a, x, y \in H$. By $a * x \preceq a * y$ we mean that for every $c \in a * x$ there exists $d \in a * y$ such that $c \preceq d$.

Papers written on the topic of ordered hyperstructures are numerous; for a collection of some results see a recently published book Davvaz [105].

[25] In [43], Chvalina gave another similarly looking construction of a hyperoperation, where $a \circ b$ is defined as $R_a \cup L_b$, where $R_a = \{b \in H \mid (a, b) \in \rho\}$ and $L_b = \{b \in H \mid (b, a) \in \rho\}$ for a binary relation ρ on $H \times H$. Such hypergroupoids were studied by De Salvo and Lo Faro [120, 121].

[26] Notice that since [146] was published in 2011 and a number of papers by various authors have emerged since, the terminology and definitions have not been standardized yet.

In [107], Davvaz, Corsini and Changphas discussed a question of whether there exists a strongly regular relation ρ on an ordered semihypergroup S such that $S/_\rho$ is an ordered semigroup. As far as the issue of hyperlattices seen as ordered (hyper)structures is concerned, notice a discussion on the topic included in Rasouli and Davvaz [270] and Rosenberg [283]. In [112], Davvaz and Omidi introduce the concept of an *ordered semihyperring*. In Section 2.6 we discuss their definition and show that the fact that they misquote Vougiouklis and his classification included as Definition 1.1.14 is crucial for our considerations regarding relation of EL–semihyperrings of Section 2.5 and ordered semihyperrings of theirs.

Finally, one must not forget the issue of *hyper BCK–algebras* which are generalizations of BCI– or rather *BCK–algebras* introduced in 1966 by Iséki [159] and brought to a shape by Iséki and Tanaka in papers such as [160]. For a collection of results on BCI– / BCK–algebras see Huang [158] or Meng and Jun [219]; notice that the original motivation for introducing such structures lies in combinatory logic and propositional calculus.

Definition 1.2.6. An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI–algebra* if it, for all $x, y, z \in X$, satisfies the following conditions:

1. $((x * y) * (x * z)) * (z * y) = 0$,
2. $(x * (x * y)) * y = 0$,
3. $x * x = 0$,
4. simultaneous validity of $x * y = 0$ and $y * x = 0$ implies that $x = y$.

Remark 1.2.7. In the theory of BCI–algebras, the standard notation "$*$" stands for a binary *operation* on X. In the context of hyperstructure theory this is rather inconvenient and misleading as the symbol "$*$" is usually reserved for a *hyperoperation*. Notice that – to increase the possible confusion – in the definition of a *hyper BCK–algebra*, i.e. Definition 1.2.9, the same symbol is used for a *hyperoperation*.

In BCI–algebras we can define, for all $x, y \in X$, a relation "$\leq$", called a *BCI–ordering*, by setting

$$x \leq y \text{ whenever } x * y = 0. \tag{1.31}$$

It is easy to show that "$\leq$" is a partial ordering on X. However, when attempting to show that the BCI–ordering "$\leq$" is compatible with the operation "$*$" (which is crucial for our further considerations), we run into difficulties as, for all $x, y, z \in X$, the fact that $x \leq y$ implies that $x * z \leq y * z$

yet $z * x \geq z * y$ (given as Huang [158], Proposition 1.1.4). Thus we see that $(X, *, \leq)$ are not partially ordered semigroups.

The nature of BCI–algebras is such that the operation "$*$" can often *intentionally* be neither associative nor commutative in the usual sense of the word. In the special case of *associative BCI–algebras*, i.e. BCI–algebras $(X; *, 0)$ such that, for all $x, y, z \in X$ there is $(x * y) * z = x * (y * z)$, we have that $x * y = y * x$ and $0 * x = x$ for all $x, y \in X$ (all these statements are in fact equivalent), which means that the BCI–ordering "$\leq$" *is* compatible with the operation "$*$". However, this result is of no practical use because from the fact that $x * y = y * x$ and definition (1.31), there is in this case $x \leq y$ and also $y \leq x$, which means that $x = y$ because the BCI–ordering is antisymmetric.

Therefore, there cannot be much common ground between the broadest general case of BCI–algebras and the topic of this book. However, this is not true for the case of *lower BCK–semilattices*.

Definition 1.2.8. A BCI–algebra is called a *BCK–algebra* if, for all $x \in X$, there is $0 * x = 0$. A BCK–algebra is called *commutative* if, for all $x, y \in X$, there is $x * (x * y) = y * (y * x)$. A BCK–algebra is called a *lower BCK–semilattice* if $(X, \leq)$, where "$\leq$" is a BCI–ordering, is a lower semilattice.

Hyper BCK–algebras were introduced by Bolurian and Hasankhani [30].

Definition 1.2.9. The set X with a hyperoperation "$*$" and a constant "0", i.e. $(X, *, 0)$, is called a *hyper BCK–algebra* if it, for all $x, y, z, t \in X$, satisfies the following conditions:

1. $(x * y) * z = (x * z) * y$,
2. $t \in x * (x * y) \Rightarrow t \leq y$,
3. $x \leq y \Rightarrow x * z \leq y * z$ and $z * y \leq z * x$,
4. $x \leq x$,
5. $0 \leq x$,
6. $x \leq y, y \leq x \Rightarrow x = y$,

where $x \leq y$ is defined by $0 \leq x * y$ and $A \leq B$ means that, for all $b \in B$, there exists $a \in A$ such that $a \leq b$ for all $x, y \in X$ and $A, B \subseteq X$.

From the above definition it can be easily proved that from simultaneous validity of $x \leq y$ and $y \leq z$ we get that $x \leq z$, i.e. that the relation

"$\leq$" in hyper BCK–algebras is a partial ordering (because, by definition, it is reflexive and symmetric). This, combined with the results on ordered hyperstructures, enables us to study hyper BCK–algebras and their relation to hyperlattices.

One gets partial ordering in *MV–algebras* introduced by Chang [39] in 1958 as well. Inspired by this approach, Ghorbani, Hasankhani and Eslami [138] introduced the concept of *hyper MV–algebras*, which was recently developed by e.g. Jun, Kang and Kim [171]. Notice that in [138] one can find definitions of some related concepts which have not been mentioned above such as *hyper K–algebras*. Since in hyper MV–algebras the partial ordering is preserved, one can again study their relations to ordered hyperstructures and hyperlattices. For an example of this see Borzooei, Radfar and Niazian [34].

The connection between hyperstructure theory and relations has also been studied by Chvalina, Corsini, Cristea, Spartalis, Ştefănescu and others in papers such as [63, 88, 97, 100, 286, 287]. For more references see respective sections. Also see a footnote on page 40 which makes reference to Phanthawimol and Kemprasit [261] and the hyperoperation on equivalence classes of a group.

Chapter 2

EL–hyperstructures

In his book on functional graphs, quasi-ordered sets and commutative hypergroups [44], published in 1995, Chvalina proposed several universal constructions of hypergroups from sets endowed with ordering. One of these, that of *quasi-order hypergroups*, has been briefly mentioned in Section 1.2. Further on we focus on another construction, that has drawn wider attention only recently.

2.1 The “Ends lemma”

Lemma 2.1.1. ([44], Theorem 1.3, p. 146) *Let $(S, \cdot, \leq)$ be a partially ordered semigroup. Binary hyperoperation $* : S \times S \to \mathcal{P}^*(S)$ defined by*

$$a * b = [a \cdot b)_{\leq} = \{x \in S \mid a \cdot b \leq x\} \tag{2.1}$$

is associative. The semihypergroup $(S, *)$ is commutative if and only if the semigroup $(S, \cdot)$ is commutative.

Lemma 2.1.2. ([44], Theorem 1.4, p. 147) *Let $(S, \cdot, \leq)$ be a partially ordered semigroup. The following conditions are equivalent:*

1^0 *For any pair $a, b \in S$ there exists a pair $c, c' \in S$ such that $b \cdot c \leq a$ and $c' \cdot b \leq a$.*

2^0 *The semi-hypergroup $(S, *)$ defined by (2.1) is a hypergroup.*

In groups, condition 1^0 of Lemma 2.1.2 holds trivially as it is sufficient to put $c = b^{-1} \cdot a$ and $c' = a \cdot b^{-1}$. Thus, the following corollary holds.

Corollary 2.1.3. *If, in Lemma 2.1.2, $(S, \cdot, \leq)$ is a partially ordered group, then $(S, *)$, constructed by means of (2.1), is a hypergroup.*

Apart from these, Chvalina proved the following lemma, which turns to be very useful in further considerations.

Lemma 2.1.4. *Let* $(S, \cdot, \leq)$ *be a partially ordered semigroup. Let* $a * b = [a \cdot b)_{\leq}$ *for every pair of elements* $a, b \in S$*. Then for every triple of elements* $a, b, c \in S$ *there is*

$$a * (b * c) = [a \cdot b \cdot c)_{\leq} = (a * b) * c. \tag{2.2}$$

Finally, Račková, one of Chvalina's students, proved the following:

Lemma 2.1.5. ([268], Theorem 4) *Let* $(H, \cdot, \leq)$ *be a partially*[1] *ordered group. Then* $(H, *)$*, constructed by means of* (2.1)*, is a transposition hypergroup.*

The following corollary is obvious because join spaces are commutative transposition hypergroups.

Corollary 2.1.6. $(H, *)$*, constructed from a commutative partially ordered group* $(H, \cdot, \leq)$ *by means of* (2.1)*, is a join space.*

This set of lemmas had been used – without any further study – in a number of papers by Chvalina and his colleagues such as Chvalinová, Hošková, Račková, Dehghan Nezhad, Borzooei, Varasteh or Hasankhani, dealing with hyperstructures of transformation / differential / integro-differential operators and / or hyperstructure generalizations of automata. A few easily accessible examples include [35, 50, 55, 59, 69, 117, 149, 151, 155]. For the context see Section 4.1 and 4.2.

Since this ad-hoc use became rather inefficient, there emerged the need to provide a theoretical basis for a growing number of somewhat similar results obtained in the above papers. It was Chvalina himself that coined the Czech term "koncové lemma" to refer to the above results. The term translates to English literally as "Ends lemma",[2] which is the name used in the title of Novák [243, 246]. Semihypergroups constructed by means of Lemma 2.1.1 had originally been called "Ends lemma–based hyperstructures" which changed to "EL–hyperstructures" in Novák [241]. Since it was this name that was used in Novák [244], the first paper on the topic aimed at wide international audience, the term *EL-hyperstructures* (recently used by e.g. Ghazavi, Anvariyeh and Mirvakili [135–137]) seems to have a chance of becoming a standard name for hyperstructures constructed using the above Chvalina's construction.

[1]In [268] Račková assumes quasi-ordered sets after she misquotes Lemma 2.1.1. However, as will be shown later, the assumption of a *quasi*-ordered instead of a *partially* ordered semigroup is not incorrect.

[2]The incorrect translation "Ending lemma" might have been used for some time.

2.2 Examples

The construction of EL–hyperstructures seems to be both *general* and *simple* enough to be applied in a sufficiently wide range of contexts. In this section we provide several examples so that one can see that this context may be straightforward as well as sophisticated.

One of the earliest occurrances of EL–semihypergroups (of course, not using this name) can be found in 1967 when Pickett [259] gives Example 2.2.1. We give an exact quote of this example including its parts which are not relevant for our present considerations (the term *multigroup* is equivalent to an *n–ary hypergroup*, here $n = 2$, i.e. we have a hypergroup).

Example 2.2.1. Let $(X, \wedge, \vee, \leq)$ be a lattice and define $a \cdot b = \{x \mid x \geq a \wedge b\}$, $a \circ b = \{x \mid x \leq a \vee b\}$. Both $(X, \cdot)$ and $(X \circ)$ are commutative multigroups and every element is a unit. The only coset decomposition is determined by X, for if Y is determined a coset decomposition for $(X, \cdot)$, say, then if a and b are any two elements of X, $Y \cdot (a \wedge b)$ meets both a and b. Hence they are in the same coset.

Some other examples of EL–hyperstructures include:

Example 2.2.2. Consider the set $\mathbb{N}$ of all natural numbers (excluding 0). Obviously $(\mathbb{N}, \cdot, \leq)$, where "$\cdot$" is the usual multiplication and "$\leq$" is the natural ordering of natural numbers by size, is a partially ordered semigroup. Thus if we define $a * b = [a \cdot b)_{\leq} = \{x \in \mathbb{N} \mid a \cdot b \leq x\}$, for all $a, b \in \mathbb{N}$, then $(N, *)$ is a commutative semihypergroup.

Example 2.2.3. If we regard the divisibility relation "$|$" in Example 2.2.2, then $(N, *)$, where $a * b = [a \cdot b)_{|} = \{x \in \mathbb{N} \mid a \cdot b | x\}$, for all $a, b \in \mathbb{N}$, is a semihypergroup.

Example 2.2.4. Since $(\mathbb{N}, \gcd, \leq)$, where "gcd" stands for the greatest common divisor of natural numbers and "$\leq$" is the usual ordering of natural numbers by size, is a partially ordered group, we can define $a * b = \{x \in \mathbb{N} \mid \gcd\{a, b\} \leq x\}$ for all $a, b \in \mathbb{N}$, and get that $(\mathbb{N}, *)$ is a commutative semihypergroup.

Example 2.2.5. If we consider the set $\mathbb{R}$ of all real numbers and the usual addition and ordering of real numbers, then $(\mathbb{R}, +, \leq)$ is a partially ordered group. Thus if we define $a * b = [a + b)_{\leq} = \{x \in \mathbb{R} \mid a + b \leq x\}$, for all $a, b \in \mathbb{R}$, we get that $(\mathbb{R}, *)$ is a join space. Obviously, the same holds for $(\mathbb{Q}, +, \leq)$ and $(\mathbb{Q}, *)$ or $(\mathbb{Z}, +, \leq)$ and $(\mathbb{Z}, *)$.

Notice that in the following example the interval may represent probabilities while the single-valued operation "$\cdot$" may represent simultaneous probability of independent events. Also, "min" and "max" may represent the event, probability of which is smaller or greater, respectively.

Example 2.2.6. The set of all real numbers from the interval $\langle 0, 1\rangle$ together with the operation of multiplication and the usual ordering of real numbers by size is a partially ordered semigroup. Thus if we define $a * b = [a \cdot b)_\leq = \{x \in \langle 0, 1\rangle \mid a \cdot b \leq x\}$, for all $a, b \in \langle 0, 1\rangle$, we get that $(\langle 0, 1\rangle, *)$ is a commutative semihypergroup. The same holds for intervals $(0, 1\rangle$ or $\langle 0, 1)$ or $(0, 1)$.

Example 2.2.7. It is easy to verify that an arbitrary interval $\langle a, b\rangle$ of real numbers with operations "min" or "max" and the natural ordering of real numbers by size is a partially-ordered semigroup, i.e. can be taken as a basis for constructing EL–semihypergroups.

In their study of *braid groups*, Al Tahan and Davvaz [4] use the idea of the "Ends lemma" to construct a cyclic hypergroup of an arbitrary braid group B_n of n strands.[3] In this respect it is important to notice that they take an abstract structure and a particular property of their elements – in their case the shortest presentation of the product of elements $\sigma_i^{\eta_i} \in B_n$. This technique can be generalized: if one can describe properties of elements of a given set by means of natural / whole / rational / real, etc. numbers, one can construct examples such as the following – intentionally simplistic (!) – Example 2.2.8.

Example 2.2.8. Let S be a set of apple trees and $p(s)$ the average weight of apples collected from a given tree s. For arbitrary $r, s \in S$ define $r * s = \{t \in S \mid p(r) + p(s) \leq p(t)\}$. Then $(S, *)$ is a commutative hypergroup.

For a more mathematical example of the previous reasoning, one can consider e.g. an automaton with a given input alphabet I and denote by $l(a)$ the length of a word constructed from the letters of this alphabet. In this case $l(a) + l(b) = l(a\&b)$, where $a\&b$ is a word constructed by the opereation of *catenation* of two words a, b. Obviously, $(I^*, \&)$, where I^* is a set of words defined over I, is a monoid, where the neutral element is the empty word. For some results of this approach see [44], chapter 6 or Massouros and Mittas [218].

[3] Braid groups were introduced to algebra by Artin [12]; for further reading on the topic see e.g. Birman's book [27].

Example 2.2.9. Denote $I = \langle 0, 1\rangle$ and $C(I)$ the set of all real continuous functions on $C(I)$ and for two functions $f, g \in C(I)$ define that $f \leq g$ if for all $x \in I$ there is $f(x) \leq g(x)$. Since $(C(I), +, \leq)$ is a commutative partially ordered group, we get that $(C(I), *)$, where $f * g = [f + g)_{\leq} = \{h \in C(I) \mid f + g \leq h\}$, for all $f, g \in C(I)$, and "+" denotes the usual pointwise addition of functions, is a join space.

Phathawimol and Kemprasit [261] (based on Corsini [92]) as well as Antampoufis, Dramalidis and Vougiouklis [7, 9] use hyperoperations which are in fact based on the "Ends lemma" construction even though they, naturally, do not use the name or the lemma itself. For more details see footnote at page 40 and Remark 2.4.4.

In Subsection 2.5.5 on page 122 some natural associative operations and partial orderings on the set of square matrices will be used, which will result in the construction of semihypergroups and join spaces of square matrices.

Example 2.2.10. Regard an arbitrary set S and its power set $\mathcal{P}(S)$. The operations "$\cap$", "$\cup$" of set intersection and set union are associative, thus $(\mathcal{P}(S), \cap)$ and $(\mathcal{P}(S), \cup)$ are semigroups. The relation "$\subseteq$" on $\mathcal{P}(S)$ is obviously reflexive and transitive and for arbitrary $A, B, C \in \mathcal{P}(S)$ such that $A \subseteq B$ there is $A \cap C \subseteq B \cap C$ and $A \cup C \subseteq B \cup C$. Thus if we define hyperoperations "$\oplus$", "$\bullet$" for arbitrary $A, B \in \mathcal{P}(S)$ by

$$A \oplus B = [A \cup B)_{\subseteq} = \{X \in \mathcal{P}(S) \mid A \cup B \subseteq X\} \tag{2.3}$$

and

$$A \bullet B = [A \cap B)_{\subseteq} = \{Y \in \mathcal{P}(S) \mid A \cap B \subseteq Y\}, \tag{2.4}$$

we get commutative semihypergroups $(\mathcal{P}(S), \oplus)$ and $(\mathcal{P}(S), \bullet)$.

Example 2.2.11. If in Example 2.2.10 we regard $\mathcal{P}^*(S) = \mathcal{P}(S)\backslash\emptyset$, we get another pair of examples of commutative semihypergroups. In Subsection 2.4.7 we will see how the fact that $(\mathcal{P}^*(S), \cup)$ is not a monoid (while $(\mathcal{P}(S), \cup)$, $(\mathcal{P}(S), \cap)$ and $(\mathcal{P}^*(S), \cap)$ are) will affect our considerations in proving whether $(\mathcal{P}^*(S), \oplus)$ and $(\mathcal{P}^*(S), \bullet)$ are hypergroups.

Theoretical studies of EL–hyperstructures have primarily been motivated by the need to provide a theoretical basis for examples scattered through a number of different papers such as [35, 50, 55, 59, 69, 117, 149, 151, 155]. In most of these, the elements of the sets in question were n–tuples (mostly pairs or triples) and the single-value operation "$\cdot$" was performed by means of their components. In those papers, both the single-valued operation and the necessary ordering "$\leq$" were motivated by the specific needs of the mathematical model (see e.g. Remark 4.1.3 on page 179). Below we include two examples of this type; for a motivation concerning Example 2.2.12 see Section 4.1.

Example 2.2.12. (included in Chvalina and Chvalinová [50], used to demonstrate some results of Novák [244]) Consider the relation of hyperstructures and homogeneous second order linear differential equations

$$y'' + p(x)y' + q(x)y = 0, \tag{2.5}$$

such that $p \in C_+(I)$, $q \in C(I)$, where $C^k(I)$ denotes the commutative ring of all continuous real functions of one variable defined on an open interval I of reals with continuous derivatives up to order $k \geq 0$ (instead of $C^0(I)$ we write only $C(I)$), and $C_+(I)$ denotes its subsemiring of all positive continuous functions. The set of nonsingular ordinary differential equations (2.5) is denoted $\mathbb{A}_2$. The pair of functions p, q is denoted $[p, q]$, $D = \frac{\mathrm{d}}{\mathrm{d}x}$ and Id is the identity operator. The notation $L(p, q)$ is reserved for the differential operator $L(p, q) = D^2 + p(x)D + q(x)Id$, i.e. the notation $L(p, q)(y) = 0$ is the equation (2.5). The set

$$\mathbb{LA}_2(I) = \{L(p, q) : C^2(I) \to C(I); [p, q] \in C_+(I) \times C(I)\} \tag{2.6}$$

is the set of all such operators. Finally, for an arbitrary $r \in \mathbb{R}$ the notation $\chi_r : I \to \mathbb{R}$ stands for the constant function with value r.

Proposition 1 of [50] states that if we define multiplication of operators by

$$L(p_1, q_1) \cdot L(p_2, q_2) = L(p_1p_2, p_1q_2 + q_1) \tag{2.7}$$

and if we define that $L(p_1, q_1) \leq L(p_2, q_2)$ if

$$p_1(x) = p_2(x),\ q_1(x) \leq q_2(x) \text{ for any } x \in I, \tag{2.8}$$

then $(\mathbb{LA}_2(I), \cdot, \leq)$ is a noncommutative partially ordered group with the unit element (identity) $L(\chi_1, \chi_0)$. Using the "Ends lemma" we get that if we put

$$\begin{array}{c} L(p_1, q_1) * L(p_2, q_2) = \\ = \{L(p, q) \in \mathbb{LA}_2(I); L(p_1, q_1) \cdot L(p_2, q_2) \leq L(p, q)\} = \\ = \{L(p_1p_2, q); q \in C(I), p_1q_2 + q_1 \leq q\} \end{array}, \tag{2.9}$$

then $(\mathbb{LA}_2(I), *)$ is a transposition hypergroup (which is given in [50] as Theorem 3).

Example 2.2.13. Consider the function of the Gaussian-shaped pulse signal $v(t) = a\exp(-2\pi t^2)$, where $a \in \mathbb{R}^+$. When regarding the second order linear differential equation in the Jacobi form, i.e. $v''(t) + p(t)v(t) = 0$, where

p is a continuous function, and creating hyperstructures of the respective linear differential operators using the "Ends lemma" following the pattern of Example 2.2.12, we see that we get a one-parametric system, i.e. an analogy of the simple Examples 2.2.2 or 2.2.5, as the operators have the form $L(0, \varphi(a))$, where φ stands for a suitable function of $a \in \mathbb{R}^+$.

More examples of this type can be constructed using the following (rather trivial) lemmas included in Novák and Křehlík [250], which can be easily generalized for vectors consisting of n components. Of course, what we present below, is a tiny selection as there exist infinitely many ways of defining associative operations on an arbitrarily chosen set H. In this way, Lemma 2.2.15, Lemma 2.2.16 and Lemma 2.2.17 serve as space saving tools for the verification of Examples 2.2.18, 2.2.19 and 2.2.20 and as tools to generate further examples.

Definition 2.2.14. For all $a = (a_1, a_2), b = (b_1, b_2) \in H$, where H is a suitable set, define $\cdot_i : H \times H \to H$, $i \in \{1, 2, 3\}$ by

1. $a \cdot_1 b = (a_1 + a_2 + b_1 + b_2, a_1 \oplus a_2 \oplus b_1 \oplus b_2)$,
2. $a \cdot_2 b = (a_1 + b_1, a_2 \oplus b_2)$,
3. $a \cdot_3 b = (k_1(a_1) + k_2(b_1), l_1(a_2) \oplus l_2(b_2))$, where k_j, l_j, $j = 1, 2$ are suitable functions.

where "$+$","$\oplus$" are suitable operations applied on components of elements of H.

Lemma 2.2.15. *Let H be a set of elements of the form $a = (a_1, a_2)$ endowed with operations "$\cdot_i$" as defined in Definition 2.2.14.*

1. *$(H, \cdot_1)$ is a semigroup if operations "$+$", "$\oplus$" are identical and simultaneously they are associative, commutative and idempotent.*
2. *$(H, \cdot_2)$ is a semigroup if and only if both operations "$+$", "$\oplus$" are associative.*
3. *If $k_1 = k_2$ (denote these by k), $l_1 = l_2$ (denote these by l), then $(H, \cdot_3)$ is a semigroup if and only if*

$$k(k(a_j) + k(b_j)) + k(c_j) = k(a_j) + k(k(b_j) + k(c_j)) \qquad (2.10)$$

for $j = 1, 2$ and all $(a_1, a_2), (b_1, b_2)$, $(c_1, c_2) \in H$ (and likewise for l).

Proof. Suppose an arbitrary triplet of elements $a, b, c \in H$ such that $a = (a_1, a_2), b = (b_1, b_2), c = (c_1, c_2)$. We need to prove that $a \cdot_i (b \cdot_i c) = (a \cdot_i b) \cdot_i c$ for $i \in \{1, 2, 3\}$.

1. As far as the left-hand side is concerned, we get that

$$b \cdot_1 c = (b_1 + b_2 + c_1 + c_2, b_1 \oplus b_2 \oplus c_1 \oplus c_2)$$

and

$$a \cdot_1 (b \cdot_1 c) = (a_1 + a_2 + (b_1 + b_2 + c_1 + c_2) + (b_1 \oplus b_2 \oplus c_1 \oplus c_2), \\ a_1 \oplus a_2 \oplus (b_1 + b_2 + c_1 + c_2) \oplus (b_1 \oplus b_2 \oplus c_1 \oplus c_2)).$$

As far as the right-hand side is concerned,

$$a \cdot_1 b = (a_1 + a_2 + b_1 + b_2, a_1 \oplus a_2 \oplus b_1 \oplus b_2)$$

and

$$(a \cdot_1 b) \cdot_1 c = ((a_1 + a_2 + b_1 + b_2) + (a_1 \oplus a_2 \oplus b_1 \oplus b_2) + c_1 + c_2, \\ (a_1 + a_2 + b_1 + b_2) \oplus (a_1 \oplus a_2 \oplus b_1 \oplus b_2) \oplus c_1 \oplus c_2).$$

One can easily see that if the operations "$+$" and "$\oplus$" are identical and on top of that if they are associative, commutative and idempotent, both sides of the equality reduce to $(a_1 + a_2 + b_1 + b_2 + c_1 + c_2, a_1 + a_2 + b_1 + b_2 + c_1 + c_2)$.

2. We get that

$$(a \cdot_2 b) \cdot_2 c = ((a_1 + b_1) + c_1, (a_2 \oplus b_2) \oplus c_2)$$

while

$$a \cdot_2 (b \cdot_2 c) = (a_1 + (b_1 + c_1), a_2 \oplus (b_2 \oplus c_2)).$$

Obviously these are equal if and only if both operations "$+$" and "$\oplus$" are associative.

3. If we apply the same reasoning as above, we get that

$$a \cdot_3 b = (k_1(a_1) + k_2(b_1), l_1(a_2) \oplus l_2(b_2))$$

and

$$(a \cdot_3 b) \cdot_3 c = (k_1(k_1(a_1) + k_2(b_1)) + k_2(c_1), l_1(l_1(a_2) \oplus l_2(b_2)) \oplus l_2(c_2))$$

while

$$b \cdot_3 c = (k_1(b_1) + k_2(c_1), l_1(b_2) \oplus l_2(c_2))$$

and

$$a \cdot_3 (b \cdot_3 c) = (k_1(a_1) + k_2(k_1(b_1) + k_2(c_1)), l_1(a_2) \oplus l_2(l_1(b_2) \oplus l_2(c_2))).$$

And it is obvious that the equality $(a \cdot_3 b) \cdot_3 c = a \cdot_3 (b \cdot_3 c)$ does not hold in general. Yet if $k_1 = k_2$ and $l_1 = l_2$, then "$\cdot_3$" is associative if and only if the condition of the lemma holds.

$\square$

Notice that special cases of operation "$\cdot_3$" include e.g. k, l being constant mappings and "$+$" and "$\oplus$" associative or k, l being idempotent homomorphisms. Also, in the definition of the operation "$\cdot_3$", functions k_i, l_i may be substituted by real numbers which changes the definition of "$\cdot_3$" to $a \cdot_4 b = (k_1 a_1 + k_2 b_1, l_1 a_2 \oplus l_2 b_2)$, where "$+$" and "$\oplus$" are suitable operations performed on components of a, b. In the following lemma notice that H is an arbitrary set (not necessarily a vector space).

Lemma 2.2.16. *Let H be a set of elements of the form $a = (a_1, a_2)$ endowed with operations "$\cdot_4$" defined by $a \cdot_4 b = (k_1 a_1 + k_2 b_1, l_1 a_2 \oplus l_2 b_2)$, where $k_i, l_i \in \mathbb{R}$, $i = 1, 2$. Then $(H, \cdot_4)$ is a semigroup if and only if the multiplication by k_i, for $i = 1, 2$, is distributive over "$+$" (and likewise multiplication by l_i for $i = 1, 2$ distributive over "$\oplus$"), there is $k_i, l_i \in \{0, 1\}$ for $i = 1, 2$, and $k_i(k_j a_m) = (k_i k_j) a_m = k_i k_j a_m$ for $i, j, m \in \{1, 2\}$ (and likewise for l).*

Proof. In this new context the reasoning included in the proof of Lemma 2.2.15 changes (under the condition of distributivity) to

$$(a \cdot b) \cdot c = (k_1 k_1 a_1 + k_1 k_2 b_1 + k_2 c_1, l_1 l_1 a_2 + l_1 l_2 b_2 + l_2 c_2)$$

while

$$a \cdot (b \cdot c) = (k_1 a_1 + k_2 k_1 b_1 + k_2 k_2 c_1, l_1 a_2 + l_2 l_1 b_2 + l_2 l_2 c_2).$$

Since we must secure that multiplication of coefficients k_i, l_i, $i = 1, 2$ is idempotent, the choice of their values is obvious. $\square$

Moreover, by setting $k_1 = f, k_2 \equiv 0$, $l_1 \equiv 0, l_2 = g$, the operation "$\cdot_3$" changes its definition to $a \cdot_5 b = (f(a_1), g(b_2))$, where f, g are suitable functions.

Lemma 2.2.17. *Let H be a set of elements of the form $a = (a_1, a_2)$ endowed with operations "$\cdot_5$" defined by $a \cdot_5 b = (f(a_1), g(b_2))$. Then $(H, \cdot)$ is a semigroup if and only if f, g are idempotent.*

Proof. Follows immediately from the proof of Proposition 2.2.15, part 3, by setting $k_1 = f, k_2 \equiv 0$, $l_1 \equiv 0, l_2 = g$. □

The above lemmas enable us to generate a number of new examples of semigroups which can be taken (after a meaningful ordering has been devised) as a basis for constructing respective EL–semihypergroups. The following examples are a small selection.

Example 2.2.18. Let $H = \{(P, R) \mid P, R \subseteq \mathcal{P}(S)\}$, where S is a suitable set. For $(A, B), (C, D) \in H$ define

$$(A, B) \cdot_1 (C, D) = (A \cap B \cap C \cap D, A \cap B \cap C \cap D).$$

and

$$(A, B) \cdot_4 (C, D) = (A \cup C, D).$$

Then $(H, \cdot_1)$ and $(H, \cdot_4)$ are semigroups.

Example 2.2.19. Let $H = \{(r, s) \mid r, s \in \mathbb{R}\}$. For $(x_1, x_2), (y_1, y_2) \in H$ define

$$(x_1, x_2) \cdot_2 (y_1, y_2) = (x_1 + y_1, x_2 y_2),$$

and

$$(x_1, x_2) \cdot_3 (y_1, y_2) = (|x_1| + |y_1|, |x_2| + |y_2|),$$

where in both cases "+" is the usual addition of real numbers, and

$$(x_1, x_2) \cdot_5 (y_1, y_2) = (|x_1|, \operatorname{sgn}(y_2)).$$

Then $(H, \cdot_2), (H, \cdot_3), (H, \cdot_5)$ are semigroups.

In the following example we give an example of the relation "$\leq$" needed to construct an EL–semihypergroup. Notice the crucial difference between this and Example 2.2.12: the result of the multiplication is either "of the same type" (a fully meaningful operator as in Example 2.2.12) or "of different quality" (a vector of matrices which are however only formal matrices, because the potential of the operation "$\cdot_5$" lies in the fact that the matrices can be treated as numbers). For more ideas for (and obstacles when) defining EL–hyperstructures on sets of matrices cf. e.g. [187, 268].

Example 2.2.20. Let $H_M = \{(\mathbf{A}_1, \mathbf{A}_2) \mid \mathbf{A}_1, \mathbf{A}_2 \in \mathbb{M}_{n,n}(\mathbb{R})\}$, where $\mathbb{M}_{n,n}(\mathbb{R})$ is the set of all square matrices over $\mathbb{R}$ (regardless of size), i.e. e.g. $\mathbf{M}_{2,2} \in \mathbb{M}_{n,n}(\mathbb{R})$ as well as $\mathbf{M}_{3,3} \in \mathbb{M}_{n,n}(\mathbb{R})$. For $(\mathbf{M}_1, \mathbf{M}_2), (\mathbf{N}_1, \mathbf{N}_2) \in H_M$ define

$$(\mathbf{M}_1, \mathbf{M}_2) \cdot (\mathbf{N}_1, \mathbf{N}_2) = ((\det(\mathbf{M}_1)), (\mathrm{tr}(\mathbf{N}_2))).$$

Then $(H_M, \cdot)$ is a semigroup. Further, for $(\mathbf{M}_1, \mathbf{M}_2), (\mathbf{N}_1, \mathbf{N}_2) \in H_M$ define

$$(\mathbf{M}_1, \mathbf{M}_2) \leq (\mathbf{N}_1, \mathbf{N}_2) \Leftrightarrow \det(\mathbf{M}_1) = \det(\mathbf{N}_1) \text{ and } \mathrm{tr}(\mathbf{M}_2) \leq \mathrm{tr}(\mathbf{N}_2)$$

and define

$$(\mathbf{M}_1, \mathbf{M}_2) * (\mathbf{N}_1, \mathbf{N}_2) = [(\mathbf{M}_1, \mathbf{M}_2) \cdot (\mathbf{N}_1, \mathbf{N}_2))_{\leq}.$$

Then, by Lemma 2.1.1, we get that $(H_M, *)$ is a semihypergroup.

Notice that the properties of determinant or trace of matrices enable us to construct a variety of EL–semihypergroups. E.g. if we regard the *Kronecker product* of matrices "$\otimes$", then $\mathrm{tr}(\mathbf{X} \otimes \mathbf{Y}) = \mathrm{tr}(\mathbf{X}) \cdot \mathrm{tr}(\mathbf{Y})$, which means that $(\mathbb{M}_{n,n}(\mathbb{R}), \otimes, \leq)$, where we put $\mathbf{A} \leq \mathbf{B}$ whenever $\mathrm{tr}(\mathbf{A}) \leq \mathrm{tr}(\mathbf{B})$, is a noncommutative quasi-ordered semigroup.

Examples of operations on sets of the above type, which in spite of being "logical picks" are *not associative* include definitions of "$\cdot$" such that e.g.:

1. $a \cdot b = (f(a_1 + a_2), g(b_1 \oplus b_2))$, where f, g are functions; not even for $f \equiv g$, "$+$" equaling "$\oplus$",
2. $a \cdot b = (f(a_1 + b_1), g(a_2 \oplus b_2))$, i.e. functions applied on the result of component-wise operations; if f, g are not homomorphisms, then this is associative only in very special contexts,
3. "combining components" such as e.g. $a \cdot b = (b_2, a_1)$ or $a \cdot b = (f(a_1), g(b_1))$.

For more details concerning this topic the reader is advised to consult any standard introductory book on the semigroup theory such as e.g. Clifford and Preston [78].

Obviously, the relation "$\leq$" need not be based on analogies of "ordering by size". E.g. in Chvalina, Novák and Křehlík [62, 70] a relation between a special kind of operators is studied. The single-valued operation "$\cdot$" needed in the "Ends lemma" is defined in a similar manner as in Example 2.2.12 while the relation of operators is defined by $V_1 \leq V_2$ whenever there exists $m \in \mathbb{N}$ and an operator V_0 such that $V_2 = V_0^m \cdot V_1$, where $V_0^m = \underbrace{V_0 \cdot \ldots \cdot V_0}_{m}$.

When discussing examples of EL–hyperstructures, one must also mention that, on an intuitive level, one can suggest that the set H could be a set of individuals, the operation "$\cdot$" could be interpreted as mating and the ordering "$\leq$" could be interpreted as comparing indices of generations, i.e. $a \leq b$ means that a is a descendant (or, alternatively, an ancestor) of b. However, when formalizing this idea, a number of practical issues occurs. In Section 3.3, these obstacles will lead us to the study of *modified EL–hyperstructures* on a partitioned set H; also see introduction to Chapter 4 and references to papers on biological inheritance mentioned on page 176.

For more examples of partially ordered (semi)groups see any standard textbook on the topic such as e.g. Fuchs [133].

2.3 Proposed questions

There are several questions that naturally arise in connection with the "Ends lemma".

1. Can the number of assumptions (which in the original wording include *reflexivity*, *transitivity* and *antisymmetry* of the relation "$\leq$", *associativity* of the operation "$\cdot$" and the fact that the operation "$\cdot$" is *compatible* with the relation "$\leq$") be reduced?

2. What are some equivalent conditions of condition 1^0 of Lemma 2.1.2, i.e. on what conditions do semigroups, which are not groups, create hypergroups?

3. Can the "Ends lemma" be used to create more specialized hypergroups such as canonical hypergroups?

4. How can the "Ends lemma" be used to create hyperstructures with two (hyper)operations including hyperstructure generalizations of lattices?

5. What are the properties of EL–hyperstructures?

6. Can the construction be applied in the n–ary context, i.e. on n–ary hypergroupoids and / or n–ary relations?

7. What is the relation of EL–hyperstructures to other concepts which combine the ideas of ordered sets and hyperoperations?

In this and the following chapter we are going to include results, most of which had been obtained by the author in [187, 240–244, 246–250] when giving answers to the above questions. In Section 3.1 the results obtained by the author will be combined with results of Ghazavi and Anvariyeh [135] while Section 3.2 will communicate results obtained by Ghazavi, Anvariyeh and Mirvakili [136].

2.4 Construction from quasi-ordered semigroups

2.4.1 Partially ordered or quasi-ordered semigroups? Converting the lemma.

Most results included in this subsection were collected from Novák [241, 246].

First of all we will consider the issue of *reduction of assumptions* and *conversion* of the “Ends lemma”. For this we need to recall the original proofs included in Chvalina [44].

Proof. Proof of Lemma 2.1.1: Suppose $a, b, c \in S$ arbitrary. First of all, it is useful to show that the following equality holds:

$$\bigcup_{t \in [b \cdot c)_{\leq}} [a \cdot t)_{\leq} = \bigcup_{x \in [a \cdot b)_{\leq}} [x \cdot c)_{\leq}.$$

Suppose therefore an abitrary $s \in \bigcup_{t \in [b \cdot c)_{\leq}} [a \cdot t)_{\leq}$. This means that $s \geq a \cdot t_0$ for a suitable $t_0 \in S$, $t_0 \geq b \cdot c$. Then $a \cdot t_0 \geq a \cdot (b \cdot c) = (a \cdot b) \cdot c$ and if we set $x_0 = a \cdot b$, we get that $x_0 \cdot c \leq s$, $x_0 \in [a \cdot b)_{\leq}$, i.e. $s \in [x_0 \cdot c)_{\leq} \subseteq \bigcup_{x \in [a \cdot b)_{\leq}} [x \cdot c)_{\leq}$. The other inclusion may be proved in the analogous way. Now we get that

$$a * (b * c) = \bigcup_{t \in b * c} a * t = \bigcup_{t \in [b \cdot c)_{\leq}} [a \cdot t)_{\leq} = \bigcup_{x \in [a \cdot b)_{\leq}} [x \cdot c)_{\leq} = \bigcup_{x \in a * b} x * c = (a * b) * c,$$

which completes the proof of associativity. Obviously, if $(S, \cdot)$ is commutative, then also $(S, *)$ is commutative. On the other hand, if $(S, *)$ is commutative, then for an arbitrary pair of elements $a, b \in S$ we have that $a * b = b * a$, i.e. $[a \cdot b)_{\leq} = [b \cdot a)_{\leq}$, which means that $a \cdot b \leq b \cdot a$ and simultaneously $b \cdot a \leq a \cdot b$, i.e. – given the fact that “$\leq$” is a partial order – means that $a \cdot b = b \cdot a$. □

One can see that the only place in the above proof, where antisymmetry of the relation “$\leq$” is used, is one of the implications on commutativity. Therefore, the “Ends lemma” may be rewritten in the following way.

Corollary 2.4.1. *Let $(S, \cdot, \leq)$ be a quasi-ordered semigroup. Binary hyperoperation $* : S \times S \to \mathcal{P}^*(S)$ defined by $a * b = [a \cdot b)_{\leq}$ is associative. If the semigroup $(S, \cdot)$ is commutative, then the semihypergroup $(S, *)$ is commutative as well.*

In the following examples we construct EL–semihypergroups using an equivalence relation, i.e. a quasi-ordering which is moreover *symmetric*.[4]

Example 2.4.2. Let $(\mathbb{N}, \cdot, \equiv)$ be the multiplicative semigroup of natural numbers and "$\equiv$" the relation of congruence modulo m. Obviously, $(\mathbb{N}, \cdot, \equiv)$ is a quasi-ordered semigroup (which is not a group) and "$\equiv$" is not antisymmetric. If we, for a fixed $m \in \mathbb{N}$, define, for arbitrary $a, b \in \mathbb{N}$, that $a * b = \{x \in \mathbb{N} \mid a \cdot b \equiv x \pmod{m}\}$, then $(\mathbb{N}, *)$ is an EL–semihypergroup. In this way $a * b$ is a class of all $x \in \mathbb{N}$ equivalent to $a \cdot b$.

Example 2.4.3. On the set $\mathbb{C}$ of all complex numbers regard a binary operation "$\cdot_{|z|}$" defined as multiplication of absolute values, i.e. for all $z_1, z_2 \in \mathbb{C}$ define $z_1 \cdot_{|z|} z_2 = |z_1| \cdot |z_2|$, and a relation "$\leq_{|z|}$" defined as equality of absolute values, i.e. for all $z_1, z_2 \in \mathbb{C}$ put $z_1 \leq_{|z|} z_2$ whenever $|z_1| = |z_2|$. Obviously, $(\mathbb{C}, \cdot_{|z|}, \leq_{|z|})$ is a quasi-ordered semigroup (and "$\leq_{|z|}$" is not antisymmetric, yet it is symmetric). Thus if we define, for all $z_1, z_2 \in \mathbb{C}$, $z_1 * z_2 = \{x \in \mathbb{C} \mid |z_1| \cdot |z_2| = |x|\}$, we get that $(\mathbb{C}, *)$ is an EL–semihypergroup. In this way $z_1 * z_2$ is a circle in the Gaussian plane with center $z = 0$ and a diameter equal to $|z_1| \cdot |z_2|$.

Remark 2.4.4. Antampoufis, Dramalidis and Vougiouklis in papers such as [7, 9, 125, 127] investigate various kinds of geometric hyperoperations (for a nice visualization of some of them see [127]). Antampoufis [7] defines a hyperoperation on $\mathbb{C} \setminus \{0\}$ by $a * b = \{z \in \mathbb{C} \setminus \{0\} \mid |z| = |ab|\}$, for all $a, b \in \mathbb{C} \setminus \{0\}$ and proves that $(\mathbb{C} \setminus \{0\}, *)$ is a commutative hypergroup. In [9] Antampoufis, Vougiouklis and Dramalidis show that along with multiplication of complex numbers we can consider also their addition and show the relation of such a hyperoperation to what they call "constant arc hyperoperation" on the set of complex numbers. In [9] some urban applications are modelled using these hyperoperations.

Antisymmetry is essential in proving that commutativity of the hyperoperation implies commutativity of the single-valued operation. In the proof of Corollary 2.4.1 we used the fact that $[a)_\leq = [b)_\leq$ implies $a = b$. However, this is true only *on condition of antisymmetry*. Indeed, suppose a simple two element set $S = \{a, b\}$ on which the relation "$\leq$" is defined as $a \leq a, a \leq b, b \leq a, b \leq b$. This reflexive and transitive relation "$\leq$" is obviously not antisymmetric and there holds $[a)_\leq = [b)_\leq$ yet $a \neq b$.

[4] In this respect notice Phanthawimol and Kemprasit [261], where hyperoperation $x \circ_N y = xyN$ for all $x, y \in G$, where G is a group and N is its normal subgroup, i.e. a hyperoperation making use of equivalence classes of G, is used and studied. Notice that this hyperoperation was originally introduced by Corsini and is included in [92], p. 11. Also see Remark 2.4.4.

Example 2.4.5. Let $S = \{a, b, c\}$ and define operation "$\cdot$" on S by the following table:

$\cdot$	a	b	c
a	a	a	a
b	a	b	c
c	a	a	a

For an arbitrary pair $x, y \in S$ define that $x \leq y$, i.e. "$\leq$" $= S \times S$, and $x * y = [x \cdot y)_{\leq}$. Thus we get that $[a)_{\leq} = [b)_{\leq} = [c)_{\leq}$. Yet the fact that $b * c = [c)_{\leq} = [a)_{\leq} = c * b$ does not imply that $c = a$.

Example 2.4.6. Regard the EL–semihypergroup $(\mathbb{N}, *)$ constructed in Example 2.4.2 from $(\mathbb{N}, \cdot, \equiv)$, in which we set $m = 3$. In this case

$$[4)_{\equiv} = \{x \in \mathbb{N} \mid x \equiv 4 \pmod 3\ \} = [7)_{\equiv},$$

yet the fact that $[4)_{\equiv} = [7)_{\equiv}$ does not imply that $4 = 7$. Also, in Example 2.4.3, the fact that two complex numbers have the same absolute value does not mean that they are the same.

The original "Ends lemma" is a way to create semihypergroups. The following theorem, included in [246], is meant as a converse of the construction.

Theorem 2.4.7. *Let $(H, \cdot)$ be a non-trivial groupoid and "$\leq$" a partial ordering on H such that for an arbitrary triple of elements $a, b, c \in H$ such that $a \leq b$ there is $c \cdot a \leq c \cdot b$ and $a \cdot c \leq b \cdot c$. Define a hyperoperation $* : H \times H \to \mathcal{P}^*(H)$ for an arbitrary pair of elements $a, b \in H$ by $a * b = [a \cdot b)_{\leq} = \{x \in H \mid a \cdot b \leq x\}$. If the hyperoperation "$*$" is associative, then the single–valued operation "$\cdot$" is associative too. Moreover, if there exists an element $e \in H$ such that for all $a \in H$ there holds $a * e = e * a = [a)_{\leq}$, then $(H, \cdot)$ is a monoid with with the neutral element e.*

Proof. 1. If the hyperoperation "$*$" defined in the theorem is associative, then the fact that an arbitrary element $x \in (a * b) * c$ implies that $x \in a * (b * c)$. Conversely, the fact that an arbitrary element $y \in a * (b * c)$ implies that $y \in (a * b) * c$.

(a) If there holds $x \in (a * b) * c$, then there exists an element $x_1 \in a * b$ such that $x \in x_1 * c$, i.e. there exists an element $x_1 \in H$ such that $a \cdot b \leq x_1$ and $x_1 \cdot c \leq x$. Thanks to the assumed properties of the relation "$\leq$" we get that $(a \cdot b) \cdot c \leq x_1 \cdot c \leq x$, i.e. $(a \cdot b) \cdot c \leq x$, which means that $x \in [(a \cdot b) \cdot c)_{\leq}$.

(b) Furthermore, we know that $x \in a * (b * c)$, i.e. by analogous reasoning we get that $x \in [a \cdot (b \cdot c))_{\leq}$.

Since x is an arbitrary element of H and since the same reasoning holds for the arbitrary above mentioned $y \in H$, we get that $[(a \cdot b) \cdot c)_{\leq} = [a \cdot (b \cdot c))_{\leq}$. However, on condition of antisymmetry of the relation "$\leq$" this implies that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, which means that the operation "$\cdot$" is associative.

2. If there exists an element $e \in H$ such that for all $a \in H$ there holds $a * e = e * a = [a)_{\leq}$ then there for all $a \in H$ holds $[a \cdot e)_{\leq} = [e \cdot a)_{\leq} = [a)_{\leq}$, which on condition of antisymmetry of the relation "$\leq$" means that $a \cdot e = e \cdot a = a$, i.e. e is the identity of $(H, \cdot)$. Obviously, the element satisfying the condition of the theorem is unique.

□

The fact that, in the context of Theorem 2.4.7, the commutativity of the hyperoperation "$*$" implies the commutativity of the operation "$\cdot$" is included already in Lemma 2.1.1.

As follows from the original proof of Lemma 2.1.2, antisymmetry of "$\leq$" is not needed in it.

Corollary 2.4.8. *Lemma 2.1.2 holds for quasi-ordered semigroups $(S, \cdot, \leq)$ as well.*

Proof. It is sufficient to include the original proof of Lemma 2.1.2. Since it was not given on page 27, we include it now.

$1^0 \Rightarrow 2^0$**:** Suppose $t \in S$ arbitrary. Since $t * S \subseteq S$ and $S * t \subseteq S$ obviously holds, we will prove the converse inclusions. Suppose $s \in S$ arbitrary. We assume that for the pair $s, t \in S$ there exists a pair $c, c' \in S$ such that $t \cdot c \leq s$, $c' \cdot t \leq s$, i.e.

$$s \in [t \cdot c)_{\leq} \cap [c' \cdot t)_{\leq} = (t * c) \cap (c' * t) \subseteq \left(\bigcup_{x \in S} t * x \right) \cap \left(\bigcup_{x \in S} x * t \right) = \\ = (t * S) \cap (S * t),$$

which means that $S \subseteq t * S$ and $S \subseteq S * t$.

$2^0 \Rightarrow 1^0$**:** Suppose that $(S, *)$ is a hypergroup and $a, b \in S$ are arbitrary. Since there is $b * S = S * b = S$, there is

$$a \in b * S = \bigcup_{t \in S} b * t = \bigcup_{t \in S} [b \cdot t)_{\leq},$$

which means that $a \in [b \cdot c)_{\leq}$ for a suitable element $c \in S$, i.e. $b \cdot c \leq a$. In an analogous way, $a \in S * b$, i.e. $c' \cdot b \leq a$ for a suitable element $c' \in S$, which is 2^0.

$\square$

In [268], Račková includes Lemma 2.1.5 (quoted in [268] as Theorem 4). However, she mistakingly assumes that $(H, \cdot, \leq)$ is a partially ordered group instead of a quasi-ordered group. However, thanks to Corollary 2.4.8, this mistake is a technicality only. What she needs to prove is the join space implication (1.11) only.[5] And her proof is based on using the compatibility condition and multiplication by inverse elements only.

Proof. Proof of Lemma 2.1.5: Suppose a hypergroup $(H, *)$ constructed from a partially ordered group $(H, \cdot, \leq)$. Assume that for some $a, b, c, d \in H$ we have $b \setminus a \approx c/d$. This means that there exists an element $x \in H$ such that $a \in b * x$ and $c \in x * d$. This means that $b \cdot x \leq a$ and $x \cdot d \leq c$. In partially ordered groups this is equivalent to $a^{-1} \cdot b \leq x^{-1}$ and $d \cdot c^{-1} \leq c^{-1}$. When multiplied first by a and then by c, these turn into $b \cdot c \leq a \cdot x^{-1} \cdot c$ and $a \cdot d \leq a \cdot x^{-1} \cdot c$, which means that $a \cdot x^{-1} \cdot c \in b * c \cap a * d$, i.e. $a * d \approx b * c$. $\square$

Therefore, with the exception of one implication on commutativity, i.e. that commutativity of the hyperoperation implies commutativity of the single-valued operation, *the "Ends lemma" can be used for quasi-ordered semigroups as well.* Of course, we must be careful not to use concepts which are defined for partial ordering only – such as e.g. the concept of the *greatest*, *smallest*, *maximal* or *minimal* element. In this respect notice Definition 1.1.35 on page 19, which introduces *EL–maximal elements*, and its motivation.

Remark 2.4.9. It is easy to show that if we, instead of (2.1), regard the dual definition of a hyperoperation "$*_d$"

$$a *_d b = {}_{\leq}(a \cdot b] = \{x \in S \mid x \leq a \cdot b\} \tag{2.11}$$

for all $a, b \in S$, then the proof included on page 39 can be (without adding any further assumptions) dualized and Corollary 2.4.8 is valid for "$*_d$" as well. This and other hyperoperations similar to the "Ends lemma" construction will be used in Subsection 2.5.5 where lattices are discussed.

[5] Notice that while Definition 1.1.11 works in the commutative context, Račková discusses the non-commutative case. In her case, the assumption of (1.11) is therefore $b \setminus a \approx c/d$ instead of $a/b \approx c/d$.

Proof. We have

$$(a *_d b) *_d c = \bigcup_{t \in _{\leq}(a \cdot b]} t *_d c = \bigcup_{t \in _{\leq}(a \cdot b]} {}_{\leq}(t \cdot c]$$

while

$$a *_d (b *_d c) = \bigcup_{s \in _{\leq}(b \cdot c]} a *_d s = \bigcup_{s \in _{\leq}(b \cdot c]} {}_{\leq}(a \cdot s]$$

Now, the fact that $r \in (a *_d b) *_d c$ means that $r \in \bigcup_{t \in _{\leq}(a \cdot b]} {}_{\leq}(t \cdot c]$, i.e. that there is $r \leq t_0 \cdot c$ for some $t_0 \in {}_{\leq}(a \cdot b]$, i.e. for such t_0 that $t_0 \leq a \cdot b$. Yet, since $(S, \cdot, \leq)$ is a quasi-ordered semigroup, the fact that $t_0 \leq a \cdot b$ means that $t_0 \cdot c \leq (a \cdot b) \cdot c$ for all $c \in S$. From transitivity of "$\leq$" we have that $r \leq (a \cdot b) \cdot c = a \cdot (b \cdot c)$. Now, denote $s_0 = b \cdot c$. From reflexivity of "$\leq$" we have that $s_0 \in {}_{\leq}(b \cdot c] = \{x \in S \mid x \leq b \cdot c\}$. Since $r \leq a \cdot s_0$, there is $r \in {}_{\leq}(a \cdot s_0] = \{y \in S \mid y \leq a \cdot s_0\} \subseteq \bigcup_{s \in _{\leq}(b \cdot c]} (a \cdot s]$, i.e. $r \in a *_d (b *_d c)$. The other inclusion is proved in an analogous way. □

2.4.2 Polygroups and more specialized hyperstructures

Now we need to set limits to our considerations. Suppose that the sequence of hyperstructures from the most general to the more specialized ones is given as: *hypergroupoids – semihypergroups – hypergroups – quasi-canonical hypergroups (i.e. polygroups) – canonical hypergroups*, where the last concept is a hyperstructure equivalent of abelian group, i.e. a structure used to construct rings and fields. It is therefore natural to ask about conditions under which the "Ends lemma" enables us to construct canonical hypergroups. Obtaining the answer is rather a simple task.

Theorem 2.4.10. *Let $(S, \cdot, \leq)$ be a non-trivial quasi-ordered semigroup such that "$\leq$" is not the identity relation. Then no element of the EL–semihypergroup $(S, *)$ is a scalar identity.*

Proof. Suppose that in $(S, *)$ there exists a scalar identity e. This means that for all $a \in S$ there holds $[a \cdot e)_{\leq} = \{a\} = [e \cdot a)_{\leq}$. Since the relation "$\leq$" is a reflexive one, there is $x \in [x)_{\leq}$ for an arbitrary $x \in S$, i.e. we get that $a \cdot e = a = e \cdot a$ for all $a \in S$. However, this implies that e is the identity of $(S, \cdot)$. As a result $[a)_{\leq} = \{a\}$ for all $a \in S$, which means that "$\leq$" is the identity relation. □

Corollary 2.4.11. *Let $(S, \cdot, \leq)$ be a non-trivial quasi-ordered semigroup such that "$\leq$" is not the identity relation. Then – regardless of commutativity – the EL–semihypergroup $(S, *)$ cannot be a polygroup or a canonical hypergoup.*

Proof. Obvious because both quasi-canonical hypergroups (polygroups) and canonical hypergroups are defined by means of scalar elements. □

Thus we can see that the "Ends lemma" cannot be used to construct canonical hypergroups or hyperstructures built on them (such as e.g. *Krasner hyperrings*) or those which use the idea of scalar elements such as e.g. *hypernearrings* of Dašić [101].[6]

Massouros, Mittas and Jantosciak [170, 215, 216, 218] studied the concept of *fortification* in transposition hypergroups which is based on the existence of *strong identities*, i.e. minimal identities in extensive hypergroups.[7] As mentioned in [170], if a transposition hypergroup with a strong identity has the property that each nonidentity element has unique nonidentity right and left inverses, which are identical, it is said to be a *fortified transposition hypergroup*. When commutative, such hypergroups have application to the theory of languages and automata and have been known and studied as *fortified join hypergroups*.

In the proof of the following theorem we use the concept of the greatest and smallest elements. Therefore, we assume that "$\leq$" is partial ordering. However, given the nature of the proof and of strong identities, it is obvious that the effort to construct EL–semihypergroups with strong identities out of non-antisymmetric relations "$\leq$" is vain.

Theorem 2.4.12. *Let $(S, *)$ be the EL–semihypergroup of a partially ordered semigroup $(S, \cdot, \leq)$. The only cases when $(S, *)$ has strong identities are either $S = \{a\}$, or $S = \{a, b\}$, where $a \leq b$, or such S that the relation "$\leq$" on $S \setminus \{g\}$, where g is the greatest element of $(S, \leq)$ and simultaneously the neutral element of $(S, \cdot)$, is the identity relation.*

[6] The notion of *hypernearring* was introduced by Dašić in [101]. Since Dašić was new to hyperstructure theory at the time of publication of the paper, he stated that the hypernearring is based on a "hypergroup" while the axioms of his Definition 1 indicate that it is a "quasi-canonical hypergroup" instead. When including the definition in [111], Davvaz and Leoreanu corrected this misprint, see p. 126. A similar slip happened years later e.g. in papers dealing with matroids over hyperfields such as Baker and Bowler [16] where "hypergroup" is used in the sense of "canonical hypergroup" without explicitly mentioning this.

[7] On page 3 we made a remark on the rather unusual *additive notation* used in some older works on hyperstructure theory. Notice that notation might be somewhat misleading even in newer papers. When defining strong identities, Jantosciak and Massouros [170] write "$x \approx ex = xe \subseteq x \cup e$". Compare this to Definition 1.1.8 on page 5.

Proof. According to the definition, an element $e \in S$ is a strong identity of S if $x * e = e * x \subseteq \{x, e\}$ for all $x \in S$. In our case this means that $[x \cdot e)_{\leq} = [e \cdot x)_{\leq} \subseteq \{x, e\}$ for all $x \in S$. There are two possible cases:

1. *$x \cdot e = e \cdot x$ for all $x \in S$:* In this case there obviously is $[x \cdot e)_{\leq} = [e \cdot x)_{\leq}$ and should these sets be a subset of $\{x, e\}$, there must, thanks to reflexivity of "$\leq$", be either $x \cdot e = e \cdot x = x$ or $x \cdot e = e \cdot x = e$. Now, the fact that $x \cdot e = e \cdot x = x$ for all $x \in S$ means that e is the neutral element of S and therefore the condition rewrites to $[x \cdot e)_{\leq} = [e \cdot x)_{\leq} = [x)_{\leq} \subseteq \{e, x\}$ for all $x \in S$. Again, thanks to reflexivity of "$\leq$" we have that $x \in [x)_{\leq}$. Therefore, there must be $x \leq e$ for all $x \in S$, i.e. e is the greatest element of S, while all $y, z \in S$ such that $y \neq e, z \neq e$ are incomparable, i.e. "$\leq$" is an identity relation on $S \setminus \{e\}$. On the other hand, the fact that $x \cdot e = e \cdot x = e$ for all $x \in S$ means that e is an absorbing element of S. Thus the condition rewrites to $[x \cdot e)_{\leq} = [e \cdot x)_{\leq} = [e)_{\leq} \subseteq \{e, x\}$ and since $e \in [e)_{\leq}$, there must be $[e)_{\leq} = \{e, x\}$ for all $x \in S$. Yet this means that S can be a two element set only as $e \leq x_1$ and $e \leq x_2$ means that $[e)_{\leq} = \{e, x_1, x_2\}$ for $x_1 \neq x_2$.

2. *The above condition does not hold:* In this case we have that for a given e and all $x \in S$ there is either $[x \cdot e)_{\leq} = [e \cdot x)_{\leq} = \{x\}$ or $[x \cdot e)_{\leq} = [e \cdot x)_{\leq} = \{e\}$ or $[x \cdot e)_{\leq} = [e \cdot x)_{\leq} = \{e, x\}$. Since the relation "$\leq$" is reflexive, the last case is the only possible one, or $e = x$ for all $x \in S$, which is again a trivial case. The fact that $[x \cdot e)_{\leq} = [e \cdot x)_{\leq}$ means that $x \cdot e \leq e \cdot x$ and at the same time $e \cdot x \leq x \cdot e$. Moreover, thanks to reflexivity of "$\leq$", there must be either (a) $x \cdot e = x$ and $e \cdot x = e$ or (b) $x \cdot e = e$ and $e \cdot x = x$, and that for all $x \in S$. Yet both of these cases result in simultaneous validity of $x \leq e$ and $e \leq x$ for all $x \in S$, which means that e is simultanouesly the greatest and the smallest element of $(S, \leq)$, i.e. $S = \{e\}$.

□

Notice, that on the intuitive level, the above results are no surprise as the "Ends lemma" relies on the concept of *principal ends*, i.e. "cones of elements greater than a given element". If these cones are to be restricted to one- or two-element sets for a multiplication by *all* elements of the given set, than this can naturally hold in very special contexts only.

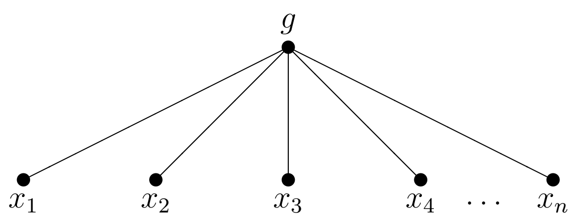

Figure 2.1: To Theorem 2.4.12: "$\leq$" on $S \setminus \{g\}$ is an identity relation

Even though we give up the study of canonical or fortified join hypergroups, it is to be pointed out that these are not the only important special types of hypergroups. In Section 2.4.5 we e.g. mention *cyclic hypergroups* introduced as early as 1937 by Wall [307].

2.4.3 Identities, inverses, zero scalars and idempotent elements

Most results included in this subsection were published by European Journal of Combinatorics (WoS Q2) as Novák [242].

We have seen that describing scalar or strong identities is pointless in EL–hyperstructures. However, this is not the case of "ordinary" identities or other special elements.

Theorem 2.4.13. *Let $(S, *)$ be the EL–semihypergroup of a quasi-ordered monoid $(S, \cdot, \leq)$ with the neutral element u. An element $e \in S$ is an identity of $(S, *)$ if and only if $e \leq u$.*

Proof. "$\Rightarrow$": If $e \in S$ is an identity of an EL–semihypergroup $(S, *)$, then there holds $e \cdot a \leq a$ and $a \cdot e \leq a$ for all $a \in H$. Specifically, this holds for $a = u$. In this case we get $e \leq u$.
"$\Leftarrow$": Suppose that $e \leq u$. Since $(S, \cdot)$ is a quasi-ordered semigroup, this is equivalent to $e \cdot a \leq a$ for any $a \in S$, which means that for any $a \in S$ we have that $a \in [e \cdot a)_{\leq} = e * a$. In an analogous way we get that $a \in a * e$, i.e. e is an identity of $(S, *)$. $\square$

Corollary 2.4.14. *Let $(S, *)$ be the EL–semihypergroup of a quasi-ordered monoid $(S, \cdot, \leq)$. The neutral element of $(S, \cdot)$ is an identity of $(S, *)$.*

Proof. Obvious. $\square$

Lemma 2.4.15. *Let $(H,*)$ be the EL–join space of a quasi-ordered commutative group $(H,\cdot,\leq)$. If an element $e \in H$ is an identity of $(H,*)$, then $e \leq e^{-1}$.*

Proof. If $e \in H$ is the neutral element of $(H,\cdot)$, the implication is obviously true. Therefore study the case that e is not the neutral element of $(H,\cdot)$. If e is an identity of $(H,*)$, then for all $a \in H$ there holds that $a \in a*e = [a\cdot e)_{\leq}$, i.e. $a \cdot e \leq a$. In a similar way there holds $e \cdot a \leq a$, i.e. $a \leq e^{-1} \cdot a$. This implies that for all $a \in H$ there is $a \cdot e \leq a \leq e^{-1} \cdot a$, which in a commutative quasi-ordered group means that $e \leq e^{-1}$. □

Recall that we denote $i(a)$ the set of inverses of an element $a \in H$. Further on we will use the notation a' for *an* inverse of a in a hyperstructure while a^{-1} will be the notation reserved for *the* inverse of a – either in a single-valued structure or in a hyperstructure (we will see that such cases are very rare).

Lemma 2.4.16. *Let $(H,*)$ be the EL–transposition hypergroup of a quasi-ordered group $(H,\cdot,\leq)$. For every $a \in H$, its inverse a^{-1} in $(H,\cdot)$ is its inverse in $(H,*)$.*

Proof. Denote u the neutral element of $(H,\cdot)$. Since a and a^{-1} are inverse elements in $(H,\cdot)$, there is $a\cdot a^{-1} = a^{-1}\cdot a = u$, i.e. $a * a^{-1} = a^{-1} * a = [u)_{\leq}$. Since $u \in [u)_{\leq}$ and u is an identity of $(H,*)$, we get that a^{-1} is the inverse of a in $(H,*)$. □

Theorem 2.4.17. *Let $(H,*)$ be the EL–transposition hypergroup of a quasi-ordered group $(H,\cdot,\leq)$. Then for an arbitrary $a \in H$ there is*

$$i(a) = \{a' \in H \mid a' \leq a^{-1}\} = {}_{\leq}(a^{-1}],$$

where a, a^{-1} are inverses in $(H,\cdot)$.

Proof. Inverse elements to $a \in H$ in $(H,*)$ are defined as such elements a' for which there exists an identity e in $(H,*)$ such that $e \in a * a'$ and simultaneously $e \in a' * a$, i.e. $a \cdot a' \leq e$ and $a' \cdot a \leq e$ for EL–hyperstructures. In order to prove the theorem, we have to prove the following implications:

1. If $a' \leq a^{-1}$, then a' is an inverse of a in $(H,*)$.
   Suppose that $a' \leq a^{-1}$. This means that $a' \cdot a \leq a^{-1} \cdot a = u$, where u is the neutral element of $(H,\cdot)$. It does not matter whether we multiply from the left or from the right. Since u is an identity of $(H,*)$, a' is an inverse of a in $(H,*)$.

2. If $a' \in H$ is an inverse of a in $(H, *)$, then $a' \leq a^{-1}$.
   Since $a, a' \in H$ are inverses in $(H, *)$, there exists an identity $e \in H$ such that $e \in a * a' \cap a' * a$. This means that there simultaneously holds $a \cdot a' \leq e$ and $a' \cdot a \leq e$. Denote u the neutral element of $(H, \cdot)$. Since from Theorem 2.4.13 there follows that $e \leq u$, and since "$\leq$" is transitive, we altogether get that $a \cdot a' \leq u$ and $a \cdot a' \leq u$, which implies $a' \leq a^{-1}$.

□

Remark 2.4.18. Theorem 2.4.13 and Theorem 2.4.17 suggest that in a general case both the set of identities and $i(a)$ are more than one-element sets, which violates defining axioms of some hyperstructure concepts including a *canonical hypergroup*. In this respect cf. Jantosciak [169], Proposition 6, the application of which (in case that *card* $i(a) > 1$ for any $a \in H$, where H is an *arbitrary EL*–transposition hypergroup) suggests another line of proof of Theorem 2.4.10. Furthermore, if we regard definition of a *fortified join space* the above theorems confirm the result given by Theorem 2.4.12 that fortification is possible in very special contexts only.

Remark 2.4.19. If the quasi-ordering "$\leq$" is an equivalence, we get that $i(a)$ is the set of such $a' \in H$ that are equivalent to a^{-1}. If we now define an operation "$\odot$" on the set of equivalence classes $H/_{\leq}$, i.e. define $A \odot B = C$, where C is the set of such elements of H that are equivalent to $a \cdot b$, where $a \in A$ and $b \in B$, we get that $(H/_{\leq}, \odot)$ is a group. Recall now page 7 and notice that one of the motivations for introducing H_v*–structures* was to obtain structures with desired properties in cases when "all but some problematic" elements have the desired property. See Corsini and Vougiouklis [96,300] and (for further reference) Vougiouklis [296].

Studying scalar elements is of no use in EL–hyperstruture. However, this is not exactly the case of *zero scalars* (often called *absorbing elements*).

Theorem 2.4.20. *Let $(S, *)$ be the EL–semihypergroup of a non-trivial quasi-ordered semigroup $(S, \cdot, \leq)$. Then $(S, *)$ has zero scalars if and only if $(S, \cdot, \leq)$ has an element which is simultaneously EL–maximal with respect to "$\leq$" and absorbing with respect to "$\cdot$".*

Proof. In the "Ends lemma" context, the defining condition of a zero scalar element $e \in S$ modifies to $[x \cdot e)_{\leq} = \{e\} = [e \cdot x)_{\leq}$ for all $x \in S$. If we realize that "$\leq$" must be reflexive, the proof is obvious. □

Corollary 2.4.21. *Let $(S, *)$ be the EL–semihypergroup of a non-trivial quasi-ordered semigroup $(S, \cdot, \leq)$. Then $(S, *)$ has at most one zero scalar* *element. To be more precise, if $(S, \cdot)$ is a monoid, then it is its neutral element that can be the only zero scalar of $(S, *)$. If $(S, \cdot)$ is not a monoid, then $(S, *)$ is without zero scalars.*

Proof. An obvious rewording of Theorem 2.4.20. □

Later on, in Subsection 2.4.7, we will work with semihypergroups without zero scalars and based on results obtained by Jafarabadi et al. [162, 163] we will prove some results concerning hyperideals of $(S, *)$.

The first of our results regarding hyperstructure *idempotent elements* is rather straightforward. We see that if the carrier structure of an EL–hypergroup is a (quasi-ordered) group, then the notions of hyperstructure idempotent elements and identities coincide.

Theorem 2.4.22. *Let $(H, *)$ be the EL–hypergroup of a quasi–ordered group $(H, \cdot, \leq)$. An element $a \in H$ is idempotent in $(H, *)$ if and only if it is an identity of $(H, *)$.*

Proof. Idempotent elements are defined as such $a \in H$ that $a \in a * a$, i.e. in the "Ends lemma" context, $a \cdot a \leq a$. In a group, this means that $a \leq u$, where u is the neutral element of $(H, \cdot)$. According to Theorem 2.4.13 this is equivalent to the fact that a is an identity of $(H, *)$. □

In the following theorem denote by a^n the hyperproduct of n elements a, i.e. $a^n = \underbrace{a * \ldots * a}_{n}$.

Theorem 2.4.23. *Let $(S, *)$ be the EL–semihypergroup of a quasi-ordered semigroup $(S, \cdot, \leq)$. Then for an arbitrary idempotent element a in $(S, \cdot)$ we have:*

(i) *a is an idempotent of $(S, *)$,*

(ii) *$a * a$ is a subsemihypergroup of $(S, *)$,*

(iii) *$[a)_{\leq} = a^2 = a^3 = \ldots = a^n$ for all $n \in \mathbb{N}, n \geq 2$.*

Proof. In the proof a will denote an arbitrary idempotent element of $(S, \cdot)$.

(i) Trivial, because since $a = a \cdot a$, there is also $[a)_{\leq} = [a \cdot a)_{\leq}$ and since the relation "$\leq$" is reflexive, there is $a \in [a)_{\leq} = [a \cdot a)_{\leq}$, i.e. $a \in a * a$.

(ii) This follows from Chattopadhyay [42], Corollary 2.18, which states that for an arbitrary idempotent element of $(S,*)$, $a*a$ is a subsemihypergroup of $(S,*)$ if and only if $a*a*a = a*a$, or from Lemma 2.1.4. Since a is a single-valued idempotent, there is

$$a*(a*a) = [a\cdot a\cdot a)_{\leq} = [a\cdot a)_{\leq} = a*a,$$

or without the need of reference to [42]

$$(a*a)*(a*a) = a*(a*(a*a)) = a*(a*a) = a*a$$

which again means that $a*a$ is a subsemihypergroup of $(S,*)$.

(iii) In the proof of *(ii)* we have seen that $a^3 = a^2$ and $a^4 = a^2$. Thus $a^4 = a^3$. However, this can be also shown in the following way:

$$a*(a*(a*a)) = [a\cdot a\cdot (a\cdot a))_{\leq} = [a\cdot (a\cdot a))_{\leq} = a*(a*a),$$

which can be easily expanded by "adding" further a. Also as above in *(i)* there is $a^2 = [a)_{\leq}$.

$\square$

Remark 2.4.24. Notice that *(ii)* follows also from the fact that for a single-valued idempotent $a\in S$ there is $a*a = [a)_{\leq}$. Thus the condition $(a*a)*(a*a)\subseteq a*a$ simplifies to $[a)_{\leq}*[a)_{\leq}\subseteq [a)_{\leq}$. Yet $[a)_{\leq}*[a)_{\leq} = \bigcup\limits_{x,y\in[a)_{\leq}} [x\cdot y)_{\leq}$, the relation "$\leq$" is reflexive and $(S,\cdot,\leq)$ is a quasi-ordered semigroup. Thus for an arbitrary $x,y\in[a)_{\leq}$, i.e. $x,y\in S$ such that $a\leq x$, $a\leq y$, there must be $a\cdot a\leq x\cdot y$, and since a is idempotent, there is $a\leq x\cdot y$. Thus not only $x\cdot y\in[a)_{\leq}*[a)_{\leq}$ but also $x\cdot y\in[a)_{\leq}$.

Unfortunately, in the general case of a quasi-ordering "$\leq$", from the validity of *(i)*, *(ii)* or *(iii)* of Theorem 2.4.23 there does not follow that a is an idempotent element of $(S,\cdot)$ which would help to give a complete answer of the question of whether an EL–hypergroup was constructed from a proper semigroup or from a group. This issue will be discussed later in Subsection 2.4.8. Obviously, if "$\leq$" is a partial ordering, then we get the following corollary.

Corollary 2.4.25. *Let $(S,*)$ be the EL–semihypergroup of a partially ordered semigroup $(S,\cdot,\leq)$. If, for some $a\in S$, there is $[a)_{\leq} = a^2 = a^3 = \ldots = a^n$ for all $n\in\mathbb{N}, n\geq 2$, then a is an idempotent element of $(S,\cdot)$.*

Example 2.4.26. Suppose the EL–semihypergroup $(\langle 0,1\rangle, *)$ constructed from the quasi-ordered semigroup $(\langle 0,1\rangle, \cdot, \leq)$, where "$\cdot$" and "$\leq$" are the usual multiplication and ordering of real numbers. In this case, by Theorem 2.4.13, every element is an identity. Further, $u = 1$ is a zero scalar, 0 and 1 are idempotent elements and $0 * 0$ and $1 * 1$ are (trivial) subsemihypergroups of $(\langle 0,1\rangle, *)$.

Example 2.4.27. Suppose the EL–semihypergroup $(\mathbb{Z}, *)$ constructed from the quasi-ordered semigroup $(\mathbb{Z}, +, \leq)$, where "$+$" and "$\leq$" are the usual addition and ordering of integers. Here, by Theorem 2.4.13, every negative integer is an identity, for every negative number $e \in \mathbb{Z}$ there obviously is $e \leq -e$ and for every $a \in \mathbb{Z}$ there is $i(a) = {}_{\leq}(-a]$. Also, $(\mathbb{Z}, *)$ has no zero scalars, and by Theorem 2.4.22, the idempotents of $(\mathbb{Z}, *)$ are the negative integers.

Example 2.4.28. Suppose the EL–semihypergroup $(\mathbb{N}, *)$ constructed from the quasi-ordered semigroup $(\mathbb{N}, \gcd, |)$, where "gcd" stands for the greatest common divisor and "$|$" is the usual divisibility relation. In this case, every element of $(\mathbb{N}, *)$ is idempotent. By Theorem 2.4.23, every set

$$a * a = \{x \in \mathbb{N} \mid \gcd\{a,a\}|x\} = \{x \in \mathbb{N} \mid a|x\}$$

is a subsemihypergroup of $(\mathbb{N}, *)$. Indeed, if e.g. $a = 3$, then obviously e.g. $12 \in 3 * 3$, $18 \in 3 * 3$. Now,

$$12 * 18 = \{x \in \mathbb{N} \mid \gcd\{12,18\}|x\} = \{x \in \mathbb{N} \mid 6|x\} \subseteq 3 * 3 = \{x \in \mathbb{N} \mid 3|x\}.$$

2.4.4 The notion of a subhyperstructure

Results of this subsection were published by APLIMAT – Journal of applied mathematics as Novák [246] and used in Novák [244], published by European Journal of Combinatorics (WoS Q2) .

Now we are going to discuss the issue of *subhyperstructures* of EL–hyperstructures. Since EL–hyperstructures rely on the idea of "cones of elements", we must, first of all, clarify the concept of a principal end generated by an element, which lies in the subset in question. Obviously, two approaches are possible. For an arbitrary element $g \in G$, where $G \subseteq S$, we may write

$$[a)_{\leq_G} = \{x \in G \mid a \leq x\}$$

as well as

$$[a)_{\leq_S} = \{x \in S \mid a \leq x\}.$$

Given this notation we may distinguish between $(G, *_G)$ based on the hyperoperation "$*_G$" such that for an arbitrary pair of elements $a, b \in G$ we set

$$a *_G b = [a \cdot b)_{\leq_G} = \{x \in G \mid a \cdot b \leq x\} \tag{2.12}$$

and $(G, *_S)$, where $a *_S b$ is defined by

$$a *_S b = [a \cdot b)_{\leq_S} = \{x \in S \mid a \cdot b \leq x\}. \tag{2.13}$$

Example 2.4.29. Suppose the linearly ordered semigroup $(\mathbb{N}, +, \leq)$ and its linearly ordered subsemigroup $(\mathbb{S}, +, \leq)$ of all even numbers. In this context we have

$$2 *_G 4 = \{x \in \mathbb{S} \mid 6 \leq x\} = \{6, 8, 10, \ldots\} \tag{2.14}$$

while

$$2 *_S 4 = \{x \in \mathbb{N} \mid 6 \leq x\} = \{6, 7, 8, \ldots\}. \tag{2.15}$$

Obviously, neither of these is "the only correct approach, while the other one is incorrect", as it is a matter of pure personal and applicational preference which of them we adopt as the one deserving our attention. Since the notation "$*_S$" reflects the idea of "cones of elements" better, we are going to start the study of implications of (2.13). Further on, instead of "$*_S$" and "$\leq_S$" the usual notation "$*$" and "$\leq$" is going to be used.

It will be useful to utilize the concept of an *upper set* known from the order theory. However, in the following definition we prefer using the term *upper end of a set* in order to visually relate the concept to the idea of the "Ends lemma". Also, we choose not to use the term *upper set* because it is used in the context of partially ordered sets while we will work in a more general context of quasi-ordered sets. Furthermore, identifying the elements which "spoil" the property of being an upper end of a set will be useful.[8]

Definition 2.4.30. Let $(S, \cdot, \leq)$ be a quasi-ordered semigroup and let G be a nonempty subset of S. If for an arbitrary element $g \in G$ there holds $[g)_\leq \subseteq G$, we call G an *upper end* of S. If there exists an element $g \in G$ such that there exists an element $x \in S \setminus G$ such that $g \leq x$ (i.e. $x \in [g)_\leq$), we say that G *is not an upper end of* S *because of the element* x.

Using Definition 2.4.30 we can clarify the issue of subhypergroupoids.

Lemma 2.4.31. *Let $(S, *)$ be the EL–semihypergroup of a quasi-ordered semigroup $(S, \cdot, \leq)$ and $G \subseteq S$ nonempty. If $(S, \cdot)$ is a monoid, denote its neutral element by u. Further suppose that $(G, \cdot)$ is a subgroupoid of $(S, \cdot)$.*

[8]In the below definition, since $x \notin G$ we could also write "$g < x$ and $x \in [g)_\leq \setminus \{g\}$".

1. *If G is an upper end of S, then $(G, *)$ is a subhypergroupoid of $(S, *)$.*

2. *If G is not an upper end of S and there holds $u \in G$, then $(G, *)$ is not a subhypergroupoid of $(S, *)$.*

3. *The statement in part 2 holds even in case that $u \notin G$ (or u does not exist) yet for some $a, b \in G$ there holds that $a \cdot b = c$, where $c \in G$ is such that there exists an element x_i because of which G is not an upper end of S such that $c \leq x_i$.*

4. *On simultaneous validity of conditions that*

   (a) *u does not exist or $u \notin G$*

   (b) *G is not an upper end of H because of elements $x_i, i \in I$*

   (c) *for every $a, b, c \in G$ there holds $a \cdot b = c$ and all the triples are such that for no x_i there holds $c \leq x_i$*

   the couple $(G, *)$ is a subhypergroupoid of $(S, *)$.

Proof. 1. Since "$\cdot$" is an operation on G, for an arbitrary pair $a, b \in G$ there holds $a \cdot b = c$, where $c \in G$. Thus $a * b = [a \cdot b)_{\leq} = [c)_{\leq}$, which is a subset of G because G is an upper end of S. Therefore we have that $G * G \subseteq G$, which means that $(G, *)$ is a subhypergroupoid of $(S, *)$.

2. If G is not an upper end of S, then there exists an element $g \in G$ such that there exists an element $x \in S \setminus G$ such that $g \leq x$. If furthermore $u \in G$, then if we consider the above mentioned element g, then $g * u = [g \cdot u)_{\leq} = [g)_{\leq} \nsubseteq G$ (because of the element x, the existence of which is assumed), which means that $G * G \nsubseteq G$, i.e. $(G, *)$ is not a subhypergroupoid of $(S, *)$.

3. Obvious since $a * b = [a \cdot b)_{\leq} = [c)_{\leq}$, for which there by definition holds $[c)_{\leq} \nsubseteq G$, i.e. $(G, *)$ is not a subhypergroupoid of $(S, *)$. Elements a, b, c have the meaning defined in part 3.

4. In this case for $\forall a, b \in G$ we have that $a * b = [a \cdot b)_{\leq} = [c)_{\leq}$, where $c \in G$ is such that $[c)_{\leq} \subset G$, i.e. we have that $G * G \subset G$, which means that $(G, *)$ is a subhypergroupoid of $(S, *)$.

□

Remark 2.4.32. In fact, parts 2 and 3 of Lemma 2.4.31 may be written as one. Yet they are included separately because of uniqueness of the neutral element u. Instead of $c \leq x_i$ we could write $c < x_i$ because we suppose $c \in G$

while $x_i \notin G$, which means that c and x_i cannot be equal. Finally, notice that if "$\cdot$" is not an operation on G, then $(G, *)$ is not a subhypergroupoid of $(S, *)$. Indeed, in this case there exists a triple a, b, c, where $a, b \in G$ while $c \notin G$, such that $a \cdot b = c$. This means that $a * b = [a \cdot b)_{\le} = [c)_{\le}$. However since the relation "$\le$" is reflexive and $c \notin G$, we get that $[c)_{\le} \not\subseteq G$, i.e. $G * G \not\subseteq G$.

Example 2.4.33. The set $\mathbb{N} \setminus \{1, 2, 3, 4, 5, 7, 9\} \subset \mathbb{N}$ with the operation "$+$" and the usual ordering of numbers is an example of a set constructed under Lemma 2.4.31, part 4. Indeed, $\mathbb{N}\setminus\{1, 2, 3, 4, 5, 7, 9\} = \{6, 8, 10, 11, 12, 13, 14 \ldots\}$ is not an upper end of $\mathbb{N}$ because of elements 7 and 9 (since e.g. $7 \in [6)_{\le}$ but $7 \notin \mathbb{N} \setminus \{1, 2, 3, 4, 5, 7, 9\}$). Yet for no couple $a, b \in \{6, 8, 10, 11, 12, 13, 14 \ldots\}$ there holds $a + b \le 7$ or $a + b \le 9$.

Lemma 2.4.31 gives a complete description of an arbitrary subset of an arbitrary EL-semihypergroup. Since subsemihypergroups, subhypergroups and other concepts are defined as special classes of subhypergroupoids, the lemma gives a complete list of candidates for various types of subhyperstructures of EL-semihypergroups. We start with examining the case of subsemihypergroups.

Theorem 2.4.34. *Let $(H, *)$ be the EL–semihypergroup of a partially ordered semigroup $(S, \cdot, \le)$. Suppose that G is either an upper end of S or such a subset of S that assumptions of Lemma 2.4.31, part 4 are fulfilled. Then*

1. *$(G, \cdot)$ is a subsemigroup of $(S, \cdot)$ if and only if $(G, *)$ is a subsemihypergroup of $(S, *)$.*

If furthermore $(S, \cdot)$ is a monoid, then

2. *$(G, \cdot)$ is a submonoid of $(S, \cdot)$ if and only if there exists an element $u \in G$ such that for all $g \in G$ there holds $g * u = u * g = [g)_{\le}$.*

Proof. Suppose that $(S, *)$ is the EL–semihypergroup of a partially ordered semigroup $(S, \cdot, \le)$ and G is a nonempty subset of S.

1. **"$\Rightarrow$"** The fact that $(G, *)$ is a subhypergroupoid of $(S, *)$ follows from Lemma 2.4.31, parts 1 and 4 respectively. For both types of G, the associativity of $(G, *)$ follows from the first part of the "Ends lemma", Theorem 2.1.1 – notice that the proof (included on page 39) may be applied without any changes even when G is not an upper end of S.

"⇐" Suppose that $(G, *)$ is a subsemihypergroup of $(S, *)$. First we have to prove that G is closed with respect to the operation "$\cdot$" of S. Yet for arbitrary elements $a, b \in G$ the fact that $a * b \subseteq G$ implies that $[a \cdot b)_{\leq} \subseteq G$, i.e. any element $x \in S$ such that $a \cdot b \leq x$ belongs to G. Since the relation "$\leq$" is reflexive, we get that $a \cdot b \in G$. As a result $(G, \cdot)$ is a groupoid. The fact that it is associative is granted by the reasoning of the proof of Theorem 2.4.7, part 1 on page 41. Altogether we get that $(G, \cdot)$ is a subsemigroup of $(S, \cdot)$.

2. **"⇒"** Denote u the neutral element of $(S, \cdot)$. If $(G, \cdot)$ is a submonoid of $(S, \cdot)$, the fact that $g * u = u * g = [g)_{\leq}$ is obvious.

   "⇐" Cf. part 2 of the proof of Theorem 2.4.7, which may be literally repeated.

□

From the proof of Theorem 2.4.34, part 1 "⇐", we directly get an obvious statement equivalent to the one included in Remark 2.4.32. Notice that its validity does not depend on the fact whether G is an upper end of S.

Corollary 2.4.35. *Let $(S, *)$ be the EL–semihypergroup of a partially ordered semigroup $(S, \cdot, \leq)$ and G a nonempty subset of S. If $(G, *)$ is a subhypergroupoid of $(S, *)$, then $(G, \cdot)$ is a subgroupoid of $(S, \cdot)$.*

Also notice that implications "⇒" in Theorem 2.4.34 hold for quasi-ordered semigroups as well. The issue of subhypergroups seems to be a bit more complicated.

Theorem 2.4.36. *Let $(S, *)$ be the EL–semihypergroup of a quasi-ordered semigroup $(S, \cdot, \leq)$. Suppose that G is an upper end of S. If $(G, \cdot)$ is a subgroup of $(S, \cdot)$, then $(G, *)$ is a subhypergroup of $(S, *)$.*

Proof. Since we assume that $(G, \cdot)$ is a subgroup of $(S, \cdot)$, we have that for an arbitrary $a, b \in G$ there holds $a \cdot b^{-1} \in G$, $b^{-1} \cdot a \in G$. Therefore if elements $c = b^{-1} \cdot a$ and $c' = a \cdot b^{-1}$ are regarded, Theorem 2.1.2 may be directly applied, or rather, its proof literally copied. Since we are proving only "⇒", we are bypassing the problem causes by the lack of antisymmetry, i.e. the theorem holds for the more general type of quasi-ordered semigroups. □

As follows from the proof of Theorem 2.1.5, the subhypergroup is a transposition hypergroup or (if it is commutative) a join space.

Remark 2.4.37. Notice that $(G, *)$, where G is such as defined in the assumptions of Lemma 2.4.31, part 4, can never be a subhypergroup of $(S, *)$. In this case the inclusion $G \subseteq a * G$ of the reproductive law is problematic. Indeed, suppose an arbitrary element $a \in G$ and any element $g \in G$ for which there holds $g \leq x_i$, where x_i is an arbitrary of those elements because of which G is not an upper end of S, i.e. $x_i \notin G$. In other words, g is such that there holds $[g)_\leq \nsubseteq G$. Then we have that $a * g = [a \cdot g]_\leq = [b)_\leq$ and thanks to the assumption of Lemma 2.4.31, part 4, especially that we consider an arbitrary $a \neq u$, we have that $g \notin [b)_\leq$, which means that $G \nsubseteq a * G$.

Example 2.4.38. Consider the set $(\mathbb{R}^n, +, \leq)$ of n–tuples of real numbers with the usual componentwise addition and lexicographic order. $(\mathbb{R}^n, +)$ is a group with the identity $(0, \ldots, 0)$. Sets $(\mathbb{R}^n_{1,\ldots,k}, +)$, where $k \leq n$, in which components $1, \ldots, k$ are arbitrary real numbers while all other components are equal to zero, are obvious examples of subgroups of $(\mathbb{R}^n, +)$. Furthermore, relation "$\leq$" is linear ordering.[9] If we now define the hyperoperation for any two n–tuples $u, v \in \mathbb{R}^n$ as

$$u *_{\mathbb{R}^n} v = [u + v)_\leq,$$

we get that $(\mathbb{R}^n, *_{\mathbb{R}^n})$ is a join space. If we regard the above subgroups $(\mathbb{R}^n_{1,\ldots,k}, +)$, then only $(\mathbb{R}^n_n, +)$ is the upper end of $(\mathbb{R}^n, +, \leq)$. All other subgroups cannot qualify because they do not meet the condition "for an arbitrary element $g \in G$ there holds $[g)_\leq \subseteq G$" from Definition 2.4.30. This is because concerning the subgroups we regard the relation "$\leq$" among all elements of the set $(\mathbb{R}^n, +, \leq)$ (i.e. we regard hyperoperation of the type (2.13) which is not restricted on elements of the subsets G).

Proposition 2.4.39. *Let $(S, *)$ be the EL–semihypergroup of a partially ordered semigroup $(S, \cdot, \leq)$ and $G \subseteq S$ nonempty. If $(G, *)$ is a subhypergroup of $(S, *)$, then $(G, \cdot)$ is a subsemigroup of $(S, \cdot)$ and G is an upper end of S such that for any pair $a, b \in G$ there exists a pair $c, c' \in G$ such that $b \cdot c \leq a$ and $c' \cdot b \leq a$.*

Proof. Thanks to Lemma 2.4.31, Remark 2.4.32, Theorem 2.4.36 and Remark 2.4.37 it is obvious that all EL–subhypergroups $(G, *)$ of $(S, *)$ are such that G is an upper end of S. Since every hypergroup is a semihypergroup, we get that $(G, *)$ is a subsemihypergroup of $(S, *)$. Yet according to Theorem 2.4.34, part 1, $(G, \cdot)$ is in this case a subsemigroup of $(S, \cdot)$.

[9] To explain the notation, e.g. $(\mathbb{R}^8_{2,4,5}, +)$ stands for the set of all 8-tuples, where the second, fourth and fifth components are arbitrary real numbers while all other components are zero, i.e. the set of $(0, n, 0, m, l, 0, 0, 0)$, where $n, m, l \in \mathbb{R}$.

The proposition for the arbitrary pair $a, b \in G$ is a copy of condition 1^0 of Theorem 2.1.2. □

What remains to be proved is whether $(G, \cdot)$ in the above proposition is a subgroup of $(S, \cdot)$. This is still an open question. Notice that Proposition 2.4.39 does not guarantee the existence of $u \in G$ such that u is the neutral element of $(S, \cdot)$.

Remark 2.4.40. Suppose that $(H, \cdot, \leq)$ is a partially ordered group with identity u and G is a non-empty subset of H. If $(G, \cdot)$ is simultaneously a subgroup of $(H, \cdot)$ and an upper end of H, then notice the following:

If we take an arbitrary $x \in H$ such that $x < g$, where $g \in G$ is arbitrary, then $x < g$ implies $u < x^{-1} \cdot g$ and since $(G, \cdot)$ is a subgroup of $(H, \cdot)$, which is a group, and simultaneously G is an upper end of H, we get that $x^{-1} \cdot g \in G$. Yet since $g \in G$, there is also $x^{-1} \in G$, which implies that $x \in G$.

As a result we get that if $(H, \cdot, \leq)$ is a linearly ordered group, there do not exist any proper subhypergroups associated to subgroups of $(H, \cdot)$ because there are no proper subgroups $(G, \cdot)$ of $(H, \cdot)$, where G is an upper end of H. Theorem 2.4.36 is thus of no practical use for linearly ordered groups. Also cf. Remark 2.4.37, which states that it is upper ends that are the only candidates for subhypergroups.

However, if $(H, \cdot)$ is a monoid only, then $x < g$ does not imply $u < x^{-1} \cdot g$ (and consequently $x \in G$) because x need not have the inverse element.

If we define the principal end generated by an element $a \in G$, where $G \subseteq H$, as $[a)_{\leq_G} = \{x \in G \mid a \leq x\}$, i.e. instead of (2.13) regard (2.12), problems of "holes" caused by elements $x \in H \setminus G$ in Definition 2.4.30 will not come up. Technically speaking there are two distinct hyperoperations in the following theorem: "$*$" and "$*_G$". Therefore, the (hyper)structures on G are not called sub(hyper)structures.

Theorem 2.4.41. *Let $(S, *)$ be the EL–semihypergroup of a partially ordered semigroup $(S, \cdot, \leq)$. Further, let $G \subseteq S$ be non-empty and such that $(G, \cdot)$ is a subgroupoid of $(S, \cdot)$ and the relation "$\leq_G$" be a restriction of "$\leq$" on G, i.e. for arbitrary elements $a, b \in G$ let $a \leq b \Rightarrow a \leq_G b$. Finally, if $(S, \cdot)$ is a monoid, denote its neutral element by u. Define a new hyperoperation $*_G : G \times G \to P^*(G)$ for arbitrary elements $a, b \in G$ by*

$$a *_G b = [a \cdot b)_{\leq_G} = \{x \in G \mid a \cdot b \leq_G x\}.$$

Then

1. *$(G, \cdot)$ is a semigroup if and only if $(G, *_G)$ is a semihypergroup.*

2. *$(G,\cdot)$ is a monoid if and only if $(G,*_G)$ is a semihypergroup and $u \in G$.*

3. *If $(G,\cdot)$ is a group, then $(G,*)$ is a transposition hypergroup.*

4. *If $(G,*)$ is a hypergroup, then $(G,\cdot)$ is a semigroup such that for any pair $a,b \in G$ there exists a pair $c,c' \in G$ such that $b\cdot c \leq a$ and $c'\cdot b \leq a$.*

Proof. The theorem is a simple corollary to the "Ends lemma", Theorem 2.4.7 and Theorem 2.1.2. □

Remark 2.4.42. The fact that G is closed with respect to "$\cdot$" is again essential: suppose a triple a,b,c such that $a,b \in G$ and $c \in H \setminus G$. If now $a *_G b$ was constructed, we would get $a *_G b = [a\cdot b)_{\leq_G} = [c)_{\leq_G}$, which is difficult to be assigned with any sense since due to reflexivity of "$\leq_G$" there must hold $c \in [c)_{\leq_G}$, i.e. $c \in \{x \in G \mid c \leq x\}$ yet we suppose that $c \notin G$.

2.4.5 Some properties of EL–semihypergroups

Most results of this subsection (with the exception of results on cyclicity) were published by European Journal of Combinatorics (WoS Q2) as Novák [244].

The study of properties of EL–hyperstructures was motivated by the fact that in numerous papers and conference contributions such as e.g. [35,50,55, 70,117,155,267] these properties were proved ad hoc for (semi)hypergroups or join spaces which had been constructed using the "Ends lemma". Therefore, in Novák [244] this issue was studied in detail from the theoretical point of view and the following results were proved.

Notice that unless stated otherwise, the notion of a subhypergroupoid $(G,*)$ of a semihypergroup $(S,*)$ is defined by means of (2.13), i.e. using the concepts of "upper ends" and Lemma 2.4.31.

Theorem 2.4.43. *Let $(H,*)$ be the EL–hypergroup of a quasi-ordered group $(H,\cdot,\leq)$ and $(G,\cdot)$ its subgroup such that G is an upper end of H. Then the hypergroup $(G,*)$ is invertible and closed in H.*

Proof. (invertibility) First of all we need to rewrite the condition of invertibility in the language of the "Ends lemma". The fact that $y \in G * x$ is equivalent to the fact that $y \in \bigcup\limits_{g\in G} g * x$, i.e. $y \in \bigcup\limits_{g\in G} [g\cdot x)_{\leq}$, which means that there exists an element $g_0 \in G$ such that $g_0 \cdot x \leq y$. In a similar way we get that $x \in G * y$ means that there exists an element $g_1 \in G$ such that $g_1 \cdot y \leq x$. Therefore, if G is a non-empty subset of H and $(H,*)$ is an EL–hyperstructure, we have to prove that for an arbitrary pair of elements $x,y \in H$ the fact that there exists an element $g_0 \in G$ such that $g_0 \cdot x \leq y$

implies that there exists an element $g_1 \in G$ such that $g_1 \cdot y \leq x$ (and in a similar way for invertibility on the right).

Suppose that $(H, \cdot, \leq)$ is a *commutative* quasi-ordered group, $G \subseteq H$, $G \neq H$, $x, y \in H$ arbitrary. Then

1. $g_0 \cdot x \leq y$ is equivalent to $g_0 \cdot x \cdot x \leq y \cdot x$, which is equivalent to $y^{-1} \cdot g_0 \cdot x \cdot x \leq x$, i.e. $y^{-1} \cdot g_0 \cdot x \cdot x \cdot y^{-1} \leq x \cdot y^{-1}$, which is equivalent to $y^{-1} \cdot g_0 \cdot x \cdot x \cdot y^{-1} \cdot y \leq x$. Therefore, if we denote $g_1 = y^{-1} \cdot g_0 \cdot x \cdot x \cdot y^{-1}$, we must examine whether $g_1 \in G$. Yet if $(H, \cdot)$ is commutative, we may write $g_1 = x \cdot y^{-1} \cdot x \cdot y^{-1} \cdot g_0$. Furthermore $g_0 \cdot x \leq y$ is equivalent to $g_0 \leq y \cdot x^{-1}$, which – since G is an upper end of H – means that $y \cdot x^{-1} \in G$. Yet $(G, \cdot)$ is also a group, thus $(y \cdot x^{-1})^{-1} = x \cdot y^{-1} \in G$. Now $g_1 = (x \cdot y^{-1}) \cdot (x \cdot y^{-1}) \cdot g_0$ is a product of elements of G. Therefore $g_1 \in G$, which means that G is invertible on the left.

2. The proof of invertibility on the right is analogous; the element a'_1 in question would be $a'_1 = y^{-1} \cdot y^{-1} \cdot x \cdot a'_0 \cdot x$. However, since "$\cdot$" is a commutative operation, "$*$" is a commutative hyperoperation, which itself alone completes the proof of invertibility.

Later, we will see that in case $(H, \cdot)$ is *not commutative*, the theorem is valid as well (see Corollary 2.4.56 on page 66).

(closedness) Suppose arbitrary elements $x, y \in G$ and $a \in H$. Further suppose that there holds $x \in a * y$, i.e. $x \in [a \cdot y)_{\leq}$, i.e. $a \cdot y \leq x$. Since $(H, \cdot, \leq)$ is a quasi-ordered group, this means that $y \leq a^{-1} \cdot x$. Since G is an upper end of H and $y \in G$, we get that $a^{-1} \cdot x \in G$. Since $x \in G$ and G is a subgroup of H, we get that $a^{-1} \in G$, which means that $a \in G$, thus G is closed from the left in H. The fact that it is closed also from the right may be proved in an analogous way. Altogether, G is closed in H. Or, in a commutative case, we could make a reference to Corsini and Leoreanu [95], chapter 1, 37 (iii), from which the theorem follows immediately. □

Remark 2.4.44. Chvalina in [44], p. 157, uses a different idea of a closed subhypergroup of a hypergroup – the one that is related to [128, 298]. He states that: "*the subhypergroup* $(G, *)$ *of a hypergroup* $(H, *)$[10] *is called closed if there holds:*

$$H * (G \setminus H) = G \setminus H = (G \setminus H) * H. \tag{2.16}$$

It can be proved that any nonempty intersection of an arbitrary system of closed subhypergroups is a closed subhypergroup of this hypergroup and then

[10] In fact, the notation G and H is switched in [44] as G is a hypergroup and H its subhypergroup. Our notation here is used for consistency reasons.

a closure $\langle A \rangle$ may be defined for every nonempty subset of $(H, *)$ in a usual way by means of their closed subhypergroups." Notice that for hyperstructures constructed using Theorem 2.4.36 (i.e. for upper ends) Theorem 2.4.43, part on closedness, is valid under this definition too. Here we have to prove that $\bigcup\limits_{g \in G, h \in H \setminus G} [g \cdot h)_{\leq} = H \setminus G = \bigcup\limits_{h \in H \setminus G, g \in G} [h \cdot g)_{\leq}$. The first equality can be proved as follows:

"$\subseteq$:" Suppose $g \in G, h \in H \setminus G$ arbitrary. Further suppose an arbitrary $x \in [g \cdot h)_{\leq}$. Since $x \in [g \cdot h)_{\leq}$, there is $g \cdot h \leq x$ and since H is a quasi-ordered group we have that $g \leq x \cdot h^{-1}$. Since G is an upper end of H, there must hold $x \cdot h^{-1} \in G$. Suppose now that $x \in G$. Since $x \cdot h^{-1} \in G$, there is $h^{-1} \in G$, i.e. $h \in G$. However, this is contradiction to the initial assumption that $h \in H \setminus G$. Therefore there must hold $x \notin G$, i.e. $x \in H \setminus G$.

"$\supseteq$:" Suppose an arbitrary $x \in H \setminus G$. If we denote g the neutral element of $(H, \cdot)$ and write $h = x$, then $x = g \cdot h$. Since the relation "$\leq$" is reflexive, we have that $x \in [g \cdot h)_{\leq}$, i.e. $x \in \bigcup\limits_{g \in G, h \in H \setminus G} [g \cdot h)_{\leq}$.

The second equality can be proved in an analogous way.

Notice that Jantosciak in [169] starts with a definition of a closed (sub)set (i.e. not a closed subhypergroup) — formulated in the "extensions" language of transposition hypergroups — and then proves that in any hypergroup if G is closed, then G is a subhypergroup of H, and shows that intersection of any family of closed sets is closed, and defines the operator $\langle A \rangle$ denoting the least closed set containing A by means of closed sets (i.e. not subhypergroups as Chvalina does.)

In Theorem 2.4.36 we have seen that if in a semigroup S we take a subsemigroup G which is a group and apply the "Ends lemma" on both S and G (in case of G using (2.13), i.e. in the same way as e.g. in Theorem 2.4.43) we get that if G is an upper end of S, then $(G, *)$ is a subhypergroup of $(S, *)$. If we assume that S is a group, we get the following theorem and corollaries.

Theorem 2.4.45. *Let $(H, *)$ be the EL–hypergroup of a quasi-ordered group $(H, \cdot, \leq)$ and $(G, *)$ its arbitrary subhypergroup associated to a subgroup $(G, \cdot)$ of $(H, \cdot)$, where G is an upper end of H. Denote the neutral element of $(H, \cdot)$ by u. Then G is ultraclosed if and only if for any $h \in H$ such that $h \leq u$ there follows that $h \in G$.*

Proof. According to Corsini and Leoreanu [95], chapter 1, 34, a subhypergroup $(G,*)$ of a hypergroup $(H,*)$ is ultraclosed if and only if it is closed and contains I_p, where I_p is the set of partial identities of $(H,*)$, i.e. $I_p = \{e \in H \mid \exists x \in H \mid x \in e * x \cup x * e\}$. Obviously, for EL–hypergroups associated to groups we have that $I_p = \{e \in H \mid \exists x \in H \mid e \cdot x \leq x \text{ or } x \cdot e \leq x\} = \{e \in H \mid e \leq u\}$. Furthermore, according to Theorem 2.4.43 every subhypergroup in question is closed. As a result we get the theorem. □

Remark 2.4.46. All subsets G discussed in Theorem 2.4.45 are upper ends (and groups), i.e. for an arbitrary element $g \in G$ there holds $[g)_{\leq} \subseteq G$, i.e. for an arbitrary element $e \in H$, $g \leq e$ it follows $e \in G$. If we now regard an arbitrary element $e \leq u$, where u is the identity of $(H,\cdot)$, then $u \leq e^{-1}$ and since $u \in G$, we get that $e^{-1} \in G$. This means that in this context the request that "$h \leq u \Rightarrow h \in G$" is equivalent to the fact that an arbitrary element, which is in relation with u, is in G. In other words, all ultraclosed subhypergroups associated to subgroups of $(H,\cdot)$ contain all elements, which are in relation with the identity of $(H,\cdot)$. As far as linear ordered groups $(H,\cdot,\leq)$ are concerned, this means that in them there are no proper ultraclosed subhypergroups associated to their proper subgroups. This corresponds to Remark 2.4.40, which states that in the case of linear ordered groups there do not exist any proper subhypergroups associated to proper subgroups defined in the above sense.

Corollary 2.4.47. *Under the assumptions of Theorem 2.4.45 assume that $G \neq H$. If $(H,\cdot,\leq)$ has the smallest element, then $(G,*)$ is not ultraclosed.*

Proof. If we continue with the reasoning of Remark 2.4.46 and take into account that if $s \in H$ is the smallest element (with this we assume that "$\leq$" is a partial order), then obviously $s \leq u$, then since all subhypergroups in question are such that G is an upper end of H, i.e. since $a \in G \Rightarrow [g)_{\leq} \subseteq G$, we get that all ultraclosed subhypergroups $(G,*)$ of $(H,*)$ are such that $[s)_{\leq} = H \subseteq G$, thus $G = H$. □

Corollary 2.4.48. *Let $(H,*)$ be the EL–hypergroup of a quasi-ordered group $(H,\cdot,\leq)$ and $(G,*)$ its arbitrary subhypergroup associated to a subgroup $(G,\cdot)$ of $(H,\cdot)$, where G is an upper end of H. Denote the neutral element of H by u. If $(H,\cdot)$ or $(H,*)$ is commutative, then G is a complete part of H if and only if, for all $h \in H$ such that $h \leq u$, there is $h \in G$.*

Proof. Follows immediately from Theorem 2.4.45 and Corsini and Leoreanu [95], chapter 1, 70, because in this case $(H,*)$ is a join space. □

Corollary 2.4.49. *The EL–hypergroup of an arbitrary quasi-ordered group $(H, \cdot, \leq)$ is regular. Also, if $(H, \cdot)$ is a semigroup, then the EL–hypergroup of an arbitrary subgroup of $(H, \cdot)$ is regular.*

Proof. Follows immediately from Lemma 2.4.16 on page 48. □

Notice that in the above theorem it makes no difference whether (2.12) or (2.13) is used to define the hyperoperation on S. The same is true also for the following theorem which states that a substructure of the underlying single-valued structure possessing the property of a normal subgroup creates a *normal subhypergroup* of its EL–hypergroup.

Theorem 2.4.50. *Let $(H, *)$ be the EL–hypergroup of a quasi-ordered group $(H, \cdot, \leq)$ and $(G, *)$ its arbitrary subsemihypergroup associated to a subsemigroup $(G, \cdot)$ of $(H, \cdot)$. If, for arbitrary $x \in H$ and $g \in G$, there holds $x \cdot g \cdot x^{-1} \in G$, then $(G, *)$ is normal.*

Proof. The condition of hyperstructure normality states that for an arbitrary $x \in H$ there must be $x * G = G * x$. When proving the inclusions in the language of EL–semihypergroups, we get that $a \in x * G$ is equivalent to $a \in \bigcup\limits_{g \in G} x * g$, i.e. there exists an element $g_0 \in G$ such that $a \in x * g_0$, i.e. $x \cdot g_0 \leq a$. Similarly, $a \in G * x$ is equivalent to the existence of an element $g_1 \in G$ such that $g_1 \cdot x \leq a$. Now, $x \cdot g_0 \leq a$ is in a group equivalent to $(x \cdot g_0 \cdot x^{-1}) \cdot x \leq a$ and if we denote $g_1 = x \cdot g_0 \cdot x^{-1}$, we get the assumption of the theorem. Testing the other inclusion results in $x \cdot (x^{-1} \cdot g_1 \cdot x) \leq a$. However, since we suppose that $x \cdot g \cdot x^{-1} \in G$ holds for an arbitrary $x \in H$, it holds also for an element $y = x^{-1}$, i.e. we could have also written $x^{-1} \cdot g \cdot x \in G$. □

Corollary 2.4.51. *Let $(H, *)$ be the EL–hypergroup of a quasi-ordered group $(H, \cdot, \leq)$ and $(G, \cdot)$ its normal subgroup such that G is an upper end of H. Then $(G, *)$ is reflexive.*

Proof. If we realize that $(H, *)$ is a transposition hypergroup (for this see Lemma 2.1.5), the corollary follows immediately from Theorem 2.4.43 and Jantosciak [169], Proposition 9, which states that, in a transposition hypergroup, a normal closed set is reflexive. □

Chvalina and Chvalinová in [53], which makes an important part of chapter 6 of Corsini and Leoreanu [95], construct state hypergroups of automata (not using the "Ends lemma" but using a similar construction) and then prove that an automaton is connected if and only if its state hypergroup is inner irreducible. For EL–hyperstructures we have the following result on inner irreducibility.

Theorem 2.4.52. *Let $(H,*)$ be the EL–hypergroup of a partially ordered commutative group $(H,\cdot,\leq)$. If for every $x \in H$ such that x, x^{-1} are incomparable with respect to "$\leq$" there is either $[x)_{\leq} \cap [x^{-1})_{\leq} \neq \emptyset$ or ${}_{\leq}(x] \cap {}_{\leq}(x^{-1}] \neq \emptyset$, then $(H,*)$ is inner irreducible.*

Proof. Let $(H,*)$ be the EL–hypergroup of a partially ordered group $(H,\cdot,\leq)$ and G_1, G_2 an arbitrary pair of its subhypergroups such that $G_1 * G_2 = H$. First of all, rewrite the condition imposed on $(H,*)$. In our case there is $H = G_1 * G_2 = \bigcup\limits_{g_1 \in G_1, g_2 \in G_2} g_1 * g_2 = \bigcup\limits_{g_1 \in G_1, g_2 \in G_2} [g_1 \cdot g_2)_{\leq}$, i.e. to an arbitrary element $x \in H$ there must exist elements $g_1 \in G_1$, $g_2 \in G_2$ such that $g_1 \cdot g_2 \leq x$.

Notice the case of $x = u$, where u is the neutral element of $(H,\cdot)$. Obviously, there are these possibilities:

1. $u \in G_1, u \in G_2$. Then $G_1 \cap G_2 \neq \emptyset$.

2. u belongs to exactly one of the subhypergroups G_1, G_2. Let us denote them so that $u \in G_1$, $u \notin G_2$.

   In a group the neutral element u may be obtained only as a product of an arbitrary element $x \in H$ and its inverse $x^{-1} \in H$. If there exists an element $x \in G_2$ such that $x^{-1} \in G_2$ then – since $(G_2,\cdot)$ is a groupoid – there is $x \cdot x^{-1} = u \in G_2$, which is a negation of the assumed fact. Therefore G_2 is such a subset of H that $x \in G_2 \Rightarrow x^{-1} \notin G_2$.

   The fact that $H = G_1 * G_2$ means that there exist $g_1 \in G_1$, $g_2 \in G_2$ such that $g_1 \cdot g_2 \leq u$. In a partially ordered group this is equivalent to

   $g_1 \leq g_2^{-1}$, which – since G_1 is an upper end of H[11] – means that $g_2^{-1} \in G_1$

   as well as to

   $g_2 \leq g_1^{-1}$, which – since G_2 is an upper end of H – means that $g_1^{-1} \in G_2$.

   However, with respect to the nature of G_2 the latter implies that $g_1 \notin G_2$. This means that there cannot be $g_1^{-1} \leq g_1$, in other words there may be either $g_1 < g_1^{-1}$ (admitting $g_1 = g_1^{-1}$ would mean that $g_1^{-1} = u$ which would be a contradiction to the assumption that $u \notin G_2$) or the elements g_1, g_1^{-1} are incomparable. Yet

[11] This as well as the statement below concerning G_2 follows from Proposition 2.4.39.

- since G_1 is an upper end of H, the fact that $g_1 < g_1^{-1}$ implies that $g_1^{-1} \in G_1$ which means that $g_1^{-1} \in G_1 \cap G_2$,
- if g_1, g_1^{-1} are incomparable, then thanks to the assumptions of the theorem there exists an element $a \in H$ such that $a \leq g_1, a \leq g_1^{-1}$ or an element $b \in H$ such that $g_1 \leq b, g_1^{-1} \leq b$. In the former case we have that $g_1^{-1} \leq a^{-1}$, $g_1 \leq a^{-1}$, which means that $a^{-1} \in G_1 \cap G_2$ because we already know that $g_1^{-1} \in G_2$ and $g_1 \in G_1$ and G_1, G_2 are upper ends of H. In the latter case we for the same reason have that $b \in G_1 \cap G_2$.

3. $u \notin G_1, u \notin G_2$. Then for no $a \in G_1$, $b \in G_2$ there holds $a \leq u$, $b \leq u$, i.e. for every $a \in G_1$, $b \in G_2$ there holds

$$u \leq a \text{ or } u, a \text{ are incomparable}$$

and

$$u \leq b \text{ or } u, b \text{ are incomparable.}$$

Now examine the case of $a = g_1$, $b = g_2$, i.e. of those elements for which there is $g_1 \cdot g_2 \leq u$.

- The fact that $u \leq g_1$ is in a partially ordered group equivalent to $g_1^{-1} \leq u$, which – since $g_1^{-1} \in G_2$ and G_2 is an upper end of H – means that $u \in G_2$, which is a contradiction.
- If u, g_1 are incomparable (yet since both are elements of G_1), then thanks to Proposition 2.4.39[12] there exists an element $c \in G_1$ such that $g_1 \cdot c \leq u$, which is equivalent to the fact that $c \leq g_1^{-1}$. However, since $c \in G_1$ and G_1 is an upper end of H, we get that $g_1^{-1} \in G_1$. However, if both $g_1, g_1^{-1} \in G_1$, then also $u \in G_1$, which is a contradiction.

□

Corollary 2.4.53. *Let $(H, *)$ be the EL–hypergroup of a partially ordered commutative group $(H, \cdot, \leq)$. If $(H, \leq)$ is a linearly ordered set or if $(H, \leq)$ has the smallest or the greatest element, then $(H, *)$ is inner irreducible.*

Proof. Obvious. □

[12]If antisymmetry were not required in the proposition, the theorem would hold for quasi-ordered groups $(H, \cdot, \leq)$ too. However, the requirement follows from the idea "$[a)_\leq = [b)_\leq \Rightarrow a = b$", which is not possible without antisymmetry.

Theorem 2.4.54. *Let $(H,*)$ be the EL–hypergroup of a quasi-ordered group $(H,\cdot,\leq)$. Then $(H,*)$ is reversible.*

Proof. If we rewrite the definition of reversibility into the "Ends lemma" notation, we get that for arbitrary $a,x,y\in H$ there must simultaneously be

1. $a\cdot x\leq y$ implies that there exists an inverse a' of a in $(H,*)$ such that $a'\cdot y\leq x$
2. $x\cdot a\leq y$ implies that there exists an inverse a'' of a in $(H,*)$ such that $y\cdot a''\leq x$

Now focus on 1. The fact that $a\cdot x\leq y$ is in quasi-ordered groups equivalent to $x\cdot y^{-1}\leq a^{-1}$ while the fact that $a'\cdot y\leq x$ is equivalent to $a'\leq x\cdot y^{-1}$. Thus condition 1 may be rewritten as "$x\cdot y^{-1}\leq a^{-1}$ implies the existence of an inverse a' of a such that $a'\leq x\cdot y^{-1}$". Now denote $a'=x\cdot y^{-1}$ (which both exists because $(H,\cdot)$ is a group and is in relation "$\leq$" with $x\cdot y^{-1}$ because "$\leq$" is reflexive). If $a'=x\cdot y^{-1}$ is an inverse of a in $(H,*)$, the proof is complete. Yet thanks to Theorem 2.4.17, a' is an inverse of a.

Validity of condition 2 may be proved in an analogous way with $a''=y^{-1}\cdot x$. □

Example 2.4.55. As a very simple example showing the validity of Theorem 2.4.54 regard the chain $(\mathbb{Z},+,\leq)$ of all integers with the usual addition and ordering of numbers and its EL–hypergroup $(\mathbb{Z},*)$. For an arbitrary triple $a,x,y\in\mathbb{Z}$ such that $a+x\leq y$, the element $a'=x-y$ and the requirement $a'+y\leq x$ turns into $(x-y)+y\leq x$. Obviously, such elements a,a' are inverses in $(\mathbb{Z},*)$ as required since $a+x\leq y$ is equivalent to $x-y+a\leq 0$, i.e. $a'+a\leq 0$, which means that a,a' are inverses in $(\mathbb{Z},*)$ because 0 is a unit of $(\mathbb{Z},*)$. If e.g. $a=5,x=15,y=40$, then obviously $a+x=15\leq 40=y$. Since $a'=x-y$, there is $a'=-25$ and we must examine whether $a'=-25$ is an inverse of $a=5$. Yet since $a'=-25\leq -a=-5$, we get that -25 is an inverse of 5 in $(\mathbb{Z},*)$.

Corollary 2.4.56. *Theorem 2.4.43 holds even in a non-commutative case.*

Proof. Follows immediately from Corsini and Leoreanu [95], chapter 1, 49, and Theorem 2.4.43 because $(H,*)$ is a regular reversible hypergroup. □

In Corsini and Corsini and Leoreanu [92, 95] there is included a brief study of K_H–hypergroups introduced by De Salvo [119]. Thanks to results of this subsection theorems proved on p. 19 of [95] may be directly applied on EL–semihypergroups. Notice that Dramalidis [126] discusses H_v–structures derived from K_H–hypergroups.

Definition 2.4.57. Let $(H,*)$ be a hypergroupoid and let $\{A(x)\}_{x\in H}$ be a family of pairwise disjoint non-empty sets. Let $K_H = \bigcup\limits_{x\in H} A(x)$ and let us define for an arbitrary $a \in K_H$ that $g(a) = x$ if and only if $a \in A(x)$. For an arbitrary pair $a, b \in K_H$ we define a hyperoperation "$\Box$" on K_H by

$$a \Box b = \bigcup_{z \in g(a)*g(b)} A(z). \tag{2.17}$$

As an immediate corollary of [95], chapter 1, 100–106, we get the following theorem:

Theorem 2.4.58. *Let $(H,*)$ be the EL–semihypergroup of a quasi-ordered semigroup $(H,\cdot,\leq)$ and let $(K_H,\Box)$ be the hypergroupoid constructed using Definition 2.4.57. Then $(K_H,\Box)$ is a semihypergroup. If $(H,\cdot)$ is moreover a group, then $(K_H,\Box)$ is a regular reversible hypergroup.*

Cyclic hypergroups are a hyperstructure analogy of *cyclic groups*. The first remark on cyclicity in hyperstructures can be traced back to Wall [307]. Based on his approach, Vougiouklis [178, 297] initiated the deeper study of cyclicity introducing concepts such as *period of a cyclic hypergroup* or *single-power cyclic hypergroup*.[13]

Definition 2.4.59. A hypergroup H is called *cyclic* if, for some $h \in H$, there is

$$H = h^1 \cup h^2 \cup \ldots \cup h^n \cup \ldots, \tag{2.18}$$

where $h^1 = \{h\}$ and $h^m = \underbrace{h \cup \ldots \cup h}_{m}$. If there exists $n \in \mathbb{N}$ such that (2.18) is finite, we say that H is a cyclic hypergroup with *finite period*; otherwise H is a cyclic hypergroup with *infinite period*. The element $h \in H$ in (2.18) is called *generator* of H, the smallest power n for which (2.18) is valid is called *period* of h. If all generators of H have the same period n, then H is called *cyclic with period* n. If, for a given generator h, (2.18) is valid but no such n exists (i.e. (2.18) cannot be finite), then H is called *cyclic with infinite period*. If we can, for some $h \in H$, write

$$H = h^n, \tag{2.19}$$

then the hypergroup H is called *single-power cyclic* with a generator h. If (2.18) is valid and for all $n \in \mathbb{N}$ and, for a fixed $n_0 \in \mathbb{N}$, $n \geq n_0$ there is

$$h^1 \cup h^2 \cup \ldots \cup h^{n-1} \subsetneq h^n, \tag{2.20}$$

[13] The definition given below follows the wording of Vougiouklis; on contrary Corsini in his definition included e.g. in [92, 95] stresses the connection between cyclic *hypergroups* and cyclic *groups*.

then we say that H is a *single-power cyclic hypergroup with infinite period for* h.

The "Ends lemma" can be used to construct cyclic hypergroups of different types.

Example 2.4.60. Regard multiplication on the interval $I = (0, 1)$ – see Example 2.2.6 on page 30, where closed interval is used. Then $(I, *)$, where $a * b = [a \cdot b)_\leq = \{x \in I \mid a \cdot b \leq x\}$, is a semihypergroup. In the following subsection we will show that, by Theorem 2.4.71, $(I, *)$ is a hypergroup. Now, for an arbitrary $a \in I$ there is $a^n \subsetneq a^{n+1}$ and, obviously, $(I, *)$ is a single power cyclic hypergroup with infinite period for an arbitrary $a \in I$.

Example 2.4.61. Suppose $(\mathbb{Z}, +, \leq)$, with the usual addition and ordering of integers. Since $(\mathbb{Z}, +, \leq)$ is a partially ordered group, $(\mathbb{Z}, *)$, where $a * b = \{x \in \mathbb{Z} \mid a + b \leq x\}$ is a hypergroup. For an arbitrary negative $a \in \mathbb{Z}$ and $k > 1$ we have $a^k = \underbrace{a * \ldots * a}_{k} = \{x \in \mathbb{Z} \mid -ka \leq x\}$, i.e. $\mathbb{Z} = a \cup a^2 \cup a^3 \cup \ldots \cup a^n \cup \ldots$ and $(\mathbb{Z}, *)$ is a single-power cyclic hypergroup with infinite period for infinitely many (yet not all) generators. In this respect notice that all infinite cyclic *groups* are isomorphic to the additive group $(\mathbb{Z}, +)$.

In the "Ends lemma" we often regard *idempotent operations* such as "min", "max", "$\cup$", "$\cap$", etc. In this case we get, for an arbitrary $h \in H$,

$$h^k = \underbrace{h * \ldots * h}_{k} = [\underbrace{h \cdot \ldots \cdot h}_{k})_{\ldots} = [h)_\leq = \{x \in H \mid h \leq x\}, \tag{2.21}$$

which means that the only chance that (2.18) becomes valid is in case that $[h)_\leq = H$ (notice that, due to reflexivity of "$\leq$", there is always $h \in [h)_\leq$). In other words, if $(H, *)$ is a hypergroup (and, by a forthcoming Theorem 2.4.71 of the following subsection, extensivity of the hyperoperation is sufficient to achieve this), then H is single power cyclic with period 2. Of course, the only generator of H is the smallest element of H (provided "$\leq$" is partial ordering). If "$\leq$" is not antisymmetric, the word "smallest" cannot be used and there can be more generators.

Example 2.4.62. Regard the quasi-ordered semigroup $(\mathbb{N}, \min, \leq)$ and its EL–semihypergroup $(\mathbb{N}, *)$, where $a * b = \{x \in \mathbb{N} \mid \min\{a, b\} \leq x\}$ for all $a, b \in \mathbb{N}$. For the same reason as in Example 2.4.60, $(\mathbb{N}, *)$ is a hypergroup. Since $[1)_\leq = \mathbb{N}$, we get that $1 * 1 = \mathbb{N}$ and 1 is the only generator of a single power cyclic hypergroup $(\mathbb{N}, *)$ with period 2. (Or 0 instead of 1 if we consider $\mathbb{N}_0$). If we change $\mathbb{N}$ to $\mathbb{Z}$, we get that the hypergroup $(\mathbb{Z}, *)$ is not cyclic because there is no integer $a \in \mathbb{Z}$ such that $[a)_\leq = \mathbb{Z}$.

Sets in all of the above examples were infinite. Therefore, the following example involves a finite set, or rather a class of finite sets.

Example 2.4.63. Regard $(\mathbb{N}, \gcd, |)$, where "gcd" stands for the greatest common divisor of natural numbers (zero excluded) and "|" is the divisibility relation. Since $(\mathbb{N}, \gcd, |)$ is a partially ordered semigroup, $(\mathbb{N}, *)$, where $a * b = \{x \in \mathbb{N} \mid \gcd\{a, b\}|x\}$, for all $a, b \in \mathbb{N}$, is a semihypergroup. Since $\{a, b\} \subseteq a * b$ for all $a, b \in \mathbb{N}$, $(\mathbb{N}, *)$ is, by a forthcoming Theorem 2.4.71, a hypergroup. Now, obviously $1 * 1 = [\gcd\{1, 1\})_{\leq} = [1)_{\leq} = \mathbb{N}$. Thus $(\mathbb{N}, *)$ is single power cyclic with period 2 and one generator 1. Of course, instead of $\mathbb{N}$ we can regard the set of divisors of an arbitrary $n \in \mathbb{N}$ and get a class of finite single power cyclic hypergroups with period 2.

Finally, we include a result concerning *homomorphisms* of EL–semihypergroups, which was proved already in Chvalina and Novák [70].[14] Recall that *isotone mappings* $f : (G, \leq_G) \to (H, \leq_H)$ are mappings which preserve the relation, i.e. $x \leq_G y$ implies $f(x) \leq_H f(y)$.

Theorem 2.4.64. *Let $(G, *_G)$ and $(H, *_H)$ be EL–semihypergroups of quasi-ordered semigroups $(G, \cdot, \leq_G)$ and $(H, \cdot, \leq_H)$, respectively, $f : (G, \cdot) \to (H, \cdot)$ a homomorphism and $f : (G, \leq_G) \to (H, \leq_H)$ an isotone mapping. Then $f : (G, *_G) \to (H, *_H)$ is a homomorphism.*

Proof. Suppose that the mapping $f : (G, \cdot, \leq_G) \to (H, \cdot, \leq_H)$ is such that $f : (G, \leq_G) \to (H, \leq_H)$ is isotone and $f : (G, \cdot) \to (H, \cdot)$ is a semigroup homomorphism. Then it is easy to see that for any element $x \in G$ we have $f([x)_{\leq_G}) \subset [f(x))_{\leq_H}$. Since $f(a \cdot b) = f(a) \cdot f(b)$ for arbitrary $a, b \in G$, we obtain that $a, b \in G$ implies

$$f(a *_G b) = f([a \cdot b)_{\leq}) \subset [f(a \cdot b))_H = f(a) *_H f(b),$$

i.e. $f : (G, *_G) \to (H, *_H)$ is a semihypergroup homomorphism. □

2.4.6 The role of extensivity

Results of this subsection were, together with the results of Section 3.3, published by Soft Computing (WoS Q2) as Novák and Křehlík [249].

In this subsection we briefly discuss one special class of EL–hyperstructures. Motivated by Chvalina [43, 44] we, in the following definition, use the name "extensivity". However, it needs to be pointed out that some

[14] Notice that in [70] the term "inclusion homomorphism" is used in the sense of "homomorphism" of Definition 1.1.29 on page 16.

other authors, such as Massouros, use a much more suitable name "closed" as this can be easily contrasted with "open". For basic definitions see e.g. Massouros [206].[15] Our results included in this subsection can be linked to those of Subsection 3.3, as in fact they were published together as [249].

Definition 2.4.65. A hyperoperation "$*$" on H is called *extensive* if for all $a, b \in H$ there is $\{a, b\} \subseteq a * b$. A hypergroupoid $(H, *)$ with an extensive hyperoperation is called an *extensive hypergroupoid*.

Notice that if the hyperoperation "$*$" is defined by $a * b = \{a, b\}$ for all $a, b \in H$, we speak of *minimal extensive hypergroupoids*. Both extensivity and minimal extensivity (for certain elements) have their meaning in real-life situations. Suppose e.g. that $a*b$ means the path consisting of points between a and b. In this case extensivity means that the endpoints are included in the path while the fact that $a * b = \{a, b\}$ means that the points are either next to each other or that there is no path between them.

Example 2.4.66. EL–semihypergroups constructed from $(S, \min, \leq)$, where $S \in \{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\} \cup \{\langle a, b\rangle\}$, where $\langle a, b\rangle$ is an arbitrary interval of real numbers, and "$\leq$" is the usual ordering of numbers by size, are extensive.

Example 2.4.67. As has been shown in Example 2.4.63, the EL–semihypergroup constructed from quasi-ordered semigroup $(\mathbb{N}, \gcd, |)$, is extensive. However, if we instead of the greatest common divisor regard minimum of natural numbers – see Example 2.2.3, then the respective EL–semihypergroup is not extensive.

Example 2.4.68. If we denote by $\mathcal{P}(S)$ a power set of an arbitrary set S and construct EL–semihypergroups in the same way as in Example 2.2.10, we get that $(\mathcal{P}(S), \oplus)$, where

$$A \oplus B = [A \cup B)_{\subseteq} = \{X \in \mathcal{P}(S) \mid A \cup B \subseteq X\}$$

is extensive while $(\mathcal{P}(S), \bullet)$, where

$$A \bullet B = [A \cap B)_{\subseteq} = \{Y \in \mathcal{P}(S) \mid A \cap B \subseteq Y\},$$

is not extensive.

[15] Notice that some authors, such as Polat [262], use the name "closed" too yet they use it in a more loose definition of *closedness* where only $a \in a * b$ is required. Such a definition is motivated by the need to study geometry motivated join spaces. In the sense of Massouros [206] this is *left closedness* and it should not be confused with *extensivity*.

Example 2.4.69. If we in Example 2.4.66 change "min" to "max" or to "+", then $(S, *)$ are not extensive. Also, the transposition hypergroup $(\mathbb{LA}_2(I), *)$ from Example 2.2.12 is not extensive. The semihypergroup $(\mathbb{N}, *)$ from Example 2.4.73, which makes use of the divisor function, is not extensive.

Example 2.4.70. Pickett's lattice hyperoperations used in Example 2.2.1 on page 29 are extensive.

If we consider the condition given by Lemma 2.1.2, i.e. "for any pair $a, b \in S$ there exists a pair $c, c' \in S$ such that $b \cdot c \leq a$ and $c' \cdot b \leq a$", then we immediately get the following theorem.

Theorem 2.4.71. *Every extensive EL–semihypergroup is a hypergroup.*

Proof. Obvious because extensivity in EL–semihypergroups means that $a \cdot b \leq a$ for all $a, b \in S$. Thus it is sufficient to set $c = c' = a$ and apply Lemma 2.1.2. □

By this, the issue of "reaching" hypergroups has been settled. What has not been settled by Theorem 2.4.71, though, is the issue of transposition hypergroups and join spaces as in Lemma 2.1.5 we need groups to construct these.

Even though there is in fact no more to say regarding extensivity and the issue of constructing hypergroups, we include the following results which will be very useful in our future considerations in Section 3.3. In order to prepare ground for them, let us change the definition of the EL–hyperoperation from (2.1) to

$$a *_m b = \{a, b\} \cup [a \cdot b)_{\leq}. \tag{2.22}$$

Since we are discussing extensive hyperoperations at the moment, this is just a technicality because $\{a, b\} \subseteq a * b$ for all $a, b \in S$ anyway. Nevertheless, we will use "$*_m$" whenever we will want to make the extensivity of "$*$" explicit. In this respect, "m" will stand for *modified*. Later on, in Section 3.3, we will discuss situations when the hyperoperation "$*$" is not necessarily extensive and the modification provided by (2.22) will secure it. In that case, the below included results will remain valid.

Theorem 2.4.72. *Every binary EL–hyperstructure, where the hyperoperation is defined by* (2.22) *instead of* (2.1)*, is an H_v–group.*

Proof. We have to show that $(H, *_m)$ is *weak associative* and that it satisfies the *reproductive law*. In order to test weak associativity we have to show that sets $a *_m (b *_m c)$ and $(a *_m b) *_m c$ have non-empty intersection for all

$a,b,c \in H$. Yet it is obvious that $\{a,b,c\} \in a *_m (b *_m c) \cap (a *_m b) *_m c$. For the reproductive law, we must test that $a *_m H = H *_m a = H$ for all $a \in H$. Yet $a *_m H = \bigcup\limits_{x\in H} a *_m x = \bigcup\limits_{x\in H} \Big(\{a,x\} \cup [a\cdot x)_\leq\Big)$ and since obviously $\bigcup\limits_{x\in H} \{a,x\} = H = \bigcup\limits_{x\in H} \{x,a\}$, the reproductive law holds. □

In the above theorem we intentionally write "EL–hyperstructure" (we could also write "EL–hypergroupoids"). This is because its proof does not require that the relation "$\leq$" and single-valued operation "$\cdot$" are compatible. It simply says that if we apply (2.22) on an *arbitrary* groupoid H, we get an H_v–group.

Example 2.4.73. Consider multiplication of natural numbers, i.e. a monoid $(\mathbb{N},\cdot)$, and regard a relation given by values of the divisor function. To be more precise, put $a \leq_\sigma b$ whenever $\sigma_0(a) \leq \sigma_0(b)$, i.e. whenever the number of divisors of a is smaller or equal to the number of divisors of b (or, alternatively, strictly smaller). If we define, for all $a,b \in \mathbb{N}$, that $a * b = \{a,b\} \cup \{x \in \mathbb{N} \mid \sigma(a\cdot b) \leq \sigma(x)\}$, we obtain an H_v–group $(\mathbb{N},*)$. As an example we have that $3 * 5 = \{3,5\} \cup \{6,8,10,12,14,15,16,18,\dots\}$ because $\sigma_0(15) = 4$. Notice that e. g. $\sigma_0(2) = \sigma_0(3) = \sigma_0(11) = 2$ and $\sigma_0(4) = 3$ while $\sigma_0(2\cdot 2) = 3$, $\sigma_0(2\cdot 11) = 4$ or $\sigma_0(3\cdot 8) = 8$, $\sigma_0(4\cdot 8) = 6$, which means that "$\leq_\sigma$" and "$\cdot$" are not compatible.

Lemma 2.4.74. *Let S be a set endowed with an operation "$\cdot$" and a relation "$\leq$". Binary hyperoperation $* : S \times S \to \mathcal{P}^*(S)$ defined by $a * b = [a\cdot b)_\leq$ is associative if and only if binary hyperoperation $*_m : S\times S \to \mathcal{P}^*(S)$ defined by $a *_m b = \{a,b\} \cup [a\cdot b)_\leq$ is associative.*

Proof. We have

$$a *_m (b *_m c) = \bigcup_{x\in b *_m c} a *_m x = \bigcup_{x\in\{b,c\}\cup[b\cdot c)_\leq} \Big(\{a,x\} \cup [a\cdot x)_\leq\Big)$$

while

$$(a *_m b) *_m c = \bigcup_{y\in a *_m b} y *_m c = \bigcup_{y\in\{a,b\}\cup[a\cdot b)_\leq} \Big(\{y,c\} \cup [y\cdot c)_\leq\Big).$$

Yet in the first case we have

$$\bigcup_{x\in\{b,c\}\cup[b\cdot c)_\leq} \{a,x\} = \{a,b,c\} \cup [b\cdot c)_\leq$$

and

$$\bigcup_{x\in\{b,c\}\cup[b\cdot c)_\leq} [a\cdot x)_\leq = [a\cdot b)_\leq \cup [a\cdot c)_\leq \cup \bigcup_{x\in[b\cdot c)_\leq} [a\cdot x)_\leq$$

while in the second case we have

$$\bigcup_{y\in\{a,b\}\cup[a\cdot b)_\leq} \{y,c\} = \{a,b,c\} \cup [a\cdot b)_\leq$$

and

$$\bigcup_{y\in\{a,b\}\cup[a\cdot b)_\leq} [y\cdot c)_\leq = [a\cdot c)_\leq \cup [b\cdot c)_\leq \cup \bigcup_{y\in[a\cdot b)_\leq} [y\cdot c)_\leq.$$

Thus we see that $a *_m (b *_m c) = (a *_m b) *_m c$ if and only if

$$\bigcup_{x\in[b\cdot c)_\leq} [a\cdot x)_\leq = \bigcup_{y\in[a\cdot b)_\leq} [y\cdot c)_\leq. \tag{2.23}$$

Yet this is exactly $a*(b*c) = (a*b)*c$ using the original hyperoperation (2.1) and conditions of Lemma 2.1.1. □

And Theorem 2.4.71 can now be considered also as a simple corollary to Theorem 2.4.72 and Lemma 2.4.74.

Corollary 2.4.75. *Every EL–hypergroup constructed from a quasi-ordered group, where the hyperoperation is defined by* (2.22) *instead of* (2.1)*, is a transposition hypergroup.*

Proof. Straightforward transfer of the proof of Lemma 2.1.5 included on page 43. □

Notice that extensivity of the EL–hyperoperation has some important implications for properties of EL–hyperstructures. Once again, the following results will also hold for EL–semihypergroups, where the hyperoperation is defined by (2.22) instead of (2.1) or those where the hyperoperation defined by (2.1) is such that (2.1) is extensive. For short, we will refer to these as mEL–hyperstructures (to be more precise, we could refer to them as to mEL–hypergroups because given Theorem 2.4.71 they are always hypergroups but because of the forthcoming Remark 2.4.81 we do not). It is important to realize that later on, in Section 3.3, we will utilize the fact mentioned in Remark 2.4.81 and we will denote by mEL–hyperstructures (or, to be more precise, mEL–hypergroupoids, since, at the moment, we are considering one hyperoperation only) a more general class of hyperstructures – those where the hyperoperation is defined by (2.22) (or extensive (2.1)) yet such that Lemma 2.1.1 cannot be used because we drop the condition that the semigroup is quasi-ordered. However, since this change in definition will have no impact on the below included results, we already use the term *mEL–hyperstructures* at this place – it is completely irrelevant whether by mEL–hyperstructures we mean extensive EL–semihypergroups or hyperstructures defined using Definition 3.3.10 on page 164.

Theorem 2.4.76. *In an mEL–hypergroupoid $(H, *_m)$:*

1. *Every element is an identity.*[16]

2. *No element is the scalar identity.*

3. *No element is a zero scalar.*

4. *$i(a) = H$ for all $a \in H$, where $i(a)$ is the set of inverses of $a \in H$.*

   The set of inverses of an arbitrary element $a \in H$ equals H.

Proof. For a fixed yet arbitrary $e \in H$ we have that $\{e, x\} \subseteq e *_m x$ for all $x \in H$. Thus e is an identity of $(H, *_m)$. If $e \in H$ is fixed, then there can obviously never be $\{e\} = e *_m x$ for all $x \in H$ because this can happen only in the special case of $x = e$. The same is true for zero scalar elements. The statement concerning the set of inverses is obvious. □

Corollary 2.4.77. *No mEL–hypergroupoid is a canonical hypergroup.*

Proof. Obvious because one of the defining axioms of canonical hypergroups is having the scalar identity. □

Theorem 2.4.78. *Every mEL–hypergroupoid is regular and reversible.*

Proof. Obvious if we recall that regularity of a hypergroupoid is defined as the fact that every element has at least one inverse; reversibility means that $y \in a \circ x$ implies that there exists an inverse a' of a such that $x \in a' \circ y$ (and likewise for $y \in x \circ a$) for all $a, x, y \in (H, \circ)$. It is enough to set $a' = x$. □

Theorem 2.4.79. *In an mEL–hypergroupoid $(H, *_m)$:*

1. *Every subset $A \subseteq H$ is invariant.*

2. *Every subhypergroup K of H has the defining property of ultraclosedness.*[17]

[16] In this respect recall Example 2.2.1 on page 29 which states the same in its special context.

[17] The concept of an *ultraclosed subhypergroup* is usually defined in hypergroups only. However, in the proof one can see that the desired property holds regardless of the nature of H.

3. Every subhypergroup K of a hypergroup H is invertible and closed.

Proof. *Invariantness* of a subset A of H, where $(H, \circ)$ is a hypergroupoid, is defined by validity of $x \circ A = A \circ x$ for all $x \in H$. Yet if we realize that

$$x *_m A = \bigcup_{a \in A} \{x, a\} \cup [x \cdot a)_{\leq} = \bigcup_{a \in A} \{x, a\} \cup \{b \in A \mid x \cdot a \leq b\} = \bigcup_{a \in A} \{x, a\} = A \cup \{x\}$$

and the same for $A *_m x$, the proof of invariantness becomes a simple corollary of Theorem 2.4.72.

A subhypergroup K of $(H, \circ)$ is called *ultraclosed* if for all $x \in H$ there is $K \circ x \cap (H - K) \circ x \neq \emptyset$ and $x \circ K \cap x \circ (H - K) \neq \emptyset$. Yet in our case x always belongs to these intersections. The fact that K is always *invertible* follows immediately from Corsini and Leoreanu [95], 37. Theorem, part (ii), p. 8, while the fact that K is always *closed*, follows immediately from [95], 37. Theorem, part (iii), p. 8 (in both cases because K is ultraclosed and we suppose that $(H, *_m)$ is a hypergroup). □

Corollary 2.4.80. *In an mEL–hypergroup if $\{A_i\}_{i \in I}$ is a family of subhypergroups, then $A = \bigcap_{i \in I} A_i$ is a subhypergroup.*

Proof. Follows immediately from Theorem 2.4.78, from Corsini and Leoreanu [95], 43. Theorem, p. 10, and Theorem 2.4.79. □

Remark 2.4.81. Notice that some properties of the original EL–hyperstructures (including the above ones) were discussed in Subsection 2.4.5. However, the results of Subsection 2.4.5 assume that the single-valued structure $(H, \cdot, \leq)$, which is used for construction of the EL–hyperstructure, is a *quasi-ordered group*. Yet nowhere in the proof of Theorem 2.4.76, Corollary 2.4.77, Theorem 2.4.78, Theorem 2.4.79 or Corollary 2.4.80 the fact that $(H, \cdot, \leq)$ is a quasi-ordered group, or its inverse elements, are used. As a result, given our modification of the hyperoperation (or its extensivity), the results are now valid for $(H, \cdot, \leq)$ being an *arbitrary semigroup* with a relation "$\leq$".

Remark 2.4.82. Here we must mention a small slip done by Corsini [92] which is repeated in Corsini and Leoreanu [95]. In [92], p. 39, 85. Theorem, and later in [95], p. 8, 34. Theorem, the authors say that for a hypergroup H, its subhypergroup A and the set of partial identities of H, denoted as I_p, there holds "A is ultraclosed $\Leftrightarrow$ A is closed and A contains I_p". However, the property of being ultraclosed means that there cannot be $A = H$ (because this would mean that we test the intersection of $x \circ H$ and $x \circ \emptyset$, which has no meaning). Yet mEL–hyperstructures are such that $I_p = H$. In this case, words "A contains I_p", i.e. $I_p \subseteq A$, result in a contradiction because they mean that $A = I_p = H$. Thus, the theorem is valid only for hypergroups where $I_p \neq H$. If one does not realize this (which is possible because [95] gives no proof of this theorem – as it is included in the "Basic notions" chapter), one can run into difficulties because the following can be – *incorrectly!* – proved.

Let $(H, *_m)$ be the mEL–hypergroup of a quasi-ordered semigroup $(H, \cdot, \leq)$. Then $(H, *_m)$ has no proper subhypergroups.

Indeed, thanks to extensivity of "$*$", H is a hypergroup. Yet [95], 34. Theorem, p. 8, states that in a hypergroup H a subhypergroup K is ultraclosed if and only if it is closed and contains the set of partial identities. For extensive hyperoperations, the concepts of partial identity and identity coincide. However, by Theorem 2.4.76, the set of (partial) identities equals H and by Theorem 2.4.79, every subhypergroup of an mEL–hyperstructure is both closed and ultraclosed. Therefore, under our assumptions, $H \subseteq K$, which means that $K = H$.

The following example shows an mEL–hypergroup, constructed from a quasi-ordered semigroup, and one of its proper subhypergroups.

Example 2.4.83. For a fixed $n \in \mathbb{N}$ suppose the set $\mathbb{M}_{n,n}(\mathbb{R}^+)$ of square matrices with entries of positive real numbers. Regard $(\mathbb{M}_{n,n}(\mathbb{R}^+), +, \leq_M)$, where "$+$" is the usual addition of matrices and, for all $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{n,n}(\mathbb{R}^+)$, set

$$\mathbf{A} \leq_M \mathbf{B} \text{ if } b_{ij} \leq a_{ij} \text{ for all } i, j \in \{1, 2, \ldots, n\}. \tag{2.24}$$

Obviously, $(\mathbb{M}_{n,n}(\mathbb{R}^+), +, \leq_M)$ is a partially ordered semigroup. If we define, using the "Ends lemma", hyperoperation "$*$" by

$$\mathbf{A} * \mathbf{B} = \{\mathbf{C} \in \mathbb{M}_{n,n}(\mathbb{R}^+) \mid \mathbf{A} + \mathbf{B} \leq_M \mathbf{C}\} \tag{2.25}$$

for all $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{n,n}(\mathbb{R}^+)$, then $(\mathbb{M}_{n,n}(\mathbb{R}^+), *)$ is an extensive EL–semihypergroup. Since the hyperoperation "$*$" is obviously extensive, we have that $(\mathbb{M}_{n,n}(\mathbb{R}^+), *)$ is a hypergroup, i.e. an mEL–hypergroup. Now, regard $\mathbb{M}^d_{n,n}(\mathbb{R}^+) \subset \mathbb{M}_{n,n}(\mathbb{R}^+)$, the set of all diagonal matrices from $\mathbb{M}_{n,n}(\mathbb{R}^+)$, i.e. such matrices that $a_{ij} = 0$ for all $i \neq j$ (in other words the null matrix is included in $\mathbb{M}^d_{n,n}(\mathbb{R}^+)$). Since the sum of diagonal matrices is again a diagonal matrix and, by Definition 2.4.30 on page 53, $\mathbb{M}^d_{n,n}(\mathbb{R}^+)$ is an upper end of $\mathbb{M}_{n,n}(\mathbb{R}^+)$, we have that $(\mathbb{M}^d_{n,n}(\mathbb{R}^+), *)$ is a subhypergroupoid of $(\mathbb{M}_{n,n}(\mathbb{R}^+), *)$. Since we can apply the above reasoning concerning the "Ends lemma" and extensivity also on $(\mathbb{M}^d_{n,n}(\mathbb{R}^+), *)$, we get that the $(\mathbb{M}^d_{n,n}(\mathbb{R}^+), *)$ is a proper subhypergroup of $(\mathbb{M}_{n,n}(\mathbb{R}^+), *)$.

2.4.7 When the underlying structure is not a group. Proper semihypergroups.

Results of this subsection were published by European Journal of Combinatorics (WoS Q2) as Novák [242].

In the previous subsection we saw that regardless of having a semigroup or a group, the extensivity of the hyperoperation results in $(H, *)$ being a hypergroup. Now, let us focus on cases when $(H, \cdot)$ is a *semigroup which is not a group* and look for conditions equivalent to the one mentioned in Lemma 2.1.2 in cases when the hyperoperation is *not extensive*. Also, we will be interested in cases when we are able to show that the EL–semihypergroup is *not* a hypergroup.

Definition 2.4.84. A semigroup which is not a group is called a *proper semigroup*. A semihypergroup which is not a hypergroup is called a *proper semihypergroup*.

Chattopadhyay in his paper [42], which is a part of the investigation of Γ–*hypergroups* initiated by Sen and Saha's paper [281], studies semihypergroups with inverse elements, which is an extension of the usual concept of a regular hypergroup onto semihypergroups.

Definition 2.4.85. ([42], Definition 2.7) A semihypergroup $(S, *)$ is called a *regular hypergroup* or simply an r*–hypergroup* if it has an identity e and for each $a \in S$ there exists an inverse $a' \in S$.

For these r–hypergroups (which in spite of their name are – given the definition – still only *semi*hypergroups) he then proves the following result.

Lemma 2.4.86. ([42], Theorem 2.8) *Every r–hypergroup is a hypergroup.*

If we combine this result with Theorem 2.4.13, the condition of Lemma 2.1.2 may be simplified considerably.

Theorem 2.4.87. *Let $(S, *)$ be the EL–semihypergroup of a quasi-ordered monoid $(S, \cdot, \leq)$ with the identity u. If for an arbitrary $a \in S$ there exists an element $b \in S$ such that $a \cdot b \leq u$ and $b \cdot a \leq u$, then $(S, *)$ is an r–hypergroup, and consequentely, a hypergroup.*

Proof. In order to prove the theorem, we are going to use Definition 2.4.85 and Theorem 2.4.13. Therefore, we will concentrate on the issue of inverses of an arbitrary element $a \in S$.

An element $a' \in S$ is defined as an inverse of an arbitrary $a \in S$ if there exists an identity e of $(S, \circ)$ such that $a \circ a' \ni e \in a' \circ a$. In the "Ends lemma" context this means that there must simultaneously be $a \cdot a' \leq e$ and $a' \cdot a \leq e$. However, Theorem 2.4.13 states that all identities of $(S, *)$ are such that $e \leq u$. Since the relation "$\leq$" is transitive, we have that there must be $a \cdot a' \leq u$ and $a' \cdot a \leq u$. Thus, if such $a' \in S$ (or $b \in S$ in the terminology of Theorem 2.4.87) exists for every $a \in S$, then every element of S has an inverse in $(S, *)$, which means that $(S, *)$ is an r–hypergroup. Thus according to Lemma 2.4.86, it is a hypergroup. □

Remark 2.4.88. Notice that in a general context Lemma 2.4.86 is not an equivalence; in Chattopadhyay [42] an appropriate example, which makes use of a hypergroup with no identity, is included. Even in the "Ends lemma" context the converse of Lemma 2.4.86 does not hold in general. An EL–hypergroup $(H, *)$ may either be based on a group or on a semigroup. If it is based on a group, then the converse does hold – trivially, because since the neutral element u of $(H, \cdot)$ is an identity of $(H, *)$, a, a^{-1} are mutual inverses in $(H, *)$ for all $a \in H$. However, if it is based on a semigroup, we cannot decide.

Yet for mEL–semihypergroups, i.e. once extensivity of "$*$" is involved, we get, by Theorem 2.4.79, the following obvious corollary.

Corollary 2.4.89. *Every mEL–semihypergroup is an r–hypergroup.*

Now we turn our attention to *conditions under which a proper semigroup gives rise to a proper EL–semihypergroup.*

Theorem 2.4.90. *Let $(S, *)$ be the EL–semihypergroup of a quasi-ordered semigroup $(S, \cdot, \leq)$ with at least two distinct elements. If $(S, \cdot)$ is a monoid, denote its identity u. The semihypergroup $(S, *)$ is not a hypergroup if and only if there exists $x \in S$, such that for some $a \in S$:*

(i) x is incomparable with all products $a \cdot b$ and $b \cdot a$ for all $b \in S$ and $x \neq u$, or

(ii) there is $x < a \cdot b$ and $x < b \cdot a$ for all $b \in S$ (with explicit exclusion of cases $a = x, b = u$ and $a = u, b = x$).

Proof. The reproductive law states that for every $a \in S$ there must be $a * S = S * a = S$. If this holds, the semihypergroup $(S, *)$ becomes a hypergroup. The law holds if all of the below inclusions hold simultaneously:

1. $a * S \subseteq S$ and $S \subseteq a * S$,
2. $S * a \subseteq S$ and $S \subseteq S * a$.

Validity of $a * S \subseteq S$ or $S * a \subseteq S$ is obvious in the "Ends lemma" context because $a * S = \bigcup_{b \in S} [a \cdot b)_{\leq}$, where $[a \cdot b)_{\leq} = \{x \in S; a \cdot b \leq x\}$, i.e. $x \in a * S$ (as well as $x \in S * a$) always implies $x \in S$.

Therefore concentrate on the other two inclusions. Suppose that $S \subseteq a*S$ does *not* hold for some $a \in S$. This means that for some $a \in S$ there exists an $x \in S$ such that $x \notin a * S$. Since in the "Ends lemma" context $a * S = \bigcup_{b \in S} [a \cdot b)_{\leq}$, we have that the condition in fact means that for some $a \in S$ there is $x \notin [a \cdot b)_{\leq}$ for all $b \in S$. In an analogous way, the fact that $S \subseteq S * a$ does *not* hold for some $a \in S$ implies that $x \notin [b \cdot a)_{\leq}$ for all $b \in S$. Yet $x \in [a \cdot b)_{\leq}$ means that $a \cdot b \leq x$. Thus the fact that the reproductive law does not hold for $(S, *)$ is equivalent to the fact that this condition (as well as $b \cdot a \leq x$) may not hold for all $b \in S$ for some $a \in S$. If we consider that the relation "$\leq$" is by definition reflexive (which excludes cases $a = x, b = u$ and $a = u, b = x$ for which $a \cdot b \leq x$ and $b \cdot a \leq x$ must by definiton always hold), we get the theorem. $\square$

Remark 2.4.91. Notice that

(i) If $(S, \cdot)$ is a monoid, under condition *(i)* of Theorem 2.4.90 we shall first of all test existence of an element $x \in S$, $x \neq u$, incomparable with all $s \in S$, $x \neq s$.

(ii) In *(ii)* of Theorem 2.4.90 there cannot be $a = x = u$. Indeed, if we then took $a = b = x = u$, we would require $x < x$, which is not possible.

(iii) If there exists an element $x \in S$ such that it is incomparable with all products $a \cdot b$ for all $a, b \in S$, condition *(i)* of Theorem 2.4.90 holds trivially. The same holds if $x < a \cdot b$ for all $a, b \in S$.

Example 2.4.92. Consider the ordered monoid $(\mathbb{N}_0, +, \leq)$ of all non-negative integers. Then its EL–semihypergroup $(\mathbb{N}_0, *)$, where $a * b = [a+b)_{\leq} = \{x \in \mathbb{N}_0 \mid a + b \leq x\}$, is not a hypergroup because $p = 0 \in \mathbb{N}_0$ has the property *(ii)* for an arbitrary nonzero $a \in \mathbb{N}_0$. Simultaneously, as has been shown in Remark 2.4.91, *(ii)*, case $a = 0$ is not possible.

In Examples 2.4.93 to 2.4.95 we use a rather simple example of a potent set of an arbitrary set S to demonstrate various strategies of determining whether a proper semigroup creates an EL–semihypergroup or an EL–hypergroup.

Example 2.4.93. Let S be an arbitrary set and $\mathcal{P}^*(S)$ the system of its non-empty subsets. Then $(\mathcal{P}^*(S), \cup)$ is a proper semigroup. From Example 2.2.10

we already know that $(\mathcal{P}^*(S), *)$, where the hyperoperation "$*$" is on $\mathcal{P}^*(S)$ defined by

$$A * B = [A \cup B)_{\subseteq} = \{X \in \mathcal{P}^*(S); A \cup B \subseteq X\} \tag{2.26}$$

for an arbitrary pair of subsets A, B of S, is a semihypergroup. Since $(\mathcal{P}^*(S), \cup)$ is not a group, we cannot use Corollary 2.1.3 to decide whether $(\mathcal{P}^*(S), *)$ is a hypergroup.

$(\mathcal{P}^*(S), \cup)$ is not a monoid because the empty set, which would have been the neutral element with respect to set union, is by definition not included in $\mathcal{P}^*(S)$. Theorem 2.4.87 thus cannot be used.

As far as Theorem 2.4.90 is concerned, according to *(i)* we must look for such subset $X \subseteq S$ that, for some subset $A \subseteq S$ and all subsets $B \subseteq S$, X is incomparable with $A \cup B$, where $X, A, B \neq \emptyset$. Yet this is not possible because every subset $X \subseteq S$ is always comparable at least with S.

According to *(ii)* we look for such a subset $X \subseteq S$ that for some subset $A \subseteq S$ there is $X \subsetneqq A \cup B$ for all subsets $B \subseteq S$. Yet if we take $A = S$ then obviously for all $X \neq S$ there is $X \subsetneqq S \cup B$ for all $B \subseteq S$. Thus $(\mathcal{P}^*(S), *)$ is not a hypergroup.

Notice that the conclusion reached in Example 2.4.93 is compatible with Corollary 2.1.3. Indeed, if we consider e.g. A a one element subset of S, B a two element subset of S, then obviously no subsets $C, C' \subseteq S$ such that $B \cup C \subseteq A$ and $C' \cup B \subseteq A$ exist.

Example 2.4.94. If in Example 2.4.93 we regard $\mathcal{P}(S)$ instead of $\mathcal{P}^*(S)$, i.e. include the empty set, $(\mathcal{P}(S), \cup)$ has the neutral element $\emptyset$ and we can consider using Theorem 2.4.87. In our case it says that if to an arbitrary subset $A \subseteq S$ there exists a subset $B \subseteq S$ such that $A \cup B \subseteq \emptyset$, then $(\mathcal{P}(S), *_{\cup})$, the EL–semihypergroup of $(\mathcal{P}(S), \cup, \subseteq)$, is a hypergroup, i.e. it obviously cannot be used. Reasoning concerning Theorem 2.4.90 remains the same as in Example 2.4.93 with the exception that the condition $X, A, B \neq \emptyset$ need not be imposed. This makes no difference, though.

Example 2.4.95. If in Example 2.4.94 we change the single-valued operation to intersection, i.e. regard an arbitrary set S, system of its subsets $\mathcal{P}(S)$, the partially ordered semigroup $(\mathcal{P}(S), \cap, \subseteq)$ with neutral element S and construct the hyperoperation analogically as in (2.26), only changing set union to set intersection, we have that $(\mathcal{P}(S), *_{\cap})$ is a semihypergroup. To an arbitrary pair of subsets $A, B \subseteq S$ there always exists a subset $C \subseteq S$ such that $B \cap C \subseteq A$ as we can set $C = \emptyset$. Thus, by Lemma 2.1.2, $(\mathcal{P}(S), *_{\cap})$ is a hypergroup.

However, to every subset $A \subseteq S$ there also exists a subset $B \subseteq S$ such that $A \cap B \subseteq \emptyset$ as we can set $B = \emptyset$ for an arbitrary A. Thus we get the same result using Theorem 2.4.87.

Reasoning concerning Theorem 2.4.90, *(i)* remains the same as in Example 2.4.93. As for Theorem 2.4.90, *(ii)*, consider an arbitrary subset $A \subseteq S$ and examine all subsets $B \subseteq S$. If we take $B = S \setminus A$, then $A \cap B = \emptyset$ and we are looking for a subset $X \subseteq S$ such that $X \subsetneqq \emptyset$. This is obviously impossible for all A, B. Thus by Theorem 2.4.90 we again reach the same conclusion.

Example 2.4.96. In Example 2.4.92 we used the ordered monoid $(\mathbb{N}_0, +, \leq)$. If we change the single-valued operation from addition to meet, it is easy to verify that we get another example of a semigroup (not monoid) such that its associated hyperstructure is a hypergroup. To be more precise, suppose $(\mathbb{N}_0, \wedge, \leq)$ and define

$$m * n = [m \wedge n)_{\leq} = \{p \in \mathbb{N}_0 \mid \min\{m, n\} \leq p\}.$$

In general, the construction may be applied to any lower semilattice $(S, \wedge)$ to obtain the same result.

Chattopadhyay [42] also works with the concept of a *regular semihypergroup* which is taken over from the theory of semigroups and in fact is a generalization of the idea of regularity in semigroups[18] (*not* of regularity in hypergroups defined on page 16). Recall that a semigroup $(S, \cdot)$ is *regular* if for an arbitrary element $a \in S$ there exists an element $b \in S$ such that $a \cdot b \cdot a = a$.

Definition 2.4.97. ([42], Definition 2.19) Let $(S, *)$ be a semihypergroup and $a \in S$. The element a is called *regular* if there exists an element $b \in S$ such that $a \in a * b * a$. The semihypergroup S is called *regular* if every element of S is regular.

Lemma 2.4.98. ([42], Theorem 2.20) *Every r–hypergroup is a regular semihypergroup.*

This has a direct corollary in the EL–context.[19]

[18] In fact, the notion of *regularity* was introduced by von Neumann in his paper [236] for *rings*. In linear algebra the straigthforward transfer of the definition of a regular element links to a very important concept of a *pseudo-regular matrix*. For details on *regular ordered semigroups* see e.g. Chvalina [46].

[19] Obviously, every mEL–semihypergroup is regular by default because it is an r–hypergroup.

Corollary 2.4.99. *Let $(S,*)$ be the EL–semihypergroup of a quasi-ordered monoid $(S,\cdot,\leq)$ with the neutral element u. If for an arbitrary $a \in S$ there exists $b \in S$ such that $a \cdot b \leq u$ and $b \cdot a \leq u$, then $(S,*)$ is a regular semihypergroup.*

Proof. Follows immediately from Theorem 2.4.87 and its proof. □

In Corollary 2.4.99 the monoid does not need to be regular. If it is, regularity of its EL–semihypergroup is secured.

Theorem 2.4.100. *The EL–semihypergroup $(S,*)$ of a regular quasi-ordered monoid $(S,\cdot,\leq)$ is regular.*

Proof. Chvalina in [44], Theorem 1.7, p. 149, proves that in a semihypergroup associated to a regular partially ordered monoid $(S,\cdot,\leq)$ with the neutral element u there exists to every $a \in S$ an element $b \in S$ such that $a*b*a = a*u$. If we realize that $a*u = [a \cdot u)_\leq = [a)_\leq$ for all $a \in S$ and the neutral element u of $(S,\cdot)$ and that $a \in [a)_\leq$ for an arbitrary $a \in S$, we immediately get the theorem. □

Let us now concentrate on the issue of *hyperideals* in EL–semihypergroups and hypergroups. Since we are working in hyperstructures with one hyperoperation, this is to be distinguished from hyperideals in ring-like hyperstructures as studied in e.g. the book by Davvaz and Leoreanu–Fotea [111] and papers stemming from it.

Recall that using the "Ends lemma" we construct semihypergroups, some of which are hypergroups. If we start with a quasi-ordered group $(H,\cdot,\leq)$, we get a hypergroup. Under Corollary 2.4.99 we also get a hypergroup, because the condition given in the corollary is a special case of condition given in Lemma 2.1.2 on page 27. Yet describing hyperideals of hypergroups is of no point since the smallest left hyperideal containing an element $a \in H$ is in this case $H * a$ ($a * H$ for a right hyperideal) which in hypergroups equals H (thanks to the reproductive law).

Yet Theorem 2.4.90 describes proper EL–semihypergroups. In these cases we may apply Theorem 2.4.101. The theorem is a simple corollary of results on hyperideals included in Chattopadhyay [42]. Other results contained in [42] may be directly applied to EL–semihypergroups described by Theorem 2.4.90 as well.[20]

[20]Notice that this particular paper deals with various types of hyperstructures with binary relations including Chvalina's *quasi-order hypergroups* introduced in [44], studied in e.g. [43,151] and included in [95]. For more on these, see Section 1.2 and Subsection 2.6.3.

Theorem 2.4.101. *Let $(S,*)$ be the EL–semihypergroup of a quasi-ordered semigroup $(S,\cdot,\leq)$ and I a hyperideal of S. If $(S,*)$ is regular, then I is regular and any hyperideal J of I is a hyperideal of S. Moreover, in this case, for any one-sided hyperideal I of S there is $I*I=I$.*

Proof. The theorem is a corollary to Theorem 2.25 and Theorem 2.27 of [42]. □

Corollary 2.4.102. *A nonempty subset I of a regular EL–semihypergroup $(S,*)$ is a hyperideal of S if and only if it is an idempotent subsemihypergroup of $(S,*)$.*

Proof. Obvious because $I*I\subseteq I$ is a defining property of a subsemihypergroup while $I\subseteq I*I$ is a defining property of an idempotent subset. Moreover, by definition every hyperideal is a one-sided ideal. □

Jafarabadi et al. studied various kinds of semihypergroups related to our topic. In [163] and later in [162] they discuss *zero scalar elements* and distinguish between semihypergroups with and without zero scalars. A special class of semihypergroups without zero scalars is called a *simple semihypergroup*. Notice that "proper hyperideal" means "other than itself" in the following definition.

Definition 2.4.103. ([163], Definition 2) A semihypergroup without zero scalars is called *simple* if it has no proper hyperideals.

Lemma 2.4.104. ([162], Corollary 1) *A semihypergroup is simple if and only if for all $a\in S$ there is $S=S*a*S$.*

The following result has already been mentioned in plain text on page 82.

Lemma 2.4.105. ([162], Proposition 2) *Every hypergroup is a simple semihypergroup.*

This leads us to an interesting question: *Are proper EL–semihypergroups simple?* In the following theorem notice that by Corollary 2.4.21 on page 49 we know that in EL–semihypergroups there is at most one zero scalar element, which – if it exists – is, within the single-valued structure, simultaneously EL–maximal and absorbing. Also notice that a complete description of proper EL–semihypergroups has been given by Theorem 2.4.90.

Theorem 2.4.106. *Let $(S,*)$ be a proper commutative semihypergroup. Furthermore, let $(S,*)$ be without zero scalars. Then $(S,*)$ is not simple.*

Proof. By Theorem 2.4.90, the proper semihypergroup $(S,*)$ is associated to a quasi ordered semigroup $(S,\cdot,\le)$ such that there exists $x\in S$ such that for some $a\in S$

(i) x is incomparable with all products $a\cdot b$ and $b\cdot a$ for all $b\in S$ and $x\neq u$, or

(ii) there is $x<a\cdot b$ and $x<b\cdot a$ for all $b\in S$ (with the explicit exclusion of cases $a=x, b=u$ and $a=u, b=x$),

where u is the neutral element of $(S,\cdot)$ – in case it exists. In the "Ends lemma context" the condition given by Lemma 2.4.104 is

$$S*a*S=\bigcup_{t\in \bigcup\limits_{s\in S}[s\cdot a)_\le}[t\cdot r)_\le$$

for all $r\in S$. This may be rewritten to

$$\bigcup_{r\in S}\left[\bigcup_{t\in S}([t\cdot a)_\le)\cdot r\right)_\le=\bigcup_{r\in S}\left[\bigcup_{t\in S}([t\cdot a)_\le\cdot r)\right)_\le$$
$$=\bigcup_{r\in S}\left[\bigcup_{t\in S}[t\cdot a\cdot r)_\le\right)_\le=\bigcup_{r\in S}\left[\bigcup_{t\in S}[t\cdot a\cdot r)_\le\right)=\bigcup_{t,r\in S}[t\cdot a\cdot r)_\le$$

The inclusion $S\subseteq S*a*S$ means that for all $y\in S$ there must exist $t,r\in S$ such that $y\in[t\cdot a\cdot r)_\le$, i.e. $t\cdot a\cdot r\le y$.

If we now restrict on *commutative* semigroups $(S,\cdot)$, we may rewrite this as $a\cdot(t\cdot r)\le y$. Yet it is obvious that none of elements $x\in S$ mentioned in *(i)* or *(ii)* are such that $a\cdot(t\cdot r)\le x$. Thus for such elements x there is $x\notin S*a*S$ and as a result $S\neq S*a*S$, which means that the semihypergroup $(S,*)$ is not simple. □

Item *(iii)* of Remark 2.4.91 may be used to describe some non-commutative cases.

Theorem 2.4.107. *Let $(S,*)$ be the EL–semihypergroup of a non-commutative quasi-ordered semigroup $(S,\cdot,\le)$ with at least two distinct elements. Furthermore, let $(S,*)$ be without zero scalars. In each of the following cases $(S,*)$ is not simple:*

(i) $(S,\cdot)$ is a monoid with the neutral element u and there exists an element $x\in S$, $x\neq u$, incomparable with all $s\in S$, $x\neq s$.

(ii) There exists an element $x\in S$ such that it is incomparable with all products $a\cdot b$ for all $a,b\in S$.

(iii) There exists an element $x \in S$ such that $x < a \cdot b$ for all $a, b \in S$.

Proof. The proof is the same as the proof of Theorem 2.4.107 up to the requirement of existence of elements $t, r \in S$ such that $t \cdot a \cdot r \leq y$. From this place on it is obvious. $\square$

Thus we see that all proper EL–semihypergroups of *commutative* semigroups *do* have proper hyperideals and we are able to identify at least some more cases in the non-commutative context.

To conclude this subsection, let us use the potent set discussed in Examples 2.4.93, 2.4.94 and 2.4.95 to demonstrate some other concepts mentioned above.

Example 2.4.108. Since for an arbitrary subset A of a set S there is $A \cup \emptyset \cup A = A$ and $A \cap S \cap A = A$, we have (by definition for $b = \emptyset$, or $b = S$, respectively) that both $(\mathcal{P}(S), \cup)$ and $(\mathcal{P}(S), \cap)$ are regular. Moreover, since also $A = A \cup A = A \cap A$, we have that an arbitrary $A \subseteq S$ is an idempotent element of both $(\mathcal{P}(S), \cup)$ and $(\mathcal{P}(S), \cap)$. Furthermore, since for every subset $B \subseteq \mathcal{P}^*(S)$ there is $B \subseteq [B)_{\subseteq} = [B \cup B)_{\subseteq} = [B \cap B)_{\subseteq}$, we have that B is an idempotent subset of both $(\mathcal{P}(S), *_{\cup})$ and $(\mathcal{P}(S), *_{\cap})$. Thus, by Theorem 2.4.23, every subset $A \subseteq S$ is an idempotent element of both $(\mathcal{P}(S), *_{\cup})$ and $(\mathcal{P}(S), *_{\cap})$ and there is $[A)_{\subseteq} = A^2 = A^3 = \ldots = A^n$ for all $n \in \mathbb{N}$, $n \geq 2$. Also, by Corollary 2.4.102, every nonempty subset $B \subseteq \mathcal{P}(S)$ is a hyperideal of both $(\mathcal{P}(S), *_{\cup})$ and $(\mathcal{P}(S), *_{\cap})$. The sets $(\mathcal{P}^*(S), \subseteq)$ and $(\mathcal{P}(S), \subseteq)$ have one maximal element, the set S. Thus, by Theorem 2.4.20, it is easy to decide whether they are without zero scalars. The property of being an absorbing element rewrites to $A \cup S = S = S \cup A$ or $A \cap S = S = S \cap A$ for all $A \subseteq S$, respectively. Thus $(\mathcal{P}^*(S), *)$ and $(\mathcal{P}(S), *_{\cup})$ have a zero scalar element S while $(\mathcal{P}(S), *_{\cap})$ is without zero scalars. Since $(\mathcal{P}(S), *_{\cap})$ is commutative, it is, by Theorem 2.4.106, not simple, i.e. has proper hyperideals – which is what we have already established.

2.4.8 The origins of EL–hypergroups

Results of this short subsection were included in Novák [244], published by European Journal of Combinatorics (WoS Q2) .

We already know that when the “Ends lemma” is applied on quasi-ordered semigroups, we get semihypergroups, i.e. Lemma 2.1.1 secures associativity of the hyperoperation. We also know that when the lemma is applied on quasi-ordered groups, we get hypergroups, or rather transposition hypergroups or join spaces. This is thanks to Lemma 2.1.2, which gives a condition equivalent to validity of the reproductive law, and Corollary 2.1.3, which says that this condition holds in groups. We get hypergroups if the EL–semihypergroup is extensive.

In this subsection we will be dealing with a very natural issue. *What are other equivalent conditions of Lemma 2.1.2? If we know that an EL–hyperstructure is a hypergroup, how can we determine whether the original structure, from which it was (or could have been) constructed, is a group or a semigroup?* Also, we will be interested in semigroups giving rise to EL–hypergroups and in the study of the transfer of some properties of the original single-valued structure to EL–hypergroups.

Obviously, if we want to use Lemma 2.1.1 to construct semihypergroups, we know whether the structure we are working with is a semigroup or a group. However, this is *not* the motivation for the results included in this subsection. Imagine we have a semigroup and by means of Lemma 2.1.1 and Corollary 2.1.3 (or thanks to the fact that the hyperoperation is extensive) we construct its EL–hypergroup. Now, consider subhypergroups of this hypergroup. For example, in situations when we need to describe all subhypergroups without actually constructing them or to decide validity of a statement concerning an arbitrary subhyperstructure. Are the subhypergroups associated to subgroups of the original group or to its subsemigroups? Both cases are possible and even the mixture of cases is possible as some of the subhypergroups may be associated to subgroups while some may be associated to proper subsemigroups. Yet proofs of some results (such as e.g. Theorem 2.4.52) could simplify if we knew the answer. Moreover, some results regarding EL–hyperstructures (see Subsection 2.4.5 on page 59 or Novák [246]) could be generalized as they assume subhypergroups associated to subgroups yet this need not be a comprehensive list of subhypergroups.

First of all, we are going to include a few results concerning *idempotent* elements, i.e. elements which (if we ignore the neutral element) exist in semigroups only. We will be looking for negative results, i.e. for statements "$(H,\cdot)$ is not a group" because this would mean that it is a semigroup.

Theorem 2.4.109. *Let $(H,*)$ be the EL–semihypergroup of a quasi–ordered semigroup $(H,\cdot,\leq)$. For an arbitrary element $a\in H$ there holds*

$$a*a=\{a\} \Leftrightarrow a \text{ is idempotent and simultaneously } EL\text{–maximal element of } (H,\cdot,\leq).$$

Proof. Suppose the suggested structures $(H,\cdot,\leq)$ and $(H,*)$ and an element $a\in H$.

"⇒" According to the definition of hyperoperation "$*$" there is $a*a=[a\cdot a)_\leq$. Since for the element $a\in H$ there holds $a*a=\{a\}$, there also holds

$[a \cdot a)_\leq = \{a\}$. Since the relation "$\leq$" is reflexive, there must be $a \cdot a \in [a \cdot a)_\leq$, i.e. altogether we get that $a \cdot a = a$, i.e. a is an idempotent of $(H, \cdot)$. Furthermore, there does not exist any element $x \neq a$ such that $a \cdot a = a \leq x$, i.e. a is an EL–maximal element of $(H, \cdot, \leq)$.

"$\Leftarrow$" If a is an idempotent element of $(H, \cdot)$, then $a \cdot a = a$. Since "$\leq$" is reflexive, there is $a \leq a$, i.e. $a \cdot a \leq a$, i.e. $a \in a * a$. Since a is EL–maximal, we have that $[a)_\leq = \{a\}$. Then $a * a = [a \cdot a)_\leq = [a)_\leq = \{a\}$.

□

Corollary 2.4.110. *Let $(H, *)$ be the EL–hypergroup of such a quasi-ordered semigroup $(H, \cdot, \leq)$ that at least two distinct elements $a, b \in H$ are in relation "$\leq$" (i.e. "$\leq$" is not trivial). If there exists an element $a \in H$ such that $a * a = \{a\}$, then $(H, \cdot)$ is not a group.*

Proof. Suppose that $(H, \cdot, \leq)$ is a quasi-ordered group and that there exists an element $a \in H$ such that $a * a = \{a\}$. This means that a is idempotent, which in a group means that $a = u$, where u is the neutral element of the group. However, the fact that $u * u = \{u\}$ also means that there does not exist an element $b \in H$, $b \neq u$, such that $u \leq b$, i.e. for all $b \in H$ there is either $b \leq u$ or u, b are incomparable. Yet $b \leq u$ implies $u \leq b^{-1}$, which is not possible for $u \neq b$, if there should simultaneously be $u < b$ and $u * u = \{u\}$. Also, if $c, d \in H$ are such that $c \leq d$, then $u \leq c \cdot d^{-1}$, which is again not possible for $c \neq d$. Therefore, simultaneous validity of the fact that $(H, \cdot, \leq)$ is a quasi-ordered group and existence of an element $a \in H$ such that $a * a = \{a\}$ implies triviality of the relation "$\leq$". Thus non-triviality of the relation "$\leq$" implies that $(H, \cdot, \leq)$ is not a quasi-ordered group (i.e. it is a quasi-ordered semigroup) or that $a * a = \{a\}$ holds for no $a \in H$. If we suppose non-triviality and existence of the element a, then $(H, \cdot)$ may not be a group. □

The proof of the following corollary is rather obvious.

Corollary 2.4.111. *Let $(H, *)$ be the EL–hypergroup of a quasi-ordered semigroup $(H, \cdot, \leq)$ and $(G, *)$ a subhypergroup of $(H, *)$.*[21]

1. *Denote u the neutral element of $(H, \cdot)$. If u is an EL–maximal element of $(G, \leq)$ and at least two distinct elements $a, b \in G$ are in relation "$\leq$", then $(G, \cdot)$ is a subsemigroup of $(H, \cdot)$ yet not a subgroup of $(H, \cdot)$.*

[21] In Subsection 2.4.4 we made distinctions between "$*_G$" and "$*_S$", to be more precise between (2.12) and (2.13), in defining subhyperstructures. This distinction is not relevant here as the corollary is valid for both cases.

2. *If for two distinct elements $a, b \in H$ there holds $a*a = \{a\}$, $b*b = \{b\}$, then H does not have the greatest element. Moreover, $(H,\cdot)$ is not a group.*

Remark 2.4.112. Notice that in a commutative case the fact stated in Corollary 2.4.110 may be deduced also from Lemma 2.4.15, which states that if $(H,\cdot)$ and $(H,*)$ are commutative and $(H,\cdot)$ a group, then for an identity a of $(H,*)$ there holds that $a \leq a^{-1}$. According to Theorem 2.4.22 the fact that $a \in a*a$ is equivalent to the fact that a is an identity of $(H,*)$, i.e. $a \leq a^{-1}$, i.e. $a \cdot a \leq u$, where u is the neutral element of $(H,\cdot)$. This means that $u \in a*a$. Since in Corollary 2.4.110 we suppose that $a*a$ is a singleton and both a and u belong to $a*a$, they must be equal. Yet this results in triviality of the relation "$\leq$".

After mentioning Theorem 2.4.23 we said that unfortunately, from the validity of *(i)*, *(ii)* or *(iii)* of Theorem 2.4.23 there does not follow that a is an idempotent element of $(S,\cdot)$. Yet we can include at least the following Theorem 2.4.113. Notice that the assumption of *antisymmetry* of "$\leq$" is crucial. Also notice that $(H,\cdot)$ need not be a monoid. If it is, one of the elements assumed in the theorem is obvious.

Theorem 2.4.113. *Let $(H,*)$ be the EL–hypergroup of a partially ordered semigroup $(H,\cdot,\leq)$. If there exists a pair of distinct elements $a, b \in H$ such that there holds $[a)_\leq = a*a$, $[b)_\leq = b*b$, then $(H,\cdot)$ is not a group.*

Proof. Straightforward because $a*a = [a\cdot a)_\leq$ and since the relation "$\leq$" is a partial order, the equality $[a)_\leq = [a\cdot a)_\leq$ implies that $a = a\cdot a$. The same holds for the element b. Thus in $(H,\cdot)$ there are two distinct idempotent elements which means that $(H,\cdot)$ cannot be a group. $\square$

2.4.9 Ideals in EL–semihypergroups. A link to Γ–semihypergroups.

This section is taken over from Ghazavi, Anvariyeh and Mirvakili [137].

In [137], Ghazavi, Anvariyeh and Mirvakili study the issue of *ideals* in EL–semihypergroups. So far, on page 83, we have included a brief discussion of the issue of whether proper EL–semihypergroups do have proper hyperideals (i.e. whether they are simple or not). Notice that the property of "being simple", i.e. "not having a proper hyperideal", is equivalent to the fact that $H*x*H = H$ for all $x \in H$. Also notice that simplicity of a semigroup is defined analogically to the simplicity of a semihypergroup.

Definition 2.4.114. A semigroup $(S, \cdot)$ is called *simple* if it has no proper ideals, i.e. if $a \cdot S = S \cdot a = S$ for all $a \in S$.

Theorem 2.4.115. *Let $(S, *)$ be the EL–semihypergroup of a quasi-ordered semigroup $(S, \cdot, \leq)$. If $(S, \cdot)$ is simple, then also $(S, *)$ is simple.*

Proof. See [137], proof of Theorem 3.12. □

The converse of the above theorem is not true; for a counterexample see [137], Example 3.14.

Among the many concepts of the ideal theory, recall the following.

Definition 2.4.116. Let $(S, \cdot, \leq)$ be a quasi-ordered semigroup and I an (left / right) ideal of S. I is called a *bi-ideal* of S if $I \cdot S \cdot I \subseteq I$ and an *interior ideal* if $S \cdot I \cdot S \subseteq I$. Finally, I is called (m, n)*–ideal* of S if $I^m \cdot S \cdot I^n \subseteq I$. In all of these cases, I is called *ordered* if for all $a \in I$ the fact that $b \leq a$ implies that $b \leq I$. In such a case we write $(I]_{\leq} = I$.

In [137], Ghazavi, Anvariyeh and Mirvakili proved the following.

Theorem 2.4.117. *Let $(S, *)$ be the EL–semihypergroup of a quasi-ordered semigroup $(S, \cdot, \leq)$ and $I \subseteq S$ non-empty.*

1. *If I is a left / right (ordered) ideal of S, which is an upper end of S, then I is a left / right hyperideal of $(S, *)$.*

2. *If S is a monoid, then I is a left / right ideal of S, which is an upper end of S, if and only if I is a left / right hyperideal of $(S, *)$.*

3. *I is maximal / minimal among all left (or right) ideals of S, which are also upper ends of S, if and only if I is maximal among all left (or right) hyperideals of $(S, *)$.*

Proof. See [137], proof of Theorem 3.3, Theorem 3.9 and Theorem 3.19. □

In a similar fashion, the properties of (ordered) bi-hyperideals, interior ideals or (m, n)–ideals can be transferred to the hyperstructure context. For details see [137]. Also, in Section 3.2 we discuss the construction of EL^2–*hyperstructures*, where the authors of [137] applied the "Ends lemma" on quasi-ordered *hyperstructures*; in Theorem 3.2.4 of the section we show how hyperideals of the original hyperstructure are transferred to hyperideals of the resulting hyperstructure.

Finally, Ghazavi, Anvariyeh and Mirvakili show that the "Ends lemma" can be used to construct Γ–*semihypergroups*. Recall that these are a hyperstructure analogy of Γ–*semigroups* introduced by Sen [280], which themselves are based on Nobusawa's Γ–*rings* [239].

Definition 2.4.118. Let S and Γ be two non-empty sets. Then S is called a Γ*–semigroup* if there exists a map $S \times \Gamma \times S \to S$, denoted as $x\gamma y$ for all $x, y \in S$, such that, for all $a, b, c, \in S$ and $\gamma, \delta \in \Gamma$, there is $(a\gamma b)\delta c = a\gamma(b\delta c)$. If moreover $(S, \leq)$ is a partially ordered set and the relation $x \leq y$ implies that $x\gamma y \leq y\gamma z$ and $z\gamma x \leq z\gamma y$ for all $x, y, z \in S$ and $\gamma \in \Gamma$, then S is called a *partially ordered* Γ*–semigroup*. If each $\gamma \in \Gamma$ is a hyperoperation on S, i.e. $x\gamma y \subseteq S$ for all $x, y \in S$, then S is called a Γ*–semihypergroup*.

Theorem 2.4.119. *The EL–semihypergroup of a partial ordered* Γ*–semigroup* $(S, \cdot, \leq)$ *is a* Γ*–semihypergroup.*

Proof. See [137], proof of Theorem 4.7. □

For examples of the above concepts on some finite element sets cf. [137].

2.4.10 A special case: hypersemilattices and hyperlattices

Results of this subsection were published in Proceedings of APLIMAT 2017 (SCOPUS) as Novák, Cristea and Křehlík [248].

In Subsection 2.6, where we discuss relation of EL–hyperstructures to some other concepts of hyperstructure theory in which ordering is used, we include a rather obvious statement that every semilattice can be used to construct an EL–semihypergroup. In this subsection we will include some results connected to H_v*–semilattices* and *hyperlattices* constructed using the "Ends lemma". Later on, in Subsection 2.5.5, we will define some orderings on the set of matrices and make use of the "Ends lemma" to construct semihypergroups on the sets of matrices which – together with some results on lattices – will eventually lead us to examples of H_v*–rings* and H_v*–matrices* over these H_v–rings.

In this subsection we will study hyper(semi)lattices from the perspective of *algebraic structures*. Naturally, one can also approach the topic from the point of view of *ordered hyperstructures*. For a synthesis of these approaches see recent works of Soltani Lashkenari, Rasouli and Davvaz such as [270,283].

First of all we show the connection between EL–(semi)hypergroups and hypersemilattices / H_v–semilattices. For the respective definitions cf. page 12 and 13.

Theorem 2.4.120. *Let* $(L, *)$ *be the* EL*–semihypergroup of a quasi-ordered semigroup* $(L, \cdot, \leq)$.

1. *If "$\cdot$" is commutative and $(L, \cdot)$ is a proper semigroup, then the condition that for all $a \in L$ there holds $a \cdot a \leq a$ is equivalent to the fact that $(L, *)$ is a hypersemilattice.*

2. *If "$\cdot$" is not commutative and "$\leq$" is antisymmetric, then $(L, *)$ is neither a hypersemilattice nor an H_v–semilattice.*

3. *If $(L, \cdot)$ is a non-trivial group and "$\leq$" is antisymmetric, then $(L, *)$ is neither a hypersemilattice nor an H_v–semilattice.*

Proof. Condition 3 of Definition 1.1.19 (in its strong associative version) is secured by default. Therefore, the question of whether the "Ends lemma" construction gives rise to hypersemilattices, is for *commutative* "$*$" equivalent to the question of validity of condition in statement 1. Moreover, in the "Ends lemma" context, the idempotency condition 1 means that $a \cdot a \leq a$ should hold for all $a \in L$.

If $(L, \cdot)$ is a proper semigroup, this has no special implications and we obtain statement 1.

However, if $(L, \cdot)$ is a group, then this is equivalent to $a \leq u$ for all $a \in L$, where u is the neutral element of $(L, \cdot)$. On condition of antisymmetry of "$\leq$" this means that u is the greatest element of $(L, \leq)$. Yet $a \leq u$ is in a partially ordered group equivalent to $u \leq a^{-1}$ for all $a \in L$, which is possible only if $u = a^{-1}$. Yet since this should hold for all $a \in L$, there is $L = \{u\}$ and we obtain statement 3.

Finally, if "$\leq$" is antisymmetric, then $(L, \cdot, \leq)$ is a partially ordered semigroup and commutativity of $(L, *)$ is equivalent to commutativity of $(L, \cdot)$ and we obtain statement 2. □

Remark 2.4.121. Further on, in Corollary 2.4.126 and in Corollary 2.4.127 we assume that $(L, *)$ is a semihypergroup constructed from a *commutative semigroup* $(L, \cdot)$. Notice that this means that $(L, *)$ is commutative. However, commutativity of $(L, *)$ might be secured also for non-commutative carrier semigroups $(L, \cdot)$ – in case "$\leq$" is a quasi-ordering which is not partial ordering. Of course, only *proper semigroups* $(L, \cdot)$ are relevant in this respect.

Example 2.4.122. If we denote by $|\mathbb{C}|_0^1$ the set of all complex numbers such that their absolute value is smaller than or equal to one 1 (i.e. we regard a unit disc of the Gaussian plane) and regard "$\cdot_{|z|}$" multiplication of absolute values and set that $z_1 \leq_{|z|} z_2$ whenever $|z_1| \leq |z_2|$, then we get that $(|\mathbb{C}|_0^1, \cdot_{|z|}, \leq_{|z|})$ is a proper quasi-ordered semigroup. Moreover, "$\leq_{|z|}$" is not antisymmetric. We define a hyperoperation on $|\mathbb{C}|_0^1$ by

$$z_1 * z_2 = [z_1 \cdot_{|z|} z_2)_{\leq_{|z|}} = \{x \in |\mathbb{C}|_0^1 \mid |z_1| \cdot |z_2| \leq |x|\}.$$

Since $z \in z * z$ for all $z \in |\mathbb{C}|_0^1$, we have that $(\mathbb{C}, *)$ is a hypersemilattice.

The case of commutative quasi-ordered groups, where "$\leq$" is *not* antisymmetric is not discussed in Theorem 2.4.120.

Example 2.4.123. Regard the additive group of complex numbers $(\mathbb{C}, +)$ and define, for all $z_1, z_2 \in \mathbb{C}$, relation "$\leq_{|z|^{-1}}$" by $z_1 \leq_{|z|^{-1}} z_2$ whenever $|z_1| \geq |z_2|$, where $|z|$ stands for the absolute value of $z \in \mathbb{C}$. It is easy to verify that $(\mathbb{C}, +, \leq_{|z|^{-1}})$ is a commutative quasi-ordered group, where "$\leq_{|z|^{-1}}$" is obviously not antisymmetric. If we define, for all $z_1, z_2 \in \mathbb{C}$, that $z_1 * z_2 = \{x \in \mathbb{C} \mid |x| \leq |z_1 + z_2|\}$, then, by Definition 1.1.19 and the "Ends lemma", $(\mathbb{C}, *)$ is a hypersemilattice. Indeed, $|z| \leq |z + z|$ for all $z \in \mathbb{C}$, i.e. $z \in z * z$ (and the rest is obvious). However, if we regard "$\leq_{|z|}$" such that $z_1 \leq_{|z|} z_2$ whenever $|z_1| \leq |z_2|$ instead of $\leq_{|z_1|^{-1}}$, then "$*$" is no longer idempotent, i.e. $(\mathbb{C}, *)$ is neither a hypersemilattice nor an H_v–semilattice.

The following corollary is more or less obvious. Only notice that the relation "$\leq$" need not be antisymmetric.

Corollary 2.4.124. *In Theorem 2.4.120, an idempotent quasi-ordered semigroup $(L, \cdot, \leq)$ always creates a hypersemilattice.*

Proof. Obvious since "$\leq$" is reflexive. □

Example 2.4.125. The set union and set intersection are idempotent operations. Therefore, if we regard $(\mathcal{P}^*(S), \cap)$ or $(\mathcal{P}^*(S), \cup)$, then condition 1 of Definition 1.1.19 holds in both cases. Moreover, both $(\mathcal{P}^*(S), \cap, \subseteq)$ and $(\mathcal{P}^*(S), \cup, \subseteq)$ are proper quasi-ordered semigroups. Therefore, if we set $A * B = \{X \subseteq \mathcal{P}^*(S) \mid A \cap B \subseteq X\}$ or $A * B = \{X \subseteq \mathcal{P}^*(S) \mid A \cup B \subseteq X\}$, then $(\mathcal{P}^*(S), *)$ is in both cases a hypersemilattice. The same is true if we consider $\mathcal{P}(S) = \mathcal{P}^*(S) \cup \{\emptyset\}$ in either of the cases.

The results of Theorem 2.4.120 and Corollary 2.4.124 may be worded also in terms of *idempotent elements* and *idempotent sets*. However, one shall not confuse the concept of idempotency in single-valued structures, i.e. $a \cdot a = a$, and in hyperstructure theory, i.e. $a \in a * a$.

Corollary 2.4.126. *Let $(L, *)$ be the EL–semihypergroup of a quasi-ordered commutative semigroup $(L, \cdot, \leq)$. Then $(L, *)$ is a hypersemilattice if and only if it is an idempotent semihypergroup.*

Proof. Obvious, since $(L, *)$ is both commutative and associative thanks to Lemma 2.1.1 ("Ends lemma") and the condition $a \in a * a$ for all $a \in L$, equivalent to $a \cdot a \leq a$ is present in definitions of both concepts. □

The following statement may be easily proved on its own, yet it may be also regarded as a corollary to the fact that $(S, *)$ mentioned below is a hypersemilattice. In the case of regular $(S, *)$ compare the following statement to Corollary 2.4.102 on page 83.

Corollary 2.4.127. *Let $(S, *)$ be the commutative EL–semihypergroup of a quasi-ordered semigroup $(S, \cdot, \leq)$. If for every $s \in S$ there holds $s \cdot s \leq s$, then every subset A of $(S, *)$ is idempotent.*

Proof. Obvious thanks to Corollary 2.4.126, also Corollary to Dehghan Nezhad and Davvaz [116], Proposition 3.1. □

Example 2.4.128. Denote the closed interval of real numbers $\langle 0, 1\rangle$ by L. Obviously, $(L, \cdot, \leq)$, where "$\cdot$" is the usual multiplication and "$\leq$" the usual ordering of real numbers, is a proper quasi-ordered semigroup. Also obviously, the condition that, for all $a \in L$, there is $a \cdot a \leq a$ holds in L. Therefore, the EL–semihypergroup $(L, *)$, is a hypersemilattice. More importantly, it is an example of a hypersemilattice constructed from a non-idempotent quasi-ordered semigroup. Moreover, since "$*$", defined on $(L, \cdot, \leq)$, is extensive, $(L, *)$ is, by Theorem 2.4.71 on page 71, a hypergroup.

Remark 2.4.129. In [116], Proposition 3.3, Dehghan Nezhad and Davvaz take the additive partially ordered group of all integers $(\mathbb{Z}, +, \leq)$, define the hyperoperation in a way similar to the "Ends lemma" by $k * l = \{u \in \mathbb{Z} \mid k + l \leq 2u\}$ for all $k, l \in \mathbb{Z}$, and show that $(\mathbb{Z}, *)$ is an H_v-semilattice which is not a hypersemilattice. Notice that the choice of *double sum* instead of *sum* in the definition of the hyperoperation is crucial as it secures idempotency of "$*$" (even though at a price of breaking its strong associativity). On the other hand, choosing *sum* instead of *double sum*, i.e. regarding an EL–semihypergroup, would secure strong associativity of "$*$" yet prevent $(\mathbb{Z}, *)$ from being an H_v–semilattice or a hypersemilattice as $(\mathbb{Z}, +, \leq)$ is a partially ordered group. In general, number domains with operations of sum or product of numbers are problematic as a basis for construction of EL–hypersemilattices as they themselves or their important subsets are (given the usual ordering by size) mostly (partially ordered) groups. For an application of $(\mathbb{Z}, *)$ defined as an EL–hypergroup see Hošková and Chvalina [151], Example 1. Also notice, that introducing the concept of *double-sum* in [116] (or any other multiple of the sum) is possible only because $(\mathbb{Z}, +, \cdot)$ is a ring as $k + l \leq 2u$ is a definition that relies on two single-valued operations, i.e. cannot be generalized so easily as the definition used in the "Ends lemma".

In Subsection 2.4.6 we discussed the role of extensivity of the hyperoperation "$*$". In Example 2.4.128 we included an example of a hypersemilattice such that the hyperoperation is extensive. If we now change the definition of the hyperoperation from $a * b = [a \cdot b)_\leq$ to $a *_m b = \{a, b\} \cup [a \cdot b)_\leq$ (see (2.22) on page 71 and Subsection 2.4.6), there is $a \in a *_m a$ for all $a \in L$ by default, i.e. axiom 1 of Definition 1.1.19 always holds. Also, again by default, axiom 3 of the definition is secured for this hyperoperation. Moreover, by Lemma 2.4.74, the new hyperoperation is associative if and only if the original hyperoperation is associative. As a result we have the following theorem and corollary.

Theorem 2.4.130. *Let $(L, *_m)$ be the mEL–semihypergroup of a proper commutative quasi-ordered semigroup $(L, \cdot, \leq)$, where "$\leq$" is not antisymmetric. Then $(L, *_m)$ is a hypersemilattice.*

Proof. Obvious. □

In case of extensive hyperoperations such that the quasi-ordering "$\leq$" is also antisymmetric, commutativity becomes an equivalent condition to L being a hypersemilattice.

Corollary 2.4.131. *Let $(L, *_m)$ be the mEL–semihypergroup of a proper partially ordered semigroup $(L, \cdot, \leq)$. Then $(L, *_m)$ is a hypersemilattice if and only if "$\cdot$" is commutative.*

Proof. Idempotency and weak associativity are obvious. Since $(L, \cdot, \leq)$ is a partially ordered semigroup, validity of the remaining axiom of Definition 1.1.19, commutativity of the hyperoperation "$*_m$", is equivalent to the commutativity of the single-valued operation. □

The list of important elements of H_v–semilattices and hypersemilattices includes *absorbent* and *fixed* elements. Notice that, for commutative semihypergroups, the definition of *fixed* elements is equivalent to the definition of *zero scalar elements*.

Definition 2.4.132. ([116], Definition 3.4) Let $(L, *)$ be an H_v–semilattice. An element $a \in L$ is called an *absorbent element* of L if it satisfies condition that $c \in a * c$ for all $c \in L$. An element $b \in L$ is called a *fixed element* of L if it satisfies condition that $b * c = \{b\}$ for all $c \in L$.

The following theorem lets us identify absorbent and fixed elements easily. Notice that due to Theorem 2.4.120 the study of groups $(L, \cdot)$ is irrelevant if "$\leq$" is antisymmetric, i.e. the cancellation law cannot be used in such cases in the forthcoming Theorem 2.4.133, statement 1.

Theorem 2.4.133. *Let $(L, *)$ be the EL–hypersemilattice of a quasi-ordered semigroup $(L, \cdot, \leq)$. Then:*

1. *$a \in L$ is an absorbent element of $(L, *)$ if and only if $a \cdot c \leq c$ holds for all $c \in L$.*

2. *If $a \in L$ is an absorbent element of $(L, *)$, then all elements $b \in L$ such that $b \leq a$ are absorbent.*

3. *$b \in L$ is a fixed element of $(L, *)$ if and only if b is an EL–maximal element of $(L, \leq)$ such that $b \cdot c = c \cdot b = b$ holds for all $c \in L$.*

4. *If $(L, \cdot, \leq)$ is non-trivial, then no element of L can be simultaneously absorbent and fixed element of $(L, *)$.*

Proof. Statement 1 is obvious.

In statement 2 suppose an arbitrary element $b \in L$ such that $b \leq a$. In a quasi-ordered semigroup $(L, \cdot, \leq)$ this means that, for all $c \in L$, there is $b \cdot c \leq a \cdot c$. Yet $a \in L$ is an absorbent element of $(L, *)$. Thus $a \cdot c \leq c$ for all $c \in L$. From transitivity we get $b \cdot c \leq c$ for all $c \in L$, i.e. b is also absorbent.

As regards statement 3: according to the definition, $b \in L$ is a fixed element of $(L, *)$ if for all $c \in L$ there holds $b * c = \{b\}$, i.e., in the "Ends lemma" context, $\{b\} = b * c = \{x \in L \mid b \cdot c \leq x\}$ for all $c \in L$. Moreover, in hypersemilattices, commutativity is assumed, i.e. in our case there must be $b * c = c * b$ for all $c \in L$. However, since "$\leq$" is reflexive, two simultaneous conclusions are equivalent with this: $b \cdot c = c \cdot b = b$ must hold for all $c \in L$ and on top of that, b must be an EL–maximal element of $(L, \leq)$.

In statement 4, if a is absorbent, then $a \cdot c \leq c$ for all $c \in L$. If a is fixed, then $a \cdot c = a$ for all $c \in L$. Thus $a \leq c$, for all $c \in L$. Yet as follows from statement 3, fixed elements are also EL–maximal ones. Thus we get a contradiction or $(L, \cdot, \leq)$ is trivial. □

Theorem 2.4.134. *Let $(L, *)$ be the EL–hypersemilattice of a quasi-ordered semigroup $(L, \cdot, \leq)$. Denote $AE(L)$ the set of all absorbent elements of $(L, *)$. Then:*

1. *$(AE(L), \cdot)$ is a subsemigroup of $(L, \cdot)$.*

2. *If $(L, \cdot)$ is a monoid, then $AE(L)$ is always non-empty.*

3. *$(AE(L), *)$, where "$*$" is defined in the same way as in $(L, *)$ yet restricted on elements of $AE(L)$, is a hypersemilattice.*

Proof. Suppose that $a, b \in L$ are absorbent elements of $(L, *)$. Then $b \cdot c \leq c$ for all $c \in L$. In a quasi-ordered semigroup this implies $a \cdot b \cdot c \leq a \cdot c$ for all $c \in L$. However, since also $a \in L$ is absorbent, we get from transitivity that $a \cdot b \cdot c \leq c$ for all $c \in L$, which means that also $a \cdot b$ is absorbent.

Further, denote u the neutral element of $(L,\cdot)$, if it exists. Obviously, since "$\leq$" is reflexive, $u\cdot c = c \leq c$ for all $c \in L$, which means that u is absorbent.

Finally, the proof of statement 3 is obvious. □

Remark 2.4.135. Notice that in Dehghan Nezhad and Davvaz [116], p. 384, plain text, an H_v–subsemilattice of $(L,*)$ is defined as an arbitrary non-empty subset $M \subseteq L$ such that $a*b \in \mathcal{P}^*(M)$, i.e. such that $M*M = M$. However, in our case of $(AE(L),*)$ we have a different situation as the equality $AE(L)*AE(L) = AE(L)$ is all but self-evident. $(AE(L),*)$ can be – but equally well need not be – a subsemihypergroup of $(L,*)$. For details see Subsection 2.4.4, or Example 2.4.136. Notice that this has also important influence on the study of hyperideals of EL–hypersemilattices.

Example 2.4.136. If we continue with Example 2.4.128, where we considered $L = \langle 0,1\rangle$ with the usual multiplication and ordering of real numbers, then we see that all elements of L are absorbent and no element of L is fixed.

If we denote by $L(Q)$ the set of all rational numbers from $L=\langle 0,1\rangle$, we get an example of what has been discussed in Remark 2.4.135: $(L(Q),\cdot)$ is a subsemigroup of $(L,\cdot)$, thus we may use $(L(Q),\cdot,\leq)$ to define the hyperoperation using the "Ends lemma". However, e.g. $\frac{1}{4}*\frac{1}{5} = \{x \mid \frac{1}{20} \leq x\}$ yet we have to decide whether we take $x \in L$ or $x \in L(Q)$ as both of these choices are equally justifiable and meaningful. Only based on answer to this question can we consider cases such as fractions of π and thus decide in what relation $(L(Q),*)$ to $(L,*)$ is.

Example 2.4.137. If we continue with Example 2.4.125, where we considered the power set of an arbitrary set S, and examine the set intersection, then we see that $A \cap C \subseteq C$ for all $C \in \mathcal{P}(S)$ holds for all $A \in \mathcal{P}(S)$ while condition $B \cap C = B$ for all $C \in \mathcal{P}(S)$ holds for $B = \emptyset$ only. On the other hand, if we examine the set union, we see that $A \cup C \subseteq C$ for all $C \in \mathcal{P}(S)$ holds for $A = \emptyset$ only while $B \cup C = B$ for all $C \in \mathcal{P}(S)$ holds for $B = S$ only.

Thus in $(\mathcal{P}(S),*)$, constructed from $(\mathcal{P}(S),\cap,\subseteq)$, all elements are absorbent and none is fixed – because $\emptyset$ is not an EL–maximal element of $(\mathcal{P}(S),\subseteq)$. However, if we construct $(\mathcal{P}(S),*)$ from $(\mathcal{P}(S),\cup,\subseteq)$, then only $\emptyset$ is absorbent and only S is fixed.

In Remark 2.4.9 on page 43 we showed that if we use hyperoperation $*_d : S\times S \to \mathcal{P}(S)$ defined by

$$a *_d b = {}_{\leq}(a\cdot b] = \{x \in S \mid x \leq a\cdot b\} \tag{2.27}$$

instead of $a*b = [a\cdot b)_{\leq}$, the Lemma 2.1.1 ("Ends lemma") remains valid. This means that our results are (with respect to duality) valid also for the hyperoperation "$*_d$" and its extensive modification $a *_{dm} b = \{a,b\} \cup_{\leq} (a \cdot b] = \{a,b\} \cup \{x \in S \mid x \leq a \cdot b\}$.

Let us now move from hypersemilattices to *hyperlattices* which were defined in Definition 1.1.18 and the context of which has been discussed on page 12. We will adopt terminology used on page 12 and study *strong join hyperlattices* (defined by Definition 1.1.18, i.e. *hyperlattices* in the sense of the original definition of Konstantinidou and Mittas [182]) and *join hyperlattices* (defined by axioms 1–4 of Definition 1.1.18). In order to remain on the "safe ground", assume that "$\leq$" is partial ordering. This assumption will also allow us to link our results to those obtained by Davvaz et al. on partially ordered (semi)hypergroups.[22] Given Theorem 2.4.120 this means that $(L, \cdot, \leq)$ must be a *proper partially ordered semigroup.*

Hyperlattices of Konstantinidou and Mittas [182] (seen as algebraic structures) are hyperstructures with one hyperoperation and one single-valued operation. Suppose that the hyperoperation "$\bigvee$" is the "Ends lemma" hyperoperation; "$\wedge$" will be arbitrary. In other words, let us study EL–hyperstructures $(L, *, \wedge)$.

First of all, we must explain the meaning of Definition 1.1.18 in this context of ours. Since axioms 1–3 (with respect to "$\bigvee$", i.e. with respect to "$*$") are defining axioms of hypersemilattices, we can use our results on hypersemilattices. In axiom 4, every element $a \in L$ must be, for all $a, b \in L$, in the intersection of $a \bigvee (a \wedge b)$ and $a \wedge (a \bigvee b)$. Yet since "$\bigvee$", i.e. "$*$" in our case, is defined as

$$a \bigvee b = a * b = [a \cdot b)_{\leq} = \{x \in L \mid a \cdot b \leq x\}, \tag{2.28}$$

we have that $a \in a \bigvee (a \wedge b)$ is equivalent to $a \in [a \cdot (a \wedge b))_{\leq}$ which is equivalent to $a \cdot (a \wedge b) \leq a$, for all $a, b \in L$. On the other hand, the fact that $a \in a \wedge (a \bigvee b)$ is equivalent to $a \in a \wedge [a \cdot b)_{\leq}$ which is equivalent to $a \in \{a \wedge x \mid x \in [a \cdot b)_{\leq}\}$, i.e. $a \in \{a \wedge x \mid a \cdot b \leq x\}$. This means that, for all $a, b \in L$, there exists an element $x \in L$ such that $a = a \wedge x$, where $a \cdot b \leq x$. Axiom 5 is in our context equivalent to the fact that $a \cdot b \leq a \Rightarrow b = a \wedge b$, for all $a, b \in L$.

As can be seen in the following examples, the EL–hyperoperation (2.28) can be used to construct both join hyperlattices and strong join hyperlattices.

Example 2.4.138. Let $L = \mathbb{N}$, operation "$\cdot$" be defined as the minimum of natural numbers and operation "$\wedge$" be defined as the maximum of natural numbers. Finally, "$\leq$" is the usual ordering of natural numbers by size. In this case, for all $a, b \in \mathbb{N}$, $a \bigvee b = a * b = \{x \in \mathbb{N} \mid \min\{a, b\} \leq x\}$ and

[22] For a collection of these see e.g. recently published book Davvaz [105]

$a \wedge b = \max\{a, b\}$. Obviously, $(\mathbb{N}, *)$ is a hypersemilattice and $(\mathbb{N}, *, \wedge)$ is a join hyperlattice (for all $a \in \mathbb{N}$ the desired $x \in \mathbb{N}$ equals a) which is not strong.

Example 2.4.139. Suppose that $L = \mathcal{P}^*(S)$, where $\mathcal{P}^*(S)$ is the system of all non-empty subsets of an arbitrary set S. For an arbitrary subsets $A, B \subseteq \mathcal{P}^*(S)$ define that

$$A \bigvee B = A * B = \{X \in \mathcal{P}^*(S) \mid A \cup B \subseteq X\},$$

i.e. the operation "$\cdot$" is defined as union of subsets of S and the relation is defined as inclusion, and $A \wedge B = A \cap B$ for all $A, B \subseteq \mathcal{P}^*(S)$. Then obviously, $A \cup (A \cap B) \subseteq A$ for all $A \subseteq \mathcal{P}^*(S)$ and for $X = S$ we have that $A = A \cap X$, where $A \cup B \subseteq X$ for all $A, B \subseteq \mathcal{P}^*(S)$. Thus axiom 4 of Definition 1.1.18 holds and $(\mathcal{P}^*(S), *, \wedge)$ is a join hyperlattice. Moreover, $A \cup B \subseteq A$ is, for all $A, B \subseteq \mathcal{P}^*(S)$, equivalent to the fact that $B \subseteq A$ and in this case $B = A \cap B$. Therefore, the join hyperlattice $(\mathcal{P}^*(S), *, \wedge)$ is strong.

Therefore, we can ask: *If we want $(L, *, \wedge)$ to be a join hyperlattice, how shall the single-valued operations "$\cdot$" and "$\wedge$" be linked?*

Theorem 2.4.140. *Let $(L, *)$ be an EL–hypersemilattice and let "$\wedge$" be an idempotent, commutative and associative operation on L.*

1. *If, for all $a, b \in L$, there is $a \cdot (a \wedge b) \leq a$ and $a \cdot b \leq a$, then $(L, *, \wedge)$ is a join hyperlattice.*
2. *If, for all $a, b \in L$, there is $a \cdot (a \wedge b) \leq a$ and $a = a \wedge 1$, where 1 is the greatest element of $(L, \leq)$, then $(L, *, \wedge)$ is a join hyperlattice.*

Proof. Since $(L, *)$ is a hypersemilattice and we assume that "$\wedge$" is an idempotent, commutative and associative operation on L, we verify axiom 4 of Definition 1.1.18 only.

1. As has been mentioned above, the left-hand side of the axiom is equivalent to the condition $a \cdot (a \wedge b) \leq a$. In the right-hand side it is enough, for all $a \in L$, to set $x = a$ and we obtain that, for all $a, b \in L$, there must be $a = a \wedge a$ (which is true because "$\wedge$" is idempotent), where $a \cdot b \leq a$ (which is what we suppose).
2. The left-hand side of the axiom is equivalent to the condition $a \cdot (a \wedge b) \leq a$. In the right-hand side we assume that $a = a \wedge 1$ and since 1 is the greatest element of $(L, \leq)$, the fact that $a \cdot b \leq 1$ holds, for all $a, b \in L$, trivially.

□

Example 2.4.141. Example 2.4.138 is an example of a join hyperlattice constructed using Theorem 2.4.140, statement 1. However, as Example 2.4.139 suggests, the converse of this condition is not true as $A \cup B \subseteq A$ does not hold for all $A, B \in \mathcal{P}^*(S)$. Notice that the condition of statement 2 holds in $(\mathcal{P}^*(S), *, \wedge)$ from Example 2.4.139.

Example 2.4.142. Suppose the partially ordered semigroup $(\mathbb{N}, \gcd, |)$, where "gcd" stands for the greatest common divisor of natural numbers and "$|$" is the usual relation of divisibility. By Corollary 2.4.124, $(N, *)$, where $a*b = \{x \in \mathbb{N} \mid \gcd\{a,b\}|x\}$, for all $a, b \in \mathbb{N}$, is a hypersemilattice. Set "$\bigvee$" = "$*$" and $a \wedge b = \min\{a, b\}$. Since there is $\gcd\{a,b\}|a$ and $\gcd\{a, \min\{a,b\}\}|a$, for all $a, b \in \mathbb{N}$, we have, by Theorem 2.4.140, that $(\mathbb{N}, *, \min)$ is a join hyperlattice. Since e.g. $\gcd\{16, 24\} = 16 \not\Rightarrow 24 = \min\{16, 24\}$, the join hyperlattice is not strong.

If we use the extensive modification of the EL–hyperoperation (see Subsection 2.4.6), i.e. assume that for all $a, b \in L$ there is

$$a \bigvee b = a *_m b = \{a,b\} \cup [a \cdot b)_{\leq} = \{a,b\} \cup \{x \in L \mid a \cdot b \leq x\}, \qquad (2.29)$$

we have $a \in a \bigvee (a \wedge b)$ by default. As a result, no special link between operations "$\cdot$" and "$\wedge$" is needed under conditions of Theorem 2.4.140.

Corollary 2.4.143. *Let $(L, *_m)$ be an mEL–hypersemilattice and let "$\wedge$" be an idempotent, commutative and associative operation on L. If, for all $a, b \in L$, there is $a \cdot b \leq a$, then $(L, *_m, \wedge)$ is a join hyperlattice.*

Proof. Obvious because in Theorem 2.4.140, statement 1, condition $a \cdot (a \wedge b) \leq a$ of the left-hand side of axiom 4 becomes reduntant and on the right-hand side of axiom 4 it is enough to set $x = a$ for all $a \in L$. □

Finally, it is to be noted that in her study of hyperlattices, i.e. strong join hyperlattices in the above sense, Konstantinidou-Seramifidou [180, 181] studied also *distributive*, *complemented* and *modular hyperlattices*.

2.5 Construction from quasi-ordered semirings

2.5.1 Setting the ground

In Section 2.4 we discussed semihypergroups constructed from quasi-ordered semigroups. Now it is natural to ask whether – and if yes, how – the lemma can be used to construct hyperstructures with more than one hyperoperation.

Prior to that, however, we need to establish the meaning of the phrase "more than one hyperoperation". "More than one" will mean "two" yet as far as hyperstructure generalizations are concerned, the issue is a bit more complex. Starting from page 8 we included a short discussion on various approaches to the issue of algebraic hyperstructures with two (hyper)operations and some historical remarks aimed at clarifying the rather confused terminology of the topic. Therefore, the reader should first recall Definition 1.1.13, Definition 1.1.14, Definition 1.1.15 and Definition 1.1.17 as well as all the context mentioned on pages starting from page 8.

Generally speaking, in our case we can make use of the "Ends lemma" in three different ways:

1. Let $(S,+)$ and $(S,\cdot)$ be two single-valued structures. We can define a hyperoperation using one of the operations "+" or "$\cdot$" such as e.g. $a * b = [a+b)_{\leq}$ – thus we get an EL–semihypergroup $(S,*)$. The hyperstructure in question will then be a triplet $(S,*,\cdot)$ where "$*$" is the hyperoperation based on the single-valued operation "+".

2. Let $(S,+)$ and $(S,\cdot)$ be two single-valued structures. We can define two hyperoperations, each based on one single-valued operation, i.e. for an arbitrary pair $a,b \in S$ we can define $a*b = [a+b)_{\leq}$ and $a \circ b = [a \cdot b)_{\leq}$. Thus we get a triplet $(S,*,\circ)$, where "$*$" and "$\circ$" are hyperoperations.

3. However, we can also start with a single single-valued structure $(S,\cdot)$ and using it define a hyperoperation "$*$" by $a * b = [a \cdot b)_{\leq}$. The hyperstructure in question will then be a triplet $(S,*,\cdot)$, where "$*$" is the hyperoperation based on the single-valued operation "$\cdot$".

Given the variety of concepts studied in the hyperstructure theory (see the above mentioned definitions), every approach will lead to a meaningful application.

The title of this section is "Hyperstructures constructed from quasi-ordered semirings". By a *semiring* we mean a structure *without* the explicit inclusion of the axiom of annihilation, or rather without the requirement for existence of elements 0 and 1. In other words, we use the following definition included e.g. in Hebisch and Weinert [143] with a remark regarding commutativity.

Definition 2.5.1. ([143], Definition 2.1) Let $S \neq 0$ be a set and "+" and "$\cdot$" binary operations on S named *addition* and *multiplication*. Then $(S,+,\cdot)$ is called a *semiring* if the following conditions are satisfied:

1. $(S,+)$ is a commutative semigroup,

2. $(S, \cdot)$ is a semigroup,

3. Both operations are linked by the *distributive laws* $a \cdot (b+c) = a \cdot b + a \cdot c$ and $(a+b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, c, \in S$.

In particular, a semiring is said to be *commutative* if also $(S, \cdot)$ is commutative, and $(S, +, \cdot)$ is called a *ring* if $(S, +)$ is a commutative group. We will call a semiring, which is not a ring, a *proper semiring*.

Remark 2.5.2. In Definition 1.1.14 the semihypergroups need not be commutative. If one wanted to achieve a complete parallel of Definition 1.1.14 in Definition 2.5.1, one would have to assume that $(S, +)$ is a semigroup (not necessarily commutative) and define commutative semirings as structures where $(S, +)$ is explicitely commutative. However, if one realizes that our aim is to construct hyperstructures of Definition 1.1.14 (and also that Theorem 2.5.4 holds), this is no more than reduntant formalism. Also notice results of Section 2.1 and Subection 2.4.1 regarding commutativity. However, if commutativity is needed in the proofs of some of the following theorems, we will explicitely mention it.

Naturally, the potential existence of an element which annihilites S with respect to multiplication, shall not be ignored. We will discuss it later on in Remark 2.5.12 on page 105.

Definition 2.5.3. By a *quasi-ordered* (or *partially ordered*) *semiring* we mean a semiring $(S, +, \cdot)$ such that "$\leq$" is a quasi– (or partial) ordering on S and

$$a \leq b \quad \Rightarrow \quad a + c \leq b + c \text{ and } c + a \leq c + b \tag{2.30}$$

$$a \leq b \quad \Rightarrow \quad a \cdot c \leq b \cdot c \text{ and } c \cdot a \leq c \cdot b \tag{2.31}$$

for all $a, b, c \in S$.

The following theorem is an immediate corollary of Theorem 2.4.10 on page 44.

Theorem 2.5.4. *The "Ends lemma" cannot be used to construct Krasner hyperrings or quasi-canonical hyperrings.*

Proof. Definition 1.1.13 (of Krasner hyperrings $(H, \oplus, \odot)$) as well as the definition of *quasi-canonical hyperrings* in Davvaz and Leoreanu–Fotea [111] assumes that $(H, \oplus)$ is a canonical hypergroup, i.e. a hypergroup with the scalar identity. Yet Theorem 2.4.10 says that non-trivial EL–semihypergroups do not have scalar identities. □

Thus, further on, we shall concentrate on more general hyperstructures which were defined in Definition 1.1.14, Definition 1.1.15 and Definition 1.1.17.

2.5.2 The issue of distributivity

Results of this subsection were published by South Bohemia Mathematical Letters as Novák [243].

Since we know that the "Ends lemma" constructs semihypergroups from semigroups, it is obvious that – given our list of objectives on page 100 – we must concentrate on the issue of distributivity. Therefore, we will assume that we have a semiring $(S,+,\cdot)$, i.e. a distributive structure, and study whether we can obtain an (inclusively) distributive hyperstructure from it. Since by the "Ends lemma" we always get semihypergroups, we can use results of Section 2.4 to decide whether $(S,\oplus)$ or $(S,\odot)$ are hypergroups, i.e. whether $(S,\oplus,\odot)$ is a stronger hyperstructure than a proper semihyperring.

In the following theorems, $(S,\oplus)$ will be the EL–semihypergroup of a quasi-ordered semigroup $(S,+,\leq)$ and $(S,\odot)$ will be the EL–semihypergroup of a quasi-ordered semigroup $(S,\cdot,\leq)$.

Lemma 2.5.5. *Let $(S,\oplus)$ and $(S,\odot)$ be EL–semihypergroups of quasi-ordered semigroups $(S,+,\leq)$ and $(S,\cdot,\leq)$, respectively. Then*

$$a\cdot(b+c)=a\cdot b+a\cdot c\Rightarrow a\odot(b\oplus c)\subseteq a\odot b\oplus a\odot c$$
$$(a+b)\cdot c=a\cdot c+b\cdot c\Rightarrow (a\oplus b)\odot c\subseteq a\odot c\oplus b\odot c$$

Proof. For an arbitrary triple $a,b,c\in S$ the fact that

$$a\odot(b\oplus c)\subseteq a\odot b\oplus a\odot c$$

may be rewritten as

$$\bigcup_{h\in[b+c)_\leq}[a\cdot h)_\leq\subseteq\bigcup_{x\in[a\cdot b)_\leq,y\in[a\cdot c)_\leq}[x+y)_\leq,$$

while the second inclusion may be rewritten in an analogous way as

$$\bigcup_{h\in[a+b)_\leq}[h\cdot c)_\leq\subseteq\bigcup_{x\in[a\cdot c)_\leq,y\in[b\cdot c)_\leq}[x+y)_\leq.$$

Suppose $g\in\bigcup_{h\in[b+c)_\leq}[a\cdot h)_\leq$ arbitrary. This means that there exists $h_0\in[b+c)_\leq$ such that $g\in[a\cdot h_0)_\leq$, i.e. $a\cdot h_0\leq g$. The fact that $h_0\in[b+c)_\leq$ is equivalent to the fact that $b+c\leq h_0$ which in a quasi-ordered semigroup implies $a\cdot(b+c)\leq a\cdot h_0$ for an arbitrary $a\in S$. We assume distributivity, i.e. there is $a\cdot b+a\cdot c\leq a\cdot h_0$, i.e. $a\cdot b+a\cdot c\leq g$, i.e. $g\in[a\cdot b+a\cdot c)_\leq$. Since the relation "$\leq$" is reflexive, there is $a\cdot b\in[a\cdot b)_\leq$ and $a\cdot c\in[a\cdot c)_\leq$, which means that if we denote $x=a\cdot b$ and $y=a\cdot c$, we get that $g\in\bigcup_{x\in[a\cdot b)_\leq,y\in[a\cdot c)_\leq}[x+y)_\leq$.

Obviously, the second inclusion may be proved in an analogous way. □

Corollary 2.5.6. *Let $(S, \oplus)$ be the EL–semihypergroup of a quasi-ordered semigroup $(S, +, \leq)$ and let $(S, \cdot, \leq)$ be a quasi-ordered semigroup. Then*

$$a \cdot (b + c) = a \cdot b + a \cdot c \Rightarrow a \cdot (b \oplus c) \subseteq a \cdot b \oplus a \cdot c$$
$$(a + b) \cdot c = a \cdot c + b \cdot c \Rightarrow (a \oplus b) \cdot c \subseteq a \cdot c \oplus b \cdot c$$

Proof. Obvious as this is a special case of Lemma 2.5.5. The proof may be repeated without the necessity of reasoning concerning the fact that $a \cdot b \in [a \cdot b)_{\leq}$ and $a \cdot c \in [a \cdot c)_{\leq}$. □

Lemma 2.5.7. *Let $(S, \oplus)$ be the EL–semihypergroup of a quasi-ordered semigroup $(S, +, \leq)$ and let $(S, \cdot, \leq)$ be a quasi-ordered group. Then*

$$a \cdot (b + c) = a \cdot b + a \cdot c \Rightarrow a \cdot (b \oplus c) \supseteq a \cdot b \oplus a \cdot c$$
$$(a + b) \cdot c = a \cdot c + b \cdot c \Rightarrow (a \oplus b) \cdot c \supseteq a \cdot c \oplus b \cdot c$$

Proof. Let $x \in a \cdot b \oplus a \cdot c = [a \cdot b + a \cdot c)_{\leq}$ be arbitrary. This means that $a \cdot b + a \cdot c \leq x$, i.e. (since we assume distributivity) $a \cdot (b + c) \leq x$. Now regard an element $h = a^{-1} \cdot x$, which must exist because $(S, \cdot)$ is a group. Since $(S, \cdot, \leq)$ is a quasi-ordered group, the fact that $a \cdot (b + c) \leq x$ implies $b + c \leq a^{-1} \cdot x$, i.e. $h = a^{-1} \cdot x \in [b + c)_{\leq}$. If we now regard the product $a \cdot h$, we get that $a \cdot h = a \cdot (a^{-1} \cdot x) = x$, i.e. $x \in \bigcup\limits_{h \in [b+c)_{\leq}} a \cdot h = a \cdot (b \oplus c)$, which means that the first inclusion holds. The other inclusion can be proved in an analogous way. □

Lemma 2.5.8. *Let $(S, \oplus)$ and $(S, \odot)$ be EL–semihypergroups of quasi-ordered semigroups $(S, +, \leq)$ and $(S, \cdot, \leq)$, respectively. Moreover, let $(S, \cdot)$ be a group. Then*

$$a \cdot (b + c) = a \cdot b + a \cdot c \Rightarrow a \odot (b \oplus c) \supseteq a \odot b \oplus a \odot c$$
$$(a + b) \cdot c = a \cdot c + b \cdot c \Rightarrow (a \oplus b) \odot c \supseteq a \odot c \oplus b \odot c$$

Proof. The lemma is a generalization of Lemma 2.5.7. The proof may be copied. The lemma holds because the relation "$\leq$" is reflexive, i.e. $[a)_{\leq} \neq \emptyset$ for all $a \in S$. We prove inclusions

$$\bigcup_{h \in [b+c)_{\leq}} [a \cdot h)_{\leq} \supseteq \bigcup_{x \in [a \cdot b)_{\leq}, y \in [a \cdot c)_{\leq}} [x + y)_{\leq},$$

and

$$\bigcup_{h \in [a+b)_{\leq}} [h \cdot c)_{\leq} \supseteq \bigcup_{x \in [a \cdot c)_{\leq}, y \in [b \cdot c)_{\leq}} [x + y)_{\leq}.$$

If we regard $g \in \bigcup\limits_{x \in [a \cdot b)_\leq, y \in [a \cdot c)_\leq} [x+y)_\leq$ arbitrary, then there exist $x_0 \in [a \cdot b)_\leq$, i.e. $a \cdot b \leq x_0$, and $y_0 \in [a \cdot c)_\leq$, i.e. $a \cdot c \leq y_0$, such that $g \in [x_0 + y_0)_\leq$, i.e. $x_0 + y_0 \leq g$. Obviously $a \cdot b + a \cdot c \leq x_0 + y_0 \leq g$, i.e. thanks to distributivity $a \cdot (b + c) \leq g$. In a quasi-ordered group this means that $b + c \leq a^{-1} \cdot g$, i.e. $h_0 = a^{-1} \cdot g \in [b + c)_\leq$ and if we regard the product $a \cdot h_0$, we get that $a \cdot (a^{-1} \cdot g) = g$, i.e. $g \in \bigcup\limits_{h \in [b+c)_\leq} [a \cdot h)_\leq$, which means that the inclusion holds. The other inclusion can be proved in an analogous way. $\square$

Before the following lemma notice that by the sum of sets A, B we mean the set $A + B = \{a + b \mid a \in A, b \in B\}$.

Lemma 2.5.9. *Let $(S, \odot)$ be the EL–semihypergroup of a quasi-ordered semigroup $(S, \cdot, \leq)$ and let $(S, +, \leq)$ be a quasi-ordered group. Then*

$$a \cdot (b + c) = a \cdot b + a \cdot c \Rightarrow a \odot (b + c) = a \odot b + a \odot c$$
$$(a + b) \cdot c = a \cdot c + b \cdot c \Rightarrow (a + b) \odot c = a \odot c + b \odot c$$

Proof. Once again, we are going to demonstrate validity of one equality only – validity of the other one can be proved in an analogous way.

"$\subseteq$" Regard an arbitrary $x \in a \odot (b + c)$. This is equivalent to the fact that $x \in [a \cdot (b + c))_\leq$, which – since we assume distributivity – is equivalent to the fact that $x \in [a \cdot b + a \cdot c)_\leq$, i.e. $a \cdot b + a \cdot c \leq x$. Now regard two elements, $x_0 = a \cdot b$ and $y_0 = x - a \cdot b$, which obviously exist because $(S, +)$ is a group. There obviously holds that $a \cdot b \in [a \cdot b)_\leq$ and $x_0 + y_0 = x$. Furthermore, since $a \cdot b + a \cdot c \leq x$, there is $a \cdot c \leq x - a \cdot b$, i.e. $y_0 = x - a \cdot b \in [a \cdot c)_\leq$. Thus $x \in [a \cdot b)_\leq + [a \cdot c)_\leq = a \odot b + a \odot c$.

"$\supseteq$" Regard an arbitrary $x \in a \odot b + a \odot c$, i.e. an arbitrary $x \in [a \cdot b)_\leq + [a \cdot c)_\leq$. This means that there exist $x_0 \in [a \cdot b)_\leq$ and $y_0 \in [a \cdot c)_\leq$ such that $x = x_0 + y_0$. However, this is equivalent to the fact that $a \cdot b \leq x_0$ and $a \cdot c \leq y_0$, which in a quasi-ordered set implies $a \cdot b + a \cdot c \leq x_0 + y_0$, i.e. $a \cdot b + a \cdot c \leq x$. This means that $x \in [a \cdot b + a \cdot c)_\leq$, which – since we assume distributivity – is equivalent to the fact that $x \in [a \cdot (b + c))_\leq$, i.e. $x \in a \odot (b + c)$.

$\square$

Lemma 2.5.10. *Let $(S, \odot)$ be the EL–semihypergroup of a quasi-ordered semigroup $(S, \cdot, \leq)$ such that "$\cdot$" is a commutative idempotent operation. Then*

$$a \cdot (b \odot c) \subseteq a \cdot b \odot a \cdot c$$
$$(a \odot b) \cdot c \subseteq a \cdot c \odot b \cdot c$$

Proof. The proof copies the proof of Lemma 2.5.5 with respect to the fact that the operation "$\cdot$" is commutative and idempotent, i.e. $[a \cdot b \cdot a \cdot c)_{\leq} = [a \cdot b \cdot c)_{\leq}$. □

Remark 2.5.11. Whether also the converse inclusion holds in Lemma 2.5.10, is still an open question. The proof of Lemma 2.5.7 cannot be repeated because the fact that $(S, \cdot)$ is a group and the fact that "$\cdot$" is an idempotent operation either contradict each other or result in trivialities. In fact, the issue of validity of the converse inclusion in the general case depends on finding conditions for validity of the following statement, where $a, b \in S$ is arbitrary and $(S, \cdot, \leq)$ is at least a quasi-ordered semigroup:

For an arbitrary $x \in [a \cdot b)_{\leq}$ *there exists* $h \in [b)_{\leq}$ *such that* $a \cdot h = x$.

On condition that $(S, \cdot, \leq)$ is a quasi-ordered group we get Lemma 2.5.7.

In ring-like structures $(R, +, \cdot)$ the neutral element of $(R, +)$ denoted 0 has the absorbing property, i.e. for an arbitrary element $a \in R$ there holds $a \cdot 0 = 0 \cdot a = 0$. This property either follows from the defining axioms of the structure (such as in rings) or is defined as an axiom itself (such as in semirings by some authors). However, in our definition of a semiring on page 100 we chose the approach without absorbing elements. Therefore, we must now check what happens if we regard monoids or groups $(S, +)$ in all the above lemmas and corollaries. Further on, we will denote the neutral element of $(S, +)$, which annihilates S with respect to "$\cdot$", by 0.

Remark 2.5.12. If we regard the absorbing element $a = 0$ (or $c = 0$ in second inclusions) in Lemma 2.5.5, Corollary 2.5.6 or Lemma 2.5.10, the inclusions obviously hold because we get $[0)_{\leq} \subseteq \bigcup\limits_{x,y \in [0)_{\leq}} [x+y)_{\leq}$ or $\{0\} \subseteq [0)_{\leq}$ and the relation "$\leq$" is reflexive.

The issue of Lemma 2.5.7 is a complex one.

1. If $(S, +)$ is a group with the neutral element 0, then the fact that $(S, \cdot)$ is a group and the fact that distributivity law holds imply that 0 is an absorbing element of $(S, \cdot)$. However, this means that $(S, \cdot)$ is trivial, thus of no interest to our considerations.

2. If $(S, +)$ is a proper semigroup (be it a monoid with the neutral element 0 or not), then $(S, \cdot)$ need not have an absorbing element. If it does not have it, we experience no problems. If it does, $(S, \cdot)$ is trivial, thus of no interest to our considerations.

3. If we restrict ourselves to a group $(S \setminus \{0\}, \cdot)$, where 0 is an absorbing element of $(S, \cdot)$, then we get

$$\bigcup_{h \in [b+c)_{\leq}} a \cdot h \supseteq [a \cdot b + a \cdot c)_{\leq}$$

which for absorbing $a = 0$ means that $\{0\} \supseteq [0)_{\leq}$ which obviously does not hold for general relation "$\leq$". The relation must therefore be a special one – such that 0 is an EL–maximal element of $(S, +, \leq)$.

   (a) Yet if $(S, +)$ is a group, then for an arbitrary $a, b \in S$ such that $a < b$ we have $0 < b - a$, which is not possible if there should simultaneously hold $[0)_{\leq} = \{0\}$. The relation "$\leq$" is thus, in such a case, trivial. In other words, considering skew-fields $(S, +, \cdot)$ in Lemma 2.5.7 is of no sense.

If we regard the absorbing element $a = 0$ in Lemma 2.5.8, we get that

$$\bigcup_{x \in [0)_{\leq}, y \in [0)_{\leq}} [x + y)_{\leq} \subseteq \bigcup_{h \in [b+c)_{\leq}} [0 \cdot h)_{\leq} = [0)_{\leq}.$$

However,

$$\bigcup_{x \in [0)_{\leq}, y \in [0)_{\leq}} [x + y)_{\leq} = \bigcup_{0 \leq x, 0 \leq y} [x + y)_{\leq} = [0)_{\leq},$$

i.e. unlike for Lemma 2.5.7 the implication holds without any further assumptions.

It can be easily verified that Lemma 2.5.9 holds for the absorbing element $a = 0$ (or $c = 0$ in the second equation) as well.

Remark 2.5.13. In Subsection 2.5.5 we will include one more theorem on distributivity, where both $(S, +, \leq)$ and $(S, \cdot, \leq)$ can be semihypergroups only.

2.5.3 EL–ring-like hyperstructures

Results of this subsection are an adapted version of results from Novák [243].[23] *They have also been summarized in Novák and Cristea [247], accepted for publication by Hacettepe Journal of Mathematics and Statistics (WoS Q4) .*

Now that we have discussed the issue of distrubtivity, we can shift our attention to ring-like hyperstructures. Suppose we have a quasi-ordered semiring $(S, +, \cdot, \leq)$ which we use to construct a hyperstructure $(S, \oplus, \odot)$, where

[23] In Novák [243] the term "quasi-ordered semiring" is not used.

$(S, \oplus)$ and $(S, \odot)$ are EL–semihypergroups. Theorems of this subsection describe the great variety of ring-like hyperstructures that we can obtain. Recall Definition 1.1.14 from page 9.

Theorem 2.5.14. *Let $(S, +, \cdot, \leq)$ be a quasi-ordered semiring and $(S, \oplus)$ and $(S, \odot)$ EL–semihypergroups constructed from quasi-ordered semigroups $(S, +, \leq)$ and $(S, \cdot, \leq)$, respectively.*

1. *$(S, \oplus, \odot)$ is a semihyperring in the general sense.*
2. *If $(S, \cdot)$ is a group, then $(S, \oplus, \odot)$ is a good semihyperring.*
3. *If $(S, +)$ is a group or if $(S, \oplus)$ is a hypergroup, then $(S, \oplus, \odot)$ is a hyperring in the general sense.*
4. *If $(S, +)$ is a group with neutral element 0 and $(S \setminus \{0\}, \cdot)$ is a group, then $(S, \oplus, \odot)$ is a good hyperring in the general sense.*

Proof. The theorem is an immediate corollary of lemmas on distributivity included in Subsection 2.5.2. □

In item 3 of Theorem 2.5.14 notice that hypergroups can be constructed from proper semigroups as well. See the following example.

Example 2.5.15. Regard an arbitrary set K and its power set $\mathcal{P}(K)$. The operations "$\cap$", "$\cup$" of set intersection and set union are associative, thus $(\mathcal{P}(K), \cap)$ and $(\mathcal{P}(K), \cup)$ are semigroups. The relation "$\subseteq$" on $\mathcal{P}(K)$ is obviously reflexive and transitive and for arbitrary $A, B, C \in \mathcal{P}(K)$ such that $A \subseteq B$ there is $A \cap C \subseteq B \cap C$ and $A \cup C \subseteq B \cup C$. Thus if we define hyperoperations "$\oplus$", "$\odot$" for arbitrary $A, B \in \mathcal{P}(K)$ by

$$A \oplus B = [A \cup B)_{\subseteq} = \{X \in \mathcal{P}(K) \mid A \cup B \subseteq X\} \tag{2.32}$$

and

$$A \odot B = [A \cap B)_{\subseteq} = \{Y \in \mathcal{P}(K) \mid A \cap B \subseteq Y\}, \tag{2.33}$$

we get semihypergroups $(\mathcal{P}(K), \oplus)$ and $(\mathcal{P}(K), \odot)$. Moreover, as set intersection is distributive with respect to set union, $(\mathcal{P}(K), \oplus, \odot)$ is a semihyperring in the general sense. Finally, as follows from Example 2.4.94, $(\mathcal{P}(K), \oplus)$ is a hypergroup. Altogether we get that $(\mathcal{P}(K), \oplus, \odot)$ is a hyperring in the general sense.

On page 11 we discussed the differences in terminology between Rota (followed by Davvaz and Leoreanu–Fotea [111]) on one hand and Spartalis and Vougiouklis on the other as far as *multiplicative hyperrings* are concerned. The following theorem clarifies the issue.

Theorem 2.5.16. *Let $(S, +, \cdot, \leq)$ be a quasi-ordered semiring and $(S, \odot)$ the EL–semihypergroup constructed from a quasi-ordered semigroup $(S, +, \leq)$. If $(S, +)$ is a group, then $(S, +, \odot)$ is a good multiplicative hyperring in the sense of Spartalis and Vougiouklis. If moreover $(S, +)$ is commutative, i.e. $(S, +, \cdot)$ is a ring, then $(S, +, \odot)$ is a strongly distributive multiplicative hyperring in the sense of Rota.*

Proof. In the case of Spartalis, the theorem is an immediate corollary of Lemma 2.5.9. In the case of Rota, we need, on top of that, to assume commutativity of $(S, +)$ and prove the validity of

$$a \odot (-b) = (-a) \odot b = -(a \odot b)$$

for all $a, b \in S$. However, in the context of the "Ends lemma" this means

$$[a \cdot (-b))_{\leq} = [(-a) \cdot b)_{\leq} = [-(a \cdot b))_{\leq}$$

for all $a, b \in S$. Yet this obviously holds because

$$a \cdot (-b) = (-a) \cdot b = -(a \cdot b)$$

for all $a, b \in S$, is a standard ring property and we suppose that $(S, +, \cdot)$ is a ring. □

In other words, even though the "Ends lemma" cannot be used to construct Krasner hyperrings, it can be used to construct "the other class of hyperrings", *multiplicative hyperrings*. For a basic introduction to the topic of multiplicative hyperrings see book [111], chapter 4; for some results on strongly distributive multiplicative hyperrings see Rota [275]. Below we include two simple examples of such hyperstructures.

Example 2.5.17. Suppose the commutative ring $(\mathbb{Z}, +, \cdot)$ and define the hyperoperation on $\mathbb{Z}$ by $a \odot b = [a \cdot b)_{\leq}$ for all $a, b \in \mathbb{Z}$, where "$\leq$" is the usual ordering of integers. Then $(\mathbb{Z}, \odot)$ is an EL–semihypergroup and $(\mathbb{Z}, +, \odot)$ is a strongly distributive multiplicative hyperring (as well as a good multiplicative hyperring).

Example 2.5.18. Already in [275] Rota uses the set of integer residue classes as a tool to construct a multiplicative hyperring. Suppose the commutative ring $(\mathbb{Z}, +, \cdot)$ and define the hyperoperation on $\mathbb{Z}$ by $a \odot b = [a \cdot b)_{\equiv}$ for all $a, b \in \mathbb{Z}$, where, for a fixed $m \in \mathbb{Z}$, "$\equiv$" is the relation of congruence modulo m. In this way $a \odot b$ is a residue class – that one in which the product $a \cdot b$ belongs to. Then $(\mathbb{Z}, \odot)$ is an EL–semihypergroup and it is easy to show that $(\mathbb{Z}, +, \odot)$ is a strongly distributive multiplicative hyperring (as well as a good multiplicative hyperring).

In Definition 1.1.14 we followed the classification of Vougiouklis and defined semihyperrings as hyperstructures $(S, \oplus, \odot)$ with two hyperoperations. However, we can also use the classification based on the original Krasner's idea, where only "$\oplus$" is a hyperoperation. Thus we get the following classification used e.g. in works by Davvaz, Ameri or Hedayati related to their study of hyperideals such as e.g. [103, 144], or works by Chaopraknoi, Hobuntud, Kemprasit and others in their study of semigroups admitting semihyperrings with zero such as e.g. [41]. Notice that by "hyperring", the Krasner hyperring of Definition 1.1.13 is meant.

Definition 2.5.19. [144] A hyperalgebra $(S, \oplus, \cdot)$ is called a *semihyperring* if and only if

(i) $(S, \oplus)$ is a semihypergroup;

(ii) $(S, \cdot)$ is a semigroup;

(iii) $\forall a, b, c \in S$, $a \cdot (b \oplus c) = a \cdot b \oplus a \cdot c$ and $(b \oplus c) \cdot a = b \cdot a \oplus c \cdot a$

If we replace *(iii)* by

$$\forall a, b, c \in S, a \cdot (b \oplus c) \subseteq a \cdot b \oplus a \cdot c \text{ and } (b \oplus c) \cdot a \subseteq b \cdot a \oplus c \cdot a$$

we say that S is a *weak*[24] *distributive semihyperring*. A semihyperring is called *with zero element*, if there exists a unique element $0 \in S$ such that $0 \oplus x = x = x \oplus 0$ and $0 \cdot x = 0 = x \cdot 0$ for all $x \in S$. [...] A semihyperring is called a *hyperring* provided $(S, +)$ is a canonical hypergroup.

Given such a definition, we immediately get the following:

Theorem 2.5.20. *Let $(S, +, \cdot, \leq)$ be a quasi-ordered semiring and $(S, \oplus)$ the EL–semihypergroup constructed from a quasi-ordered semigroup $(S, +, \leq)$. Then $(S, \oplus, \cdot)$ is a semihyperring in the sense of Definition 2.5.19.*

Proof. The theorem is an immediate corollary to lemmas on distributivity included in Subsection 2.5.2. □

Given our objective 3 on page 100, we get the following result.

Theorem 2.5.21. *Let $(S, \odot)$ be the EL–semihypergroup of a quasi-ordered semigroup $(S, \cdot, \leq)$ such that "$\cdot$" is a commutative idempotent operation. Then $(S, \odot, \cdot)$ is a weak distributive semihyperring in the sense of Definition 2.5.19.*

[24] Hedayati and Ameri indeed use the word "weak" even though, given our discussion in Remark 1.1.16 on page 10, it is rather misleading as what they describe is "inclusive" or "non-strong" distributivity.

Proof. The theorem is an immediate corollary of Lemma 2.5.10. □

EL–semihyperrings of Definition 2.5.19 will always be without zero elements because the condition "$0 \oplus x = x = x \oplus 0$" is the property of scalar identities which, by Theorem 2.4.10 on page 44, do not exist in EL–semihypergroups. This blocks the road to Krasner hyperrings because the semihypergroup $(S, \oplus)$ can never be a canonical hypergroup. However, it still can be a commutative hypergroup in which the transposition axiom holds, i.e. a join space. Such hyperstructures $(H, \oplus, \cdot)$, introduced by Massouros and Mittas [218] to facilitate the study of automata, are called *hyperringoids*.

Definition 2.5.22. A *hyperringoid* is a structure $(H, \oplus, \cdot)$ where $(H, \oplus)$ is a join space, $(H, \cdot)$ is a semigroup and the multiplication "$\cdot$" is bilaterally distributive over "$\oplus$".

Lemma 2.5.23. ([95], p. 12) *A commutative hypergroup is canonical if and only if it is a join space with a scalar identity.*

In other words, hyperringoids are semihyperrings of Definition 2.5.19 such that "$\oplus$" is commutative and in $(H, \oplus)$ both reproductive (1.5) and transposition laws (1.11) hold. Yet how to enable this in the context of the "Ends lemma" was discussed in Section 2.4. The obvious suggestion to get hyperringoids is to assume that both $(S, \oplus)$ and $(S, \cdot)$ are groups – because by Lemma 2.1.5 the transposition law holds in EL–semihypergroups constructed from quasi-ordered groups and in Lemma 2.5.7 we need $(S, \cdot)$ to be a group. However, this would result in trivialities. Yet we can use hypergroups $(S, \oplus)$ constructed from proper semihypergroups in which the transposition law holds. Extensive EL–semihypergroups of Subsection 2.4.6 are an example of such hyperstructures.

Theorem 2.5.24. *Let $(S, +, \cdot, \leq)$ be a quasi-ordered semiring and $(S, \oplus)$ the mEL–semihypergroup constructed from a commutative extensive quasi-ordered semigroup $(S, +, \leq)$. Moreover, let $(S, \cdot)$ be a group. Then $(S, \oplus, \cdot)$ is a hyperringoid.*

Proof. By Theorem 2.4.71, $(S, \oplus)$ is a hypergroup (commutativity is not needed). By Corollary 2.4.75, the transposition law holds in $(S, \oplus)$ (commutativity is still not needed). If we suppose commutativity of "$\oplus$", we conclude that $(S, \oplus)$ is a join space. Distributivity follows from Lemma 2.5.5 and Lemma 2.5.7. In Remark 2.5.12 we mention that the fact that the (potentially existing) neutral element of $(S, +)$ is absorbing in $(S, \cdot)$ results in trivialities – yet this is only to be remarked as $(S, +)$ need not be a monoid. □

If we suppose that the quasi-ordered semiring $(S, +, \cdot, \leq)$ is a *field*, all we can get is an *inclusively distributive hyperringoid*. Indeed, by supposing that both $(S, +)$ and $(S \setminus \{0\}, \cdot)$ are groups we, by Remark 2.5.12, loose Lemma 2.5.7. However, in such a case we need not assume extensivity of "+".

Finally, notice that Massouros brothers [209] arrived to a more general definition as they define hyperringoids as hyperstructures $(H, +, \cdot)$, where $(H, +)$ is a hypergroup only while they call hyperringoids of Definition 2.5.22 *join hyperringoids*. This has no influence on the validity of Theorem 2.5.24, which assumes extensivity of "+". However, it would enable us to get a weak distributive hyperringoid – in the sense of [209] – even in case that $(S, +, \cdot)$ was not a field. However, as this a matter of terminology and proper application of lemmas on distributivity only, we do not seek to explore this topic.

Remark 2.5.25. Throughout Subsection 2.5.2, the results of which are crucial for our considerations in this subsection, we proved implications in the form "single-valued distributivity implies". This was because we wanted to show what the single-valued semirings result in when the "Ends lemma" is applied on them. Naturally, the validity of hyperstructure distributivity may be achieved by means of other conditions.

2.5.4 A special case: composition in EL–semihyperrings

Results of this subsection were accepted for publication in Hacettepe Journal of Mathematics and Statistics (WoS Q4) as Novák and Cristea [247].

Out of the many special classes of (semi)hyperrings we will mention *composition hyperrings* introduced by Cristea and Jančić-Rašović [98] and motivated by the study of a hyperring of polynomials in Jančić-Rašović [166]. This choice of topic is purely arbitrary as we want to show an example of a special property in the context of hyperrings in the sense of Spartalis and Vougiouklis.

The "hyperring" of the following definition is the "good hyperring in the general sense".

Definition 2.5.26. ([98], Def. 3.1) A *composition hyperring* is an algebraic structure $(R, \oplus, \odot, \circ)$, where $(R, \oplus, \odot)$ is a commutative hyperring and the hyperoperation "$\circ$" satisfies the following properties, for all $x, y, z \in R$:

1. $(x \oplus y) \circ z = x \circ z \oplus y \circ z$
2. $(x \odot y) \circ z = (x \circ z) \odot (y \circ z)$

3. $x \circ (y \circ z) = (x \circ y) \circ z$.

The binary hyperoperation "$\circ$" having the above properties is called the *composition hyperoperation* of the hyperring $(R, \oplus, \odot)$.

Composition hyperrings are multi-valued generalizations of *composition rings* introduced in Adler [1]. To follow [1, 98] we further on regard *commutative* hyperoperations only (recall that hyperstructures of Subsection 2.5.3 need not be commutative; also "commutative hyperring" in the above definition means that both "$\oplus$" and "$\odot$" are commutative). Notice that in the "Ends lemma" context, commutativity of the single-valued operation implies commutativity of the hyperoperation and antisymmetry of "$\leq$" turns this implication into equivalence. If $x \circ y$ is a one-element set for all $x, y \in R$, we will say "operation" rather than "hyperoperation" even though it will have to be at certain point applied in an element-wise manner on sets (see below e.g. (2.42) on page 117). We will continue with the approach suggested in Remark 2.5.25 and study properties of hyperstructures which follow from the properties of the single-valued structures.

To start with, we recall the precise meaning of symbols "$\oplus$" and "$\odot$". When applied on elements $a, b \in R$, they have the usual "Ends lemma" meaning

$$a \oplus b = [a + b)_{\leq} = \{x \in R \mid a + b \leq x\} \tag{2.34}$$

and

$$a \odot b = [a \cdot b)_{\leq} = \{y \in R \mid a \cdot b \leq y\}. \tag{2.35}$$

For sets $A, B \subseteq R$ there is

$$A \oplus B = \bigcup_{\substack{a \in A \\ b \in B}} [a + b)_{\leq} = \bigcup_{\substack{a \in A \\ b \in B}} \{x \in R \mid a + b \leq x\} \tag{2.36}$$

and

$$A \odot B = \bigcup_{\substack{a \in A \\ b \in B}} [a \cdot b)_{\leq} = \bigcup_{\substack{a \in A \\ b \in B}} \{y \in R \mid a \cdot b \leq y\}. \tag{2.37}$$

First of all we discuss a rather trivial case of *constant composition*.

Definition 2.5.27. If there is $x \circ y = r \circ s$ for an arbitrary quadruple of elements $x, y, r, s \in R$, we call the composition operation (hyperoperation) "$\circ$" *constant composition operation* (*hyperoperation*).

For the rest of this subsection recall that semihyperrings in the general sense are the weakest and most general type of hyperstructures mentioned in Theorem 2.5.14 on page 107.

Theorem 2.5.28. *Let $(S, \oplus, \odot)$ be a semihyperring in the general sense constructed using the "Ends lemma" from idempotent quasi-ordered semigroups $(S, +, \leq)$ and $(S, \cdot, \leq)$. Consider $r \in S$ arbitrary. Then "$\circ$" defined by*

$$a \circ b = [r)_{\leq} \tag{2.38}$$

for all $a, b \in S$, is a constant composition hyperoperation on $(S, \oplus, \odot)$. It is a constant operation if "$\leq$" is antisymmetric and r is the greatest element of $(S, \leq)$.

Proof. In our notation, the left-hand side of Definition 2.5.26, property 1, reads $(x \oplus y) \circ z$. This is

$$[x + y)_{\leq} \circ z = \bigcup_{\substack{\text{number of elements} \\ \text{of } [x+y)_{\leq} \text{ -times}}} [r)_{\leq} = [r)_{\leq}.$$

The right-hand side reads $(x \circ z) \oplus (y \circ z)$, which is

$$[r)_{\leq} + [r)_{\leq} = \bigcup_{a,b \in [r)_{\leq}} [a + b)_{\leq} = \bigcup_{\substack{r \leq a \\ r \leq b}} [a + b)_{\leq}.$$

Since $r \leq a$, $r \leq b$ implies $r + r \leq a + b$ and the relation "$\leq$" is reflexive, there is $[r)_{\leq} + [r)_{\leq} = [r + r)_{\leq}$. For idempotent "+" there is $r + r = r$, i.e. $[r)_{\leq} + [r)_{\leq} = [r)_{\leq}$.

The same reasoning can be applied on property 2 of the definition while property 3 holds obviously. Finally, if r is the greatest element of $(S, \leq)$, i.e. we need to assume antisymmetry of "$\leq$", then $[r)_{\leq} = \{r\}$, thus "$\circ$" is an operation rather than a hyperoperation. □

Example 2.5.29. If we continue with Example 2.5.15, where the hyperring in the general sense of the power set $\mathcal{P}(K)$ was discussed, and define

$$A \circ B = [R)_{\subseteq} = \{T \in \mathcal{P}(K) \mid R \subseteq T\}$$

for an arbitrary pair of $A, B \in \mathcal{P}(K)$ and an arbitrary $R \in \mathcal{P}(K)$, we get a constant composition hyperoperation on $\mathcal{P}(K)$. If $R = K$, then "$\circ$" becomes a constant composition operation.

Theorem 2.5.28 obviously does not hold when operations "+" or "$\cdot$" are non-idempotent. Not even one of the inclusions holds because neither $r \in [r + r)_{\leq}$ nor $r + r \in [r)_{\leq}$ in a general case. Yet we can prove the following theorem. Notice that we use the concept of maximum / minimum of a two element set in it. In order not to complicate our reasoning we assume that "$\leq$" is a partial ordering even though a way out could possibly be found even without antisymmetry.

Theorem 2.5.30. *Let $(S, \oplus, \odot)$ be a semihyperring in the general sense constructed from partially ordered semigroups $(S, +, \leq)$ and $(S, \cdot, \leq)$. If they exist, denote e_s the neutral element of $(S, +)$ and e_p the neutral element of $(S, \cdot)$.*

1. *If simultaneously $e_p \leq e_p + e_p$ and $e_s \leq e_s \cdot e_s$, then "$\circ_{\min e}$" defined by*
$$a \circ_{\min e} b = [\min\{e_s, e_p\})_{\leq} \tag{2.39}$$
for all $a, b \in S$, is a constant composition hyperoperation on $(S, \oplus, \odot)$.
2. *If simultaneously $e_p + e_p \leq e_p$ and $e_s \cdot e_s \leq e_s$, then "$\circ_{\max e}$" defined by*
$$a \circ_{\max e} b = [\max\{e_s, e_p\})_{\leq} \tag{2.40}$$
for all $a, b \in S$, is a constant composition hyperoperation on $(S, \oplus, \odot)$.

Before proving the theorem, agree that if the elements e_s, e_p are incomparable, then since their minimum does not exist, we set $a \circ_{\min e} b = \emptyset$. Moreover, if only e_s exists, then we set $\min\{e_s, e_p\} = e_s$ (and the same for e_p). And make the analogous agreement for the maxima.

Proof. We will prove the theorem for "$\circ_{\min e}$" only. The proof for "$\circ_{\max e}$" is analogous.

In our notation the left-hand-side of Definition 2.5.26, property 1, reads $(x \oplus y) \circ z$. This is
$$[x+y)_{\leq} \circ_{\min e} z = \bigcup_{\substack{\text{number of elements} \\ \text{of } [x+y)_{\leq}\text{–times}}} [\min\{e_s, e_p\})_{\leq} = [\min\{e_s, e_p\})_{\leq}$$
while the right-hand side, which reads $(x \circ z) \oplus (y \circ z)$, is
$$[\min\{e_s, e_p\})_{\leq} + [\min\{e_s, e_p\})_{\leq} = \bigcup_{\substack{\min\{e_s,e_p\} \leq a \\ \min\{e_s,e_p\} \leq b}} [a+b)_{\leq}.$$
Now, the following cases are possible:

$e_s \leq e_p$: This means that $\min\{e_s, e_p\} = e_s$; the left-hand side is $[e_s)_{\leq}$ while the right-hand side is $\bigcup\limits_{\substack{e_s \leq a \\ e_s \leq b}} [a+b)_{\leq} = [e_s + e_s)_{\leq} = [e_s)_{\leq}$, i.e. the same.

$e_p < e_s$: This means that $\min\{e_s, e_p\} = e_p$; the left-hand side is $[e_p)_{\leq}$ while the right hand side is $\bigcup\limits_{\substack{e_p \leq a \\ e_p \leq b}} [a+b)_{\leq} = [e_p + e_p)_{\leq}$. Suppose now an arbitrary $x \in [e_p)_\leq$, i.e. such $x \in S$ that $e_p \leq x$. Since we assume that $e_p < e_s$, there is also $e_p + e_p < x + e_s = x$, i.e. $x \in [e_p + e_p)_\leq$. If on the other hand we suppose an arbitrary $x \in [e_p + e_p)_\leq$, i.e. $e_p + e_p \leq x$, then on condition assumed in the theorem, i.e. $e_p \leq e_p + e_p$, there is from transitivity that $e_p \leq x$, which means that $x \in [e_p)_\leq$. Altogether $[e_p)_\leq = [e_p + e_p)_\leq$.

If neither e_s nor e_p exists or if e_s and e_p are incomparable, we end up with $\emptyset = \emptyset$. If only e_s exists, we get the same as when $e_s \leq e_p$. If only e_p exists, we get the same as when $e_p < e_s$.

The proof of Definition 2.5.26 property 2, is completely analogous. The proof of property 3 is obvious. □

Example 2.5.31. Since $(\mathbb{Z}, +, \leq)$, where "$\leq$" is the natural ordering of integers, is a partially ordered group, $(\mathbb{Z}, \cdot, \leq)$ a partially ordered semigroup and $e_s = 0$, $e_p = 1$, the hyperoperation "$\circ$" defined for all $a, b \in \mathbb{Z}$ by $a \circ b = [0)_\leq$ is an example of a constant composition hyperoperation on the hyperring in the general sense $(\mathbb{Z}, \oplus, \odot)$ in a context where the single-valued operations "$+$" and "$\cdot$" are non-idempotent. The conditions of Theorem 2.5.30 obviously hold because $1 \leq 1 + 1$ and $0 \leq 0 \cdot 0$.

The constant compositions are rather trivial and degenerated cases yet even there the limits of applying the composition property in the context of the "Ends lemma" can be seen. It is rather difficult to achieve equality in properties 1 and 2 since the addition (or multiplication) on the left-hand side is applied on elements while on the right-hand side it is (in a general case) applied on sets – and this is done in a context where neither $a \in [a + a)_\leq$ nor $a + a \in [a)_\leq$ holds generally. Let us therefore adjust the composition hyperoperation of Definition 2.5.26 to suit the "Ends lemma" better.

Definition 2.5.32. A binary operation (hyperoperation) on a semihyperring in the general sense $(S, \oplus, \odot)$ is called a *left weak composition operation* (*hyperoperation*) and denoted "$\circ_{lw}$" if, for all $x, y, z \in S$,

1. $(x \oplus y) \circ_{lw} z \subseteq (x \circ_{lw} z) \oplus (y \circ_{lw} z)$
2. $(x \odot y) \circ_{lw} z \subseteq (x \circ_{lw} z) \odot (y \circ_{lw} z)$
3. $x \circ_{lw} (y \circ_{lw} z) = (x \circ_{lw} y) \circ_{lw} z$.

or the *right weak composition operation* (*hyperoperation*) and denoted "$\circ_{rw}$" if, for all $x, y, z \in S$:

1. $(x \circ_{rw} z) \oplus (y \circ_{rw} z) \subseteq (x \oplus y) \circ_{rw} z$

2. $(x \circ_{rw} z) \odot (y \circ_{rw} z) \subseteq (x \odot y) \circ_{rw} z$

3. $x \circ_{rw} (y \circ_{rw} z) = (x \circ_{rw} y) \circ_{rw} z$.

The hyperstructure $(S, \oplus, \odot, \circ_W)$ (regardless of type) is called a *weak composition hyperstructure* (i.e. *weak composition semihyperring / weak composition hyperring* / etc.) regardless of whether "$\circ_W$" = "$\circ_{lw}$" or "$\circ_W$" = "$\circ_{rw}$" or whether "$\circ_W$" is single– or multi–valued.

Chvalina has in [43,44] and subsequent papers introduced and studied the concept of *quasi-order hypergroups* (for details see Section 1.2 and Subsection 2.6.3). In the following theorem we not only give necessary conditions for the existence of a left (right) weak composition hyperoperation but also establish a link between quasi-order hypergroups and EL–hyperstructures by defining the composition hyperoperation by $a \circ b = [a)_\leq \cup [b)_\leq$ for all $a, b \in S$, i.e. by a condition used when testing whether a hypergroupoid $(H, \circ)$ is a quasi-order hypergroup.[25] Notice that, thanks to reflexivity of relation "$\leq$", the set $[a)_\leq \cup [b)_\leq$ has for $a \neq b$ always at least two elements, i.e. the notation defines a hyperoperation.

Theorem 2.5.33. *Let $(S, \oplus, \odot)$ be a semihyperring in the general sense constructed from quasi-ordered semigroups $(S, +, \leq)$ and $(S, \cdot, \leq)$. If, for all $r \in S$, there is $r + r \leq r$ and $r \cdot r \leq r$, then there always exists a left weak composition hyperoperation "$\circ_{lw}$" on $(S, \oplus, \odot)$.*

Proof. Define $a \circ_{lw} b = [a)_\leq \cup [b)_\leq$ for all $a, b \in S$. In this context the left-hand side of property 1 of Definition 2.5.32 is

$$[x + y)_\leq \circ_{lw} z = \bigcup_{x+y \leq a} [a)_\leq \cup [z)_\leq = [x + y)_\leq \cup [z)_\leq$$

while the right-hand side is

$$(x \circ_{lw} z) \oplus (y \circ_{lw} z) = ([x)_\leq \cup [z)_\leq) \oplus ([y)_\leq \cup [z)_\leq) = \bigcup_{\substack{a \in [x)_\leq \cup [z)_\leq \\ b \in [y)_\leq \cup [z)_\leq}} [a + b)_\leq,$$

i.e. $(x \circ_{lw} z) \oplus (y \circ_{lw} z) = \{d \in S \mid a + b \leq d, (x \leq a \text{ or } z \leq a) \text{ and } (y \leq b \text{ or } z \leq b)\}$. Suppose an arbitrary $c \in [x + y)_\leq \circ_{lw} z$. There are two options: $c \in [x + y)_\leq$ or $c \in [z)_\leq$. If $c \in [x + y)_\leq$, then obviously $c \in (x \circ_{lw} z) \oplus (y \circ_{lw} z)$ because $a \in [x)_\leq$, $b \in [y)_\leq$, i.e. $x \leq a$, $y \leq b$ implies $x + y \leq a + b$ which thanks

[25] For details see e.g. Corsini and Leoreanu [95], chapter 3, §1 and (1.29) on page 23 and (2.63) on page 137.

to transitivity of "$\leq$" means that $x+y \leq c$ which is what we suppose. If $c \in [z)_{\leq}$, i.e. $z \leq c$, then if we suppose that $z+z \leq z$, we get from transitivity of "$\leq$" that $z+z \leq c$. Yet this is on the right-hand side the case of $a \in [z)_{\leq}$, $b \in [z)_{\leq}$, i.e. $z+z \leq a+b$.

The proof of property 2 is analogous, the proof of property 3 is obvious. □

Corollary 2.5.34. *If $(S,+,\leq)$ and $(S,\cdot,\leq)$ are idempotent quasi-ordered semigroups, then there always exists a left weak composition hyperoperation "$\circ_{lw}$" on $(S,\oplus,\odot)$. The same holds if $r+r \leq r$ for all $r \in S$ and $(S,\cdot,\leq)$ is an idempotent quasi-ordered semigroup or if $r \cdot r \leq r$ for all $r \in S$ and $(S,+,\leq)$ is an idempotent quasi-ordered semigroup.*

Proof. Conditions $r+r \leq r$, $r \cdot r \leq r$ included in Theorem 2.5.33 in this case turn into $r \leq r$. However, since the relation "$\leq$" is reflexive, they hold trivially. □

Theorem 2.5.35. *Let $(S,\oplus,\odot)$ be a semihyperring in the general sense constructed from a quasi-ordered semigroup $(S,+,\leq)$ and a commutative idempotent quasi-ordered semigroup $(S,\cdot,\leq)$. There always exists a right weak composition hyperoperation on $(S,\oplus,\odot)$.*

Proof. For arbitrary $A,B \subseteq S$ denote

$$A \circ_{rw} B = \{a \cdot b \mid a \in A, b \in B\}, \tag{2.41}$$

where "$\cdot$" is the single-valued product of $(S,\cdot,\leq)$. One-element sets A,B will be represented by the elements themselves, i.e. $\{a\} \circ_{rw} \{b\} = a \cdot b$, which will allow us to write

$$a \circ_{rw} b = a \cdot b \tag{2.42}$$

for all $a,b \in S$.

Now, in property 1 of Definition 2.5.32 we get on the left-hand side, which reads $(x \circ_{rw} z) \oplus (y \circ_{rw} z)$, the set $[x \cdot z + y \cdot z)_{\leq}$ which, thanks to distributivity of the single-valued structure $(S,+,\cdot)$, is $[(x+y) \cdot z)_{\leq}$. On the right-hand side, which reads $(x \oplus y) \circ_{rw} z$, we get $[x+y)_{\leq} \circ_{rw} z$, which equals $\bigcup\limits_{x+y \leq s} \{s \cdot z\}$. Yet since the relation "$\leq$" is reflexive, there is $x+y \leq x+y$ and $[(x+y) \cdot z)_{\leq} \subseteq \bigcup\limits_{x+y \leq s} \{s \cdot z\}$.

In property 2 of Definition 2.5.32 we get that (thanks to commutativity and idempotency)

$$\begin{aligned}&(x \circ_{rw} z) \odot (y \circ_{rw} z) = (x \cdot z) \odot (y \cdot z) = [x \cdot z \cdot y \cdot z)_{\leq} = \\ &= [x \cdot y \cdot z \cdot z)_{\leq} = [x \cdot y \cdot z)_{\leq}.\end{aligned}$$

On the left-hand side we get that $[x \cdot y)_{\leq} \circ_{rw} z = \bigcup\limits_{x \cdot y \leq r} \{r \cdot z\}$. Thus thanks to reflexivity of the relation "$\leq$" property 2 holds.

In property 3 of Definition 2.5.32 there is $x \circ_{rw} (y \circ_{rw} z) = x \circ_{rw} (y \cdot z) = x \cdot y \cdot z$ and $(x \circ_{rw} y) \circ_{rw} z = (x \cdot y) \circ_{rw} z = x \cdot y \cdot z$. □

The following lemma mentions a rather specific case of EL–semihypergroups constructed from linearly ordered commutative semigroups. Even though linear ordering is rather special as we assume two more properties of the relation "$\leq$" on top of reflexivity and transitivity, it is important to realize that number sets with the usual operations of addition and multiplication *are* linearly ordered and commutative; in some of these the conditions assumed in the lemma hold. Notice that unlike in Theorem 2.5.35, idempotency is not needed in this lemma.

Lemma 2.5.36. *Let $(S, \oplus, \odot)$ be a semihyperring in the general sense constructed from linearly ordered commutative semigroups $(S, +, \leq)$ and $(S, \cdot, \leq)$. If implications $a + a \leq b \Rightarrow a \leq b$ and $a \cdot a \leq b \Rightarrow a \leq b$ hold for all $a, b \in S$, then there always exists a right weak composition hyperoperation "$\circ_{rw}$" on $(S, \oplus, \odot)$.*

Proof. We will show that the weak composition hyperoperation in question will be

$$a \circ_{rw} b = [\max\{a, b\})_{\leq}. \tag{2.43}$$

Suppose arbitrary $x, y, z \in S$. First we discuss the meaning of property 1 of Definition 2.5.32 based on definitions of "$\oplus$" and "$\circ_{rw}$". In our notation the left-hand side reads $(x \circ_{rw} z) \oplus (y \circ_{rw} z)$. This is

$$[\max\{x, z\})_{\leq} \oplus [\max\{y, z\})_{\leq} = \bigcup_{\substack{a \in [\max\{x,z\})_{\leq} \\ b \in [\max\{y,z\})_{\leq}}} [a + b)_{\leq} = \bigcup_{\substack{\max\{x,z\} \leq a \\ \max\{y,z\} \leq b}} [a + b)_{\leq},$$

which results in the following four cases based on the relations between x, y and z. Notice that reasoning in cases **C** and **D** is analogous to reasoning in case **B**.

A) $x \leq z, y \leq z$**:** In this case $\max\{x, z\} = z$, $\max\{y, z\} = z$ and moreover $x + y \leq z + z$. Thus

$$\bigcup_{\substack{\max\{x,z\} \leq a \\ \max\{y,z\} \leq b}} [a + b)_{\leq} = \bigcup_{\substack{z \leq a \\ z \leq b}} [a + b)_{\leq} = \{c \in S \mid a + b \leq c, z \leq a, z \leq b\}.$$

At the same time conditions $z \leq a$, $z \leq b$ result in $z + z \leq a + b$ and from transitivity of "$\leq$" we get that $z + z \leq c$. Finally

$$(x \circ_{rW} z) \oplus (y \circ_{rW} z) = \{c \in S \mid x + y \leq c\} = \{c \in S \mid z + z \leq c\}. \tag{2.44}$$

B) $x \leq z, z \leq y$**:** In this case $\max\{x,z\} = z$, $\max\{y,z\} = y$ and moreover from transitivity of "$\leq$" there is $x \leq y$. Thus

$$\bigcup_{\substack{\max\{x,z\}\leq a\\ \max\{y,z\}\leq b}} [a+b)_{\leq} = \bigcup_{\substack{z\leq a\\ y\leq b}} [a+b)_{\leq} = \{c \in S \mid a+b \leq c, z \leq a, y \leq b\}.$$

At the same time conditions $z \leq a$, $y \leq b$ result in $z+y \leq a+b$ and from transitivity of "$\leq$" we get that $z+y \leq c$. Finally

$$(x \circ_{rW} z) \oplus (y \circ_{rW} z) = \{c \in S \mid z+y \leq c\}. \tag{2.45}$$

C) $z \leq x, y \leq z$**:** This results in $(x \circ_{rw} z) \oplus (y \circ_{rw} z) = \{c \in S \mid x+z \leq c\}$.

D) $z \leq x, z \leq y$**:** This results in

$$(x \circ_{rw} z) \oplus (y \circ_{rw} z) = \{c \in S \mid x+y \leq c\} = \{c \in S \mid z+z \leq c\}$$

The right-hand side of property 1 of Definition 2.5.32 reads $(x \oplus y) \circ_{rw} z$. Based on definitions of "$\oplus$" and "$\circ_{rw}$" this is

$$[x+y)_{\leq} \circ_{rw} z = \bigcup_{r\in[x+y)_{\leq}} [\max\{r,z\})_{\leq} = \bigcup_{x+y\leq r} [\max\{r,z\})_{\leq}.$$

However, in our case this is the same as $[\max\{x+y,z\})_{\leq}$, which is

$$\{d \in S \mid \max\{x+y,z\} \leq d\}. \tag{2.46}$$

Now we verify the inclusion in property 1 of Definition 2.5.32. Suppose an arbitrary $c \in (x \circ_{rw} z) \oplus (y \circ_{rw} z)$ and let us find out whether $c \in (x \oplus y) \circ_{rw} z$. We have to test each of the cases $\mathbf{A} - \mathbf{D}$.

ad A: The element c is such that $z+z \leq c$, $x+y \leq c$ and at the same time $x \leq z$, $y \leq z$. Thus

1. If $\max\{x+y,z\} = x+y$, then (2.46) turns into $\{d \in S \mid x+y \leq d\}$. Thus $c \in (x \oplus y) \circ_{rW} z$ obviously holds.
2. If $\max\{x+y,z\} = z$, then (2.46) turns into $\{d \in S \mid z \leq d\}$ and we have to show that $z \leq c$. Yet since $z+z \leq c$, there is – thanks to the assumption of the theorem – also $z \leq c$ and $c \in (x \oplus y) \circ_{rW} z$.

ad B: The element c is such that $z+y \leq c$ and at the same time $x \leq z$, $z \leq y$. Thus

1. If $\max\{x+y, z\} = x+y$, then (2.46) turns into $\{d \in S \mid x+y \leq d\}$. Since $x \leq z$, there is $x+y \leq z+y$ and from transitivity we get that $x+y \leq c$. Thus $c \in (x \oplus y) \circ_{rW} z$.
2. If $\max\{x+y, z\} = z$, then (2.46) turns into $\{d \in S \mid z \leq d\}$ and we have to show that $z \leq c$. Since $z \leq y$, there is $z+z \leq z+y$ and from transitivity of "$\leq$", there is $z+z \leq c$. Yet this means – thanks to the assumption of the theorem – that $z \leq c$ and $c \in (x \oplus y) \circ_{rW} z$.

ad C: The element c is such that $x+z \leq c$ and at the same time $z \leq x$, $y \leq z$. Thus

1. If $\max\{x+y, z\} = x+y$, then (2.46) turns into $\{d \in S \mid x+y \leq d\}$ and we have to show that $x+y \leq c$. Suppose on contrary that $c < x+y$. Since $y \leq z$, there is $c < x+z$. Yet since simultaneously $x+z \leq c$, we get from transitivity that $c < c$ which is impossible. Thus $x+y \leq c$ and $c \in (x \oplus y) \circ_{rW} z$.
2. If $\max\{x+y, z\} = z$, then (2.46) turns into $\{d \in S \mid z \leq d\}$ and we have to show that $z \leq c$. Since $z \leq x$, there is $z+z \leq x+z$ and from transitivity of "$\leq$", there is $z+z \leq c$. Yet this – thanks to the assumption of the theorem – means that $z \leq c$ and $c \in (x \oplus y) \circ_{rW} z$.

ad D: The element c is such that $x+y \leq c$, $z+z \leq c$ and at the same time $z \leq x$, $z \leq y$. Thus

1. If $\max\{x+y, z\} = x+y$, then (2.46) turns into $\{d \in S \mid x+y \leq d\}$ and we have to show that $x+y \leq c$. Yet this is one of our assumptions. Thus $c \in (x \oplus y) \circ_{rW} z$ holds trivially.
2. If $\max\{x+y, z\} = z$, then (2.46) turns into $\{d \in S \mid z \leq d\}$ and we have to show that $z \leq c$. Yet since $z+z \leq c$, there is also – thanks to the assumption of the theorem – that $z \leq c$ and $c \in (x \oplus y) \circ_{rW} z$.

Thus we have verified validity of property 1 of Definition 2.5.32. The proof of property 2 is completely analogous.

Verifying property 3 is rather straightforward. The left-hand side $x \circ_{rw} (y \circ_{rw} z)$ is

$$x \circ_{rw} [\max\{y,z\})_{\leq} = \bigcup_{r \in [\max\{y,z\})_{\leq}} [\max\{x,r\})_{\leq} = \bigcup_{\max\{y,z\} \leq r} [\max\{x,r\})_{\leq}$$

while the right-hand side $(x \circ_{rw} y) \circ_{rw} z$ is

$$[\max\{x,y\})_{\le} \circ_{rw} z = \bigcup_{s\in[\max\{x,y\})_{\le}} [\max\{s,z\})_{\le} = \bigcup_{\max\{x,y\}\le s} [\max\{s,z\})_{\le}.$$

Yet since the relation "$\le$" is reflexive, i.e. $\max\{y,z\} \le \max\{y,z\}$, $\max\{x,y\} \le \max\{x,y\}$, both sides equal $[\max\{x,y,z\})_{\le}$.

Thus finally (2.43) is a weak composition hyperoperation on $(R,\oplus,\odot)$ with the assumed properties. □

Remark 2.5.37. Notice that as regards number domains, the implications used in Theorem 2.5.36 which obviously hold in $\mathbb{N}$ or $\mathbb{Z}$, do not hold for other number domains. The transition to $\mathbb{Q}$ or $\mathbb{R}$ is not possible as e.g. $0.1\cdot 0.1 \le 0.02$ yet $0.1 \not\le 0.02$. Notice that if we expanded Example 2.5.40 to $R=\mathbb{R}^+$ or considered this in the theorem, then e.g. in case **C2** of the proof the conditions would not hold for multiplication and $x=0.1$, $y=0.02$, $z=0.1$.

Example 2.5.38. If we continue with Example 2.5.15 on page 107 and define

$$A \circ_{lw} B = [A)_{\subseteq} \cup [B)_{\subseteq} = \{R \in \mathcal{P}(K) \mid A \subseteq R \text{ or } B \subseteq R\}$$

for all $A,B \in \mathcal{P}(K)$, then since both set intersection and set union are idempotent, the above defines a left weak composition hyperoperation on $(\mathcal{P}(K),\oplus,\odot)$, i.e. $(\mathcal{P}(K),\oplus,\odot,\circ_{lw})$ is a weak composition hyperring in the general sense.

Example 2.5.39. If we continue with Example 2.5.15 and define $A\circ_{rw} B = A\cap B$ for all $A,B\in\mathcal{P}(K)$, then since the set intersection is both commutative and idempotent (and distributive with respect to set union), this defines a right weak composition operation on $(\mathcal{P}(K),\oplus,\odot)$, i.e. that $(\mathcal{P}(K),\oplus,\odot,\cap)$ is a weak composition hyperring in the general sense.

Examples 2.5.40 and 2.5.41 are partly motivated by the classical *interval binary hyperoperation* on linearly ordered groups discussed e.g. in Iwasava [161] and defined by

$$a*b = [\min\{a,b\})_{\le} \cap (\max\{a,b\}]_{\le} = \{x\in G \mid \min\{a,b\} \le x \le \max\{a,b\}\} \tag{2.47}$$

for all $a,b\in G$.

Example 2.5.40. Regard the ordered semiring of natural numbers, i.e. a distributive structure $(\mathbb{N},+,\cdot)$, where $(\mathbb{N},+)$ and $(\mathbb{N},\cdot)$ are semigroups and "$\le$" is the usual ordering of natural numbers with the smallest element 1.

Obviously $(\mathbb{N}, +, \leq)$ and $(\mathbb{N}, \cdot, \leq)$ are quasi-ordered semigroups, which enables us to construct EL–semihypergroups $(\mathbb{N}, \oplus)$ and $(\mathbb{N}, \odot)$. Thus we get a semihyperring in the general sense $(\mathbb{N}, \oplus, \odot)$. By Lemma 2.5.36 we get that (2.43) is a weak composition hyperoperation on $(\mathbb{N}, \oplus, \odot)$.

Example 2.5.41. One can easily show that when changing $\max\{a, b\}$ to $\min\{a, b\}$ in (2.43), we get another weak composition hyperoperation on $(\mathbb{N}, \oplus, \odot)$.

2.5.5 A corollary: from lattices to H_v–matrices

Results of this subsection were published by Analele Ştiinţifice ale Universităţii "Ovidius" Constanţa (WoS Q4) as Křehlík and Novák [187].

When one views EL–hyperstructures from the perspective of the *lattice theory*, it becomes obvious that dualizing the concept, involving one more "similar" hyperoperation or changing "$a \cdot b \leq x$" in the construction to the interval definition using two hyperoperations may lead to interesting results. Moreover, despite all the advances of the hyperstructure theory, the concept of a *matrix*, i.e. a two-dimensional scheme of $m \times n$ entries, has been studied in it only occassionaly. The exception to this rule is the concept of H_v–*matrices* used by Vougiouklis in the representation theory [299–301]. Before including its definition recall the definition of H_v–rings on page 10.

Definition 2.5.42. By an H_v–*matrix* we mean a matrix entries of which are elements of an H_v–ring or an H_v–field.

However, since entries of H_v–matrices are elements of H_v–*rings*, i.e. hyperstructures with two hyperoperations, defining and working with H_v–matrix multiplication, trace or rank of H_v–matrices or other matrical concepts is complicated or yet unexplored. In fact, when in 2009 in his overview paper [299], Vougiouklis presented "some of the open problems arising on the topic in the procedure to find representations on hypergroups", four out of the eight presented problems regard H_v–matrices.

Therefore, in this subsection, we apply the "Ends lemma" on sets of matrices. We show that our considerations naturally result in H_v–matrices which might provide a tool for better applications of this concept. Also, we show that in the context of lattices, there exist other classes of hyperstructures analogous to EL–hyperstructures. In this we expand results of Davvaz, Leoreanu-Fotea, Rosenberg or Varlet [192, 195, 196, 295].

In this subsection we denote $\mathbb{M}_{m,n}(\mathcal{S})$ the set of all $m \times n$ matrices with entries from a suitable set $\mathcal{S}$, i.e.

$$\mathbb{M}_{m,n}(\mathcal{S}) = \{\mathbf{M} = [m_{i,j}]) \mid m_{i,j} \in \mathcal{S}, i = \{1, \ldots, m\}, j = \{1, \ldots, n\}\}. \quad (2.48)$$

On $\mathbb{M}_{m,n}(\mathcal{S})$ we, for an arbitrary pair of matrices $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{m,n}(\mathcal{S})$, naturally define relation "$\leq_M$" in an entry-wise manner by

$$\mathbf{A} \leq_M \mathbf{B} \text{ if } a_{i,j} \leq_e b_{i,j} \text{ for all } i = \{1, \ldots, m\}, j = \{1, \ldots, n\}, \tag{2.49}$$

where "$\leq_e$" is a suitable relation between entries of the matrices. Suppose that $(\mathcal{S}, \inf, \sup, \leq_e)$ is a lattice and define the *minimum of matrices* $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{m,n}(\mathcal{S})$ by

$$\min\{\mathbf{A}, \mathbf{B}\} = \mathbf{C}, \text{ where } \mathbf{C} \in \mathbb{M}_{m,n}(\mathcal{S}) \text{ is such that } c_{i,j} = \inf\{a_{i,j}, b_{i,j}\} \tag{2.50}$$

for all $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$, in case of two matrices and analogically in case of more matrices; and the *maximum of matrices* $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{m,n}(\mathcal{S})$ by

$$\max\{\mathbf{A}, \mathbf{B}\} = \mathbf{D}, \text{ where } \mathbf{D} \in \mathbb{M}_{m,n}(\mathcal{S}) \text{ is such that } d_{i,j} = \sup\{a_{i,j}, b_{i,j}\} \tag{2.51}$$

for all $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$, in case of two matrices and analogically in case of more matrices.

We will show later on that the straightforwardness and suspected "simplicity" of the above definitions is in fact their advantage. Of course, we do not seek to explore concepts such as *traces*, *determinants* or *calculations of inverse matrices* in this context of lattices. Our matrix operations will be restricted to those defined above.

Lemma 2.5.43. *The operations "*min*" and "*max*" defined on $\mathbb{M}_{m,n}(\mathcal{S})$ are idempotent, commutative and associative. $(\mathbb{M}_{m,n}(\mathcal{S}), \leq_M)$ is a partially ordered set. $(\mathbb{M}_{m,n}(\mathcal{S}), \min, \leq_M)$, and $(\mathbb{M}_{m,n}(\mathcal{S}), \max, \leq_M)$, are partially ordered semigroups.*

Proof. Obvious. □

Lemma 2.5.43 allows us to make an immediate conclusion regarding the structure $(\mathbb{M}_{m,n}(\mathcal{S}), \min, \max, \leq_M)$.

Theorem 2.5.44. *$(\mathbb{M}_{m,n}(\mathcal{S}), \min, \max, \leq_M)$ is a lattice.*

Proof. Lemma 2.5.43 verifies commutativity, associativity and idempotency. The absorption laws hold thanks to the relationship between "$\leq_M$" and "$\leq_e$", expressed by (2.49), and the fact that $(\mathcal{S}, \inf, \sup, \leq_e)$ is a lattice. □

Now that we have established the context of $\mathbb{M}_{m,n}(\mathcal{S})$, we define two pairs of dual hyperoperations on $\mathbb{M}_{m,n}(\mathcal{S})$ using (2.49) and (2.50), or (2.51), respectively.

First, for an arbitrary pair of matrices $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{m,n}(\mathcal{S})$ we define[26]

$$\mathbf{A} \circ \mathbf{B} = \{\mathbf{C} \in \mathbb{M}_{m,n}(\mathcal{S}) \mid \min\{\mathbf{A}, \mathbf{B}\} \leq_M \mathbf{C}\}, \tag{2.52}$$

i.e., for all $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$,

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \circ \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \dots & \dots & \dots \\ b_{m1} & \dots & b_{mn} \end{bmatrix} =$$
$$\left\{ \begin{bmatrix} c_{11} & \dots & c_{1n} \\ \dots & \dots & \dots \\ c_{m1} & \dots & c_{mn} \end{bmatrix} \in \mathbb{M}_{m,n}(\mathcal{S}) \mid \inf\{a_{ij}, b_{ij}\} \leq_e c_{ij} \right\}$$

and dually

$$\mathbf{A} \bullet \mathbf{B} = \{\mathbf{D} \in \mathbb{M}_{m,n}(\mathcal{S}) \mid \max\{\mathbf{A}, \mathbf{B}\} \geq_M \mathbf{D}\}, \tag{2.53}$$

i.e., for all $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$,

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \bullet \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \dots & \dots & \dots \\ b_{m1} & \dots & b_{mn} \end{bmatrix} =$$
$$\left\{ \begin{bmatrix} d_{11} & \dots & d_{1n} \\ \dots & \dots & \dots \\ d_{m1} & \dots & d_{mn} \end{bmatrix} \in \mathbb{M}_{m,n}(\mathcal{S}) \mid \sup\{a_{ij}, b_{ij}\} \geq_e d_{ij} \right\}.$$

In the following lemma, recall that "$\approx$" stands for non-empty intersection.

Lemma 2.5.45. *For an arbitrary quadruple* $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4 \in \mathbb{M}_{m,n}(\mathcal{S})$ *we have* $\mathbf{A}_1 \circ \mathbf{A}_2 \approx \mathbf{A}_3 \circ \mathbf{A}_4$ *and* $\mathbf{A}_1 \bullet \mathbf{A}_2 \approx \mathbf{A}_3 \bullet \mathbf{A}_4$.

Proof. The proof for both hyperoperations is analogous, we include it only for hyperoperation "$\circ$". Suppose $\mathbf{A}_i, i \in \{1, 2, 3, 4\}$, are arbitrary elements of $\mathbb{M}_{m,n}(\mathcal{S})$. Denote $\mathbf{B} = \max\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\}$. Since $\mathbb{M}_{m,n}(\mathcal{S})$ is a lattice, there is $\mathbf{B} \in \mathbb{M}_{m,n}(\mathcal{S})$. Moreover, there is $\min\{\mathbf{A}_1, \mathbf{A}_2\} \leq_M \mathbf{B}$ and $\min\{\mathbf{A}_3, \mathbf{A}_4\} \leq_M \mathbf{B}$. As a result, $\mathbf{B} \in \mathbf{A}_1 \circ \mathbf{A}_2$ and also $\mathbf{B} \in \mathbf{A}_3 \circ \mathbf{A}_4$, which proves the lemma. $\square$

[26]The symbol "$\circ$" used in (2.52) shall not be confused with the composition hyperoperation of Subsection 2.5.4.

Example 2.5.46. Let $\mathcal{S}$ be the lattice of divisors of a suitable natural number n with $\inf\{a, b\}$ being the greatest common divisor of $a, b \in \mathbb{N}$, $\sup\{a, b\}$ being the least common multiple of a, b and $a \leq_e b$ if $a|b$. For e.g. $n = 120$, divisors of which are $1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120$, construct $\mathbb{M}_{2,2}(\mathcal{S})$ and regard an arbitrary quadruple of matrices $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4 \in \mathbb{M}_{2,2}(\mathcal{S})$, e.g. $\mathbf{A}_1 = \begin{bmatrix} 8 & 15 \\ 3 & 6 \end{bmatrix}$, $\mathbf{A}_2 = \begin{bmatrix} 10 & 12 \\ 20 & 24 \end{bmatrix}$ $\mathbf{A}_3 = \begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix}$, $\mathbf{A}_4 = \begin{bmatrix} 8 & 12 \\ 30 & 1 \end{bmatrix}$. Then $\mathbf{B} = \begin{bmatrix} 40 & 60 \\ 60 & 24 \end{bmatrix}$,

$$\mathbf{A}_1 \circ \mathbf{A}_2 = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, 2|a_{11}, 3|a_{12}, 1|a_{21}, 6|a_{22} \right\},$$
$$\mathbf{A}_3 \circ \mathbf{A}_4 = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, 1|a_{11}, 2|a_{12}, 5|a_{21}, 1|a_{22} \right\},$$

and obviously $\mathbf{B} \in \mathbf{A}_1 \circ \mathbf{A}_2 \cap \mathbf{A}_3 \circ \mathbf{A}_4$.

Theorem 2.5.47. *$(\mathbb{M}_{m,n}(\mathcal{S}), \circ)$ and $(\mathbb{M}_{m,n}(\mathcal{S}), \bullet)$ are join spaces.*

Proof. Since hyperoperations "$\circ$" and "$\bullet$" are dual, i.e. the respective proofs would be analogous, we will prove only the fact that $(\mathbb{M}_{m,n}(\mathcal{S}), \circ)$ is a join space. First of all, commutativity of the hyperoperation is obvious. Next, we immediately get that $(\mathbb{M}_{m,n}(\mathcal{S}), \circ)$ is a semihypergroup (constructed using the "Ends lemma", i.e. Lemma 2.1.1).

Reproductive law, i.e. condition $\mathbf{A} \circ \mathbb{M}_{m,n}(\mathcal{S}) = \mathbb{M}_{m,n}(\mathcal{S})$ holds for all $\mathbf{A} \in \mathbb{M}_{m,n}(\mathcal{S})$: It is evident that $\mathbf{A} \circ \mathbb{M}_{m,n}(\mathcal{S}) \subseteq \mathbb{M}_{m,n}(\mathcal{S})$, for any $\mathbf{A} \in \mathbb{M}_{m,n}(\mathcal{S})$. As far as the opposite inclusion, i.e. $\mathbb{M}_{m,n}(\mathcal{S}) \subseteq \mathbf{A} \circ \mathbb{M}_{m,n}(\mathcal{S})$, for all $\mathbf{A} \in \mathbb{M}_{m,n}(\mathcal{S})$, is concerned, notice that

$$\mathbf{A} \circ \mathbb{M}_{m,n}(\mathcal{S}) = \bigcup_{\mathbf{X} \in \mathbb{M}_{m,n}(\mathcal{S})} \mathbf{A} \circ \mathbf{X} = \bigcup_{\mathbf{X} \in \mathbb{M}_{m,n}(\mathcal{S})} \{\mathbf{C} \in \mathbb{M}_{m,n}(\mathcal{S}) \mid \min\{\mathbf{A}, \mathbf{X}\} \leq C\}.$$

For a fixed $\mathbf{A} \in \mathbb{M}_{m,n}(\mathcal{S})$ and an arbitrary $\mathbf{M} \in \mathbb{M}_{m,n}(\mathcal{S})$ the following cases are possible:

1. If $\mathbf{M} \leq_M \mathbf{A}$, then $\min\{\mathbf{A}, \mathbf{M}\} = \mathbf{M}$ and since "$\leq_M$" is reflexive, there is $\mathbf{M} \in \mathbf{A} \circ \mathbb{M}_{m,n}(\mathcal{S})$.

2. If $\mathbf{A} \leq_M \mathbf{M}$, then $\min\{\mathbf{A}, \mathbf{M}\} = \mathbf{A}$ which means that $\mathbf{M} \in \mathbf{A} \circ \mathbb{M}_{m,n}(\mathcal{S})$.

3. If $\mathbf{A}$ and $\mathbf{M}$ are not in relation "$\leq_M$", then there is $\min\{\mathbf{A}, \mathbf{M}\} \leq \mathbf{M}$, which means that $\mathbf{M} \in \mathbf{A} \circ \mathbb{M}_{m,n}(\mathcal{S})$.

Therefore, $(\mathbb{M}_{m,n}(\mathcal{S}), \circ)$ is a commutative hypergroup. Finally, the transposition axiom holds thanks to Lemma 2.5.45. □

Remark 2.5.48. Notice that in the proof of Theorem 2.5.47, when proving the validity of the reproductive law, Lemma 2.1.5 (part of the "Ends lemma", see page 28) could not be used because it assumes that the single-valued structure, in our case $(\mathbb{M}_{m,n}(\mathcal{S}), \inf, \leq_e)$, or $(\mathbb{M}_{m,n}(\mathcal{S}), \sup, \leq_e)$, is a partially ordered *group*. We could have used Theorem 2.4.71 on page 71 though, because both hyperoperations are extensive.

Now, analogous to (2.52) and (2.53) we for matrices $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{m,n}(\mathcal{S})$ define

$$\mathbf{A} * \mathbf{B} = \{\mathbf{C} \in \mathbb{M}_{m,n}(\mathcal{S}) \mid \max\{\mathbf{A}, \mathbf{B}\} \leq_M \mathbf{C}\}, \tag{2.54}$$

i.e., for all $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$,

$$\begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} * \begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \ldots & \ldots & \ldots \\ b_{m1} & \ldots & b_{mn} \end{bmatrix} =$$

$$\left\{ \begin{bmatrix} c_{11} & \ldots & c_{1n} \\ \ldots & \ldots & \ldots \\ c_{m1} & \ldots & c_{mn} \end{bmatrix} \in \mathbb{M}_{m,n}(\mathcal{S}) \mid \sup\{a_{ij}, b_{ij}\} \leq_e c_{ij} \right\}.$$

and dually

$$\mathbf{A} \star \mathbf{B} = \{\mathbf{D} \in \mathbb{M}_{m,n}(\mathcal{S}) \mid \min\{\mathbf{A}, \mathbf{B}\} \geq_M \mathbf{D}\}, \tag{2.55}$$

i.e., for all $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$,

$$\begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} \star \begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \ldots & \ldots & \ldots \\ b_{m1} & \ldots & b_{mn} \end{bmatrix} =$$

$$\left\{ \begin{bmatrix} d_{11} & \ldots & d_{1n} \\ \ldots & \ldots & \ldots \\ d_{m1} & \ldots & d_{mn} \end{bmatrix} \in \mathbb{M}_{m,n}(\mathcal{S}) \mid \inf\{a_{ij}, b_{ij}\} \geq_e d_{ij} \right\}.$$

Example 2.5.49. Suppose that $\mathcal{S}$ is a lattice of non-negative integer pairs, where we set $(a, b) \leq_e (c, d)$ if $a \leq c$ and $b \leq d$, and consider 2×2 matrices of such entries, e.g. $\mathbf{A} = \begin{bmatrix} (5,8) & (3,0) \\ (2,4) & (1,9) \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} (7,2) & (2,1) \\ (6,1) & (3,5) \end{bmatrix}$. Then the hyperproduct $\mathbf{A} * \mathbf{B}$ is

$$\mathbf{A} * \mathbf{B} = \left\{ \begin{bmatrix} (a_{11}^1, a_{11}^2) & (a_{12}^1, a_{12}^2) \\ (a_{21}^1, a_{21}^2) & (a_{22}^1, a_{22}^2) \end{bmatrix} \in \mathbb{M}_{2,2}(\mathcal{S}) \right\},$$

where the entries are such that

$$7 \leq a_{11}^1, 8 \leq a_{11}^2, 3 \leq a_{12}^1, 1 \leq a_{12}^2, 6 \leq a_{21}^1, 4 \leq a_{21}^2, 3 \leq a_{22}^1, 9 \leq a_{22}^2.$$

Theorem 2.5.50. *$(\mathbb{M}_{m,n}(\mathcal{S}), *)$ and $(\mathbb{M}_{m,n}(\mathcal{S}), \star)$ are commutative semihypergroups.*

Proof. For hyperoperation "$*$" follows directly from the "Ends lemma", i.e. from Lemma 2.1.1, and from Lemma 2.5.43; for hyperoperation "$\star$" follows from the fact that "min" and "max" are dual. □

Remark 2.5.51. Unlike semihypergroups $(\mathbb{M}_{m,n}(\mathcal{S}), \circ)$ and $(\mathbb{M}_{m,n}(\mathcal{S}), \bullet)$, the above semihypergroups do not satisfy the reproductive axiom. Since hyperoperations "$\star$" and "$*$", or rather operations "min" and "max", are dual, we will demonstrate this on $(\mathbb{M}_{m,n}(\mathcal{S}), *)$ only and show that the condition $\mathbf{A} * \mathbb{M}_{m,n}(\mathcal{S}) \neq \mathbb{M}_{m,n}(\mathcal{S})$ does not hold for all $\mathbf{A} \in \mathbb{M}_{m,n}(\mathcal{S})$. First of all,

$$\mathbf{A}*\mathbb{M}_{m,n}(\mathcal{S}) = \bigcup_{\mathbf{X}\in\mathbb{M}_{m,n}(\mathcal{S})} \mathbf{A}*\mathbf{X} = \bigcup_{\mathbf{X}\in\mathbb{M}_{m,n}(\mathcal{S})} \{\mathbf{C} \in \mathbb{M}_{m,n}(\mathcal{S}) \mid \max\{\mathbf{A}, \mathbf{X}\} \leq_M C\}.$$

Now, for matrices $\mathbf{A}, \mathbf{X} \in \mathbb{M}_{m,n}(\mathcal{S})$ such that $\mathbf{X} \leq_M \mathbf{A}$ there is $\max\{\mathbf{A}, \mathbf{X}\} = \mathbf{A}$ and if we take $\mathbf{M} \in \mathbb{M}_{m,n}(\mathcal{S})$ such that $\mathbf{M} \leq_M \mathbf{A}$, then $\mathbf{M} \notin \mathbf{A} * \mathbb{M}_{m,n}(\mathcal{S})$.

Theorem 2.5.52. *The transposition axiom holds both in $(\mathbb{M}_{m,n}(\mathcal{S}), *)$ and in $(\mathbb{M}_{m,n}(\mathcal{S}), \star)$.*

Proof. Once again, it is sufficient to prove the statement for $(\mathbb{M}_{m,n}(\mathcal{S}), *)$ only. The proof is analogous to the proof of Lemma 2.5.45, only for the matrix $\mathbf{B} = \max\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\}$ there holds $\mathbf{B} \geq_M \max\{\mathbf{A}_1, \mathbf{A}_4\}$ and $\mathbf{B} \geq_M \max\{\mathbf{A}_2, \mathbf{A}_3\}$, i.e. $\mathbf{B} \in \mathbf{A}_1 * \mathbf{A}_4$ and simultaneously $\mathbf{B} \in \mathbf{A}_2 * \mathbf{A}_3$. □

Example 2.5.53. If in Example 2.5.46 we use "$*$" instead of "$\circ$", we get that

$$\mathbf{A}_1 * \mathbf{A}_2 = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, 40|a_{11}, 60|a_{12}, 60|a_{21}, 24|a_{22} \right\},$$
$$\mathbf{A}_3 * \mathbf{A}_4 = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, 8|a_{11}, 12|a_{12}, 30|a_{21}, 3|a_{22} \right\},$$

and obviously $\mathbf{B} = \begin{bmatrix} 40 & 60 \\ 60 & 24 \end{bmatrix} \in \mathbf{A}_1 \circ \mathbf{A}_2 \cap \mathbf{A}_3 \circ \mathbf{A}_4$.

Remark 2.5.54. Notice that even though the transposition axiom is usually studied in hypergroups, its validity is neither restricted to nor follows from the validity of the reproductive law. Transposition axiom in semihypergroups which are not hypergroups has been studied e.g. by Massouros and Massouros [217].

Among the very basic notions of the hyperstructure theory there is the idea of proclaiming a line segment as the result of the hyperoperation applied on its endpoints. Inspired by this, and by the interval binary hyperoperation (2.47) on page 121, for an arbitrary pair of matrices $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{m,n}(\mathcal{S})$, we define

$$\mathbf{A} \odot \mathbf{B} = \{\mathbf{C} \in \mathbb{M}_{m,n}(\mathcal{S}) \mid \min\{\mathbf{A}, \mathbf{B}\} \leq_M \mathbf{C} \leq_M \max\{\mathbf{A}, \mathbf{B}\}\}, \tag{2.56}$$

i.e., for all $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$,

$$\begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} \odot \begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \ldots & \ldots & \ldots \\ b_{m1} & \ldots & b_{mn} \end{bmatrix} =$$

$$\left\{ \begin{bmatrix} c_{11} & \ldots & c_{1n} \\ \ldots & \ldots & \ldots \\ c_{m1} & \ldots & c_{mn} \end{bmatrix} \in \mathbb{M}_{m,n}(\mathcal{S}) \mid \inf\{a_{ij}, b_{ij}\} \leq_e c_{ij} \leq_e \sup\{a_{ij}, b_{ij}\} \right\}.$$

Notice that (2.56) is in fact also a matrix variation of a hyperoperation defined by Varlet [295], Definition 2.5.56, which is frequently used in machine learning applications and is often studied alongside with another hyperoperation introduced by Nakano [229] and studied e.g. by Comer [85], which creates join spaces from modular lattices. Varlet's ideas have been studied and used by e.g. Davvaz, Leoreanu-Fotea or Rosenberg [192, 194–196]. The nature of $(\mathbb{M}_{m,n}(\mathcal{S}), \odot)$ can be easily established with the help of results obtained by Varlet [295].

In this respect, first recall that a lattice $(\mathfrak{L}, \wedge, \vee)$ such that "$\wedge$" distributes over "$\vee$" (and dually "$\vee$" over "$\wedge$") is called *distributive.*

Theorem 2.5.55. *The lattice* $(\mathbb{M}_{m,n}(\mathcal{S}), \min, \max)$ *is distributive if and only if the lattice* $(\mathcal{S}, \inf, \sup)$ *is distributive.*

Proof. The proof is rather obvious thanks to the straightforward correspondence between relations "$\leq_M$" and "$\leq_e$" suggested by (2.49) and correspondence between the definition of minimum and maximum of matrices using infima and suprema of their entries. If $(\mathcal{S}, \inf, \sup)$ is distributive, then distributive laws are valid for all $a_{ij}, b_{ij}, c_{ij} \in \mathcal{S}$, i.e. distributive laws are valid for matrices as well, which means that $(\mathbb{M}_{m,n}(\mathcal{S}), \min, \max)$ is distributive. On the other hand, if $(\mathbb{M}_{m,n}(\mathcal{S}), \min, \max)$ is distributive, then

$$\max\{\mathbf{A}, \min\{\mathbf{B}, \mathbf{C}\}\} = \min\{\max\{\mathbf{A}, \mathbf{B}\}, \max\{\mathbf{B}, \mathbf{C}\}\}$$

for all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{M}_{m,n}(\mathcal{S})$ and thanks to the definition of the minimum and maximum of matrices we immediately have that $(\mathcal{S}, \inf, \sup)$ is distributive. $\square$

Definition 2.5.56. [295] Let $\mathfrak{L}_{\leq} = (L, \wedge, \vee)$ be a lattice with join "$\wedge$", meet "$\vee$" and order relation "$\leq$" and let, for every $a, b \in L$,

$$a \diamond b = \{x \in L \mid a \wedge b \leq x \leq a \vee b\}.$$

Theorem 2.5.57. [295] *For a lattice $\mathfrak{L}_{\leq}$ the following are equivalent:*

(1) $\mathfrak{L}_{\leq}$ is distributive,

(2) $\mathbb{L}_{\leq} = (L, \diamond)$ is join space.

Theorem 2.5.55 and Varlet's results allow us to immediately state the following corollary.

Corollary 2.5.58. $(\mathbb{M}_{m,n}(\mathcal{S}), \odot)$ *is a join space if and only if the lattice* $(\mathcal{S}, \inf, \sup)$ *is distributive.*

The above constructions naturally result in H_v–rings, i.e. as a consequence in H_v–matrices. In Theorem 2.5.55 we have already seen that sets of matrices $(\mathbb{M}_{m,n}(\mathcal{S}), \min, \max)$ are distributive lattices if and only if sets $(\mathcal{S}, \inf, \sup)$ of their entries are distributive lattices. Moreover, the following – stronger – lemma holds.

Lemma 2.5.59. *Let $(S, \oplus)$ and $(S, \odot)$ be EL–semihypergroups of partially ordered semigroups $(S, +, \leq)$ and $(S, \cdot, \leq)$, respectively. If "$\cdot$" distributes over "$+$" from both left and right (i.e. if $(S, +, \cdot, \leq)$ is a partially ordered semiring), then "$\odot$" weakly distributes over "$\oplus$" from both left and right.*

Proof. For an arbitraty $s \in S$ we will denote – as usually – the set $\{x \in S \mid s \leq x\}$ by $[s)_{\leq}$. For arbitrary $a, b, c \in S$ consider the element $a \cdot (b + c)$ which, thanks to distributivity, equals $a \cdot b + a \cdot c$. Notice that

$$a \odot (b \oplus c) = a \odot [b + c)_{\leq} = \bigcup_{x \in [b+c)_{\leq}} a \cdot x = \bigcup_{b+c \leq x} a \cdot x$$

and on the other hand

$$(a \odot b) \oplus (a \odot c) = [a \cdot b)_{\leq} \oplus [a \cdot c)_{\leq} = \bigcup_{y \in [a \cdot b)_{\leq}, z \in [a \cdot c)_{\leq}} [y + z)_{\leq} = \bigcup_{a \cdot b \leq y, a \cdot c \leq z} [y + z)_{\leq}$$

and since the relation "$\leq$" is reflexive, we immediately see that $a \cdot (b + c) = a \cdot b + a \cdot c$ is the common element of both regarded sets. Analogous reasoning can be done for the element $(a + b) \cdot c = a \cdot c + b \cdot c$. $\square$

Therefore, we straightforwardly get the following:

Theorem 2.5.60. *If the lattice* $(\mathcal{S}, \inf, \sup)$ *is distributive, then* $(\mathbb{M}_{m,n}(\mathcal{S}), \circ, *)$ *and* $(\mathbb{M}_{m,n}(\mathcal{S}), \bullet, \star)$ *are* H_v*–rings.*

Proof. Follows immediately from Theorem 2.5.47, which provides the hypergroups, Theorem 2.5.50, which provides the semihypergroups, and from Lemma 2.5.59, which provides weak distributivity (because $\mathcal{S}$ is distributive). □

This result immediately takes us to the concept of H_v*–matrices.*

Corollary 2.5.61. *If* $\mathcal{S}$ *is a distributive lattice, then* $\mathbb{M}_{m,n}(\mathcal{S})$ *is the set of* H_v*–matrices.*

Proof. If in Theorem 2.5.60 we set $m = n = 1$, then $\mathbb{M}_{m,n}(\mathcal{S})$ becomes $\mathcal{S}$, i.e. the set from which we take entries of $\mathbb{M}_{m,n}(\mathcal{S})$. Definitions of hyperoperations "$\circ, \bullet, *, \star$" simplify accordingly. □

Thus we see that using Theorem 2.5.60 and Corollary 2.5.61 we can in fact construct H_v–matrices of different classes: first of all, matrices, entries of which are elements of $\mathcal{S}$ (because, thanks to Corollary 2.5.61, $\mathcal{S}$ is an H_v–ring). We have denoted this set of matrices by $\mathbb{M}_{m,n}(\mathcal{S})$. Yet since $\mathbb{M}_{m,n}(\mathcal{S})$ is itself a lattice, which is distributive if and only if $\mathcal{S}$ is distributive, we can apply Theorem 2.5.60 and regard elements of $\mathbb{M}_{m,n}(\mathcal{S})$ as entries of matrices again. For easier future reference we can denote this set of matrices as $\mathbb{M}^2_{m,n}(\mathcal{S})$.

Remark 2.5.62. Notice that the conditions of definition of H_v–ring are rather weak for the "Ends lemma" context. All our hyperoperations are not only weakly associative but associative. Also, we do not obtain H_v–groups but hypergroups. In other words, three things remain to be secured for $(\mathbb{M}_{m,n}(\mathcal{S}), \circ, *)$ and $(\mathbb{M}_{m,n}(\mathcal{S}), \bullet, \star)$ to become *Krasner hyperrings*: existence of a scalar identity of $(\mathbb{M}_{m,n}(\mathcal{S}), \circ)$ or $(\mathbb{M}_{m,n}(\mathcal{S}), \bullet)$, existence of absorbing elements of $(\mathbb{M}_{m,n}(\mathcal{S}), *)$ or $(\mathbb{M}_{m,n}(\mathcal{S}), \star)$, and distributivity of the hyperoperations instead of weak distributivity shown by Lemma 2.5.59. However, Theorem 2.4.10 on page 44 shows that the existence of scalar identities of $(\mathbb{M}_{m,n}(\mathcal{S}), \circ)$ and $(\mathbb{M}_{m,n}(\mathcal{S}), \bullet)$ is not possible.

When setting $m = n = 1$ in $\mathbb{M}_{m,n}(\mathcal{S})$, we obtain the original lattice $\mathcal{S}$. For all $a, b, c, d \in H$, hyperoperations (2.52), (2.53), (2.54), (2.55), (2.56) in this case reduce to

$$
\begin{aligned}
a \circ b &= \{c \in \mathcal{S} \mid \inf\{a, b\} \leq_e c\} \\
a \bullet b &= \{d \in \mathcal{S} \mid \sup\{a, b\} \geq_e c\} \\
a * b &= \{c \in \mathcal{S} \mid \sup\{a, b\} \leq_e c\} \\
a \star b &= \{d \in \mathcal{S} \mid \inf\{a, b\} \geq_e d\} \\
a \odot b &= \{c \in \mathcal{S} \mid \inf\{a, b\} \leq_e c \leq_e \sup\{a, b\}\}
\end{aligned}
$$

and we immediately get the following corollary.

Corollary 2.5.63. *If* $(\mathcal{S}, \inf, \sup, \leq_e)$ *is a lattice, then*

1. $(\mathcal{S}, \circ)$ *and* $(\mathcal{S}, \bullet)$ *are join spaces,*
2. $(\mathcal{S}, *)$ *and* $(\mathcal{S}, \star)$ *are proper semihypergroups which satisfy the transposition axiom.*

Thus – in lattices – the concept of EL–hyperstructures can be not only dualized but also its natural analogy can be proved.

2.6 Relation to similar concepts

In this section we will show the relation of EL–hyperstructures to *ordered hyperstructures* of Heidari and Davvaz [146], to Chvalina's concept of *quasi-order hypergroups* [43, 44, 95] and to *lower BCK–semilattices* [158]. For the respective definitions recall Section 1.2.

2.6.1 Ordered semihypergroups

First, we are going to relate EL–semihypergroups $(S, \circ)$, constructed from quasi-ordered semigroups $(S, \cdot, \leq)$ by means of the hyperoperation "$\circ$" defined, for all $a, b \in S$, by

$$a \circ b = [a \cdot b)_{\leq} = \{x \in S \mid a \cdot b \leq x\}, \tag{2.57}$$

and ordered, i.e. partially ordered, semihypergroups $(S, \circ, \preceq)$. Recall that *compatibility in the sense of single-valued structures* means that

$$x \leq y \Rightarrow a \cdot x \leq a \cdot y \text{ and } x \cdot a \leq y \cdot a \tag{2.58}$$

for all $x, y, a \in S$, while *compatibility in the sense of hyperstructures* means that

$$x \preceq y \Rightarrow a * x \preceq a * y \text{ and } x * a \preceq y * a \tag{2.59}$$

for all $a, x, y \in S$, where by $a * x \preceq a * y$ we mean that for every $c \in a * x$ there exists $d \in a * y$ such that $c \preceq d$.

First of all, suppose that the relations "$\leq$" and "$\preceq$" are the same. Thus, when rewriting the compatibility condition (2.59), we get that

$$x \leq y \Rightarrow [a \cdot x)_{\leq} \leq [a \cdot y)_{\leq} \text{ and } [x \cdot a)_{\leq} \leq [y \cdot b)_{\leq}, \tag{2.60}$$

for all $x, y, a \in S$, i.e. the fact that $x \leq y$ implies that for all $a \in S$ we have that for every $c \in S$ such that $a \cdot x \leq c$ there must exist $d \in S$ such that $a \cdot y \leq d$ and $c \leq d$ and for every $f \in S$ such that $x \cdot a \leq f$ there must exist $g \in S$ such that $y \cdot a \leq g$ and $f \leq g$. The following lemma is obvious.

Lemma 2.6.1. *In the EL–semihypergroup $(S,\circ)$ of a quasi-ordered semigroup $(S,\cdot,\leq)$ there is*

$$x \leq y \Rightarrow a \circ y \subseteq a \circ x \text{ and } y \circ a \subseteq x \circ a \tag{2.61}$$

for all $x, y, a \in S$.

Proof. Suppose an arbitrary $k \in a \circ y = \{k \in S \mid a \cdot y \leq k\}$. Since $(S,\cdot,\leq)$ is a quasi-ordered semigroup, the fact that $x \leq y$ implies $a \cdot x \leq a \cdot y$, for all $a \in S$, and from transitivity of "$\leq$", we have that $a \cdot x \leq k$, i.e. $k \in a \circ x$. Proving the other inclusion is analogous. □

Now we need to establish, in the context of (2.57), the relation between $a \circ x$ and $a \circ y$ for $x \leq y$ as described by (2.59). For every element $c \in a \circ x$ we need to find an element $d \in a \circ y$ such that $c \leq d$ and for every element $f \in x \circ a$ we need to find an element $g \in y \circ a$ such that $f \leq g$. This is an easy and straightforward task in the following two special cases.

Lemma 2.6.2. *The EL–semihypergroup $(S,\circ)$ of a quasi-ordered semigroup $(S,\cdot,\leq)$ is an ordered semihypergroup $(S,\circ,\leq)$ if:*

1. *$(S,\leq)$ has the greatest element or,*
2. *the relation "$\leq$" is linear ordering.*

Proof. We will show the proof for elements c, d (in the sense of the above text) only as reasoning for elements f, g of the other-sided multiplication is analogous.

1. If $(S,\leq)$ has the greatest element, then, for an arbitrary $c \in S$, the desired element $d \in a \circ y$ is exactly this greatest element of $(S,\leq)$. Of course, in such a case, "$\leq$" must be partial ordering.
2. Since every two elements $x, y \in S$ are in relation "$\leq$" and, if $x \leq y$, there is $a \cdot x \leq a \cdot y$ and $a \circ y \subseteq a \circ x$, for all $a \in S$, then due to the construction of sets $a \circ x$ and $a \circ y$ the statement is obvious.

□

Thus we see that we must focus on such cases of $c \in a \circ x$, where $c \notin a \circ y$. If $c \leq a \cdot y$, then it is enough to set $d = a \cdot y$ because reflexivity of "$\leq$" provides that $d \in a \circ y$. Therefore, we must focus on cases of $c \in a \circ x$, where there is simultaneously $c \notin a \circ y$ and elements $a \cdot y$ and c are not in relation "$\leq$". Notice that this means that we focus on cases where c is not in relation with any element of $a \circ y$. In other words, that to such an element $c \in a \circ x$ with these properties there exists no element $d \in a \circ y$ such that $c \leq d$.

Given this perspective, the following theorem becomes obvious and we can see that Lemma 2.6.2 is in fact its corollary.

Theorem 2.6.3. *The EL–semihypergroup $(S, \circ)$ of a partially ordered semigroup $(S, \cdot, \leq)$ is an ordered semihypergroup $(S, \circ, \leq)$ if an arbitrary pair of elements $x, y \in S$ has an upper bound.*

Proof. Obvious because the existence of an upper bound of an arbitrary two element subset of $(S, \leq)$ prevents the situation described before the theorem. □

If for a pair of elements $x, y \in S$ there is $x \leq y$, then condition (2.59) must be valid for *all* $a \in S$. This means that if $(S, \cdot)$ is a monoid with a unit u, there must be also $u \circ x \leq u \circ y$, i.e. $[u \cdot x)_\leq \leq [u \cdot y)_\leq$ (and also $x \circ u \leq y \circ u$), which means $[x)_\leq \leq [y)_\leq$. This justifies the following theorem.

Theorem 2.6.4. *The EL–semihypergroup $(S, \circ)$ of a quasi-ordered monoid $(S, \cdot, \leq)$ is not an ordered semihypergroup $(S, \circ, \leq)$ if there exists a pair of elements such that it does not have an upper bound yet has a lower bound.*

Proof. The assumptions of the theorem are such that there exists a triple of elements $c, x, y \in S$ such that $x \leq c$, $x \leq y$ while c and y are not related and do not have an upper bound, i.e. no element from S is simultaneously greater than both c and y (see Fig. 2.6.1). Since $(S, \cdot)$ is a monoid and $x \leq y$, there must be – should $(S, \circ, \leq)$ be a partially ordered semihypergroup – also $[x)_\leq \leq [y)_\leq$. Yet to our $c \in [x)_\leq$ there obviously does not exist any element $d \in [y)_\leq$ such that $c \leq d$. Therefore, $(S, \circ, \leq)$ is not a partially ordered semigroup. □

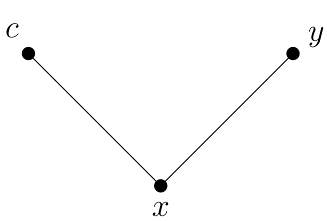

Figure 2.2: To Theorem 2.6.4: $[x)_\leq = \{x, c, y\}$ while $[y)_\leq = \{y\}$

In Section 4.2 we include a link of the above results to transformation hypergroups with phase tolerance spaces [151] and *general hyperstructures* [59] studied by Dehghan Nezhad, Chvalina and Hošková.

Remark 2.6.5. Notice that in Theorem 2.6.4 pairs of elements which do not have a lower bound need not be tested for the existence of their upper bound. Indeed, imagine that in Fig. 2.6.1 elements c and x are not related. Then $c \notin [x)_\leq$ and the problem of finding a suitable element $d \in [y)_\leq$ such that $c \leq d$ disappears.

Out of the infinitely many ways of defining relation "$\preceq$" by means of "$\leq$", one stands out. Let us, for all $x, y \in S$, define that

$$x \preceq y \text{ whenever } y \leq x. \tag{2.62}$$

This turns out to be a universal way of obtaining an ordered semihypergroup from an arbitrary EL–semihypergroup.

Theorem 2.6.6. *Let $(S, \circ)$ be the EL–semihypergroup of a partially ordered semigroup $(S, \cdot, \leq)$. For an arbitrary pair of elements $x, y \in S$ define $x \preceq y$ whenever $y \leq x$, i.e. define "$\preceq$" as the inverse relation to "$\leq$". Then the relation "$\preceq$" is compatible with the hyperoperation "$\circ$".*

Proof. After we rewrite condition (2.59) and take into account our definition of relation "$\preceq$", we get that, for all $x, y, a \in S$, the fact that $y \leq x$ implies that to every element $c \in S$ such that $a \cdot x \leq c$ there exists an element $d \in S$ such that $a \cdot y \leq d$ and $d \leq c$. However, since $(S, \cdot, \leq)$ is a partially ordered semigroup, the fact that $y \leq x$ implies that $a \cdot y \leq a \cdot x$, i.e. we can, for an arbitrary $c \in a \circ x$ set $d = a \cdot x$ because $a \cdot x \in a \circ y$. Obviously, the same reasoning can be used for multiplication by an arbitrary $a \in S$ from the right. $\square$

Remark 2.6.7. Naturally, the issue of "$\leq$" being a quasi-ordering and the properties of "$\preceq$" must be discussed separately. Obviously, the fact that "$\leq$" is a partial ordering, means that also "$\preceq$" is a partial ordering. Notice that even though Heidari and Davvaz [146] originally mention *ordered hyperstructures* only, in e.g. Ghazavi, Anvariyeh and Mirvakili [136] *quasi-ordered hyperstructures* are discussed as well. The motivation to study *partial ordering* on hyperstructures lies in the fact that including antisymmetry is suitable for description of hyperstructure generalizations of lattices. However, we do not seek to explore this topic in such a detailed way, especially because it is the relation between the compatibility conditions (2.58) and (2.59) that is relevant for us for the time being.

2.6.2 Ordered semihyperrings

In Subsection 2.5.3 on page 106 we mentioned various approaches to defining the concept of a *semiring* and a *semihyperring*. The concept of an *ordered semihyperring* was defined by Davvaz and Omidi [112] only recently. They define semihyperrings as hyperstructures with two hyperoperations $(S, \oplus, \odot)$, i.e. follow the classification of Vougiouklis, Definition 1.1.14, yet they *misquote* it by adding the requirement that $(S, \oplus)$ has a scalar identity which is moreover absorbing with respect to the hyperoperation "$\odot$". In other words, they make it a parallel to the definition of a semiring with zero. Notice that the alternative definition of a *semihyperring* we mention in Section 2.5, i.e. Definition 2.5.19 on page 109, is different because we regard a hyperstructures $(S, \oplus, \cdot)$, where "$\cdot$" is a single-valued operation. To be more precise, Davvaz and Omidi use the following definition.[27]

Definition 2.6.8. ([112], Definition 2.1) A *semihyperring* is an algebraic hyperstructure $(R, +, \cdot)$ which satisfies the following axioms:

1. $(R, +)$ is a commutative semihypergroup with a zero element 0 satisfying $x + 0 = 0 + x = \{x\}$, that is, (i) For all $x, y, z \in R$, $x + (y + z) = (x + y) + z$, (ii) For all $x, y \in R$, $x + y = x + y$, (iii) There exists $0 \in R$ such that $x + 0 = 0 + x = \{x\}$ for all $x \in R$.

2. $(R, \cdot)$ is a semihypergroup.

3. The multiplication "$\cdot$" is distributive with respect to the hyperoperation "$+$", that is $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$ for all $x, y, z \in R$.

4. The element $0 \in R$ is an absorbing element, that is $x \cdot 0 = 0 \cdot x = 0$ for all $x \in R$.

It is on such hyperstructures that Davvaz and Omidi define partial ordering.

Definition 2.6.9. ([112]) An *ordered semihyperring* $(R, +, \cdot, \leq)$ is a semihyperring of Definition 2.6.8 equipped with a partial order relation "$\leq$" such that for all $a, b, c \in R$, we have

1. $a \leq b$ implies $a + c \leq b + c$, meaning that for any $x \in a + c$, there exists $y \in b + c$ such that $x \leq y$.

[27] Since we are going to adjust this definition later on, when quoting it we use the *exact* quote with "$+$" and "$\cdot$" instead of "$\oplus$" and "$\odot$". In the forthcoming Definition 2.6.11 the usual symbols "$\oplus$" and "$\odot$" are used.

2. $a \leq b$ and $0 \leq c$ imply $a \cdot c \leq b \cdot c$, meaning that for any $x \in a \cdot c$ there exists $y \leq b \cdot c$ such that $x \leq y$. The case $c \cdot a \leq c \cdot b$ is defined similarly.

Notice that the above definition uses the definition of a semihyperring in which the hyperoperation "+" is already commutative while the hyperoperation "·" need not be so. Since the terminology has not been codified yet, notice that sometimes such ordered semihyperrings are called *positive* and in ordered semihyperring the compatibility condition with respect to "·" is required to hold for *all* $c \in R$ instead of those for which $0 \leq c$ holds. See e.g. Omidi and Davvaz [257] for a brief discussion and examples.

Lemma 2.6.10. *In the EL-semihypergroup $(S, \circ)$ of a partially ordered semigroup $(S, \cdot, \leq)$, if an element of S is absorbing with respect to "$\circ$", then it is absorbing with respect to "·" and maximal with respect to "$\leq$".*

Proof. Obvious since $x \circ 0 = 0 \circ x = \{0\}$ for all $x \in S$ means that $[x \cdot 0)_{\leq} = [0 \cdot x)_{\leq} = \{0\}$ which, thanks to reflexivity of the relation "$\leq$" means that $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$. Since $[x \cdot 0)_{\leq} = [0)_{\leq}$ is a one-element set and "$\leq$" is reflexive, 0 is a maximal element of $(S, \leq)$. □

Thus, condition 2 of Definition 2.6.9 holds trivially. Now we can see that the definition in the sense of Davvaz and Omidi included in [112] cannot be used in the case of EL-hyperstructures as it is either not possible to construct a semihyperring or as the definition of ordering leads to a degenerated (even though not completely trivial) case. However, we can come back to the roots and use the original definition of Vougiouklis without the explicit use of scalar identity which has the property of absorption with respect to the other hyperoperation, and combine it with the ordering defined by Davvaz and Omidi, yet with the compatibility condition of "·" applied on all $c \in R$. This will enable us to apply some already existing results – those where the properties of the special element 0 are not required (or where we explicitely prove them anew). Thus, the following definition can be justified in our context.

Definition 2.6.11. By an *ordered semihyperring* we mean a hyperstructure $(S, \oplus, \odot, \preceq)$ such that

1. $(S, \oplus)$ and $(S, \odot)$ are semihypergroups.
2. The hyperoperation "$\odot$" distributes over "$\oplus$", i.e. for all $x, y, z \in S$ there is $x \odot (y \oplus z) = x \odot y \oplus x \odot z$ and $(x \oplus y) \odot z = x \odot z \oplus y \odot z$.
3. The relation "$\preceq$" is a partial ordering.

4. The relation "$\preceq$" is compatible with both "$\oplus$" and "$\odot$", i.e. for all $x, y, z \in S$ there is
   (a) $a \preceq b$ implies $a \oplus c \preceq b \oplus c$, meaning that for any $x \in a \oplus c$, there exists $y \in b \oplus c$ such that $x \preceq y$ (the case $c \oplus a \preceq c \oplus b$ is defined in an analogous way).
   (b) $a \preceq b$ implies $a \odot c \preceq b \odot c$, meaning that for any $x \in a \odot c$ there exists $y \preceq b \odot c$ such that $x \preceq y$ (the case $c \odot a \preceq c \odot b$ is defined in an analogous way).

If in the distributive law inclusion holds instead of equality, the ordered semihyperring is called *ordered semihyperring in the general sense.*

Example 2.6.12. Consider $(\mathbb{Z}, \min, +, \leq)$, with the usual operations and ordering of integers. It is easy to verify that $(\mathbb{Z}, \min, \leq)$ is a partially ordered semigroup and $(\mathbb{Z}, +, \leq)$ is a partially ordered group. Distributive laws, which rewrite to $a + \min\{b, c\} = \min\{a + b, a + c\}$ and $\min\{a, b\} + c = \min\{a + c, b + c\}$ for all $a, b, c \in \mathbb{Z}$, also hold. As a result, if we define $a \oplus b = [\min\{a, b\})_{\leq}$ and $a \odot b = [a + b)_{\leq}$ for all $a, b \in \mathbb{Z}$, we get that $(\mathbb{Z}, \oplus, \odot)$ is a semihyperring in the sense of Definition 1.1.14 on page 9 and, by Lemma 2.6.2, also an ordered semihyperring in the sense of Definition 2.6.11.

Example 2.6.13. If we adjust Example 2.6.12 so that instead of $(\mathbb{Z}, \min, +, \leq$ $)$ we regard $(\mathbb{N}, \min, +, \leq)$, then $(\mathbb{N}, +)$ is a semigroup only and we get that $(\mathbb{N}, \oplus, \odot)$, where "$\oplus$" and "$\odot$" are defined in the same way as in Example 2.6.12, is – based on Definition 1.1.14 – a semihyperring in the general sense. Since "$\leq$" is a linear ordering, $(\mathbb{N}, \min, +, \leq)$ is an ordered semihyperring in the general sense.

2.6.3 Quasi-order hypergroups

Given the definition of *quasi-order* hypergroups and Proposition 1.2.4 of Section 1.2 the quest for relationship between quasi-order hypergroups and EL–hyperstructures is not so straightforward. When constructing EL–hyperstructures, we *start* with a relation "$\leq$" while in order to prove that a hyperstructure is a quasi-order hypergroup, we have to *find* it.

If $(S, \cdot, \leq)$ is an idempotent quasi-ordered semigroup, then, by Theorem 2.4.23 on page 50, $(S, *)$ is commutative and such that, for all $a \in S$, there is $a^3 = a^2$, i.e., by original Chvalina's Definition 1.2.2, $(S, *)$ is a quasi-order hypergroup. Now, the condition $a * b = a^2 \cup b^2$ of Definition 1.2.3 in this case turns into

$$[a \cdot b)_{\leq} = [a)_{\leq} \cup [b)_{\leq}. \tag{2.63}$$

If the relation "$\leq$", which we use to construct the EL–semihypergroup $(S, *)$, has this property, then $(S, *)$ can be viewed as a quasi-order hypergroup, or rather as an order hypergroup because the implication (1.28) of Definition 1.2.3 on page 22 holds, for idempotent "$\cdot$", trivially.

Example 2.6.14. Consider the EL–semihypergroup $(\mathbb{Z}, *)$ constructed from a partially ordered semigroup $(\mathbb{Z}, \min, \leq)$. Obviously, the operation "min" is idempotent and $[\min\{a, b\})_\leq = [a)_\leq \cup [b)_\leq$ for all $a, b \in \mathbb{Z}$. Thus $(\mathbb{Z}, *)$ is a quasi-order hypergroup. Notice that the same is true when we change $\mathbb{Z}$ to $\mathbb{N}$. There is no ambiguity in terminology because "$*$" is an extensive hyperoperation, i.e. $(\mathbb{Z}, *)$ and $(\mathbb{N}, *)$ are, by Theorem 2.4.71 on page 71, hypergroups.

A nice example of a link between quasi-ordered hypergroups and EL–hyperstructures can be found in [167], in which Jančić-Rašović studies hyperrings – to be more precise *hyperrings in the general sense* of Definition 1.1.14. Inspired by Chvalina [43, 44] she defines, on a quasi-ordered semigroup, one hyperoperation using the idea of (1.29) on page 23 and the other using the "Ends lemma". Of course, since the paper was submitted in early 2012, she does not speak of EL–hyperstructures. Below we include her result ([167], Corollary 3.1), rewritten in our language and notation.

Theorem 2.6.15. *Let $(H, \cdot)$ be a semigroup equipped with binary relations "$\leq_1$", "$\leq_2$" such that both $(H, \cdot, \leq_1)$ and $(H, \cdot, \leq_2)$ are quasi-ordered semigroups and "$\leq_1$" $\subseteq$ "$\leq_2$". Define, for all $a, b \in H$, hyperoperations "$+_{\leq_1}$" and "$\circ_{\leq_2}$" on H as follows:*

$$\begin{aligned} x +_{\leq_1} y &= [x)_{\leq_1} \cup [y)_{\leq_1} \\ x \circ_{\leq_2} y &= [x \cdot y)_{\leq_2} \end{aligned}$$

Then $(H, +_{\leq 1}, \circ_{\leq_2})$ is a strong hyperring in the general sense.

2.6.4 Lower BCK–semilattices

In this subsection, "$*$" stands for a BCK operation, not for a hyperoperation.

In lower BCK–semilattices $(X, *, 0)$ we denote $x \wedge y$ the greatest lower bound of an arbitrary pair of elements $x, y \in X$. Obviously, in all lower BCK–semilattices, i.e. not only in such that X is a commutative BCK–algebra, there is $x \wedge y = y \wedge x$ and $(X, \wedge)$ is a semigroup. If we define[28]

$$x \leq_n y \text{ whenever } x \wedge y = x \tag{2.64}$$

[28]Notice that the lower index "n" in "$\leq_n$" stands for "new" to distinguish (2.64) from (1.31), which is defined by means of $x * y = 0$.

for all $x, y \in X$, we see that "$\leq_n$" is compatible with the operation "$\wedge$" and we can construct the following.

Example 2.6.16. Suppose a lower BCK–semilattice $(X, \leq)$, where "$\wedge$" is the greatest lower bound of elements $x, y \in X$. For all $x, y \in X$ we define "$\leq_n$" by (2.64). It is easy to show that the relation "$\leq_n$" is a partial ordering. Indeed, since $x \wedge x = x$, the relation is reflexive. Also, if $x \leq_n y$, there is $x \wedge y = x$ and if $y \leq_n x$, there is $y \wedge x = y$, yet since $x \wedge y = y \wedge x$, there is $x = y$. Finally, if $x \leq_n y$ and $y \leq_n z$, then $x \wedge y = x$ and $y \wedge z = y$ and

$$x \wedge z = (x \wedge y) \wedge z = x \wedge (y \wedge z) = x \wedge y = x,$$

i.e. $x \leq_n z$. The compatibility condition also holds because given an arbitrary $a \in X$ and a pair $x, y \in X$ such that $x \leq_n y$ we have

$$(x \wedge a) \wedge (y \wedge a) = (x \wedge y) \wedge (a \wedge a) = x \wedge a,$$

i.e. $x \wedge a \leq_n y \wedge a$. And, thanks to commutativity of "$\wedge$", we also have that $a \wedge x \leq_n a \wedge y$. Altogether, we have that $(X, \wedge, \leq_n)$ is a partially ordered semigroup and as such it can be used to construct an EL–hyperstructure $(X, \circ)$ by defining the hyperoperation $\circ$ by

$$x \circ y = [x \wedge y)_{\leq_n} = \{z \in X \mid x \wedge y \leq_n z\} \tag{2.65}$$

for all $x, y \in X$. And we immediately have that $(X, \circ)$ is a semihypergroup.

In fact, it is irrelevant whether X is a lower BCK–semilattice as it is important that it is a *semilattice*. Example 2.6.16 is thus in fact an example supporting the following lemma, which – by a nice loop – brings us back in time to Pickett's Example 2.2.1 on page 29, the oldest one in this book that makes use of the "Ends lemma".

Lemma 2.6.17. *Every semilattice $(X, \wedge, \leq)$ or $(X, \vee, \leq)$ can be used to construct an EL–hypergroup $(X, \circ)$, where, for all $a, b \in X$ there is*

$$a \circ b = [a \wedge b)_{\leq} = \{x \in X \mid a \wedge b \leq x\}, \tag{2.66}$$

or $a \circ b = [a \vee b)_{\leq} = \{x \in X \mid a \vee b \leq x\}$ respectively.

Proof. Given the above reasoning, obvious. The fact that the semihypergroup $(X, \circ)$ is a hypergroup, follows from Theorem 2.4.71 because the hyperoperation is extensive. □

Bounded BCI/BCK–algebras are such BCI/BCK–algebras that have the greatest element, which is usually denoted by 1. Yet this fact is exactly what Lemma 2.6.2 assumes. Therefore, we easily obtain the following theorem.

Theorem 2.6.18. *An EL-semihypergroup $(X, \circ)$ constructed from a bounded lower BCK–semilattice $(X, \wedge, \leq)$ by means of (2.66) is an ordered semihypergroup.*

Proof. Follows immediately from Lemma 2.6.2. □

Example 2.6.19. In [31] Bordbar, Zahedi and Jun study ideals of BCK–algebras. They regard the set $(\mathcal{I}(X), \wedge, \leq)$, where $\mathcal{I}(X)$ is the set of all ideals of a lower BCK–semilattice X and "$\wedge$" is for an arbitrary pair of ideals $A, B \in \mathcal{I}(X)$ defined by

$$A \wedge B = \{a \wedge b \mid a \in A, b \in B\}.$$

They show that "$\wedge$" is a binary operation on $\mathcal{I}(X)$ and that $(\mathcal{I}(X), \wedge)$ is a semigroup. This enables us to proceed in the similar way as in Example 2.6.16 and define relation "$\leq_i$" on $\mathcal{I}(X)$ by putting

$$A \leq_i B \text{ whenever } A \wedge B = A$$

for all $A, B \in \mathcal{I}(X)$. It is easy to show that $(\mathcal{I}(X), \wedge, \leq_i)$ is a partially ordered semigroup. Indeed, the fact that "$\leq_i$" is reflexive, antisymmetric and transitive can be verified by means analogical to those used in Example 2.6.16. Finally, suppose that $A \leq B$ and C are arbitrary ideals of X. Then $A \wedge B = A$. So we have

$$(A \wedge C) \wedge (B \wedge C) = (A \wedge B) \wedge (C \wedge C) = A \wedge C,$$

i.e. $A \wedge C \leq B \wedge C$. Since $a \wedge b = b \wedge a$ for all $a, b \in X$, we have also that $C \wedge A \leq C \wedge B$, which means that the relation "$\leq_i$" is compatible with the operation "$\wedge$" applied on ideals of X.

Therefore, we can define a hyperoperation "$\circ$" on the set $\mathcal{I}(X)$ by setting

$$A \circ B = [A \wedge B)_{\leq} = \{C \in \mathcal{I}(X) \mid A \wedge B \leq_i C\}$$

and immediately conclude that $(\mathcal{I}(X), \circ)$ is a semihypergroup. Since $X \in \mathcal{I}(X)$ is obviously the greatest element of $(\mathcal{I}(X), \leq_i)$, we can again immediately conclude that $(\mathcal{I}(X), \circ)$ is an ordered semihypergroup. Indeed, because if $A \leq_i B$ and C are arbitrary ideals of X, and we suppose an arbitrary ideal $G \in A \circ C$, then by definition $A \wedge C \leq_i G$ and we must look for such an ideal H of X that $B \wedge C \leq_i X$ and $G \leq_i X$. Yet no matter what A, B, C, G are, we can always set $H = X$.

Chapter 3

Extensions and modifications

3.1 Extensions to n–ary case

Most results of this section (with the exception of Subsection 3.1.3) were published by Analele Ştiinţifice ale Universităţii "Ovidius" Constanţa (WoS Q4) as Novák [240].

The step from binary hyperstructures to *n–ary hyperstructures* has been done only recently: implicitly in a general case of *universal hyperalgebras* by Šlapal [282] and explicitly by Davvaz and Vougiouklis who in [115] introduced the concept of *n–ary hypergroup* (sometimes called simply *n–hypergroup*) and presented n–ary generalization of some very basic concepts of hyperstructure theory. The connection between hypergroups and n–ary hypergroups was thoroughly studied in Leoreanu–Fotea and Corsini [191]. The topic was transferred to the fuzzy context in Davvaz and Corsini [106].

Results recently obtained in the area of n–ary generalization of hyperstructures associated to binary relations fall into three groups: some, such as Cristea and Ştefănescu in e.g. [97, 99], generalize the binary relation and construct *binary hyperstructures associated to n–ary relations* while others, such as Leoreanu-Fotea and Davvaz in e.g. [193] generalize the hyperstructure and construct *n–ary hyperstructures associated to binary relations*. Finally, the third approach, presented e.g. in Anvariyeh and Momeni [6] is possible too – as one can study *n–ary hyperstructures associated to n–ary relations*.

Out of these three options we first of all, in Subsection 3.1.2, develop the approach pioneered by Leoreanu-Fotea and Davvaz in [193]. In this we make use of n–ary hyperstructure concepts defined in Ameri and Norouzi [11], Davvaz and Vougiouklis [115] or Leoreanu [190]. Then, in Subsection 3.1.3 we include some results obtained by Ghazavi and Anvariyeh [135] in the direction of Cristea and Ştefănescu [97, 99].

3.1.1 Two approaches to the n–ary extension

By Definition 1.1.1 on page 2, EL–semihypergroups that we have studied so far are hyperstructures of *arity* 2. It is thus natural to find out whether the construction can be extended to involve more than two elements.

Analogically to the standard definition (2.1), i.e.

$$a * b = [a \cdot b)_\le = \{x \in S \mid a \cdot b \le x\}, \tag{3.1}$$

we could define an n-ary hyperoperation $* : \underbrace{S \times \ldots \times S}_{n} \to \mathcal{P}^*(S)$ by

$$\underbrace{a_1 * \ldots * a_n}_{n} = [\underbrace{a_1 \cdot \ldots \cdot a_n}_{n})_\le = \{x \in S \mid \underbrace{a_1 \cdot \ldots \cdot a_n}_{n} \le x\} \tag{3.2}$$

In a standard notation used e.g. by Davvaz and Vougiouklis [115] or Leoreanu–Fotea and Davvaz [193], which is also used in Subsection 1.1.1, this would be denoted as a hyperoperation $f : S^n \to \mathcal{P}^*(S)$ (or with H instead of S if we wanted to make use of the distinction *semihypergroup* vs. *hypergroup*) defined by

$$f(a_1^n) = [\underbrace{a_1 \cdot \ldots \cdot a_n}_{n})_\le = \{x \in S \mid \underbrace{a_1 \cdot \ldots \cdot a_n}_{n} \le x\}. \tag{3.3}$$

The hypergroupoid would be an *n-ary hypergroupoid* and would be denoted in the former case by $(S, *)$ and in the latter case by (S, f).[1]

However, first of all we need to establish meaning of the very basic concepts used in (3.2) or (3.3). The result of the hyperoperation $f(a_1^n)$ applied on elements $a_1, \ldots, a_n$, $n > 2$ is the upper end of a *single* element $\underbrace{a_1 \cdot \ldots \cdot a_n}_{n} \in S$. (In further text we call such an element as *generating* the upper end.) Yet *how* does one obtain this single element? In other words, *what is the arity of the single-valued operation "$\cdot$"*? In a general case, "$\cdot$" may be a *binary* operation, an n–ary operation, or a j–ary operation for some special j such that $2 < j < n$.

In Subsection 3.1.2 we suppose that "$\cdot$" is a binary operation, i.e. that the product $\underbrace{a_1 \cdot \ldots \cdot a_n}_{n}$ is an *iterated binary operation*. This is usually defined in such a way that for $j \ge 1$, $n \ge j$ we denote by a_j^n a sequence of elements a_i,

[1]Further on we will use the standard notation, i.e. define the n–ary hyperoperation using analogies of (3.3). Analogies of notation (3.2) will be used only at places where the explicit reference to the binary hyperoperation "$*$" makes the understanding more straightforward.

$j \leq i \leq n$ and for the single-valued binary operation s_f we define two new operations s_l^{it} and s_r^{it} in the following way:

$$s_l^{it}(a_1^n) = \begin{cases} a_1 & n = 1 \\ s_f(s_l^{it}(a_1^{n-1}), a_n) & n > 1 \end{cases}$$

and

$$s_r^{it}(a_1^n) = \begin{cases} a_1 & n = 1 \\ s_f(a_n, s_r^{it}(a_1^{n-1})) & n > 1 \end{cases}$$

Obviously, in a general case $s_l^{it}(a_1^n) \neq s_r^{it}(a_1^n)$. However, if the original binary operation s_f is associative, then the two newly defined operations s_l^{it} and s_r^{it} are equal and we may write s^{it} instead. For details on iterated binary operations cf. e.g. Miller, Vandsome and McBrewster [220].

Further on we will use the notation $\underbrace{a_1 \cdot \ldots \cdot a_n}_{n}$ in the sense of $s^{it}(a_1^n)$. More precisely we should distinguish between $s_l^{it}(a_1^n)$ and $s_r^{it}(a_1^n)$ but this would be redundant because the "Ends lemma" which we attempt to generalize, i.e. Lemma 2.1.1, assumes asociativity of the single-valued operation.

Remark 3.1.1. Notice that the decision on nature of $\underbrace{a_1 \cdot \ldots \cdot a_n}_{n}$ has a number of implications. If, contrary to our assumption, one decides to consider this element as a result of an n–ary operation (as is the case of Subsection 3.1.3), then all theorems must be adjusted to work with n–ary quasi-ordered (semi)groups. These, however, must first be properly defined. Thus, from a certain point of view, our decision on the nature of $\underbrace{a_1 \cdot \ldots \cdot a_n}_{n}$ in Subsection 3.1.2 is not only naturally following from the context but also easier and more convenient to work with.

Remark 3.1.2. Just as we have considered the meaning of $\underbrace{a_1 \cdot \ldots \cdot a_n}_{n}$ and discussed whether it is a result of an n–ary or an iterated binary *single-valued operation* "$\cdot$", we may discuss the meaning of the symbol $\underbrace{a_1 * \ldots * a_n}_{n}$. Again, in a general case it could stand for both an n–ary or an iterated binary *hyperoperation*. Yet as has been suggested above, in the case of the hyperoperation we choose the n–ary option.

3.1.2 Implications of iterated binary operation

First, discuss the issue of *associativity* and *commutativity* in n–ary hyperstructures defined by (3.3). The following theorem is a parallel to the "Ends lemma", i.e. Lemma 2.1.1.

Theorem 3.1.3. *Let $(S, \cdot, \leq)$ be a quasi-ordered semigroup. n–ary hyperoperation $f : S^n \to \mathcal{P}^*(S)$ defined by (3.3), i.e. as*

$$f(a_1^n) = [\underbrace{a_1 \cdot \ldots \cdot a_n}_{n})_{\leq} = \{x \in S \mid \underbrace{a_1 \cdot \ldots \cdot a_n}_{n} \leq x\}.$$

is associative. Furthermore, it is commutative if the semigroup $(S, \cdot)$ is commutative.

□

Proof. In order to prove *associativity*, we will modify the proof of Chvalina [44], Lemma 1.6, p. 148, which shows that if we start with a quasi-ordered[2] semigroup $(S, \cdot)$ there holds $a * (b * c) = (a * b) * c = [a \cdot b \cdot c)_{\leq}$.

First of all, suppose the following: $x, y, a_i \in S$, $i = 1, \ldots, n+1$, $x \leq y$ and that $(S, \cdot, \leq)$ is a quasi-ordered semigroup. This implies that $a_i \cdot x \leq a_i \cdot y$, $x \cdot a_i \leq y \cdot a_i$ and $[a_i \cdot y)_{\leq} \subseteq [a_i \cdot x)_{\leq}$, $[y \cdot a_i)_{\leq} \subseteq [x \cdot a_i)_{\leq}$ for $i = 1, \ldots, n$ (and the same for any product of any number of elements of S in position of a_i – if we keep their order).

Second, notice that obviously for all $x \in S$ such that $a_n \cdot a_{n+1} \leq x$ there is $[\underbrace{a_1 \cdot \ldots \cdot a_{n-1}}_{n-1} \cdot x)_{\leq} \subseteq [\underbrace{a_1 \cdot \ldots \cdot a_{n+1}}_{n+1})_{\leq}$. This is easy to verify because the fact that $y \in [\underbrace{a_1 \cdot \ldots \cdot a_{n-1}}_{n-1} \cdot x)_{\leq}$ is equivalent to the fact that $\underbrace{a_1 \cdot \ldots \cdot a_{n-1}}_{n-1} \cdot x \leq y$. On the other hand, the fact that $a_n \cdot a_{n+1} \leq x$ is equivalent to $\underbrace{a_1 \cdot \ldots \cdot a_{n+1}}_{n+1} \leq \underbrace{a_1 \cdot \ldots \cdot a_{n-1}}_{n-1} \cdot x$, which due to transitivity of the relation "$\leq$" means that $\underbrace{a_1 \cdot \ldots \cdot a_{n+1}}_{n+1} \leq y$, i.e. $y \in [\underbrace{a_1 \cdot \ldots \cdot a_{n+1}}_{n+1})_{\leq}$. Naturally, it is not important whether we multiply by x from left or right, i.e. there is also $[x \cdot \underbrace{a_3 \cdot \ldots \cdot a_{n+1}}_{n-1})_{\leq} \subseteq [\underbrace{a_1 \cdot \ldots \cdot a_{n+1}}_{n+1})_{\leq}$ for all $x \in S$ such that $a_1 \cdot a_2 \leq x$.

Then consider that the proof of Lemma 1.6 of [44] goes (using the above considerations for $n = 2$ and notation a, b, c instead of a_i) as follows:

$$a * (b * c) = \bigcup_{x \in b * c} a * x = \bigcup_{x \in [b \cdot c)_{\leq}} [a \cdot x)_{\leq} = [a \cdot b \cdot c)_{\leq} \cup \bigcup_{x > b \cdot c} [a \cdot x)_{\leq} = [a \cdot b \cdot c)_{\leq}$$

[2]In [44] Chvalina in fact assumes that the semigroup is partially ordered. However, antisymmetry is not used in the proof. The reason why Chvalina assumes partially ordered semigroups is that he shows that Lemma 1.6 can be used in the proof of the "Ends lemma", where – because of the part on commutativity – he assumes partial ordering. Notice that Lemma 1.6 of [44] is included as Lemma 2.1.4 on page 28. Also see Subsection 2.4.1.

and similarly

$$(a * b) * c = \bigcup_{x \in [a \cdot b)_\leq} [x \cdot c)_\leq = [a \cdot b \cdot c)_\leq,$$

which combined means that $a * (b * c) = (a * b) * c = a * b * c$. This can be denoted as $f(a, f(b, c)) = f(f(a, b), c)$ or $f(a_1, f(a_2^3)) = f(f(a_1^2), a_3)$ using the notation (3.3) for any triple of elements of S.

In a completely analogous manner we can prove that $f(a_1, f(a_2^4)) = f(f(a_1^3), a_4) = f(a_1^4)$ for any quadruple of elements of S as well as $f(a_1, f(a_2^5)) = f(f(a_1^4), a_5) = f(a_1^5)$ for any quintuple of elements of S. Thus for arity $n = 3$ we have that

$$f(a_1^{i-1}, f(a_i^{i+2}), a_{i+3}^5) = f(a_1^{j-1}, f(a_j^{i+2}), a_{j+3}^5)$$

for all $i, j \in \{1, 2, 3\}$, which means that associativity in 3–ary EL–hypergroupoids (S, f) is secured. Obviously, this consideration can be repeated for any higher arity n.

Proving *commutativity* is rather simple: since the single-valued operation "$\cdot$" is commutative and as has been shown above also associative, then all permutations $\underbrace{a_1 \cdot \ldots \cdot a_n}_{n}$ are equal. This means that all respective upper ends $[\underbrace{a_1 \cdot \ldots \cdot a_n}_{n})_\leq$ are equal because they are generated always by the same element. In other words, all permutations of the hyperoperation f are equal, i.e. the hyperoperation f is commutative. □

Theorem 2.4.7 on page 41 is meant as a converse of the original construction. The following theorem is its n–ary extension.

Theorem 3.1.4. *Let $(S, \cdot)$ be a non-trivial groupoid and "$\leq$" a partial ordering on S such that for an arbitrary pair of elements $a, b \in S$, $a \leq b$, and for an arbitrary $c \in S$ there holds $c \cdot a \leq c \cdot b$, $a \cdot c \leq b \cdot c$. Further define an n–ary hyperoperation f (also denoted by "$*$") using notation (3.3) (or (3.2)).*

Then if the hyperoperation f (or "$*$") is associative, then the single-valued operation "$\cdot$" is associative too. Furthermore, if the hyperoperation f (or "$*$") is commutative, then the single-valued operation "$\cdot$" is commutative too.

□

Proof. The fact that the hyperoperation f (or "$*$") is *associative*, means that all permutations $f(a_1^{i-1}, f(a_i^{n+i-1}), a_{n+i}^{2n-1})$ for an arbitrary $i \in \{1, 2, \ldots, n\}$ are equal, i.e. if an arbitrary element $x \in S$ belongs to one of the permutations $f(a_1^{i-1}, f(a_i^{n+i-1}), a_{n+i}^{2n-1})$, it belongs to all other ones.

Suppose an arbitrary $x \in f(a_1^{i-1}, f(a_i^{n+i-1}), a_{n+i}^{2n-1})$ for some $i \in \{1, 2, \ldots, n\}$, e.g. for $i = 1$. This means that $x \in f(f(a_1^n), a_{n+1}^{2n-1})$, i.e. using the "$*$" notation, $x \in \underbrace{a_1 * \ldots * a_n}_{n} * \underbrace{a_{n+1} * \ldots * a_{2n-1}}_{n-1}$. This means that there exists an element $x_1 \in \underbrace{a_1 * \ldots * a_n}_{n}$ such that $x \in x_1 * \underbrace{a_{n+1} * \ldots * a_{2n-1}}_{n-1}$. In other words, for these elements x_1 and x there holds that $\underbrace{a_1 \cdot \ldots \cdot a_n}_{n} \leq x_1$ and $x_1 \cdot \underbrace{a_{n+1} \cdot \ldots \cdot a_{2n-1}}_{n-1} \leq x$. Thanks to the properties assumed in the theorem this – when combined – means that

$$(\underbrace{a_1 \cdot \ldots \cdot a_n}_{n}) \cdot (\underbrace{a_{n+1} \cdot \ldots \cdot a_{2n-1}}_{n-1}) \leq x_1 \cdot (\underbrace{a_{n+1} \cdot \ldots \cdot a_{2n-1}}_{n-1}) \leq x$$

and thanks to assumed transitivity of the relation "$\leq$" we get that

$$x \in [(\underbrace{a_1 \cdot \ldots \cdot a_n}_{n}) \cdot (\underbrace{a_{n+1} \cdot \ldots \cdot a_{2n-1}}_{n-1}))_{\leq}. \tag{3.4}$$

Yet we could have started with any other permutation $f(a_1^{i-1}, f(a_i^{n+i-1}), a_{n+i}^{2n-1})$ and apply analogous reasoning on it. E.g. for $i = 2$ we have that $x \in a_1 * (\underbrace{a_2 * \ldots * a_{n+1}}_{n}) * \underbrace{a_{n+2} * \ldots * a_{2n-1}}_{n-1}$ and conclude that

$$x \in [a_1 \cdot (\underbrace{a_2 \cdot \ldots \cdot a_{n+1}}_{n}) \cdot (\underbrace{a_{n+2} \cdot \ldots \cdot a_{2n-1}}_{n-2}))_{\leq}, \tag{3.5}$$

and since $f(a_1^{i-1}, f(a_i^{n+i-1}), a_{n+i}^{2n-1})$ are equal for $i = 1$ and $i = 2$ (just as for any other $i \in \{1, 2, \ldots, n\}$) and we supposed an *arbitrary* element $x \in f(a_1^{i-1}, f(a_i^{n+i-1}), a_{n+i}^{2n-1})$, we get that the upper ends in (3.4) and (3.5) (just as any other upper end which results from using another i) are equal too.

Since we assume that the relation "$\leq$" is antisymmetric, using implication $[a)_{\leq} = [b)_{\leq} \Rightarrow a = b$ we get that also the elements generating the upper ends are equal. As a result, the single-valued operation "$\cdot$" is associative.

Proving *commutativity* of the single-valued operation "$\cdot$" is rather straightforward. If the hyperoperation f is commutative, then $f(a_1^n)$ is the same regardless of the permutation of elements $a_1, \ldots, a_n$. By the definition of the hyperoperation f marked as (3.3), this means that all upper ends $[\underbrace{a_1 \cdot \ldots \cdot a_n}_{n})_{\leq}$ are the same regardless of the permutation of elements $a_1, \ldots, a_n$. However, on condition of antisymmetry of the relation "$\leq$", from the fact that

$[a)_\leq = [b)_\leq \Rightarrow a = b$ we immediately get that also $\underbrace{a_1 \cdot \ldots \cdot a_n}_{n}$ is the same regardless of the permutation of elements $a_1, \ldots, a_n$, which together with already proved associativity means that the single-valued operation "$\cdot$" is commutative. □

Now we can proceed to *conditions on which an n–ary EL–semihypergroup becomes an n–ary hypergroup*. Recall that the concept of n–ary hypergroup may be defined in two equivalent ways: either as Definition 1.1.5 on page 4 or by expanding the reproductive law, i.e. expanding validity of

$$x * H = H * x = H$$

for all $x \in H$, to the form

$$\underbrace{H * \ldots * H}_{i-1} * x * \underbrace{H * \ldots * H}_{n-i} = H \tag{3.6}$$

for all $x \in H$ and all $i = \{1, 2, \ldots, n\}$ using notation (3.2) or

$$f(H^{i-1}, x, H^{n-i}) = H \tag{3.7}$$

for all $x \in H$ and all $i = \{1, 2, \ldots, n\}$ using notation (3.3).

Since in the "Ends lemma" context obviously $f(H^{i-1}, x, H^{n-i}) \subseteq H$ for an arbitrary $i \in \{1, 2, \ldots, n\}$, we must concentrate on the other inclusion, i.e. secure that

$$H \subseteq \underbrace{H * \ldots * H}_{i-1} * x * \underbrace{H * \ldots * H}_{n-i}, \tag{3.8}$$

or $H \subseteq f(H^{i-1}, x, H^{n-i})$, for all $x \in H$ and $i = \{1, 2, \ldots, n\}$.

Theorem 3.1.5. *If $(H, \cdot, \leq)$ is the quasi-ordered group, then the n–ary EL–semihypergroup constructed using Theorem 3.1.3 is an n–ary hypergroup.*

□

Proof. As has been suggested above, we need to verify validity of inclusion (3.8). To do this, suppose an arbitrary element $h \in H$ and first of all suppose that we need to verify that $H \subseteq H * x$ or $H \subseteq x * H$. Obviously, $h \cdot x^{-1} \in H$ and $x^{-1} \cdot h \in H$. Thus we get that $h \cdot x^{-1} \cdot x = h \leq h$ (since "$\leq$" is reflexive) and $x \cdot x^{-1} \cdot h = h \leq h$, i.e $h \in [(h \cdot x^{-1}) \cdot x)_\leq \subseteq \bigcup\limits_{g \in H} [g \cdot x)_\leq = H * x$ as well as $h \in x * H$.

Yet instead of $h \cdot x^{-1} \in H$ we may write $h \cdot h^{-1} \cdot h \cdot x^{-1} \in H * H = \bigcup\limits_{f \in H, g \in H} [f \cdot g)_\leq$ (and instead of $x^{-1} \cdot h \in H$ we may write $x^{-1} \cdot h \cdot h^{-1} \cdot h \in H * H$) and we can repeat this for any number of instances of H. □

Theorem 3.1.6. *Let (S, f) be the n–ary EL–semihypergroup of a quasi–ordered monoid $(S, \cdot, \leq)$ with the neutral element u. Then*

1. *If $e \in S$ is an identity of (S, f), then $\underbrace{e \cdot \ldots \cdot e}_{n-1} \leq u$.*

2. *If $e \leq u$ for some $e \in S$, then e is an identity of (S, f).*

□

Proof. In order to prove part 1 suppose that $e \in S$ is an identity of (S, f), i.e. that $x \in f(\underbrace{e, \ldots, e}_{i-1}, x, \underbrace{e, \ldots, e}_{n-i})$ for all $x \in H$ and all i such that $1 \leq i \leq n$. In the context of definition of the hyperoperation f – see (3.3) – the inclusion means that $x \in [\underbrace{e \cdot \ldots \cdot e}_{i-1}, x, \underbrace{e \cdot \ldots \cdot e}_{n-i})_{\leq}$, i.e. $\underbrace{e \cdot \ldots \cdot e}_{i-1} \cdot x \cdot \underbrace{e \cdot \ldots \cdot e}_{n-i} \leq x$. Since this holds for all $x \in S$, we may e.g. set $x = u$, where u is the neutral element of $(S, \cdot)$. And we get the statement.

As far as part 2 is concerned, suppose that $e \leq u$, where u is the neutral element of $(S, \cdot)$. Since $(S, \cdot, \leq)$ is a quasi-ordered monoid, we have that also $e \cdot x \leq u \cdot x = x$ and $e \cdot e \cdot x \leq e \cdot x$ for an arbitrary $x \in S$. From transitivity of the relation "$\leq$" we get that $e \cdot e \cdot x \leq x$, i.e. $x \in [e \cdot e \cdot x)_{\leq} = f(e, e, x)$. But we could have also multiplied by x from the left and get $x \cdot e \leq x \cdot u = x$. Then from $e \cdot x \leq x$ we get that $e \cdot x \cdot e \leq x \cdot e$ and from transitivity we get that $e \cdot x \cdot e \leq x$, i.e. $x \in [e \cdot x \cdot e)_{\leq}$, i.e. $x \in f(e, x, e)$. Finally, from $x \cdot e \leq x$ and $x \cdot e \cdot e \leq x \cdot e$ we get that $x \in f(x, e, e)$, which completes the proof for arity $n = 3$. In order to prove the statement for higher arities we may obviously use the same strategies. □

Remark 3.1.7. Notice that for arity $n = 2$ Theorem 3.1.6 turns into equivalence stating that $e \in S$ is an identity of (S, f) if and only if $e \leq u$, which is Theorem 2.4.13 on page 47. Further notice that we obtain the same result for idempotent "$\cdot$" and $n > 2$.

Corollary 3.1.8. *If in Theorem 3.1.6 $(S, \cdot, \leq)$ is a quasi-ordered group, then if $e \in S$ is an identity of (S, f), then also $\underbrace{e \cdot \ldots \cdot e}_{n-1} \leq \underbrace{e^{-1} \cdot \ldots \cdot e^{-1}}_{n-1}$.*

□

Proof. We continue the proof of part 1 of Theorem 3.1.6. By $n - 1$ times repeated multiplication by e^{-1} we get that $u \leq \underbrace{e^{-1} \cdot \ldots \cdot e^{-1}}_{n-1}$ and thanks to the transitivity of the relation "$\leq$" we get the statement. □

Corollary 3.1.9. *The neutral element u of $(S,\cdot)$ is an identity of its n–ary EL–semihypergroup (S,f).*

Proof. Obvious. □

Example 3.1.10. If we regard the hypergroup $(\mathbb{R},f)$, where

$$f(a_1^n) = [\underbrace{a_1 + \ldots + a_n}_{n})_\leq = \{x \in \mathbb{R} \mid \underbrace{a_1 + \ldots + a_n}_{n} \leq x\}$$

for arbitrary real numbers $a_1,\ldots,a_n$, we get that 0 and all negative numbers are all identities of this hypergroup. Also, obviously, $\underbrace{x + \ldots + x}_{n-1} \leq 0$ for both 0 and an arbitrary negative x.

Example 3.1.11. If we regard the set $(\mathcal{P}(S),f)$, where

$$f(A_1^n) = [\underbrace{A_1 \cup \ldots \cup A_n}_{n})_\subseteq = \{X \in \mathcal{P}^*(S) \mid \underbrace{A_1 \cup \ldots \cup A_n}_{n} \subseteq X\},$$

we get that this hypergroup has the only identity $\emptyset$.

Scalar neutral elements (or *scalar identities*) are such elements, where the inclusion in Definition 1.1.4 of n–ary identities on page 3 changes to equality – see (1.4).

Definition 3.1.12. ([11], p. 380) Element e of an n–ary hypergroup (H,f) is called a *scalar neutral element* if

$$\{x\} = f(\underbrace{e,\ldots,e}_{i-1},x,\underbrace{e,\ldots,e}_{n-i}) \tag{3.9}$$

for every $1 \leq i \leq n$ and for every $x \in H$.

Remark 3.1.13. Notice that in [11] Ameri and Norouzi use a slightly different notation: instead of $f(\underbrace{e,\ldots,e}_{i-1},x,\underbrace{e,\ldots,e}_{n-i})$ they write $f(e^{(i-1)},x,e^{(n-i)})$. Also notice that sometimes, e.g. Davvaz and Vougiouklis [115], p. 168, the concept of a more general term *scalar* is used when defining that "the element $a \in H$ is called a scalar if $|f(x_1^i,a,x_{i+2}^n)| = 1$ for all $x_1,\ldots,x_i,x_{i+2},\ldots,x_n \in H$", i.e. defining that $f(\underbrace{e,\ldots,e}_{i-1},x,\underbrace{e,\ldots,e}_{n-i})$ must be a one-element set, not neccessarily the set $\{x\}$ as in the case of the scalar neutral element.

As has been done with Theorem 3.1.6, let us now permit a more general case of scalar neutral elements – in *semihypergroups* instead of hypergroups. To be consistent in naming concepts we prefer the name *scalar identity* to *scalar neutral element* further on.

Theorem 3.1.14. *Let $(S, \cdot, \leq)$ be a non-trivial quasi-ordered semigroup and (S, f) its n–ary EL–semihypergroup. If $e \in S$ is a scalar identity of (S, f), then*

$$x = \underbrace{e \cdot \ldots \cdot e}_{i-1} \cdot x \cdot \underbrace{e \cdot \ldots \cdot e}_{n-i} \tag{3.10}$$

for all $x \in S$ and all $1 \leq i \leq n$.

□

Proof. Suppose that in (S, f) there exists a scalar neutral identity e. This means that for every $x \in S$ and every i such that $1 \leq i \leq n$ there is

$$\{x\} = f(\underbrace{e, \ldots, e}_{i-1}, x, \underbrace{e, \ldots, e}_{n-i}).$$

Yet thanks to the definition of the hyperoperation f this means that

$$\{x\} = [\underbrace{e \cdot \ldots \cdot e}_{i-1} \cdot x \cdot \underbrace{e \cdot \ldots \cdot e}_{n-i})_{\leq}.$$

Since "$\leq$" is reflexive, there is

$$\underbrace{e \cdot \ldots \cdot e}_{i-1} \cdot x \cdot \underbrace{e \cdot \ldots \cdot e}_{n-i} \in [\underbrace{e \cdot \ldots \cdot e}_{i-1} \cdot x \cdot \underbrace{e \cdot \ldots \cdot e}_{n-i})_{\leq},$$

which means that $x = \underbrace{e \cdot \ldots \cdot e}_{i-1} \cdot x \cdot \underbrace{e \cdot \ldots \cdot e}_{n-i}$ for all $x \in S$ and all i such that $1 \leq i \leq n$. □

Remark 3.1.15. Obviously, if for some $x \in S$ or some $i \in \{1, \ldots, n\}$ condition (3.10) does not hold, then $e \in S$ is not a scalar identity of (S, f). This equivalent condition might be a better tool for finding scalar identities than the theorem itself.

Corollary 3.1.16. *The neutral element u of a quasi-ordered monoid $(S, \cdot, \leq)$ is a scalar identity of its n–ary EL–semihypergroup (S, f) if and only if "$\leq$" is the identity relation.*

□

Proof. By definition

$$f(\underbrace{u,\dots,u}_{i-1},x,\underbrace{u,\dots,u}_{n-i}) = [\underbrace{u\cdot\dots\cdot u}_{i-1}\cdot x\cdot\underbrace{u\cdot\dots\cdot u}_{n-i})_{\leq} = [x)_{\leq}.$$

This is equal to $\{x\}$ for reflexive "$\leq$" and all $x\in S$ if and only if "$\leq$" is the identity relation. □

Remark 3.1.17. Notice that for arity $n=2$ condition (3.10) turns into $x=e\cdot x=x\cdot e$ for all $x\in S$ which is possible only for $e=u$, where u is the neutral element of $(S,\cdot)$. And we immediately conclude that "$\leq$" must be the identity relation. As a result, there do not exist any non-trivial canonical hyperstructures constructed using the "Ends lemma", which is Corollary 2.4.11 on page 45.

Example 3.1.18. If we regard the hypergroup $(\mathbb{R},f)$ from Example 3.1.10, we see that condition (3.10) can hold for $e=0$ only, which means that $(\mathbb{R},*)$ does not have a scalar identity.

Apart from identities and scalar identities we might consider *zero elements* (or *absorbing elements*) of n–ary hyperstructures.

Definition 3.1.19. ([11], p. 380) Element 0 of an n–ary hypergroup (H,f) is called a *zero element* if

$$\{0\} = f(\underbrace{x_1,\dots,x_{i-1}}_{i-1},0,\underbrace{x_{i+1},\dots,x_n}_{n-i}) \tag{3.11}$$

for every $1\leq i\leq n$ and for every $(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)\in H^{n-1}$.

Obviously, the zero element is unique. The following theorem might be used to detect it. We see that only EL–maximal elements of $(S,\leq)$ can be zero elements. As in the case of identities and scalar identities of (S,f) we might again expand the definition onto semihypergroups. Before the following theorem recall that by *EL–maximal element* of a quasi-ordered set $(S,\leq)$ we mean such $x\in S$ that $[x)_{\leq}=\{x\}$.

Theorem 3.1.20. *Let $(S,\cdot,\leq)$ be a non-trivial quasi-ordered semigroup and (S,f) the n–ary EL–semihypergroup associated to it. If 0 is the zero element of (S,f), then 0 is an EL–maximal element of $(S,\leq)$.*

□

Proof. From (3.11) in the definition of the zero element and from the definition of the hyperoperation f we get that

$$[\underbrace{x_1 \cdot \ldots \cdot x_{i-1}}_{i-1} \cdot 0 \cdot \underbrace{x_{i+1} \cdot \ldots \cdot x_n}_{n-i})_{\leq} = \{0\} \tag{3.12}$$

for every i such that $1 \leq i \leq n$ and for every $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in S^{n-1}$. Since the relation "$\leq$" is reflexive, there is

$$\underbrace{x_1 \cdot \ldots \cdot x_{i-1}}_{i-1} \cdot 0 \cdot \underbrace{x_{i+1} \cdot \ldots \cdot x_n}_{n-i} \in [\underbrace{x_1 \cdot \ldots \cdot x_{i-1}}_{i-1} \cdot 0 \cdot \underbrace{x_{i+1} \cdot \ldots \cdot x_n}_{n-i})_{\leq},$$

which combined with (3.12) means that for a zero element 0 there must be $\underbrace{x_1 \cdot \ldots \cdot x_{i-1}}_{i-1} \cdot 0 \cdot \underbrace{x_{i+1} \cdot \ldots \cdot x_n}_{n-i} = 0$ for every i such that $1 \leq i \leq n$ and for every $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in S^{n-1}$. Yet if this holds, (3.12) reduces to $[0)_{\leq} = \{0\}$, which means that 0 is an EL–maximal element of $(S, \leq)$. □

Example 3.1.21. Since there are no EL–maximal elements in $(\mathbb{R}, +, \leq)$ there are no zero elements in $(\mathbb{R}, f)$ from Example 3.1.10.

Example 3.1.22. If we want to describe zero elements in $(\mathcal{P}(S), f)$ from Example 3.1.11, we must concentrate on the only EL–maximal element of $(\mathcal{P}(S), \cup, \subseteq)$, i.e. on $\mathcal{P}(S)$ itself. We easily verify that it is a zero element of $(\mathcal{P}(S), f)$.

Inverse elements in n–ary hyperstructures are studied e.g. in Ameri and Norouzi [11]. The property of *having a unique inverse element* required in [11] is taken over from the definition of *canonical n–ary* hypergroup included in Leoreanu [190]. Notice that canonical n–ary hypergroups are a special class of *commutative* n–ary hyperstructures (moreover, with the unique identity e having a certain further property), i.e. the definition of inverse elements included in [11], which has been taken over from [190], must be adjusted to a more general case – see Definition 1.1.7 on page 5.

Theorem 3.1.23. *Let (H, f) be an n–ary EL–hypergroup of a quasi-ordered group $(H, \cdot, \leq)$. For an arbitrary $x \in H$ there holds*

1. *If $x' \leq x^{-1}$, then x' is an inverse of x in (H, f).*

2. *If x' is an inverse of x in (H, f), then there is $a \leq x^{-1}$ for all $a \in$ $\mathrm{perm}\{x' \cdot \underbrace{e \cdot \ldots \cdot e}_{2(n-2)}\}$.*

In both cases x^{-1} denotes the inverse of $x \in H$ in $(H, \cdot)$ and e is some (unspecified) identity of (H, f).

□

Proof. Suppose that $x \in H$, $x' \in H$ are arbitrary and denote by the upper index "$^{-1}$" the inverse in $(H, \cdot)$. Finally, denote by u the neutral element of $(H, \cdot)$. Throughout the proof recall (3.3) on page 142 for the definition of the hyperoperation f using the single-valued operation "$\cdot$" and the relation "$\leq$".

ad 1: If $x' \leq x^{-1}$, then also $x' \cdot x \leq x^{-1} \cdot x = u$ and $x \cdot x' \leq x \cdot x^{-1} = u$. Moreover, we can multiply by the element u any number of times, or "insert" it anywhere "in between" x and x' or x' and x on the left side. Since according to Corollary 3.1.9 u is an identity of (H, f), we have that x' is an inverse of x.

ad 2: Suppose that x' is an inverse of x in (H, f). This means that there exists an identity $e \in H$ such that (1.9) holds. This means that

$$\underbrace{x \cdot x' \cdot e \cdot \ldots \cdot e}_{\text{arbitrary permutation of } n \text{ elements}} \leq e$$

When we multiply this by $\underbrace{e \cdot \ldots \cdot e}_{n-2}$, we get

$$\underbrace{x \cdot x' \cdot e \cdot \ldots \cdot e}_{\text{arbitrary permutation of } x, x' \text{ and } 2(n-2) \text{ instances of } e} \leq \underbrace{e \cdot \ldots \cdot e}_{n-1}.$$

However, from Theorem 3.1.6 and transitivity of the relation "$\leq$" we get that

$$\underbrace{x \cdot x' \cdot e \cdot \ldots \cdot e}_{\text{arbitrary permutation of } x, x' \text{ and } 2(n-2) \text{ instances of } e} \leq u$$

which is equivalent to

$$\underbrace{x' \cdot e \cdot \ldots \cdot e}_{\text{arbitrary permutation of } x' \text{ and } 2(n-2) \text{ instances of } e} \leq x^{-1}.$$

It can be easily verified that commutativity / non-commutativity of the single-valued operation "$\cdot$" is not relevant in the last step.

□

Remark 3.1.24. Notice that for arity $n = 2$ there is $2(n-2) = 0$, i.e. Theorem 3.1.23 turns into an equivalence which enables us to describe the set of all inverses of an arbitrary $x \in H$ (denoted as $i(x)$) in a far more elegant way by

$$i(x) = {}_{\leq}(x^{-1}] = \{x' \in G \mid x' \leq x^{-1}\}, \tag{3.13}$$

which has already been shown on page 48 as Theorem 2.4.17.

Example 3.1.25. If we regard the hypergroup $(\mathbb{R}, f)$ from Example 3.1.10, we see that all $a \in \mathbb{R}$ such that $a \leq -x$ are inverses of an arbitrary real number x in (R, f). We also see that we might set $e = 0$ and Theorem 3.1.23 turns into equivalence.

3.1.3 Implications of the "Ends lemma" being applied on n–ordered semigroups

Results of this subsection are taken over from Ghazavi and Anvariyeh [135].

In [135], Ghazavi and Anvariyeh, motivated by Novák [240], applied the idea of the "Ends lemma" in the context of *n–ordered semigroups* which were studied e.g. by Novák and Novotný [252–254] (originating from the study of ternary structures) and used in the hyperstructure context by Cristea and Ştefănescu [97, 99]. First of all, recall necessary basic definitions; to be more precise, recall Definition 1.1.33 of basic ordering concepts, and Definition 1.1.34 of the compatibility condition for n–ary relations and groupoids (for both definitions see p. 18).

One can see that the – for our purposes crucial – difference between *binary* and *n–ary* relations is that there is no definition of n–ary antisymmetry. In other words, even though some proofs which require quasi-ordering "$\leq$" can be generalized or modified (in some cases more or less straightforwardly), proofs which require *antisymmetry* on top of reflexivity and transitivity of "$\leq$" must be approached from a new perspective. Naturally, not all of such results can be transferred. Notice that in *n–ordered groupoids* the relation is required to be reflexive and n–transitive, i.e. they are an analogy of proper quasi-ordered groupoids (not "partially ordered groupoids" as the name may suggest).

Ghazavi and Anvariyeh [135] aimed at presenting such generalizations into the n–ary context that as many previous results as possible become special cases of their results for $n = 2$. For this they proposed three natural generalizations of the "Ends lemma"; they focus on the following one.

Definition 3.1.26. Suppose $(S, \cdot, \rho)$ is an n–ordered groupoid. For $a, b \in S$ we define the hyperoperation $* : S \times S \to \mathcal{P}^*(S)$ by

$$a * b = [a \cdot b)_\rho = \{t \in S \mid (a \cdot b, \underbrace{t, \ldots, t}_{n-1}) \in \rho\}. \tag{3.14}$$

The hypergroupoid $(S, *)$ is called EL_n*–hypergroupoid* associated to n–ordered groupoid $(S, \cdot, \rho)$.

Example 3.1.27. Let $n = 3$ and $S = \{a, b, c\}$ be a groupoid with the following multiplication table:

$\cdot$	a	b	c
a	a	a	a
b	a	b	a
c	a	a	c

Define the ternary relation ρ as follows:

$$\rho := \{(a, a, c), (a, c, c), (a, a, a), (b, b, b), (c, c, c)\}.$$

In this case, $[a)_\rho = \{a, c\}$, $[b)_\rho = \{b\}$ and $[c)_\rho = \{c\}$. As a result, for the hyperoperation "$*$", we get the following table:

$*$	a	b	c
a	$\{a, c\}$	$\{a, c\}$	$\{a, c\}$
b	$\{a, c\}$	$\{b\}$	$\{a, c\}$
c	$\{a, c\}$	$\{a, c\}$	$\{c\}$

It is easy to verify that $(S, *)$ is an EL_3–semihypergroup.

The above Definition 3.1.26 enables the following generalizations of earlier results concerning the "Ends lemma".

Theorem 3.1.28. *Let $(S, *)$ be an EL_n–hypergroupoid associated to an n–ordered semigroup $(S, \cdot, \rho)$. Then:*

1. *$(S, *)$ is a semihypergroup.*

2. *If the n–ordered semigroup $(S, \cdot, \rho)$ is commutative, then also $(S, *)$ is commutative.*

3. *If $(S, \cdot, \rho)$ is an n–ordered semigroup, then $(S, *)$ is a hypergroup if and only if there for every $a, b \in S$ exist $c, c' \in S$ such that*

$$(a \cdot c, \underbrace{b, \ldots, b}_{n-1}) \in \rho \text{ and } (c' \cdot a, \underbrace{b, \ldots, b}_{n-1}) \in \rho.$$

4. *If $(S, \cdot, \rho)$ is an n–ordered group, then $(S, *)$ is a transposition hypergroup.*

Proof. See [135]; the proofs mostly follow the reasoning of the respective proofs for the case $n = 2$. □

Of course, this approach reaches its limitations once proofs for $n = 2$ start requiring antisymmetry of the relation "$\leq$" which has no counterpart for n–relations ρ. This is e.g. the case of the converse of the "Ends lemma", i.e. Theorem 2.4.7 included on page 41, which says that if the hyperoperation is associative, then the single-valued operation is associative too. Also, e.g. Theorem 2.4.34 on page 55, which is an equivalence, can be generalized to an implication only (see Corollary 4.3 of [135]).

Example 3.1.29. Let $n = 3$ and $S = \{a, b, c\}$ be a groupoid with the following multiplication table.

$\cdot$	a	b	c
a	a	a	a
b	a	c	a
c	a	a	b

Now consider the ternary relation $\rho = S^3$. In this case, $(S, \cdot, \rho)$ is a 3-ordered groupoid and its associated EL_3–hypergroupoid $(S, *)$ is rather trivial as $x * y = S$ for all $x, y \in S$. However, this triviality makes it clear that $(S, *)$ is associative. Yet since $(b \cdot b) \cdot c \neq b \cdot (b \cdot c)$, the single-valued operation "$\cdot$" is *not* associative.

In [135] a number of properties included in Subsection 2.4.5 (again, mainly as straightforward generalizations of the case $n = 2$) are included for EL_n–semihypergroups. Notice that e.g. $u \in H$ is an identity of $(H, *)$ if and only if $(u, \underbrace{e, \ldots, e}_{n-1}) \in \rho$, where e is the neutral element of the n-ordered group $(H, \cdot, \rho)$, or that the set of inverses of $a \in H$ is in such a case $i(a) = \{y \in H \mid (y, \underbrace{a^{-1}, \ldots, a^{-1}}_{n-1}) \in \rho\}$, where a^{-1} is the inverse of a in $(H, \cdot)$. Compare these results to those included in Subsection 2.4.3 or in Subsection 3.1.2.

3.2 EL^2–hyperstructures

Results of this subsection are taken over from Ghazavi, Anvariyeh and Mirvakili [136].

In Section 1.2 we mentioned the approach pioneered by Heidari and Davvaz [146] in which ordering "$\leq$" is studied on hyperstructures. Notice that in *ordered hyperstructures*, the ordering is supposed to be compatible with the hyperoperation, not with the single-valued operation. Since [146], numerous papers written by collaborators and / or PhD students of Davvaz appeared on the topic; results of some of these are included in a recently published book Davvaz [105].

In [136], Ghazavi, Anvariyeh and Mirvakili took the idea of the "Ends lemma" and transferred it from *quasi-ordered semigroups* to *quasi-ordered hypergroupoids*. Since, by this transition, they construct hyperstructures from hyperstructures (not hyperstructures from single-valued structures) – yet they use the same idea – they called the resulting hyperstructures *EL^2–hyperstructures*.

Definition 3.2.1. Suppose $(H, \circ)$ is a quasi-ordered hypergroupoid. For $a, b \in H$ we define a hyperoperation $* : H \times H \to \mathcal{P}^*(H)$ by

$$a * b = [a \circ b)_{\leq} = \bigcup_{m \in a \circ b} [m)_{\leq}. \tag{3.15}$$

We call the hypergroupoid $(H, *)$ *EL^2–hypergroupoid* associated to (or of) the quasi-ordered hypergroupoid $(H, \circ, \leq)$.

For the original construction, reflexivity of the relation "$\leq$" is crucial in many proofs as it provides that $[a)_{\leq}$ is always non-empty. In EL^2–hyperstructures, reflexivity of the hyperstructure ordering "$\leq$" has a somewhat different meaning. Notice that the following proposition is used in the proof of Theorem 3.2.3, part 1.

Proposition 3.2.2. *Let $(H, *)$ be an EL^2–hypergroupoid of a quasi-ordered hypergroupoid $(H, \circ, \leq)$. Then $a \circ b \subseteq a * b$ for all $a, b \in H$.*

Proof. Let $t \in a \circ b$ be arbitrary. Because $t \leq t$, we conclude that $t \in [t)_{\leq} \subseteq \bigcup_{m \in a \circ b} [m)_{\leq} = a * b$. $\square$

Recall the role of extensivity discussed in Subsection 2.4.6. In the case of the original EL–hyperstructures, we need associativity of the single-valued operation "$\cdot$" and validity of the compatibility condition for "$\cdot$" and "$\leq$" to achieve associativity of the hyperoperation. However, if the hyperoperation is extensive, none of these are required to achieve weak associativity of the hyperoperation or the validity of the reproductive law – see the proof of Theorem 2.4.72 on page 71.

Theorem 3.2.3. *Let $(H, *)$ be an EL^2–hypergroupoid of a quasi-ordered hypergroupoid $(H, \circ, \leq)$.*

1. *If $(H, \circ)$ is weak associative, then $(H, *)$ is weak associative.*
2. *If $(H, \circ)$ is associative, then $(H, *)$ is associative.*
3. *If the reproductive law holds in $(H, \circ)$, then it holds also in $(H, *)$.*
4. *If $(H, \circ)$ is a commutative quasi-ordered hypergroup, then $(H, *)$ is a commutative hypergroup.*
5. *If in $(H, \circ, \leq)$ there for all $a, b \in H$ holds that either $a \leq b$ or $b \leq a$, then $(H, *)$ is a transposition hypergroup.*
6. *Every identity of $(H, \circ)$ becomes an identity of $(H, *)$.*
7. *The scalar identity of $(H, \circ)$ does not become the scalar identity of $(H, *)$. However, it is an identity of $(H, *)$.*

Proof. See [136], proofs of Theorem 3.4, Theorem 3.6, Corollary 3.5, Corollary 3.7, Theorem 3.8, Theorem 3.9, Theorem 5.1, Corollary 5.2 and Theorem 5.3.

We include the proof of item 1, i.e. weak associativity, only. Since, for all $a, b, c \in H$, there is $(a \circ b) \circ c \cap a \circ (b \circ c) \neq \emptyset$ and, by Proposition 3.3, also $(a \circ b) \circ c \subseteq (a * b) * c$ and $a \circ (b \circ c) \subseteq a * (b * c)$, there must exist an element which is included both in $(a * b) * c$ and in $a * (b * c)$, i.e. $(H, *)$ must be weak associative. □

In [136] the authors show that weak associativity of "$\circ$" need not imply associativity of "$*$" (see their Example 5). They also show (Example 9) a hypergroup $(H, \circ, \leq)$ which does not result in a transposition hypergroup $(H, *)$, etc.

Ghazavi, Anvariyeh and Mirvakili [136] use Definition 2.4.30 on page 53 of the *upper end* (the definition makes use of the set and relation only so it can be used for both quasi-ordered semigroups $(H, \cdot, \leq)$ and hypergroupoids $(H, \circ, \leq)$) to prove an analogy of Lemma 2.4.31 and Theorem 2.4.34 for EL^2–hyperstructures. Again, as with EL_n–hyperstructures of Ghazavi and Anvariyeh [135], the analogy of Theorem 2.4.34 has the form of implication, not of equivalence.

Moreover, in [136] there are also included the following results concerning *hyperideals* of EL^2–hyperstructures.

Theorem 3.2.4. *Let $(H, *)$ be an EL^2–hypergroupoid of a quasi-ordered hypergroupoid $(H, \circ, \leq)$.*

1. *If I is a right / left ideal (hyperideal) and, moreover, an upper end of $(H, \circ, \leq)$, then I is a right / left hyperideal of $(H, *)$.*

2. *Every right / left hyperideal of $(H, *)$ is a right / left hyperideal (not necessarily an ideal) of $(H, \circ, \leq)$.*

3. *If I is a minimal ideal (hyperideal) of $(H, \circ, \leq)$ and, moreover, an upper end of $(H, \circ, \leq)$, then I is also a minimal hyperideal of $(H, *)$.*

4. *If I is a prime ideal (hyperideal) of $(H, \circ, \leq)$ and, moreover, an upper end of $(H, \circ, \leq)$, then I is also a prime hyperideal of $(H, *)$.*

Proof. See [136], proof of Theorem 4.4, Corollary 4.5 and Corollary 4.6. □

For examples see [136].

3.3 The case of a partitioned semigroup

Results of this subsection were, together with the results of Subection 2.4.6, published by Soft Computing (WoS Q2) as Novák and Křehlík [249].

If we examine the construction of EL–hyperstructures from the practical point of view (looking for real-life examples), we find out that in some contexts the construction has several disadvantages:

1. There are too many assumptions. Out of these, the assumption of *compatibility* of the relation "$\leq$" and operation "$\cdot$" is the most difficult to achieve in real-life situations.

2. The original definition of the hyperoperation, i.e. (2.1) on page 27, does not provide that $\{a, b\} \subseteq a * b$. If one wants to use this hyperoperation for e.g. description of family relations, then if a and b are parents, the hyperperation describes children only, not the family as a whole. Also, in case of childless families we get (if parents a, b are not regarded) that $a * b = \emptyset$, which means that "$*$" is not a hyperoperation at all (as what we get are *partial hypergroupoids*). Or, if $a * b$ is meant to describe a path between nodes $a, b \in S$, then the inclusion of the endpoints is not secured by default.

3. It does not take into account the usual situation of "incompatibility" of elements. When applying the idea in e.g. genetics, one must take into account that for some a and b the hyperproduct (or the single-valued product) is meaningless as mating is possible between some individuals

only. If the original concept of (2.1) does not regard this, how can we test associativity? Or, on the other hand, when constructing e.g. the Cartesian composition of automata (or their hyperstructure generalizations; see Section 4.2) one needs the input sets to be disjoint, i.e. the resulting composition will in fact consist of elements of two types. See Example 3.3.1, Example 3.3.5 and Example 3.3.6 and Chvalina, Křehlík and Novák [61].

Out of all these reasons we need to modify the "Ends lemma" concept. In this section we will discuss the intuitive yet simple case of *two* sets T_i. However, it is not difficult to regard a general case of n sets. Notice that already in Subsection 2.4.6 we discussed the role of extensivity, i.e. case 2 of the above list.

3.3.1 New definitions, their motivation and relation to the cardinal sum

First of all, we start with two non-empty sets, T_1^o and T_2^o endowed with single-valued operations "$\cdot_1$" and "$\cdot_2$" and relations "$\leq_1$" and "$\leq_2$", respectively. Since we want to use Lemma 2.1.1 (or to relate our results to it), we assume that $(T_1^o, \cdot_1, \leq_1)$ and $(T_2^o, \cdot_2, \leq_2)$ are quasi-ordered semigroups.

Example 3.3.1. Let $(T_1^o, \cdot_1, \leq_1)$ be the set of two-dimensional vectors, components of which are real numbers, with the operation of component-wise addition, where for arbitrary $u, v \in T_1^o$ such that $u = (u_1, u_2), v = (v_1, v_2)$ we set $u \leq_1 v$ if $u_1 \leq v_1$. Let $(T_2^o, \cdot_2, \leq_2)$ be the set of 2×2 matrices of real entries with the operation of entry-wise addition, where for arbitrary $\mathbf{A}, \mathbf{B} \in T_2^o$ we set $\mathbf{A} \leq_2 \mathbf{B}$ if $tr(\mathbf{A}) \leq tr(\mathbf{B})$, where $tr(\mathbf{A})$ stands for the trace of matrix $\mathbf{A}$. Obviously, $(T_1^o, \cdot_1, \leq_1)$ and $(T_2^o, \cdot_2, \leq_2)$ are quasi-ordered semigroups.

In a case like the one used in Example 3.3.1 considering "criss-cross" operations between elements of T_1^o and T_2^o may be problematic or even need not make any sense at all. Therefore, *define* a new operation "$\cdot$" on their union so that within the respective classes T_1^o, T_2^o we use the results given by "$\cdot_1$", "$\cdot_2$", respectively, and for results of the "criss-cross" multiplication we reserve some special element, which we denote s, which has the form of elements of one of the classes yet is *artificially added* into the class. Whenever any relation "$\leq$" is concerned, s is related to itself only. In fact, what we want to make use of, is a special case of *cardinal sum* of ordered sets, or rather of relational and operational systems. Therefore, first of all, we give the general definition, included e.g. in Birkhoff [25], and then, referred to as Definition 3.3.3, introduce the special case suitable for our purposes.

Definition 3.3.2. Let $I \neq \emptyset$ be an index set and $\{(T_i^o, \leq_i) \mid i \in I\}$ a system of pairwise disjoint quasi-ordered sets. By the *cardinal sum* $(G, \leq)$ we mean the sum $\sum\limits_{i\in I}(T_i^o, \leq_i) = (\bigcup\limits_{i\in I} T_i^o, \leq)$, where for $a, b \in G = \bigcup\limits_{i\in I} T_i^o$ we put $a \leq b$ whenever there exists an index $i_0 \in I$ such that $a, b \in T_i^o$ and $a \leq_{i_o} b$.

Now, in this general construction let us assume that $\{(T_i^o, \cdot_i, \leq_i)\}$ is a system of pairwise disjoint commutative quasi-ordered semigroups, i.e. in our case we assume, on top of Definition 3.3.2, commutative binary operations "$\cdot_i$" compatible with the relations "$\leq_i$". Moreover, let our system be such that $T_z^o = \{s\}$ for exactly one $z \in I$. On such a system we define the cardinal sum $(H, \leq) = \sum\limits_{i\in I\setminus\{z\}} (T_i^o, \leq_i) + T_z^o$ and on H we define a binary operation "$\cdot$" in the following way:

$$a \cdot b = \begin{cases} a \cdot_i b & \text{if there exists } i \in I \setminus \{z\} \text{ such that } a, b \in T_i^o \\ s & \text{if } a \in T_i^o, b \in T_j^o \text{ where } i \neq j \\ s & \text{if } a = s \text{ or } b = s. \end{cases} \tag{3.16}$$

It is easy to see that $(H, \cdot, \leq)$ is a commutative quasi-ordered semigroup.

In case of I being a three-element index set we have T_1^o, T_2^o and $T_3^o = T_z^o = \{s\}$. In order to make things easier, assume that s takes a form of elements of either T_1^o or T_2^o (we can do this because in our practical examples we *artificially add* the element s to our considerations). Thus, if we *include* s into one of the classes, we in fact now have two classes instead of three. In order to describe this new context, we use notation T_1, T_2 instead (the upper index o in T_i^o stands for "original"). Thus we obtain the following – explicit – definition modified for our simple case of a three (in fact, two) element index set. And in this context we define a hyperoperation on H.

Definition 3.3.3. Let $(T_1^o, \cdot_1, \leq_1)$ and $(T_2^o, \cdot_2, \leq_2)$ be quasi-ordered commutative semigroups such that $T_1^o \cap T_2^o = \emptyset$ and $T_1^o \neq \emptyset$, $T_2^o \neq \emptyset$. For a fixed $i \in \{1, 2\}$ let $T_i = T_i^o \cup \{s\}$, where $s \notin T_1^o \cup T_2^o$ and $T_j = T_j^o$ for $j \in \{1, 2\}$, $i \neq j$. Denote $H = T_1 \cup T_2$ and on H define a binary operation "$\cdot$" by

$$a \cdot b = \begin{cases} s & \text{if } a \in T_1^o, b \in T_2^o \text{ or } a \in T_2^o, b \in T_1^o \\ a \cdot_1 b & \text{if } a, b \in T_1^o \\ a \cdot_2 b & \text{if } a, b \in T_2^o \\ s & \text{for all } a \in T_1^o \cup T_2^o, b = s \text{ or } a = s, b \in T_1^o \cup T_2^o. \end{cases} \tag{3.17}$$

Next, define a hyperoperation "$*_{m2}$" on H by

$$a *_{m2} b = \{a, b\} \cup [a \cdot b)_{\leq}, \tag{3.18}$$

where for an arbitrary pair of elements $a, b \in H$ the relation "$\leq$" is defined as

$$a \leq b = \begin{cases} a \leq_1 b & \text{if } a, b \in T_1^o \\ a \leq_2 b & \text{if } a, b \in T_2^o \end{cases} \tag{3.19}$$

and $[s)_{\leq} = \{x \in H \mid s \leq x\} = \{s\}$ and $s \notin [y)_{\leq} = \{x \in H \mid y \leq x\}$ for all $y \in T_1^o \cup T_2^o$. We call $(H, *_{m2})$ a *modified EL–hyperstructure of the second type* or an *m_2EL–hyperstructure* for short.[3]

Example 3.3.4. Let us continue with Example 3.3.1. Usually, making a componentwise sum of a vector and a matrix makes no sense because of incompatible dimensions. For this reason, denote $s = (\infty, 0)$. Formally speaking, s is a vector, yet it does not belong to T_1^o (even though it has two components, we regard real components but do not regard infinity in T_1^o). Thus, we set $T_1 = T_1^o \cup \{s\}$ and $T_2 = T_2^o$. Given our definition and arbitrary vectors and arbitrary matrices $\mathbf{A}, \mathbf{B}$, we have that $u \cdot v$ is the usual sum of vectors within T_1^o and $\mathbf{A} \cdot \mathbf{B}$ is the usual sum of matrices within T_2^o. Given the definition, $\mathbf{A} \cdot u = s$ and $u \cdot s = s \cdot u = \mathbf{A} \cdot u = s \cdot \mathbf{A} = s$ (in plain words, impossible products give meaningless results). Using "$\leq$" we relate vectors to vectors and matrices to matrices, while the "criss-cross" relation such as e.g. $\mathbf{A} \leq u$ is not permitted. Also, by our definition, s is not related to any element of H, only to itself.

Notice that sometimes defining single-valued operations within the classes may not be possible (or may be uninteresting for our purposes) while the "criss-cross" operation may have the practical use we want.

Example 3.3.5. Let $(T_1^o, \cdot_1, \leq_1)$ be the set of two-dimensional vectors, components of which are positive real numbers, with the operation of componentwise multiplication, where for arbitrary $u, v \in T_1^o$ such that $u = (u_1, u_2), v = (v_1, v_2)$ we set $u \leq_1 v$ if $u_1 \leq v_1$. Let $(T_2^o, \cdot_2, \leq_2)$ be the set of 2×2 matrices of real entries with the operation of entry-wise addition, where for arbitrary $\mathbf{A}, \mathbf{B} \in T_2^o$ we set $\mathbf{A} \leq_2 \mathbf{B}$ if $a_{11} \leq b_{11}$). In this case, operations within the classes are possible and $(T_1^o, \cdot_1, \leq_1)$ and $(T_2^o, \cdot_2, \leq_2)$ are quasi-ordered semigroups yet all of this is just a side-effect of our considerations now since we

[3] The reason why we – rather illogically – call this modification (introduced as the first one) *of the second type* is that it is historically younger as traces of the other modification defined in Definition 3.3.7 can be found already in Chvalina, Křehlík and Novák [61].

might in fact want to consider a new operation such as the usual multiplication of matrices of different dimensions – yet in this case only $u \cdot \mathbf{A}$ would be defined (also notice that in this case $u \cdot \mathbf{A} \in T_1^o$ which would not be the case if we considered e.g. 2×3 matrices).

Example 3.3.6. Suppose that T_1^o is the set of males and T_2^o is the set of females. Let "$\cdot_1$" and "$\cdot_2$" be the operation of mating. For most living organisms, mating is not possible within the same sex as a male and a female are needed. Therefore, we need a "criss-cross" operation "$\cdot$" while speaking about hypergroupoids $(T_1^o, \cdot_1)$ or $(T_2^o, \cdot_2)$ makes no sense in this context. Moreover, the relation "$\leq$" may be the descendent relation valid for all individuals regardless of sex, i.e. $a \leq b$ may mean that a is the offspring (or parent) of b. Obviously, a product of mating between individuals with distinguished sex is an individual of exactly one sex (that is, moreover, related to their parents, i.e. elements of both T_1^o and T_2^o).

Therefore, we adjust Definition 3.3.3 in the following way. Notice that by this definition we overcome the obstacles with multiplication of vectors and matrices in Example 3.3.5 because – if we denote "$\cdot_o$" the usual multiplication of $m \times n$ matrices (again, "o" can stand for "original") – we, in case of Example 3.3.5 for all vectors u and matrices $\mathbf{A}$, define $a \cdot b := u \cdot_o \mathbf{A} = c$, where c is a vector, while for $b \cdot a = \mathbf{A} \cdot_o u$, which has no meaning under standard definitions, we *define* that $b \cdot a := u \cdot_o \mathbf{A} = a \cdot b = c$, which is the same vector. Naturally, operations such that $a \cdot b \neq b \cdot a$ for some $a, b \in T_1^o \cup T_2^o$ are not possible under the following definition.

Definition 3.3.7. Let T_1^o, T_2^o be non-empty sets such that $T_1^o \neq \emptyset$, $T_2^o \neq \emptyset$. For a fixed $i \in \{1, 2\}$ let $T_i = T_i^o \cup \{s\}$, $s \notin T_1^o \cup T_2^o$, and $T_j = T_j^o$ for $j \in \{1, 2\}$ such that $j \neq i$. Denote $H = T_1 \cup T_2$ and by means of an outer operation $\cdot_o : T_1^o \times T_2^o \to T_1^o$ define a binary operation "$\cdot$" on H such that

$$a \cdot b = \begin{cases} a \cdot_o b & \text{if } a \in T_1^o, b \in T_2^o \\ b \cdot_o a & \text{if } a \in T_2^o, b \in T_1^o \\ s & \text{if } a, b \in T_1^o \\ s & \text{if } a, b \in T_2^o \\ s & \text{for all } a \in T_1^o \cup T_2^o, b = s \text{ or } a = s, b \in T_1^o \cup T_2^o. \end{cases} \tag{3.20}$$

Next, define a hyperoperation "$*_{m1}$" on H by

$$a *_{m1} b = \{a, b\} \cup [a \cdot b)_{\leq}, \tag{3.21}$$

where "$\leq$" is such a quasi-ordering on H that $[s)_{\leq} = \{x \in H \mid s \leq x\} = \{s\}$ and $s \notin [y)_{\leq} = \{x \in H \mid y \leq x\}$ for all $y \in T_1^o \cup T_2^o$. We call $(H, *_{m1})$ a *modified EL–hyperstructure of the first type* or an *m_1EL–hyperstructure* for short.

Remark 3.3.8. Notice that if either $T_1^o = \emptyset$ or $T_2^o = \emptyset$, then in Definition 3.3.3 there is no need to consider the special element s and $(H, \cdot, \leq)$ reduces to $(T_i^o, \cdot_i, \leq_i)$ for a suitable $i \in \{1, 2\}$, where $a * b = \{a, b\} \cup [a \cdot b)_\leq$ is used (instead of $a * b = [a \cdot b)_\leq$ used in Lemma 2.1.1) to define the hyperoperation on H. Therefore, in our paper, we consider not only results valid for a decomposed set H but also for the original "Ends lemma" construction using Lemma 2.1.1 which is made explicitely extensive. We will see later on that thanks to the extensivity of "$*$" we often do not need the assumption that $(H, \cdot, \leq)$ is a *quasi-ordered semigroup*. It is important to notice that even though Lemma 2.4.74 on page 72 shows that Lemma 2.1.1 holds even if the hyperoperation is made explicitely extensive, Lemma 2.1.1 (and consequentely, Lemma 2.4.74) cannot be used if the condition that "the semigroup $(H, \cdot)$ is quasi-ordered" is dropped.

Remark 3.3.9. It is easy to prove that $(H, \cdot)$ is a semigroup under both definitions. However, it is *not true* that under both definitions $(H, \cdot, \leq)$ is a *quasi-ordered semigroup*, i.e. that for an arbitrary triple of elements $a, b, c \in H$ such that $a \leq b$ there holds $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$. Notice that this condition is incompatible with formulas (3.20). Indeed, if $a \in T_1^o$ and $b \in T_2^o$ and $a \leq b$, then for $c \in T_1^o$ the expression $a \cdot c \leq b \cdot c$ means $s \leq x$, where $x \neq s$. Yet the special element s is related to itself only, i.e., by definition, $s \not\leq x$, which is a contradiction. Therefore, in the most general contexts we will be trying to avoid the assumption that $(H, \cdot, \leq)$ is a quasi-ordered semigroup. In m_2EL–hyperstructures this condition makes no problems because we do not allow relations between elements of different types, which explains our future interest in "$*_{m2}$" rather than in "$*_{m1}$". Also, when $T_1 = \emptyset$ or $T_2 = \emptyset$, the above problem does not apear.

Therefore, we introduce the term *mEL–hyperstructure* in which we stress the fact that it is *extensivity* that is a priority. Notice that if we moreover assume that one of the sets T_1^o or T_2^o is empty and $H = T_i^o$ is a quasi-ordered semigroup (where T_i^o is the non-empty of the two sets), we get a modification of the original construction of Lemma 2.1.1 (with extensivity added). This is e.g. the case of Corollary 3.3.21.

Definition 3.3.10. By a *modified EL–hyperstructure* (or *mEL–hyperstructure* for short) we mean a hypergroupoid $(H, *_m)$, where "$*_m$" is a commutative hyperoperation on H defined for all $a, b \in H$ by $a *_m b = \{a, b\} \cup [a \cdot b)_\leq$, where $(H, \cdot)$ is a semigroup and "'$\leq$" is a quasi-ordering on H.

Obviously, every m_1EL–hyperstructure as well as every m_2EL–hyperstructure is an mEL–hyperstructure. Also, hyperstructures discussed in Subsection 2.4.6 are mEL–hyperstructures of Definition 3.3.10.

Remark 3.3.11. Since writing $T_1 = T_1^o \cup \{s\}$, $T_2 = T_2^o$ (or vice versa), $H = T_1 \cup T_2$ would be rather inconvenient in some proofs, we use equivalent notation $T_1 \setminus \{s\}$, $T_2 \setminus \{s\}$, $H = T_1 \cup T_2$ in the assumptions of our theorems instead. Of course, s belongs to exactly one of the sets, so – technically speaking – one of the notations is not formally correct as $\{s\} \not\subseteq T_i$ for one $i \in \{1, 2\}$, which means that, in such a case, $T_i \setminus \{s\}$ does not comply with the usual definition of set difference.

3.3.2 Results in the context of the new definitions

In the context of mEL–hyperstructures we can now include results of Subsection 2.4.6. However, all of those results follow from the simple fact that we make the definition of the hyperoperation extensive by changing it from $a * b = [a \cdot b)_\leq$ to $a *_m b = \{a, b\} \cup [a \cdot b)_\leq$. Yet in this subsection, we regard a new context given by Definition 3.3.3, Definition 3.3.7 and Definition 3.3.10. Of course, results of Subsection 2.4.6 remain valid.

Let us now consider the set $H = T_1 \cup T_2$ such that $(T_1, *_m)$ and $(T_2, *_m)$ are constructed using Lemma 2.1.1 and Lemma 2.4.74. Since T_1 and T_2 are elements of different types, hyperoperations on T_1 and T_2 are different. Therefore, for the time being, denote them "$*_1$" and "$*_2$". In the following theorem we must discuss the associativity of "the whole of H", a rather strange concept, discussion of which could be avoided in the proof of Theorem 2.4.72 on page 71. Notice that oddity of this concept comes from the fact that in the proof we have to apply "$*_1$" on elements $a \in T_1$ and $x \in T_2$, i.e., technically speaking, we are discussing nonsense. However, this does not matter because both hyperoperations "$*_1$" and "$*_2$" were constructed using the same formulas (3.18) and the nature of the single-valued operations "$\cdot$" is irrelevant for the proof. Finally, in the assumptions of the following theorem (and further on in Theorem 3.3.15) we need to make sure that the special element s is excluded from both T_1 and T_2, even though it belongs to one of the sets only as they are disjoint – see Remark 3.3.11.

Theorem 3.3.12. *Let $(H, *_{m2})$ be an m_2EL–hyperstructure such that $H = T_1 \cup T_2$, where $T_1 \neq \emptyset$, $T_2 \neq \emptyset$ and $(T_1 \setminus \{s\}, \cdot_1, \leq_1)$ and $(T_2 \setminus \{s\}, \cdot_2, \leq_2)$ are quasi-ordered semigroups. Then $(H, *_{m2})$ is a hypergroup.*

Proof. Thanks to Theorem 2.4.72 only associativity must be tested. Since $(T_1 \setminus \{s\}, \cdot_1, \leq_1)$ and $(T_2 \setminus \{s\}, \cdot_2, \leq_2)$ fulfill assumptions of Lemma 2.1.1 (and consequently of Lemma 2.4.74), the test of associativity reduces to testing the case $a \in T_1$, $b, c \in T_2$ (or $a \in T_2$, $b, c \in T_1$, which is equivalent) for $a *_1 (b *_2 c) = (a *_1 b) *_2 c$ (or swapping "$*_1$" and "$*_2$", which is equivalent).

Since $b, c \in T_2$, there is obviously $b *_2 c \subseteq T_2$. Moreover, $\{b, c\} \subseteq b *_2 c$. Now,

$$a *_1 (b *_2 c) = \{a\} \cup \{x \in H \mid x \in b *_2 c\} \cup \{a \cdot x \mid x \in b *_2 c\}.$$

However, since $a \in T_1$ and elements of $b *_2 c$ are elements of T_2, we get that

$$a *_1 (b *_2 c) = \{a\} \cup \{x \mid x \in b *_2 c\} \cup \{s\}.$$

On the other hand, $a *_1 b = \{a, b, s\}$ and

$$\{a, b, s\} *_2 c = a *_2 c \cup b *_2 c \cup c *_2 s.$$

Yet since $a *_2 c = \{a, c\} \cup [s)_\leq = \{a, c, s\}$, $c *_2 s = \{c, s\}$ and $c \in b *_2 c$, the proof is complete. □

Before discussing the issue of whether mEL–hyperstructures are join spaces, i.e. commutative hypergroups in which the transposition axiom holds, we include the following Lemma 3.3.13 and Lemma 3.3.14 which will simplify the proof of Theorem 3.3.15. Notice that commutativity of mEL–hyperstructures obviously results in the fact that the right and left extensions a/b and $b \setminus a$ coincide. Recall that in the hyperstructure theory $b \setminus a$ is defined as $b \setminus a = \{x \in H \mid a \in b \circ x\}$ for all $a, b \in (H, \circ)$. For details on join spaces and transposition hypergroups see e.g. Corsini and Corsini and Leoreanu [92, 95] or Jantosciak [169].

Lemma 3.3.13. *Let $(H, *_m)$ be an arbitrary mEL–hyperstructure. Then for an arbitrary $a \in H$:*

1. *$a/a = H$.*
2. *$b \setminus a = a/b \neq \emptyset$ for all $a, b \in H$.*
3. *If $H = T_1 \cup T_2$ is an m_1EL– or m_2EL–hyperstructure, where $T_1 \neq \emptyset$ and $T_2 \neq \emptyset$, then $a *_m s = s *_m a = \{a, s\}$.*

Proof. 1. Obvious. Notice that in this case the fact whether $T_1 \neq \emptyset$ and $T_2 \neq \emptyset$ in $H = T_1 \cup T_2$ is irrelevant. Also, if $a = s$, the statement is obvious because $s/s = \{x \in H \mid s \in \{s, x\}\}$.

2. Suppose the contrary, i.e. that for some $a, b \in H$ there is $a \notin b *_m x$ for all $x \in H$. Yet in this case $H \not\subseteq b *_m H$, i.e. the reproduction axiom does not hold in $(H, *_m)$. Yet this is a contradiction to Theorem 2.4.72. Cases $a = s$ or $b = s$ obviously hold because $s \in \{b, s\}$ or $s \in \{a, s\}$.

3. Obvious because the hyperoperation is commutative and for an arbitrary $a \in H$ we have $a *_m s = \{a, s\} \cup [a \cdot s)_\leq = \{a, s\} \cup [s)_\leq = \{a, s\}$. □

Lemma 3.3.14. *Let $(H, *_{m2})$ be an m_2EL–hyperstructure. Denote by T_i an arbitrary of types $T_1, T_2 \subset H$. Then:*

1. *If $a, b \in T_i$, $a \neq s$, $b \neq s$, $a \neq b$, then $b \setminus a = a/b \subseteq T_i$.*

2. *$a \setminus s = s/a = \{s\} \cup T_i$, where $a \notin T_i$, $a \neq s$.*

3. *$s \setminus a = a/s = \{a\}$ for an arbitrary $a \in H \setminus \{s\}$.*

4. *For an arbitrary quadruple of elements $a, b \in T_1$ and $c, d \in T_2$ such that $a \neq s$, $b \neq s$, $c \neq s$, $d \neq s$, there is $b \setminus a \cap c/d = \emptyset$.*

5. *$a *_{m2} b = \{a, b, s\}$ for all $a \in T_i$, $b \notin T_i$.*

6. *$b \setminus a = a/b = \{a\}$ for all $a \in T_i$, $b \notin T_i$, $a \neq s$.*

Proof. 1. We have that $b \setminus a = \{x \in H \mid a \in b *_{m2} x\} = \{x \in H \mid a \in \{b, x\} \cup [b \cdot x)_\leq\}$. Suppose that a, x are of different types, e.g. $a \in T_1$ and $x \in T_2$. Then $b \cdot x = s$, i.e. $[b \cdot s)_\leq = \{s\}$ and $b *_{m2} x = \{b, x, s\}$. However, if $a = x$, then we have a contradiction because $T_1 \cap T_2 = \emptyset$. Therefore, a and x must be of the same type T_i.

2. Suppose an arbitrary $a \in T_1$. Then $a \setminus s = \{x \in H \mid s \in a *_{m2} x\} = \{x \in H \mid s \in \{a, x\} \cup [a \cdot x)_\leq\}$. If a and x are of different types, then $a \cdot x = s \in [s)_\leq$. Therefore for an arbitrary $x \in T_2$ there is $s \in a *_{m2} x$. The fact that $s \in a *_{m2} s$ is obvious.

3. Suppose an arbitrary $a \in H$. Then $s \setminus a = \{x \in H \mid a \in x *_{m2} s\} = \{x \in H \mid a \in \{x, s\}\}$. Obviously, $x = a$.

4. Follows from item 1 and the fact that $T_1 \cap T_2 = \emptyset$.

5. Obvious because in this case $[a \cdot b)_\leq = [s)_\leq = \{s\}$.

6. We have that $b \setminus a = \{x \in H \mid a \in b *_{m2} x\} = \{x \in H \mid a \in \{b, x\} \cup [b \cdot x)_{\leq}\}$. From the proof of item 1 there follows that a and x must be of the same type T_i. Yet in this case x and b are of different types which means that $[b \cdot x)_{\leq} = \{s\}$. Since $a \neq s$ and $a \in T_i$, $b \notin T_i$ and $T_1 \cap T_2 = \emptyset$, there is also $a \neq b$. Thus the condition "$x \in H$ such that $a \in \{b, x, s\}$" can be fulfilled only for $x = a$. Therefore $b \setminus a = \{a\}$. If $b = s$, we get item 3 immediately.

□

Theorem 3.3.15. *Let $(H, *_{m2})$ be an m_2EL–hyperstructure such that $H = T_1 \cup T_2$, where $T_1 \neq \emptyset$, $T_2 \neq \emptyset$ and $(T_1 \setminus \{s\}, \cdot_1, \leq_1)$ and $(T_2 \setminus \{s\}, \cdot_2, \leq_2)$ are quasi-ordered groups. Then the transposition axiom holds in $(H, *_{m2})$.*

Proof. Throughout the proof we will work with elements $a, b, c, d \in H$. Before giving the proof recall some basic facts. The hyperoperation "$*_{m2}$" and the operation "$\cdot$" are commutative. As a result $b \setminus a = a/b$ for all $a, b \in H$. Elements of the same type $T_i \subset H$ can be put into relation "$\leq$" while elements of different types cannot. The special element $s \in H$ is related to itself only and is absorbing with respect to the operation "$\cdot$", which means that $[a \cdot s)_{\leq} = \{s\}$ for all $a \in H$. We need to prove that that $b \setminus a \cap c/d \neq \emptyset$ implies $a *_{m2} d \cap b *_{m2} c \neq \emptyset$ for all $a, b, c, d \in H$.

There are 16 possible arrangements of elements $a, b, c, d \in H$ into T_1, T_2:

1. $a, b, c, d \in T_1$
2. $a, b, c, d \in T_2$
3. $a \in T_1, b, c, d \in T_2$
4. $b \in T_1, a, c, d \in T_2$
5. $c \in T_1, a, b, d \in T_2$
6. $d \in T_1, a, b, c \in T_2$
7. $a, b \in T_1, c, d \in T_2$
8. $b, c \in T_1, a, d \in T_2$
9. $a, c \in T_1, b, d \in T_2$
10. $d, a \in T_1, b, c \in T_2$
11. $b, d \in T_1, a, c \in T_2$
12. $d, c \in T_1, a, b \in T_2$
13. $b, c, d \in T_1, a \in T_2$
14. $a, c, d \in T_1, b \in T_2$
15. $a, b, d \in T_1, c \in T_2$
16. $a, b, c \in T_1, d \in T_2$

First of all, suppose that $a \neq s$, $b \neq s$, $c \neq s$, $d \neq s$, which will enable us to apply Lemma 3.3.13 and Lemma 3.3.14.

In case 1 we have that $b \setminus a = \{a\} \cup \{x \in H \mid b \cdot x \leq a\}$ while $c/d = \{c\} \cup \{x \in H \mid d \cdot x \leq c\}$. Regard now an arbitrary $x \in b \setminus a \cap c/d$. Since in a quasi-ordered group $b \cdot x \leq a$ implies $b \cdot c \leq a \cdot x^{-1} \cdot c$ and $d \cdot x \leq c$ implies $a \cdot d \leq a \cdot x^{-1} \cdot c$, we see that $a \cdot x^{-1} \cdot c \in b *_{m2} c \cap a *_{m2} d$. Case 2 is analogous.

In case 3 we have that $b \setminus a = \{a\} \subseteq T_1$ while $c/d \subseteq T_2$. Therefore, in this case the intersection is never non-empty. Case 13 is analogous.

Similarly, in case 4 we have that $b \setminus a = \{a\} \subset T_2$ and $c/d = \{c\} \cup \{x \in T_2 \mid x \leq c \cdot d^{-1}\}$. The intersection of these two sets is non-empty only on condition that $a \leq c \cdot d^{-1}$ or $a = c$. Therefore suppose that the former holds. We have that $a *_{m2} d = \{a, d\} \cup [a \cdot d)_{\leq} = \{a, d\} \cup \{x \in H \mid a \cdot d \leq x\}$ and $c *_{m2} b = \{c, b, s\}$. Yet since we suppose that $a \leq c \cdot d^{-1}$, there is $a \cdot d \leq c$, which means that $c \in a *_{m2} d$. If $a = c$, then obviously $a \in a *_{m2} d \cap c *_{m2} b$. Case 14 is analogous.

In case 5 we have that $b \setminus a \subseteq T_2$ yet $c/d = \{c\}$. Yet since $c \in T_1$, the intersection of these two sets is never non-empty. Case 15 si analogous.

In case 6 we have that $b \setminus a = \{a\} \cup \{x \in T_2 \mid x \leq a \cdot b^{-1}\}$ and $c/d = \{c\} \subset T_2$. The intersection of these two sets is non-empty on condition that $c \leq a \cdot b^{-1}$ or $a = c$. Therefore suppose that the former holds. We have that $a *_{m2} d = \{a, d, s\}$ and $c *_{m2} b = \{c, b\} \cup \{x \in H \mid c \cdot b \leq x\}$. Yet since we suppose that $c \leq a \cdot b^{-1}$, there is $c \cdot b \leq a$, which means that $a \in c *_{m2} b$. If $a = c$, then obviously $a \in a *_{m2} d \cap c *_{m2} b$. Case 16 is analogous.

In case 7 we have that $b \setminus a \subseteq T_1$ while $c/d \subseteq T_2$, which means that their intersection is never non-empty. Case 12 is analogous. The same, albeit the sets are different, happens in cases 8 and 10.

The sets in case 9 are the same as in case 8, i.e. $b \setminus a = \{a\}$ and $c/d = \{c\}$, yet now there may happen that they coincide. This happens when $a = c$. In this case there is obviously $a \in a *_{m2} d \cap c *_{m2} b$. Case 11 is analogous.

Finally, we need to verify cases when some of the elements $a, b, c, d \in H$ equal s. Yet most of such cases reduce to trivialities as either $b \setminus a$ and c/d have no intersection or $a *_{m2} d \cap b *_{m2} c$ is non-empty by default. In fact, the only non-trivial case is $a \neq b = c$, such that $a, b, c \in T_2$, and $d = s \in T_1$. Since $b/s = \{b\} \subseteq T_2$, sets $b \setminus a$ and b/s have non-empty intersection only if $b \in b \setminus a$. Since $b \setminus a = \{x \in H \mid a \in b *_{m2} x\} = \{x \in H \mid, a \in \{b, x\} \cup [b \cdot x)_{\leq}\}$ and $a \neq b$, there must be (if we want that $b \in b \setminus a$) that $a \in [b \cdot b)_{\leq}$. Since $a *_{m2} s = \{a, s\}$, then $a *_{m2} s \cap b *_{m2} c \neq \emptyset$ only if $a \in b *_{m2} c$ or $s \in b *_{m2} c$. Yet since we assume that $b = c$, this means that $a \in b *_{m2} b$ or $s \in b *_{m2} b$ and we have just shown that if $b \setminus a$ and b/s have non-empty intersection, the former holds. $\square$

Example 3.3.16. Consider the set $(\mathbb{R}^n, +, \leq)$ of n–tuples of real numbers with the usual addition by components and lexicographic order. $(\mathbb{R}^n, +, \leq)$ is a partially ordered group with the identity $(0, \ldots, 0)$. If we now define the hyperoperation "$*$" for any two n–tuples $u, v \in \mathbb{R}^n$ as

$$u * v = [u + v)_{\leq} = \{x \in \mathbb{R}^n \mid u + v \leq x\},$$

by Lemma 2.1.1 and Lemma 2.1.2 we get that $(\mathbb{R}^n, *)$ is a hypergroup. By Lemma 2.4.74 and Theorem 2.4.71 we get that if we modify the hyperoperation to

$$u *_m v = \{u, v\} \cup [u + v)_{\leq} = \{u, v\} \cup \{x \in \mathbb{R}^n \mid u + v \leq x\},$$

$(\mathbb{R}^n, *_m)$ is also a hypergroup. If we now take the union of two specific sets of this type, e.g. $H = \mathbb{R}^2 \cup \mathbb{R}^3$, we can consider applying Theorem 3.3.12 and Theorem 3.3.15. Before doing that let us agree that e.g. $s = (\infty, 0)$ and define $(u_1, u_2) + (\infty, 0) = (\infty, 0)$, $(v_1, v_2, v_3) + (\infty, 0) = (\infty, 0)$, $(u_1, u_2) + (v_1, v_2, v_3) = (\infty, 0)$, for all $u = (u_1, u_2) \in \mathbb{R}^2$ and $v = (v_1, v_2, v_3) \in \mathbb{R}^3$, and $s = (\infty, 0)$ is incomparable with all $u \in \mathbb{R}^2$ and $v \in \mathbb{R}^3$. Obviously, the sum of two elements from $\mathbb{R}^2$ and of two elements of $\mathbb{R}^3$ is meaningful and the addition is commutative, i.e. we are using formulas (3.17) to handle the sum "+". Thus, by Theorem 3.3.12 and Theorem 3.3.15, $(H, *_{m2})$, where $*_{m2}$ is defined by (3.18), is a join space.

Example 3.3.17. In Example 3.3.16 lexicographic order is used. As a result $(\mathbb{R}^n, \leq)$ is a chain. It is easy to verify that we can obtain an analogous result if we set e.g. $(u_1, u_2, \ldots u_n) \leq (v_1, v_2, \ldots v_n)$ if and only if $u_i = v_i$, for all $i = 1, 2, \ldots n - 1$, and $u_2 \leq v_2$ for all $u = (u_1, u_2, \ldots u_n)$, $v = (v_1, v_2, \ldots v_n)$.

Example 3.3.18. Obviously, the same reasoning as in Example 3.3.16 can be applied on the set $(\mathbb{M}_{n,n}(\mathbb{R}), +)$ of square matrices with real coefficients endowed with the usual componentwise sum. Here, the ordering can be defined e.g. by $\mathbf{M} \leq \mathbf{N}$ if and only if $\mathrm{tr}(\mathbf{M}) \leq \mathrm{tr}(\mathbf{N})$, for all $\mathbf{M}, \mathbf{N} \in \mathbb{M}_{n,n}(\mathbb{R})$, where $\mathrm{tr}(\mathbf{M})$ is the trace of $\mathbf{M}$. The componentwise sum of matrices is commutative. The sum of $u = (u_1, u_2, \ldots u_n)$ and a square matrix has no meaning, therefore we define that $u + \mathbf{M} = \mathbf{M} + u = s = (\infty, 0, \ldots, 0)$ for all $u \in \mathbb{R}^n$ and $\mathbf{M} \in \mathbb{M}_{n,n}(\mathbb{R})$. If we now regard $H = \mathbb{R}^n \cup \mathbb{M}_{n,n}(\mathbb{R})$, we get analogous results as in Example 3.3.16.

For details and for more examples with matrices or structures of the same type as in Example 3.3.18 see Chvalina, Křehlík and Novák [61] or Račková [268] or Section 2.2. Notice that [268] includes examples of EL–semihypergroups constructed using multiplication of symmetric matrices, i.e. matrices used e.g. for linear systems used when computing splines. For examples of EL–semihypergroups constructed from noncommutative groups motivated by linear differential operators or Laplace transform (or other tools used e.g. in signal processing) see papers by Chvalina, Hošková–Mayerová, Křehlík and Novák such as [58, 59, 62].

It is important to notice that an arbitrary interval $\langle a, b\rangle$ (or a matrix of real numbers from $\langle a, b\rangle$) together with the operation "max" or "min" and the usual ordering of real numbers by size (or its inverse) gives rise to mEL–hyperstructures. The same holds for the interval $\langle 0,1\rangle$ (or a matrix of real numbers from $\langle 0,1\rangle$) with the operation of multiplication. Thus, the regarded numbers, vectors or matrices can denote probabilities and the operations can denote the smaller or greater probability or the probability of simultaneous occurrance.

Remark 3.3.19. An example of an ICT application where all the above aspects combine are the techniques of detection and navigation such as the simultaneous localization and mapping (SLAM). For a basic overview of SLAM see e.g. Davidson et al. [102] or Vu, Aycard and Appenrodt [305]. In these we need to create a map from data obtained by means of sensors (camera / lidar) and at the same time localize ourselves in this map. Using advanced techniques of image processing, the 2D data, in which key points such as corners of buildings or edges had been identified, are back projected into a 3D map so that we get a map composed of landmarks. In case we are using lidars, the 2D map has a form of an occupancy grid, i.e. a probabilistic grid mapping obstacle occurances. The moving object (such as a car or a robot) has to localize itself in such a map as precisely as possible making use of the observed obstacles. Simultaneously, the map is iteratively updated as one source complements the other (as each sensor gives different information). Thus the basic layout of the situation is the same as in Example 3.3.16. The relation "$\leq$" in Example 3.3.17 suggests that when scanning the occupancy grid for information, i.e. reading it, we move using vertical or horizontal lines (since $u=(u_1,u_2)\in\mathbb{R}^2$ is a point in the grid; this is an example of movement, of course). Also, we need to consider probability of an obstacle occurance – yet immediately before this remark we mentioned that interval $\langle 0,1\rangle$ with operations such as "min", "max" or the ususal multiplication gives rise to (modified) EL–hyperstructures.

Of course, the description of the model we give in this example is rather simple in comparison with the complexity of SLAM as we consider points from $\mathbb{R}^2$ or $\mathbb{R}^3$ and e.g. in SLAM from cameras the landmarks are represented by vectors and the SLAM process is computed using covariant matrices; see Davidson et al. [102]. However, the structures considered in the "Ends lemma" and our modifications of it can be of any type. For more complex examples such as the "Ends lemma" being applied on the set of matrices see Subsection 2.5.5.

Remark 3.3.20. Notice that in m_2EL hyperstructures the types T_1^o and T_2^o, which make up the set H, are disjoint and are linked by the special element $s\in H$ only. This makes our considerations in the proof of Theorem 3.3.15 rather straightforward. However, if we permitted relations between elements of different types (which is the case of m_1EL–hyperstructures – see the "male – female" Example 3.3.6), we would run into difficulties very soon. Most importantly, we would not be able to work with quasi-ordered semigroups – see Remark 3.3.9.

As a corollary we get a statement for the original construction of EL–hyperstructures in which we use (3.18) instead of (2.1) to define the hyperoperation. Notice that what we get is a result analogical to Lemma 2.1.5 included on page 28.

Corollary 3.3.21. *Let $(H, *_m)$ be an mEL–hyperstructure such that $T_1 = \emptyset$ or $T_2 = \emptyset$ and $(H, \cdot, \leq)$ is a quasi-ordered group. (In other words, we assume conditions of the original "Ends lemma" construction yet instead of* (2.1) *we use* (3.18) *to define the hyperoperation.) Then $(H, *_m)$ is a join space.*

Proof. By repeating the proof of Theorem 2.4.71 on page 71, we have that $(H, *_m)$ is a hypergroup. By the proof of case 1 of Theorem 3.3.15, we know that the transposition axiom holds, which completes the proof. Making distinctions between the first and second modiciations is irrelevant because all elements are of the same type. □

Remark 3.3.22. Notice an important fact: neither in the proof of Theorem 2.4.72 nor in the proof of Theorem 3.3.15 have we used transitivity of the relation "$\leq$". However, this property is essential in proving associativity of the hyperoperation in the original construction and, as a result, also in Lemma 2.4.74 and Theorem 2.4.71. For details see the proof of Lemma 2.1.1 on page 27 or its modification for the n–ary context on page 144. This fact is an explanation of the wording of the following corollary. Although the wording is rather complicated, the corollary in fact says that in m_2EL–hyperstructures, in which we drop the assumption of transitivity of the relation "$\leq$", all join space axioms hold – only associativity is replaced by weak associativity (in the hyperstructure sense) because dropping transitivity prevents us from applying Lemma 2.1.1 (and consequently Lemma 2.4.74).

Corollary 3.3.23. *Let $H = T_1 \cup T_2$, where $T_1 \neq \emptyset, T_2 \neq \emptyset$ and $T_1 \cap T_2 \neq \emptyset$, further let $(T_1 \setminus \{s\}, \cdot_1)$ and $(T_2 \setminus \{s\}, \cdot_2)$ be commutative groups in which the operation "$\cdot$" is defined using* (3.17) *and finally let "$\leq$" be a reflexive relation on H such that for all $a, b, c \in H$ the fact that $a \leq b$ implies $c \cdot a \leq c \cdot b$. Then $(H, *_{m2})$, where "$*_{m2}$" is a hyperoperation* (3.18)*, is an H_v-group in which the transposition axiom holds.*

Proof. Follows directly from Theorem 2.4.72, from Theorem 3.3.15 and from Remark 3.3.22. □

Thus, by Theorem 3.3.15, m_2EL–hyperstructures are join spaces. If we drop transitivity of "$\leq$", associativity is replaced by weak associativity. Thanks to Corollary 2.4.77 we know that modified EL–hyperstructures (i.e. also m_2EL–hyperstructures) cannot be canonical hypergroups. As far as m_1EL–hyperstructures are concerned, by *the proof of* Theorem 2.4.72 on page 71 we know that they are H_v–groups (notice that the proof does not change under Definition 3.3.7 of m_1EL–hyperstructures).

Remark 3.3.24. Even though we have discussed the case of a semigroup partitioned into elements of two types, the nature of our definitions enable us to generalize the results for context of T_i^o, where $i \in \mathbb{N}$ is arbitrary.

Chapter 4

Applications

"*What are the applications of your theory?*" – a nightmare question for a mathematician because the obvious answer is: "*Well, engineer, let's sit down and talk for a while and I am sure we will find many of them.*" – yet the mathematician is expected to provide a comprehensive list on demand to justify his research.

In this chapter I collect a few suggestions or areas in which the theory of the previous chapters can be found, used or applied. In Section 4.1 we discuss a branch of mathematics in which the "Ends lemma" was used to obtain some mathematical results. In Section 4.2 we show how these results were (or can be) applied in some engineering sciences. Finally, in Section 4.3 we use the "Ends lemma" and related topics to construct a hyperstructure model of a specific engineering problem.

Of course, the topics included in this chapter are a *sample* only. The "Ends lemma" construction is so natural and general that it can go unnoticed in many situations or applications. Antampoufis, Dramalidis and Vougiouklis [9] were not aware of the construction when they proposed their *urban applications* yet they use it in their considerations. When Davvaz, Dehghan Nezhad and Heidari [110] noticed that algebraic hyperstructures naturally occur in genetics, they had only a rough idea in mind (see also Example 3.3.6 on page 163 for a more elaborate yet still a rough idea of a similar nature). Al Tahan and Davvaz [2] approached the topic of biological inheritance from the perspective of *ordered hyperstructures* yet in Subsection 2.6.1 we discuss the relation between *EL–hyperstructures* and *ordered hyperstructures*, i.e. their results relate to our topic. Also, when defining their hyperoperation on a *braid group* (which can be applied to the study of *fluid mechanics*; see e.g. Thiffeault and Finn [292]), Al Tahan and Davvaz [4] only briefly remark that there exists a link between their hyperoperation and the "Ends lemma". Jun and Song [172] and Flaut [134] linked *BCK–algebras* to block codes used in *channel encoding* in earlier mobile communication systems. Matrices of block codes are studied in Saeid et al. [279]. Since, in a special case, we obtain lattices (or semilattices), whenever multivalued aspects are employed we can make use of results of the hyperstructure theory and concepts and results such as those mentioned in Subsection 2.5.5.

It is not only the above mentioned biological inheritance that seems a natural motivation for hyperstructures defined in the "Ends lemma" or "quasi-order hypergroups" way. The following quote is taken from Hošková, Chvalina and Račková [153]: "Nuclear fission occurs when a heavy nucleus splits, or fissions, into two small nuclei. As a result of this fission process we can get several dozens of different combinations of two medium-mass elements and several neutrons. Another typical example of the situation when the result of interaction between two particles is the whole set of particles is the interaction between a foton with certain energy and an electron. The result of this interaction is not deterministic. A photo-electric effect or Coulomb repulsion effect or changeover of foton onto a pair electron – positron can arise. ... Another motivation for investigation of hyperstructures yields from technical processes such as a time sequence of military car repairs with respect to its roadability consequences and operational behavious." Now, the ordering which naturally arises in all these situations, can be sometimes seen from the point of view of the "Ends lemma", i.e. essential for constructing a hyperoperation, or from the point of view of *ordered hyperstructures*, i.e. as a feature of an – already existing – hyperstructure.

Many hyperstructures motivated by chemistry or physics are H_v–structures (for some examples see papers of Benvidi, Davvaz, Dehghan Nezhad, Nadjafikhah or Moosavi Nejad such as [109, 118]). Yet as has been shown in Subsection 2.4.6, securing weak associativity is rather trivial in our context.

Also worth exploring is the link between the topic of Section 4.1 and neural networks started by Chvalina and Smetana [77] in which neurons are described mathematically by means of linear differential operators of Subsection 4.1.2. Notice that [77] makes use of the mathematical approach of Buchholz [36] and builds on some results of Srivastava et al. [288].

4.1 Various kinds of operators

One of the first areas, in which EL–hyperstructures were used, was the study of various kinds of operators related to ordinary differential equations, integro-differential equations or affine transformations. A list of such papers (often conference proceedings) includes papers written by various students or collaborators of Chvalina [56–58, 60, 62, 64, 70, 74, 75]; see also a list in Hošková and Chvalina [150]. This study was initiated by Chvalina and Chvalinová [50, 52, 55] and motivated by the algebraic approach to the investigation of global properties of ordinary differential equations by Neuman [231–235] who himself continued the work of Borůvka [32]. It is to be noted that by applying the group theory to the study of ODE's, Neuman [232] achieved some fundamental results regarding global properties of differential equations. The study initiated by Chvalina and Chvalinová was motivated by the effort to enhance these results by applying the multi-valued aspect. In this respect notice book by Davvaz and Leoreanu–Fotea [111], Section 8.4, where results (albeit not making use of EL–hyperstructures) of Chvalina and Chvalinová [52] are included.

4.1.1 Notation

Following Neuman, we denote $\mathbb{A}_n$ the set of all ordinary differential equations of order n. In case of *homogeneous linear* ODE's we by $\mathbb{LA}_n(I)$ mean the set of all *linear differential operators*, i.e. left-hand sides of homogeneous ODE's such that their right-hand side is 0, with coefficients defined on interval I, where I is usually open to avoid problems with continuity at endpoints. By $\mathbb{VA}_n(I)$ we denote the set of solution spaces of all n–order linear ODE's, coefficients of which are defined on I. By L, or V, we denote one particular operator, or a solution space of one particular ODE. Given the standard form of notation of homogeneous linear ODE's, we use notation $L(p_{n-1}, \ldots, p_0)$, where p_i, $i = 0, \ldots, n-1$ are the respective coefficients, i.e. functions defined on I; whenever $L(\vec{p})$ is used, $\vec{p}$ stands for a vector of – suitably ordered – components $p_{n-1}, \ldots, p_0$. Also, in case of $n = 2$ we prefer using notation $L(p, q)$, i.e. given a second-order homogeneous linear ODE

$$y'' + x^3 y' + y \cos x = 0,$$

we write $L(p, q)(y) = 0$, where $p(x) = x^3$ and $q(x) = \cos x$. We always suppose that the coefficient at highest-order derivative is 1. We suppose that these functions are *continuous on* I, which we denote by $p_i \in C(I)$. All p_i are functions of x, i.e. – technically speaking – we should always write $p_i(x)$ instead of p_i. By $C^n(I)$ we denote the set of functions, derivatives of which are, on I, continuous up to order n; by $C^\infty(I)$ we mean set of functions derivatives of which of any order are continuous.

Whenever we work with *complex functions*, we denote by $\mathbb{C}$ the complex domain and by Ω a non-empty subset of $\mathbb{C}$, such as $\Omega = \{z \in \mathbb{C} \mid \Re(z) > 0\}$ used in Chvalina and Novák [70]. The set of complex functions of one variable is denoted by $\mathbb{C}^\Omega$, i.e. $\mathbb{C}^\Omega = \{f : \Omega \to \mathbb{C}\}$. The variable of our complex functions is z, i.e. whenever f is a function, we mean $f(z)$.

In contexts other than linear ODE's we use letters other than L to denote specific operators – such as "T" for *transformation* (see e.g. Chvalina, Moučka and Novák [64]) or "V" for *Volterra* (see e.g. Chvalina and Novák [70]; confusion with a notation of a solution space of one particular ODE is not likely thanks to different nature of their arguments). The notation such as $T(\lambda, F, \varphi)(f)$ means that the operator is *applied* on f, i.e. that it transforms f into another function, such as e.g.

$$T(\lambda, F, \varphi)(f) = \lambda F f + \varphi$$

as used in Chvalina, Moučka and Novák [64].

Further on we make use of operations on sets of various kinds of operators. In case of *non-commutative* operations, we by $C(\texttt{operator})$ denote the set of operators *commuting* with a given operator, i.e. e.g.

$$C(L(p,q)) = \{L(r,s) \mid L(r,s) \cdot L(p,q) = L(p,q) \cdot L(r,s)\} \qquad (4.1)$$

for a given operation "$\cdot$".

4.1.2 Hyperstructures motivated by ordinary differential equations and affine transformations

Motivated by Borůvka's and Neuman's works [32, 33, 231], Chvalina and Chvalinová [50, 52, 55] applied the tools of (algebraic) hyperstructure theory on the classical results achieved by the algebraic approach to the study of differential equations. One of the first tasks in this area was to test meaningful constructions of hyperstructures (semihypergroups, hypergroups or join spaces) of various kinds of operators associated to specific differential equations. In [52] Chvalina and Chvalinová, making use of *quasi-order hypergroups*, presented constructions which lead to commutative hyperstructures satisfying the reproductive law, i.e. *quasi-hypergroups*.[1] Soon after [52], Chvalina and Chvalinová [50], with the help of the "Ends lemma", constructed transposition hypergroups of linear differential operators and showed certain properties of these. Notice that in Novák [242, 244] some of the proofs included in the above mentioned papers could be simplified considerably because they turned out to be examples illustrating results of Subsection 2.4.5.

Theorem 4.1.1. [50] *Let $I \subseteq \mathbb{R}$ be an open interval and $\mathbb{LA}_2(I) = \{L(p,q) \mid p, q \in C(I), p(x) > 0$ for all $x \in I\}$. For an arbitrary pair of operators $L(p_1,q_1), L(p_2,q_2) \in \mathbb{LA}_2(I)$ define an operation "$\cdot$" by*

$$L(p_1 q_1) \cdot L(p_2, q_2) = L(p_1 p_2, p_1 q_2 + q_1) \qquad (4.2)$$

[1] Some of their results were generalized by Trimèche [293] in his use of hypergroups in the study of wavelets.

and a relation "$\leq$" by

$$L(p,q_1) \leq L(p_2,q_2) \text{ if } p_1(x) = p_2(x), q_1(x) \leq q_2(x) \text{ for all } x \in I. \tag{4.3}$$

Then $(\mathbb{LA}_2(I), \cdot, \leq)$ *is a noncommutative ordered group. If we define a hyperoperation "$*$", for all* $L(p_1,q_1), L(p_2,q_2) \in \mathbb{LA}_2(I)$, *by*

$$L(p_1,q_1) * L(p_2,q_2) = [L(p_1,q_1) \cdot L(p_2,q_2))_{\leq}, \tag{4.4}$$

$(\mathbb{LA}_2(I), *)$ *is a noncommutative transposition hypergroup.*

Moreover, if we denote by χ_1 the function $I \to \mathbb{R}$ such that $\chi(x) = 1$ for all $x \in I$, and by $\mathbb{L}_1\mathbb{A}_2(I)$ the set of all operators $L(\chi_1, q) \in \mathbb{LA}_2(I)$, then we get the following result.

Theorem 4.1.2. [50] $(\mathbb{L}_1\mathbb{A}_2(I), *)$ *is a subhypergroupoid of* $(\mathbb{LA}_2(I), *)$. *Moreover, it is a closed, invertible, reflexive and normal subhypergroup of* $(\mathbb{LA}_2(I), *)$.

Now, the proofs included in [50] are rather lengthy and complicated. However, as has been shown in Novák [244], if we use results of Subsection 2.4.5, they become obvious. Moreover, using results of Subsection 2.4.7, we can omit the assumption that $p(x)$ need to be positive functions. In Theorem 4.1.1 this is assumed because a group must be constructed to a obtain the EL–hypergroup. Yet as results of Subsection 2.4.7, or rather Novák [242], show, a hypergroup can in this case be obtained even without the existence of inverse operators. For details see [242], Example 3.

Remark 4.1.3. It is to be noted that the motivation for operation (4.2) is rather straightforward. Regard two linear functions, $f(x) = ax + b$ and $g(x) = cx + d$, and denote $f = [a,b], g = [c,d]$. Then their composition $(f \circ g)(x) = acx + ad + b$ can be denoted as $[ac, ad + b]$. For a detailed investigation of transposition hypergroups of linear functions (including the use of EL–hyperstructures) see Beránek and Chvalina [22].

The way Theorem 4.1.1 constructs an operation and a relation, i.e. using the "Ends lemma" consequently a hyperoperation, can be easily expanded from second-order linear ODE's to n–order linear ODE's. This was done by Chvalina and Chvalinová in [51]. In order to get a join space, it is enough to define the multiplication of operators $L(\vec{p}), L(\vec{q}) \in \mathbb{LA}_n(I)$ by $L(\vec{p}) \circ_m L(\vec{q}) = L(\vec{u})$, where, for all $x \in I$, the k–th component of $L(\vec{u})$ is

$$u_k(x) = p_m(x)q_k(x) + \delta_{km}p_k(x) \tag{4.5}$$

and $m \in \mathbb{N}$ is such a component of $L(\vec{p})$ that $p_m(x) > 0$ for all $x \in I$, and δ_{km} stands for the Kronecker δ. The relation of operators $L(\vec{p}) \leq L(\vec{q})$ is defined by $p_m(x) = q_m(x)$ and $p_k(x) \leq q_k(x)$ for all $k \neq m$.

Apart from the above construction, in Chvalina, Moučka and Novák [66] or Chvalina and Račková [76] one can find another construction of EL–join spaces linked to functions of one variable. Regard functions $f, g \in C^k(I)$ of one variable, where $k \in \{0, 1, \ldots\}$, their pointwise addition, and relation "$f \leq g$ whenever $f(x) \leq g(x)$ for all $x \in I$". Chvalina and Račková [76] obtained the following result.

Theorem 4.1.4. [76] *Let $C^k(I)$, where $k = \{0, 1, \ldots\}$, be the ring of functions $f : I \to \mathbb{R}$, derivatives of which are continuous up to order k. Regard the set $(C^k(I), +, \leq)$, where "$+$" is pointwise addition of functions and "$\leq$" is the relation defined above, and for an arbitrary pair of functions $f, g \in C^k(I)$ define a hyperoperation "$*$" on $C^k(I)$ by*

$$f * g = \bigcup_{[a,b] \in \mathbb{R}_0^+ \times \mathbb{R}_0^+} [af + bg)_{\leq}, \tag{4.6}$$

i.e.

$$f * g = \bigcup_{[a,b] \in \mathbb{R}_0^+ \times \mathbb{R}_0^+} \{h \in C^k(I) \mid af(x) + bg(x) \leq h(x), x \in I\}.$$

Then $(C^k(I), *)$ is a join space.

Remark 4.1.5. Notice that, since we are using the linear combination of functions, the join space $(C^k(I), *)$ is not an EL–join space for $a, b \neq 1$. Recall also Remark 2.4.129 on page 93 where the concept of multiple of a sum in the definition of the hyperoperation is discussed. Notice that the same idea has been used in Chvalina, Moučka and Novák [65] to describe hyperstructures of preference relations, a *microeconomics* concept.

In [24] Beránek and Chvalina, motivated by the one-to-one correspondence between all equations $L(p, q)(y) = 0$ and their solution spaces $V(\varphi_1, \varphi_2)$ (see Neuman [232]), construct a join space of fundamental solution systems of certain second order linear homogeneous differential equations. Below we include this result as included in Chvalina, Chvalinová and Novák [57].[2]

Suppose two linearly independent functions $\varphi_1, \varphi_2 \in C^2(I)$, i.e. functions such that their Wronski determinant is nonzero for any $x \in I$, i.e.

$$W[\varphi_1, \varphi_2] = \begin{vmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_1'(x) & \varphi_2'(x) \end{vmatrix} \neq 0 \text{ for any } x \in I.$$

[2]Notice the not-so-obvious use of the "Ends lemma" in Theorem 4.1.6.

Denote by $V(\varphi_1, \varphi_2)$ the two-dimensional linear space formed by all functions

$$y(x) = c_1\varphi_1(x) + c_2\varphi_2(x),$$

where $c_1, c_2 \in \mathbb{R}$, i.e.

$$V(\varphi_1, \varphi_2) = \{c_1\varphi_1 + c_2\varphi_2 \mid \varphi_1, \varphi_2 \in C^2(I), c_1, c_2 \in \mathbb{R}\}.$$

Regard $\mathbb{LA}_2(I)$ used in Theorem 4.1.1 and denote by $\mathbb{VA}_2(I)$ the set of two-dimensional linear solution spaces of second order linear differential equations $L(p, q)(y) = 0$. Finally, denote

$$D[\varphi_1, \varphi_2] = \begin{vmatrix} \varphi_1''(x) & \varphi_2''(x) \\ \varphi_1(x) & \varphi_2(x) \end{vmatrix}.$$

As has been mentioned above, Neuman [232] shows that there is a one-to-one correspondence between all equations $L(p, q)(y) = 0$ and their solution spaces $V(\varphi_1, \varphi_2)$. Notice that the ODE corresponding to the space $V(\varphi_1, \varphi_2)$ with linearly independent $\varphi_1, \varphi_2 \in C^2(I)$ has the form

$$y''(x) + \frac{D[\varphi_1, \varphi_2]}{W[\varphi_1, \varphi_2]}y'(x) + \frac{W[\varphi_1', \varphi_2']}{W[\varphi_1, \varphi_2]}y(x) = 0.$$

For any solution space $V(\varphi_1, \varphi_2) \in \mathbb{VA}_2(I)$ we choose an arbitrary but fixed base which will be further on called *representing fundamental solution system* of the corresponding second order linear homogeneous ODE.

Theorem 4.1.6. [24] *Let $I \subseteq \mathbb{R}$ be an open interval and $\mathbb{VA}_2(I)$ the set of solution spaces defined above. Regard a pair $V(\varphi_1, \varphi_2), V(\psi_1, \psi_2) \in \mathbb{VA}_2(I)$ such that their bases $\{\varphi_1, \varphi_2\}, \{\psi_1, \psi_2\}$ form representing fundamental solution systems of second order linear homogeneous ODE's*

$$y''(x) + \frac{D[\varphi_1, \varphi_2]}{W[\varphi_1, \varphi_2]}y'(x) + \frac{W[\varphi_1', \varphi_2']}{W[\varphi_1, \varphi_2]}y(x) = 0$$

and

$$y''(x) + \frac{D[\psi_1, \psi_2]}{W[\psi_1, \psi_2]}y'(x) + \frac{W[\psi_1', \psi_2']}{W[\psi_1, \psi_2]}y(x) = 0$$

respectively. Denote $\mathcal{F}(\varphi_1, \varphi_2, \psi_1, \psi_2)$ the set of all representing fundamental solution systems of differential equations

$$y''(x) + \frac{D[\varphi_1, \varphi_2]D[\psi_1, \psi_2]}{W[\varphi_1, \varphi_2]W[\psi_1, \psi_2]}y'(x) + v(x)y(x) = 0,$$

where $v \in C(I)$ is such that

$$\frac{D[\varphi_1, \varphi_2]W[\psi_1', \psi_2']}{W[\varphi_1, \varphi_2]W[\psi_1, \psi_2]} + \frac{W[\varphi_1', \varphi_2']}{W[\varphi_1, \varphi_2]} \leq v(x).$$

Then $(\mathbb{VA}_2(I), *)$, where "$*$" is a hyperoperation on $\mathbb{VA}_2(I)$ such that for all $V(\varphi_1, \varphi_2), V(\psi_1, \psi_2) \in \mathbb{VA}_2(I)$ we define

$$V(\varphi_1, \varphi_2) * V(\psi_1, \psi_2) = \{V(\omega_1, \omega_2) \in \mathbb{VA}_2(I) \mid \{\omega_1, \omega_2\} \in \mathcal{F}(\varphi_1, \varphi_2, \psi_1, \psi_2)\},$$

is a (noncommutative) transposition hypergroup.

In the construction of operation (4.2) we used the idea of *composition of linear transformations*. Regard now a function $f \in \mathbb{C}^\Omega$ and for an arbitrary $\lambda \in \mathbb{C}$ such that $\lambda \neq 0$ and $F, G \in \mathbb{C}^\Omega$ define an operator

$$T(\lambda, F, \varphi)(f) = \lambda F f + \varphi.$$

In other words, we define a transformation of $\mathbb{C}^\Omega$ such that a function f is multipled by a complex number λ and another function F and then shifted in the direction of φ, i.e. in a certain sense we make use of the idea of *affine transformations*. If we now denote the set of such operators $\mathcal{T}(\Omega)$ and regard composition of two operators $T(\lambda_1, F_1, \varphi_1), T(\lambda_2, F_2, \varphi_2) \in \mathcal{T}(\Omega)$, we get that

$$T(\lambda_1, F_1, \varphi_1) \circ T(\lambda_2, F_2, \varphi_2) = T(\lambda_1\lambda_2, F_1F_2, \lambda_1F_1\varphi_2 + \varphi_1). \tag{4.7}$$

In [64] Chvalina, Moučka and Novák show conditions under which these operators form a commutative quasi-ordered semigroup which is then used to construct a respective EL–semihypergroup and – under certain restrictions – an EL–hypergroup. When doing this, the idea of *centralizers*, i.e. of (4.1), is used. For a fixed $T_0 = T(\lambda_0, F_0, \varphi_0) \in \mathcal{T}(\Omega)$ we define

$$Ct_{\mathcal{T}}(T_0) = \{T(\lambda, F, \varphi) \in \mathcal{T}(\Omega) \mid T(\lambda, F, \varphi) \circ T_0 = T_0 \circ T(\lambda, F, \varphi)\}.$$

The *single-valued operation* on $Ct_{\mathcal{T}}(T_0)$ has been defined by (4.2). Regard now a *relation* on $Ct_{\mathcal{T}}(T_0)$. For this we set that

$$T_1 \leq T_2 \text{ whenever there exists } n \in \mathbb{N} \text{ such that } T_2 = T_1 \circ T_0^n, \tag{4.8}$$

where $T^n(\lambda, F, \varphi)$ is

$$T^n(\lambda, F, \varphi)(f) = \begin{cases} f + n\varphi = T(1, 1, n\varphi)(f) & \text{if } F = \lambda^{-1} \\ \lambda^n F^n f + \frac{\lambda^n F^n - 1}{\lambda F - 1}\varphi = T(\lambda^n, F^n, \frac{\lambda^n F^n - 1}{\lambda F - 1}\varphi)(f) & \text{if } F \neq \lambda^{-1} \end{cases},$$

which can be easily shown using mathematical induction. It is now a matter of routine verification to show that if we define a hyperoperation "$*_{\mathfrak{T}}$" on $Ct_{\mathfrak{T}}(T_0)$, for a fixed $n \in \mathbb{N}$, by

$$T(\lambda_1, F_1, \varphi_1) *_{\mathfrak{T}} T(\lambda_2, F_2, \varphi_2) = \{T_0^n \circ T(\lambda_2, F_2, \varphi_2) \circ T(\lambda_1, F_1, \varphi_1)\}, \quad (4.9)$$

we get that $(Ct_{\mathfrak{T}}(T_0), *_{\mathfrak{T}})$ is an EL–semihypergroup. Moreover, if we restrict the hyperoperation "$*_{\mathfrak{T}}$" on

$$S(T_0) = \{T(\lambda, F, \varphi) \in Ct_{\mathfrak{T}}(T_0) \mid \lambda \neq 0, F(z) \neq 0 \text{ for all } z \in \Omega\},$$

then $(S(T_0), "*_{\mathfrak{T}}")$ is an EL–hypergroup constructed from a group $(S(T_0), \circ)$, with the neutral element being the identity mapping $T(1,1,0) = T^0(\lambda, F, \varphi)$, such that the inverse of $T(\lambda, F, \varphi)$ is $T\left(\frac{1}{\lambda}, \frac{1}{F}, -\frac{\varphi}{\lambda F}\right)$.

In [152,153,155] Hošková, Chvalina and Račková use the idea of the "Ends lemma" to construct and study *join spaces of Fredholm integral operators*. Moreover, they construct certain subgroups of the group of Fredholm integral operators and using the decomposition of the group by these subgroups they obtain a quasi-hypergroup of blocks of operators. By doing this, the authors in fact obtain hyperstructures in a very classical way (see e.g. Dresher and Ore [128], Eaton [132] or Utumi [294]). Notice that many of the results obtained in [155] had been included in Davvaz and Leoreanu–Fotea [111]. Therefore, we discuss the topic in a rather brief way.

Definition 4.1.7. By the *Fredholm integral equation of the first kind* we mean an equation

$$\int_a^b K(x,s)\varphi(s)\mathrm{d}s = f(x), \quad (4.10)$$

while by the *Fredholm integral equation of the second kind* we mean an equation

$$\varphi(x) - \lambda \int_a^b K(x,s)\varphi(s)\mathrm{d}s = f(x), \quad (4.11)$$

where $K(x,s)$ is a real or complex valued function defined on $I \times I$, where $I = \langle a, b \rangle \subset \mathbb{R}$ and is called *kernel*, $f(x)$ defined on I is a real valued function called *free* or *absolute member*, λ is a real number and φ is an unknown function.

Closely related to Fredholm integral equations are *Volterra integral equations* which have variable upper integral limits. The corresponding integral operators are called *Fredholm*, or *Volterra* integral operators, respectively.

Thus, if we concentrate on the Fredholm integral equation of the second kind, we can write

$$F(\lambda, K, f) = \lambda \int_a^b K(x,s)\varphi(s)\mathrm{ds} + f(x) \tag{4.12}$$

In [152] Hošková, Chvalina and Račková discuss the issue of commutativity of multiplication of Fredholm integral operators. It turns out (see [152], Lemma 1) that in order to achieve this it is suitable to define the multiplation as

$$F(\lambda_1, K_1, f_1) \cdot F(\lambda_2, K_2, f_2) = F(\lambda_1\lambda_2, K_2 f_1 + K_1, f_1 f_2) \tag{4.13}$$

for all $F(\lambda_1, K_1, f_1), F(\lambda_2, K_2, f_2) \in \mathbb{F}$. Furthermore, if we define the relation "$\leq$" on the set $\mathbb{F}$ of all operators, such that $f(x) > 0$ for all $x \in I$, by $F(\lambda_1, K_1, f_1) \leq F(\lambda_2, K_2, f_2)$ whenever

$$\lambda_1 = \lambda_2 \text{ and } f_1(x) \equiv f_2(x) \text{ and } K_1(x,s) \leq K_2(x,s) \text{ for all } x \in I, \tag{4.14}$$

then we obtain the following result, included in Hošková, Chvalina and Račková [155].

Theorem 4.1.8. [155] *Regard the set $\mathbb{F}$ of Fredholm integral operators (4.12), where $f(x) > 0$ for all $x \in I$, multiplication of operators given by (4.13) and relation "$\leq$" given by (4.14). On $(\mathbb{F}, \cdot, \leq)$ define, for all operators $F(\lambda_1, K_1, f_1), F(\lambda_2, K_2, f_2)$, hyperoperation "$*$" by*

$$\begin{aligned} F(\lambda_1, K_1, f_1) * F(\lambda_2, K_2, f_2) &= \\ = \{F(\lambda, K, f) \in \mathbb{F} \mid F(\lambda_1, K_1, f_1) \cdot F(\lambda_2, K_2, f_2) &\leq F(\lambda, K, f)\}. \end{aligned} \tag{4.15}$$

Then $(\mathbb{F}, *)$ is a join space.

Just as is the case of Theorem 4.1.1 and Theorem 4.1.2, the rather lengthy proofs included in [155] can be shortened considerably using the theory included in Subsection 2.4.5. For details see Novák [242], Example 4. Notice that results of [155] were generalized by Dehghan Nezhad and Davvaz [117]. For details and further issues linked to Fredholm or Volterra integral equations (such as implications of applying *Laplace transform*) see Hošková, Chvalina and Račková [152, 153].

4.2 Generalizations of automata

In the *theory of automata* various types of automata are considered. Generally speaking, by an *automaton* we mean a quintuple consisting of an *input* *alphabet*, an *output alphabet*, a *set of states* and two functions linking these three sets, which give rules for reaching desired states or output. Automata can be classified using a number of different criteria. Further on we concentrate on *automata without outputs* (or rather a special class of it), i.e. we reduce the quintuple to a triad consisting of an *input alphabet*, a *set of states* and a function giving rules for reaching states. We work with the following definition (notice that condition 1 is sometimes omitted if we regard a semigroup instead of a monoid).

Definition 4.2.1. By a *quasi–automaton* we mean a structure $\mathbb{A} = (I, S, \delta)$ such that $I \neq \emptyset$ is a monoid, $S \neq \emptyset$ and $\delta : I \times S \to S$ satisfies the following condition:

1) There exists an element $e \in I$ such that $\delta(e, s) = s$ for any $s \in S$.

2) $\delta(y, \delta(x, s)) = \delta(xy, s)$ for any pair $x, y \in I$ and any $s \in S$.

The set I is called the *input set* or *input alphabet*, the set S is called the *state set* and the mapping δ is called *next-state* or *transition function*.

This definition dates back to 1970s and works such as Dörfler [123,124] or Gécseg and Peák [140], even though Dörfler [124] gives the triad in a swapped form of (S, I, M), which also means that the components in conditions 1 and 2 are swapped. Also worth mentioning is that the name "quasi–automaton" is not very frequent. The following is a quote from Warner [308]: "An automaton is defined by Ginzburg [139] to be quintuple $(X, Y, Q, \delta, \lambda)$ consisting of input set X, output set Y, state set Q, next-state function $\delta : X \times Q \to Q$ and next-output function $\lambda : X \times Q \to Y$. If we concentrate on changes of state rather than outputs, we consider the triple (X, Q, δ) called by Ginzburg a *semiautomaton*." Gécseg and Peák [140], p. 11[3] clarify the distinction between *automata without outputs*, *quasi-automata* and *semiautomata* in the following way: "Indeed, every automaton (A, X, δ) without outputs can be considered as a quasi–automaton with $S = F_1(X)$ [*Note:* where $F_1(X)$ is the free semigroup of X, i.e. a semigroup elements of which are all the finite sequences of zero or more elements of X with string catenation as the associative operation]. In other words, every automaton without outputs is a quasi–automaton such that its input semigroup is free. If, especially, the input semigroup S of a quasi–automaton (A, S, δ) is a right cancellative semigroup, i.e. for arbitrary $r, s, t \in S$ the implication

$$rt = st \Rightarrow r = s$$

[3]For the formalization of the first sentence of the quote see Dörfler [124], Definition 1 and Definition 2, and his remark between them.

holds, then the automaton is called a *semiautomaton*.[4]" When introducing semiautomata, Ginzburg [139], p. 40, writes: "Many physical devices have the remarkable property of tending to remain in any of a finite number of situations or *states*. The 'jumping' from one state to another (sometimes the same) is a continuous process which must be carefully considered by the designer of the device, but it can be disregarded by the user interested only in the above discrete states."

No matter how old (in the context of information technology, ancient) the works of Dörfler, Ginzburg, Gécseg and Peák or Warner may be they are still worth considering and exploring.

In our further text we are going to include some results acheived in the context of quasi-automata of Definition 4.2.1. It is to be noted that the first links between the theory of automata and the hyperstructure theory date back to Massouros and Mittas [213, 218]; some of these results are included in Corsini and Leoreanu [95].[5] In [53] Chvalina and Chvalinová use the hyperstructure theory to prove some properties of quasi-automata of Definition 4.2.1; see page 64. For consistency reasons we will use terms "quasi–automaton" and "quasi–multiautomaton" instead of "semiautomaton" and "multi–semiautomaton" even though, in many cases, the input semigroup $(S, \cdot)$ will be a group, i.e. right cancellative, which means that we could speak of semiautomata.

Remark 4.2.2. In Definition 4.2.1, Condition 1 is called *unit condition* (UC) while condition 2 is sometimes called *Mixed Associativity Condition* (MAC), even though most authors give it no name. Based on the name "MAC", we further on – see Definition 4.2.3 – use the name "GMAC" which stands for the *Generalized Mixed Associativity Condition*. This is the term that can be found in papers dealing with the hyperstructure generalizations of quasi-automata, called *quasi–multiautomata*, which we briefly study further on. This study was initiated by Chvalina and Chvalinová [53] and explored e.g. by Hošková, Chvalina and Dehghan Nezhad in [59], where also a link to Section 4.1 can be found.

If in quasi–automata we suppose that the input set I is a *semihypergroup instead of a free monoid*, we arrive at the concept of a *quasi–multiautomaton*. When defining it caution must be exercised when adjusting the conditions imposed on the transition function δ as on the left-hand side of condition 2 we get a state while on the right-hand side we get a *set of states*. However, in

[4] Automata without outputs are called semiautomata in M. Yoeli [310].

[5] Notice that in order to study certain concepts of automata theory, Massouros and Mittas introduced the notion of *hyperringoid* and *fortification* in join spaces, which are discussed on pages 110 and 45, respectively.

the dichotomy *deterministic — nondeterministic*, quasi–multiautomata still are deterministic because the range of δ is S.

Definition 4.2.3. A *quasi–multiautomaton* is a triad $\mathbb{A} = (I, S, \delta)$, where $(I, *)$ is a semihypergroup, S is a non–empty set and $\delta : \ I \times S \to S$ is a transition map satisfying the condition:

$$\delta(b, \delta(a,\ s)) \in \delta(a * b,\ s) \text{ for all } a, b \in I,\ s \in S. \tag{4.16}$$

The hyperstructure $(I, *)$ is called the *input semihypergroup* of the quasi–multiautomaton $\mathbb{A}$ (I alone is called the *input set* or *input alphabet*), the set S is called the *state set* of the quasi–multiautomaton $\mathbb{A}$, and δ is called *next-state* or *transition function*. Elements of the set S are called *states*, elements of the set I are called *input symbols*.

Example 4.2.4. Regard a simple machine (such as a player using – for a better understanding – a tape), where $I = \{\mathrm{FFWD}, \mathrm{REW}, \mathrm{STOP}\}$ and $S = \{$tape beginning, at 1:00, at 2:00, tape end at 3:00$\}$, where “FFWD” stands for “forward the tape by 1 minute”, “REW” stands for “rewind the tape by 1 minute” and “STOP” means “no action”. Now, what happens if two *conflicting commands* are issued *simultaneously*? The natural response is either “no action” or “error message” or “perform the commands one by one”. And it is the last option that can be easily described by the hyperoperation applied on elements of I. Suppose that we have $\mathrm{FFWD} * \mathrm{REW} = \{\mathrm{FFWD}, \mathrm{REW}\}$ and the current state is “tape end at 3:00”. The set on the right-hand side of the GMAC condition (4.16) is

$$\delta(\{\mathrm{FFWD}, \mathrm{REW}\}, \text{tape end at 3:00}) = \{\text{tape end at 3:00, at 2:00}\}$$

while on the left hand side we get, in case FFWD is executed first,

$$\delta(\mathrm{REW}, \delta(\mathrm{FFWD}, \text{tape end at 3:00})) = \delta(\mathrm{REW}, \text{tape end at 3:00}) = \text{at 2:00}$$

or, in case REW is executed first,

$$\delta(\mathrm{FFWD}, \delta(\mathrm{REW}, \text{tape end at 3:00})) = \delta(\mathrm{FFWD}, \text{at 2:00}) = \text{tape end at 3:00}.$$

Or, suppose that we are in state “at 1:00” and simultaneously issue commands “FFWD” and “STOP”, where $\mathrm{FFWD} * \mathrm{STOP} = \{\mathrm{FFWD}, \mathrm{STOP}\}$. In this case we get $\{\text{at 2:00}, \text{at 1:00}\}$ on the right hand side while on the left-hand side we get the state “at 2:00” regardless of which command is performed first.

Remark 4.2.5. In the above example it is natural to consider that FFWD $*$ FFWD $= \{\text{FFWD}\}$ or REW $*$ REW $= \{\text{REW}\}$. Yet, with most machines, double pressing of the same button results in a faster or repeated action. However, our set I consists of movements by one minute only. Therefore, it is obvious that, in a case like Example 4.2.4, Chvalina's Definition 4.2.3 should be changed in such a way that the transition function δ is, for such $a \in I$ that $a * a = \{a\}$, instead of $\delta(a * a, s) = \delta(a, s)$ defined by

$$\delta(a * a, s) = \delta(a, \delta(a, s)) \tag{4.17}$$

for all $s \in S$. Notice that the concept of *quasi-multiautomaton* was originally coined for contexts where the hyperoperation is built on the idea of *principal ends* – either using the "Ends lemma" construction or using the idea of *quasi-order hypergroups*. However, in such contexts, the fact that $a * a = \{a\}$ is not very common. Moreover, if $a * a = \{a\}$ for *all* elements a, the quasi-ordering is trivial (because in both cases we assume reflexivity).

So, using this simple example one can easily see the difference between the *transition function of a quasi-automaton* and the *transition function of a quasi-multiautomaton*. In quasi-automata the state achieved by applying y in a state, which is the result of application of x in s, is the same as the state achieved by applying xy in s. On the other hand, the condition (4.16) says that it is one of the many states achievable by applying any command from $x * y$ in state s.

Notice that there are many situations in which conflicting commands, or some forbidden combinations, might be issued at the same time. One such case are *flip–flop circuits*, a type of logic circuits whose output depends not only on the present value of its signals but on the sequence of past inputs.[6] In fact, the study of multi–valued aspects in quasi–automata can be related to Comer [84], who – in his Problem 15 – suggests to develop sentential logic, i.e. propositional calculus, in which the truth table of logical OR is adjusted so that $1 \vee 1 = \{0, 1\}$. Also notice the mathematical theory of *quantum computers*, the construction of which is still in its very infancy. Already in [153] the authors (when discussing hyperstructures of Fredholm integral operators), mention the basic idea of *qubits*, the units of information of quantum computers, which – unlike conventional bits – do not have, at one moment in time, exactly one of states 0 or 1.[7]

[6] Another simple real-life example is the simultaneous pushing of buttons "make call", "terminate call" on a mobile phone or an intercom.

[7] In this respect the issue of *Boolean matrices* and its link to binary relations could be worth exploring. For a straightforward explanation of the link see Aghabozorgi, Jafarpour and Davvaz [3], for some results in this area see works of Massouros and Tsitouras such as [212].

In fact, the reasoning in Example 4.2.4 is such that we somehow “store” the commands, i.e. in a certain way we imitate the functionality of *pushdown automata*, even though the concept is absolutely different. The hyperoperation “$*$” used in Example 4.2.4 was *extensive* (in fact *minimal extensive*), i.e. such that for all $a, b \in I$ there is $\{a, b\} \in a * b$ (in our case $\{a, b\} = a * b$). However, this need not be the case at all. Also, in information sciences, the single-valued operation “$\cdot$” is usually *catenation of words*, i.e. $a \cdot b = ab$, which implies the distinction between the alphabet I and the set of words of the alphabet, i.e. a free monoid $I^* = (I, \cdot)$ under catenation. In papers on quasi–multiautomata and the GMAC condition (4.16) the authors regard an *arbitrary* single-valued operation “$\cdot$” or an *arbitrary* hyperoperation performed on the input alphabet I.

Example 4.2.6. Suppose that the input alphabet is $I = \{a, b, c\}$. If we regard the set of all possible (finite or infinite) strings which can be made up of characters a, b, c, we get $I^* = \{a, b, c, aa, bb, cc, ab, ac, bc, aaa, bbb, ccc, \ldots\}$. Thus we regard the free semigroup $(I^*, \&)$, where “&” is the operation of catenation. However, instead of catentaion we can regard an *arbitrary* operation on I. In this case we work with a semigroup $(I, \cdot)$, i.e. regard that, for all $x, y \in I$, we have $x \cdot y \in I$ instead of $x \cdot y \in I^*$.

The transition function δ is a tool to obtain the next state by applying a given letter in a given state. In other words, we apply an outer operation on the set of states S. Given this perspective, one can abandon the link to physical machines or formal languages and one can view quasi–automata (or quasi–multiautomata) as another abstract algebraic structure inspired by the use of the *outer operation* of vector spaces. This is the approach used by Chvalina, Hošková (Mayerová), Moučka, Dehghan Nezhad and others in papers such as [49, 59, 64, 66, 68, 70, 73, 75, 76, 154] in which the input semihypergroups are various sets of operators constructed in Section 4.1 and the set of states is the set of functions restricted in various ways. In the context of those papers we speak of “actions of semihypergroups on given phase sets”.[8]

Definition 4.2.7. ([152], Definition 2) Let X be a set, $(G, *)$ a semihypergroup and $\pi : X \times G \to X$ a mapping such that, for each $x \in X$, $s, t \in G$,

$$\pi(\pi(x, t), s) \in \pi(x, t * s),$$

where $\pi(x, t * s) = \{\pi(x, u) \mid u \in t * s\}$. Then (X, G, π) is called a *discrete transformation semihypergroup* or an *action of the semihypergroup G on the phase set X*. The mapping π is usually said to be simply an *action*.

[8]In the following definition notice that it uses the swapped notation of Dörfler; see page 185.

As we have seen in Section 4.1, the semihypergroups often make use of the "Ends lemma" construction; the EL–hyperoperation is such that the GMAC condition (4.16) of quasi–multiautomata holds.

Instead of the various kinds of operators used in Section 4.1 or studied in papers such as Hošková, Chvalina and Račková [152, 153], one can consider operators of differential equations induced by specific modelling functions used e.g. in *signal analysis* and *signal processing*. One can see that in those applications we naturally obtain both special kinds of linear differential operators and sets of matrices or functions used in the above mentioned papers.

Since these functions are functions of time, we regard $I = \langle 0, \infty)$ and use the variable t instead of x further on.

Our first example is the *function of the muffled oscillations*

$$y(t) = a\exp(-\lambda t)\sin(bt). \tag{4.18}$$

Its first and second derivatives are $y'(t) = a\exp(-\lambda t)\,(b\cos(bt) - \lambda\sin(bt))$ and $y''(t) = a\exp(-\lambda t)\,((\lambda^2 - b^2)\sin(bt) - 2\lambda t\cos(bt))$, respectively. The relation between $y''(t), y'(t)$ and $y(t)$ is described by the linear ODE

$$y''(t) + 2\lambda y'(t) + (\lambda^2 + b^2)y(t) = 0. \tag{4.19}$$

As another example, consider the modelling time function

$$y(t) = \exp(\alpha t) - \exp(\beta t), \alpha < \beta, \tag{4.20}$$

which is in fact a two parameter time signal known as *multiexponential function of nuclei decay*. Its derivatives are $y'(t) = -\alpha\exp(-\alpha t) + \beta\exp(-\beta t)$ and $y''(t) = \alpha^2\exp(-\alpha t) + \beta^2\exp(-\beta t)$, which leads to the following differential equation

$$y''(t) - (\alpha - \beta)y'(t) + \alpha\beta y(t) = 0. \tag{4.21}$$

Finally, we mention two functions which are solutions of *differential equations in the Jacobi form*. First, the *Gaussian–shaped pulse signal*

$$v(t) = a\exp(-2\pi t^2), \tag{4.22}$$

the first and second derivatives of which are $v'(t) = -4a\pi t\exp(-2\pi t^2)$ and $v''(t) = 16a\pi^2 t^2 v(t)$, respectively. This leads to the differential equations in the Jacobi form

$$v''(t) + (4\pi - 16a\pi^2 t^2)v(t) = 0 \tag{4.23}$$

with the parameter a running through a suitable number set. Second, the *Chapman-Richard's function* (CHRF)

$$y = A[1 - \exp(-ct)]^b, \tag{4.24}$$

one of the most common functions based on the original *Bertalanffy equation* derived for growth and increment of body weight. Its first and second derivatives are $y' = Abc[1 - \exp(-ct)]^{b-1} \exp(-ct)$ and $y'' = -Abc^2[1 - \exp(-ct)]^b \frac{\exp(ct)-b}{(\exp(ct)-1)^2}$, respectively. From these we obtain a second order linear ODE in the Jacobi form

$$y''(t) + \frac{Abc^2 \cdot \exp(-ct) \cdot [b \exp(-ct) - 1]}{(1 - \exp(-ct))^2} y(t) = 0. \tag{4.25}$$

The following paragraphs are adapted from books by Bellanger and Jan [20, 164] and from Jan and Janová [165].

1. *Linear discrete systems* are described by means of input/output models. In some applications a more general state model is used. This model works with vector input and output and describes values in chosen internal points of the system and enables its users to transform the basic realization structure into structures which are (from the point of view of the input/output correspondence) equivalent. The usual form of the state model is

$$\begin{aligned} \vec{q}_{n+1} &= \mathbf{A}\vec{q}_n + \mathbf{B}\vec{x}_n \\ \vec{y}_n &= \mathbf{C}\vec{q}_n + \mathbf{D}\vec{x}_n, \end{aligned} \tag{4.26}$$

   where $\vec{q}_n = (q_n^1, q_n^2, \ldots, q_n^m)^T$ is a column vector of internal, state variables, $\vec{x}_n = (x_n^1, x_n^2, \ldots, x_n^p)^T$ is the vector of input values and $\vec{y}_n = (y_n^1, y_n^2, \ldots, y_n^l)^T$ is the vector of output values. Further, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are matrices of appropriate dimensions which define the system in question. The name of the model is derived from the concept of state variables which are parameters remembered (or rather, capable of being remembered) by the system. One can find an evident parallel between this system and the functional diagrams of finite automata since discrete systems are special cases of automata with alphabets (input, output, state ones) composed of sets of values taken by signals $\vec{x}$, $\vec{y}$, $\vec{z}$ (in case of digital systems by respective sets of admissible vectors of numbers) and transition functions and output functions are expressed by the matrix relations of the model.

2. *Discrete linear transformation* is a mapping $\{x_n\} \to \{x_k\}$ from the original domain (in general a vector space $\mathbb{C}^{N_1}$, in most cases $\mathbb{R}^{N_1}$) to the transformation domain (vector space $\mathbb{C}^{N_2}$ or $\mathbb{R}^{N_2}$) given by

$$x_k = \sum_{n=0}^{N_1-1} a_{k,n} x_n, \quad k = 0, \ldots N_2 - 1 \tag{4.27}$$

or (in vector form) by $X = \mathbf{A}\vec{x}$, where $\mathbf{A}$ is an $N_2 \times N_1$ matrix of the form

$$\mathbf{A} = \begin{pmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,N_1-1} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,N_1-1} \\ \vdots & \vdots & & \vdots \\ a_{N_2-1,0} & a_{N_2-1,1} & \cdots & a_{N_2-1,N_1-1} \end{pmatrix}$$

called the *transformation kernel*, elements of which are real or complex numbers. In most applications there is $N_1 = N_2$ and as a result $\mathbf{A}$ is a square matrix. If it is moreover regular, i.e. $det(\mathbf{A}) \neq 0$, the transformation is invertible, which means that there is a one-to-one correspondence between the original sequence $\vec{x}$ and its transformation $\vec{y}$ called *discrete spectrum*. The inverse transformation is $\vec{x} = \mathbf{A}^{-1}\vec{y}$. Based on special properties of the matrix $\mathbf{A}$ we obtain respective special transforms such as *Hadamar transform*, *Walsh transform*, *discrete Fourier transform*, *Haar transform*, etc. For details see Jan [164].

3. *Inverse filtering and noised signal recovery.* Very often, one needs to recover an unknown signal from its garbled and noise influenced form which is a result of passing through a garbling or noisy environment (such as a communication channel). As an example recall a recovery method called *Kalman filtering*. Apart from the scalar Kalman filter the vector filter is often used in applications since the state vector need not contain the delayed values of one state variable only but instead it can be compose of various variables that have various physical meaning in the modelled system. This enables a traightforward creation of well understandable models of physical systems, the internal variables of which (i.e. the original signals) can be Kalman filters estimated using external observation. Denote vectors of delayed values of the original and observed signals by

$$\vec{x}_n = (x_n^1, x_n^2, \dots, x_n^R)^T, \quad \vec{y}_n = (y_n^1, y_n^2, \dots, y_n^R)^T,$$

where R is the order of the given model. In a similar way denote the vectors of the driving noise and undesired noise $\vec{g}_n$ and ν_n. The state equation of the model describing the signal generation is

$$\vec{x}_{n+1} = \mathbf{A}_{n+1}\vec{x}_n + \vec{g}_{n+1} \tag{4.28}$$

while the output equation is

$$\vec{y}_{n+1} = \mathbf{C}_{n+1}\vec{x}_n + \vec{\nu}_{n+1}. \tag{4.29}$$

In contrast to the standard form of the state description (4.26), we have that it is the vector of undesired noise that is used in the output equation of the system (instead of the input vector of the system), otherwise there is obviously $\mathbf{B} = \mathbf{D} = \mathbf{I}$ from (4.26), i.e. the output equation of the system is

$$\vec{y}_{n+1} = \mathbf{A}_{n+1}\mathbf{C}_{n+1}\vec{x}_n + \mathbf{C}_{n+1}\vec{g}_{n+1} + \vec{\nu}_{n+1}. \tag{4.30}$$

In cases when we want to create signals of complicated structure or model complicated systems (provided we choose sufficiently high order R) we need to use vector and matrix mathematical apparatus (e.g. in finite impluse response filters).

We remark that in [61], Chvalina, Křehlík and Novák, using the above reasoning and motivation, discuss the issue of *Cartesian composition* of quasi–multiautomata and the way the GMAC condition (4.16) must be adjusted so that the resulting structure is again a quasi–multiautomaton. In this respect notice that in Subsection 2.5.5, most of which was published as Křehlík and Novák [187], we work with sets of matrices. Also, one of the motivating lines of Novák and Křehlík [249] is the need to make use of the undefined product of vectors or matrices.

Finally, it is to be noted that Chvalina's approach to the generalization of quasi–automata via the GMAC condition (4.16) is not the only possible one. In this respect notice Ashrafi and Madanshekaf [13]. Also, refer to [35], in which Borzooei, Varasteh and Hasankhani transferred Chvalina's results into the context of fuzzy sets and fuzzy automata.

4.3 Underwater wireless sensor networks

Results included in this section were accepted for publication in Proceedings of ICNAAM 2017 (WoS) as Novák, Ovaliadis and Křehlík [251].

In this section we make use of our construction in describing a mathematical model of one particular engineering problem. To be more precise, we discuss *underwater wireless sensor networks*, abbreviated as UWSN's, which are groups of sensors deployed underwater, networked via acoustic links and performing collaborative tasks.

UWSN's are often used in *environment monitoring*, where they review how human activities affect marine ecosystems, *undersea explorations* such as detecting oilfields, *disaster prevention*, e.g. when monitoring ocean currents, in *assisted navigation* for e.g. location of dangerous rocks in shallow waters, or for *disturbed tactical surveillance* for e.g. intrusion detection.

The fact that such wireless sensor networks are deployed underwater results in profound differences from terrestrial wireless sensor networks. The key aspects, which are different, include *communication method*, i.e. radio waves vs acoustic signals, *cost* (while contruction of terrestrial networks experience decreasing prices of components, underwater sensors are expensive devices), *memory capacity* (because water is a problematic medium resulting in loss of large quantities of data), *power limitations* due to the nature of the signal and longer distances handled, as well as problems related to *deployment* of the network, i.e. issues connected to static or dynamic deployment. In underwater sensor networks we commonly face challenges of *limited bandwith*, *high bit error rates*, *large propagation delays*, and *limited battery resources* caused by the fact that in the underwater environment, sensor batteries are impossible to recharge especially because no solar energy is available underwater. As the number of sensor nodes that stop working due to the power loss increases, the network topology has to be reconfigured to guarantee network connectivity and effective communication between sensor nodes. This affects the size of the UWSN coverage area, leading to a less efficient data aggregation and smaller reliability of data. Obviously, greater efficiency in battery use means prolonging network lifetime without sacrificing system performances. A key factor in keeping the UWSN alive and operational for as long as possible is the *protocols* used for discovering and maintaining the routes between sensor nodes. The most commonly used routing protocols are: *flooding*, *multipath*, *cluster* and *miscellaneous protocols* – see Wahid and Dongkyun [306]. In the *flooding approach*, the transmitters send a packet to all nodes within the transmission range. In the *multipath approach* source sensor nodes establish more than one path towards *sink nodes* on the surface. Finally, in the *clustering approach* the sensor nodes are grouped together in a *cluster*. For an easy-to-follow reading on how UWSN's work and on advantages of clustering see e.g. Domingo and Prior [122].

Recent research shows that the cluster based protocols give a great contribution towards the concept of energy efficient networks – see Ayaz et al. [15], Ovaliadis and Savage [258] or Rault, Abdelmadjid and Yacine [266]. A common cluster based network consists of a centralised station deployed at the surface of the sea called a *sink* (or *surface station*) and *sensor nodes* deployed at various *tiers* inside the sea environment. These are grouped into *clusters*. In this architecture, each cluster has a head sensor node, called *cluster head* (CH). The cluster head is assumed to be inside the transmission range of all sensor nodes that belong to its cluster. Every cluster head operates as a coordinator for its cluster, performing significant tasks such as cluster maintenance, transmission arrangements, data aggregation and data routing.

In our model, denote H the set of all elements (i.e. sensor nodes as well as sinks) of an arbitrary UWSN. Suppose that all elements are capable of handling (i.e. receiving or transmitting) data in the same way and performing the same set of tasks, i.e. that they are – from the mathematical point of view – interchangable and equal (of course, with respect to their functionality as sinks / sensor nodes). Since the aim of the system is to collect information, our elements of H must communicate the data – ideally upwards, towards the surface. As mentioned above, there are different ways of passing information. We concentrate on *multipath* and *cluster routing approach* (see Figure 1 and 2). For details concerning these see e.g. Ayaz [15] or Li et al. [198]. Multipath routing protocols forward the data packets to the sink via other nodes while in cluster based routing protocols, data packets are first aggregated to the respective cluster heads and then forwarded via other cluster heads to the sink on the surface. In our model we denote the i–th cluster by cl_i and its cluster head by CH_i. We call non-CH nodes *plain*. Sinks are treated as cluster heads.

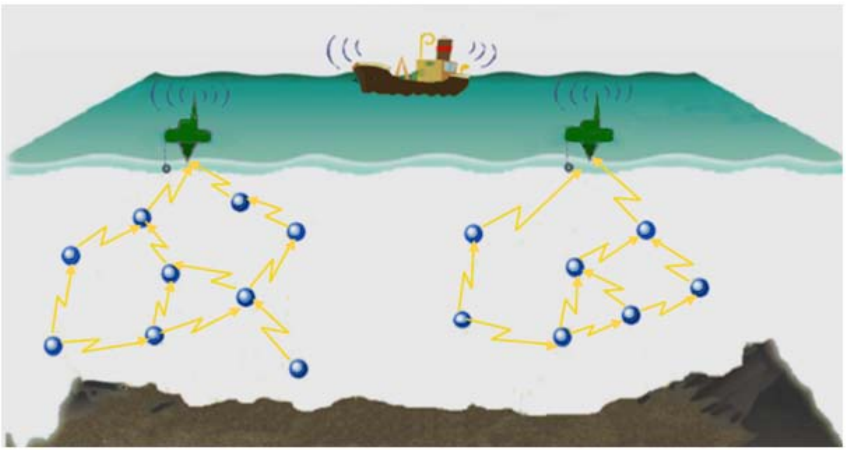

Figure 4.1: Multipath approach to UWSN data aggregation – notice the upwards oriented communication between nodes

4.3.1 Mathematical model

Now, denote by H the set of all components of an UWSN and for a given pair of elements $a, b \in H$, regard a *binary hyperoperation*, where $a * b$ is, for arbitrary $a, b \in H$, defined by:

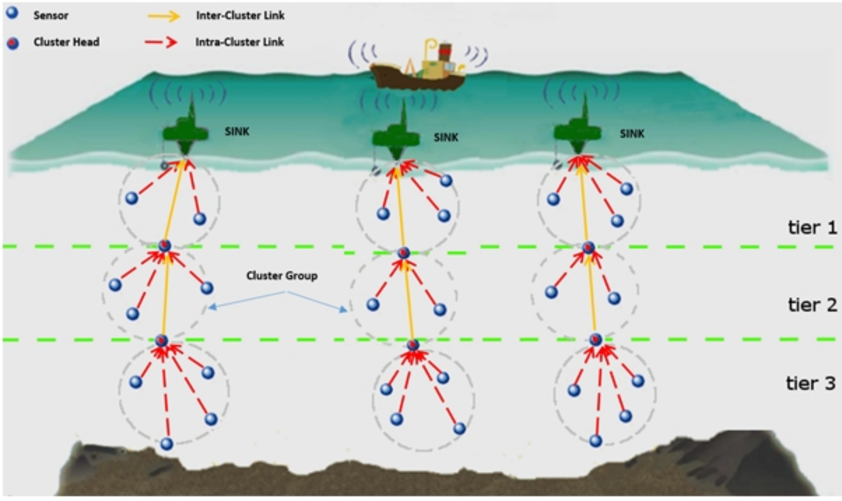

Figure 4.2: Cluster based approach to UWSN data aggregation – idealized deployment; tiers need not be horizontal, we usually regard distance towards sink instead of depth. For some more explanatory figures see e.g. Huang et al. [157]

$$a * b = \begin{cases} \{a,b\} \cup [a \cdot b)_{\leq} & \text{for } (a = CH_i, b = CH_j) \text{ or } a, b \in cl_i \\ \{a,b\} & \text{for } (a \neq CH_i \text{ or } b \neq CH_j) \\ & \text{and } (a \in cl_i, b \in cl_j, i \neq j) \end{cases} \tag{4.31}$$

Above, by $[a \cdot b)_{\leq}$ we mean a set $\{x \in H \mid a \cdot b \leq x\}$, where $a \cdot b$ is a result of a *single-valued binary operation* such that $a \cdot b$ is, for arbitrary $a, b \in H$, defined by

$$a \cdot b = \begin{cases} CH_i & \text{for } a, b \in cl_i \\ CH_k & \text{for } a = CH_i, b = CH_j, i \neq j \\ s & \text{for } a = s \text{ or } b = s \text{ or} \\ & ((a \neq CH_i \text{ or } b \neq CH_j) \text{ and } (a \in cl_i, b \in cl_j, i \neq j)) \end{cases} \tag{4.32}$$

and CH_k is such a cluster head that $CH_i \leq CH_k$, $CH_j \leq CH_k$, where $a \leq b$ is a relation between elements of H such that: (1) $s \leq s$, $s \leq CH_i$ and $CH_i \leq s$ for all clusters cl_i, (2) *within the same cluster* cl_i we have $a_j \leq CH_i$ for all $a_j \in cl_i$ while mutually different plain elements of the cluster are incomparable, (3) *between clusters* for $a = CH_i$, $b = CH_j$ the fact that $a \leq b$ means that the tier of b (measured towards the surface) is smaller than or equal to the tier of a, (4) in all other cases a and b are not related. By CH_k

above we mean a cluster head on the *closest* tier above both CH_i and CH_j. Of course, CH_k always exists yet need not be unique as there may be more cluster heads at this closest tier – in such a case we choose the most suitable one or regard all cluster heads as equal. Notice that, in our definitions, the fact that $CH_i \leq CH_j$ and simultaneously $CH_j \leq CH_i$ does not mean that $CH_i = CH_j$; it only means that CH_i and CH_j are on the same tier. If we are able to choose the most suitable cluster head, the relation "$\leq$" (restricted to $H \setminus \{s\}$) becomes *partial ordering* and we can write $CH_k = \sup\{CH_i, CH_j\}$ (with respect to the relation "$\leq$"). Finally, the element s is an element of H reserved for situations when a and b fail to communicate. It is *artificially added* to our set of elements H – or we can agree that one (given the actual sensor deployment of course carefully chosen) of elements of H will be s. In this way, technically speaking, we should in fact write $H_e = H \cup \{s\}$, where H_e could mean "expanded". Of course, if we choose the option of $s \in H$, then $H_e = H$.

Given these definitions, $a \cdot b$ is the *element in which the data from a and b meet* while $a * b$ is the *path in which the data from both a and b can spread.* The facts that $a \cdot b = s$ or $a * b = \{a, b\}$ both stand for *communication failure.*

Lemma 4.3.1. *$(H, \leq)$ is a quasi-ordered set.*

Proof. Given our assumptions, *reflexivity* is obvious. If $a \leq b$ and $b \leq c$, then either a, b, c are cluster heads or a is a plain element of cl_i and b, c are cluster heads. (Given our assumptions, no other option is possible. Of course, b could be a plain element of cl_j but then $a \leq b$ would have no meaning.) In this case, *transitivity* of "$\leq$" is obvious. $\square$

Suppose that we have arbitrary $a, b \in H$. Since the result of $a \cdot b$ is such an element of H in which the data from a and b meet, it is natural to *suppose* that $a \cdot b = b \cdot a$, i.e. that "$\cdot$" is *commutative.* However, we can suppose this only on condition that we devise such algorithms that $a \cdot b = CH_k = CH_l = b \cdot a$ for arbitrary clusters cl_k, cl_l. Further on suppose that such an algorithm exists, i.e. that $(H, \cdot)$ is a *commutative groupoid.* The following lemma is obvious.

Lemma 4.3.2. *If $(H, \cdot)$ is a commutative groupoid, then $(H, *)$ is a commutative hypergroupoid.*

Proof. Obvious because if $a \cdot b = b \cdot a$, then also $[a \cdot b)_{\leq} = [b \cdot a)_{\leq}$. $\square$

Lemma 4.3.3. *The hypergroupoid $(H, *)$ is an H_v–group, i.e. a weak associative quasi-hypergroup.*

Proof. Obvious, because $\{a, b, c\} \subseteq a*(b*c) \cap (a*b)*c$ and $a*H = \bigcup\limits_{h \in H} a*h = H = H*a$. $\square$

Lemma 4.3.4. *The quasi-ordering "$\leq$" and the operation "$\cdot$" are compatible, i.e. for all $a, b \in H$ such that $a \leq b$ and an arbitrary $c \in H$ there is $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$.*

Proof. The fact that for arbitrary $a, b \in H$ there is $a \leq b$, means that *either* (1) $a \in cl_i$ is plain and $b = CH_i$ *or* (2) $a = CH_i$, $b = CH_j$. In case (1) we, for an arbitrary $c \in H$ have that either $c \in cl_i$ or $c \notin cl_i$. If $c \in cl_i$, we get that $a \cdot c = b \cdot c = CH_i$. If $c \notin cl_i$, then $a \cdot c = b \cdot c = s$. In case (2) we have to consider the following: (a) if $c \notin cl_i$, $c \notin cl_j$, then $a \cdot c = b \cdot c = s$, (b) if $c \in cl_i$, then $a \cdot cl_i = CH_i \cdot c = CH_i$ while $b \cdot c = CH_j \cdot c = s$ and we assume $CH_j \leq s$, (c) if $c \in cl_j$, then $a \cdot c = CH_i \cdot c = s$ while $b \cdot c = CH_j \cdot c = CH_j$ and we assume $s \leq CH_j$. Moreover, in (2) we have to consider special cases of c being a cluster head. If c is a cluster head, say $c = CH_k$, then in (a) we must test whether $CH_i \cdot CH_k \leq CH_j \cdot CH_k$. But since the product of cluster heads is always a cluster head in the closest tier above both cluster heads, the inequality obviously holds. If $c = CH_i$, we in (b) have $a \cdot c = a \cdot a = CH_i$ while $b \cdot c = CH_j \cdot CH_i$ yet since we suppose that $a \leq b$, this is equal to CH_j and $a \cdot c \leq b \cdot c$. If $c = CH_j$, we in (c) we have $a \cdot c = CH_i \cdot CH_j = CH_j$ while $b \cdot c = CH_j \cdot CH_j = CH_j$ and since "$\leq$" is reflexive, the inequality holds. Since we assume commutative "$\cdot$", the other inequality holds as well. Finally, case (3) $a = s$ or $b = s$ is obvious. $\square$

Now, denote $H_{CH} \subseteq H$ the set of cluster heads. This enables us to regard both multipath and clustering based systems as the fact that $H_{CH} = H$ means that every element of H is a cluster head, i.e. the system is a multipath one. In such a case the model simplifies substantially, as there is no need for the special element s and we do not need to distinguish between communication *within* and *between* clusters. The operation "$\cdot$" defined by (4.32) reduces to $a \cdot b = c$ (we still suppose that it is commutative) and, consequently, the hyperoperation (4.31) reduces to $a * b = \{a, b\} \cup [a \cdot b)_{\leq}$, in both cases for all $a, b \in H$.

Lemma 4.3.5. *If we are able to uniquely identify CH_k in (4.32), then $(H_{CH}, \cdot, \leq)$ is a partially ordered semigroup.*

Proof. The assumption of uniquely defined CH_k means that the condition of associativity, i.e. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in H_{CH}$ turns into $\sup\{\sup\{a, b\}, c\} = \sup\{a, \sup\{b, c\}\}$, which is obviously true. Therefore, $(H_{CH}, \cdot)$ is a semigroup. The *compatibility condition* holds because this is the special case of (2a) of Lemma 4.3.4 – without the need of assumptions regarding the special element s. $\square$

Finally, we assign meaning to $x \in [a)_{\leq}$. This means that $a \leq x$, i.e. that *the data from the element* a *reach the element* x. Thus, if x is a sink, than the fact that $x \in [a)_{\leq}$ means that the data from a can be successfully collected. What we want is that, if we denote S the set of all sinks, for all $a \in H$ there exists at least one $x \in S$ such that $x \in [a)_{\leq}$, which means that *data from all elements of our network* H *can be successfully collected.*

Of course, in order to achieve this, it is crucial to have an algorithm for unique determination of CH_k in (4.32). Yet this is a task that can be solved in a number of different ways such as LEACH (used also for terrestrial networks), *cluster head selection algorithm* in DUCS, etc. For details see e.g. Domingo and Prior [122, 157] or Huang et el. [156].

Remark 4.3.6. Notice that it is worth considering to regard the data collection process as applying a transition function of a quasi-automaton. Using a quasi-multiautomaton instead of a quasi-automaton would be an advantage in cases when data are collected by a cluster head simultaneously, which is an issue discussed by Domingo and Prior [122] (see ToA techniques). In this case the next state of a quasi-multiautomaton would be equivalent to reconfiguring the network. In this respect also notice the comment on *inner irreducibility* of a state hypergroup of an automaton on page 63.

Bibliography

[1] I. Adler, Composition rings, *Duke Math. J.*, **29** (1962), 607–623.

[2] M. Al Tahan, B. Davvaz, N-ary hyperstructures associated to the genotypes of F_2–offspring, *Int. J. Biomath.*, **10** (2017), 1750118.

[3] H. Aghabozorgi, M. Jafarpour, B. Davvaz, Enumeration of Varlet and Comer hypergroups, *Electron. J. Combin.*, **18** (2011), #P131.

[4] M. Al Tahan, B. Davvaz, On a special single-power cyclic hypergroup and its automorphisms, *Discrete Math. Algorithm. Appl.*, **8**(4) (2016), 1650059.

[5] S. M. Anvariyeh, B. Davvaz, Strongly transitive geometric spaces associated to hypermodules, *J. Algebra*, **332** (2009), 1340–1359.

[6] S. M. Anvariyeh, S. Momeni, n–ary hypergroups associated with n–ary relations, *Bull. Korean Math. Soc.*, **50**(2) (2013), 507–524.

[7] N. Antampoufis, Hypergroups and H_b–groups in complex numbers, *J. Basic Science*, **4**(1) (2008), 17–25.

[8] N. Antampoufis, Š. Hošková–Mayerová, A brief survey on the two different approaches of fundamental equivalence relations on hyperstructures, *Ratio Mathematica*, **33** (2017), 47–60.

[9] N. Antampoufis, T. Vougiouklis, A. Dramalidis, Geometrical and circle hyperoperations in urban applications, *Ratio Sociologica*, **4**(2) (2011), 53–66.

[10] R. Ameri, H. Hedayati, On k–hyperideals of semihyperrings, *J. Discrete Math. Sci. Cryptogr.*, **10** (2007), 41–54.

[11] R. Ameri, M. Norouzi, Prime and primary hyperideals in Krasner (m, n)–hyperrings, *European J. Combin.*, **34** (2013), 379–390.

[12] E. Artin, Theorie der Zopfe, *Abh. Math. Sem. Univ. Hamburg*, **4** (1925), 47–72.

[13] A. R. Ashrafi, A. Madanshekaf, Generalized action of a hypergroup on a set, *Ital. J. Pure Appl. Math.*, **15**(3) (1998), 127–135.

[14] A. Asokkumar, Derivations in hyperrings and prime hyperrings, *Iran. J. Math. Sci. Inform.*, **8**(1) (2013), 1–13.

[15] M. Ayaz et al., A survey on routing techniques in underwater wireless sensor networks, *J. Netw. Comput. Appl.*, **34**(6) (2011), 1908-1927.

[16] M. Baker, N. Bowler, Matroids over hyperfields, arXiv:1601.01204v5 [math.CO] 19 Apr 2017.

[17] J. H. Bandelt, J. Hedlíková, Median Algebras, *Discrete Math.*, **45** (1983), 1–30.

[18] J. H. Bandelt, H. M. Mulder, Pseudo-median graphs are join spaces, *Discrete Math.*, **109** (1992), 13–26.

[19] Z. Bavel, The source as a tool in automata, *Information and Control*, **18** (1971), 140–155.

[20] M. Bellanger, *Digital Processing of Signals, Theory and Practice*, J. Wiley & Sons. Ltd., London, 1979.

[21] M. Benado, Asupra unei probleme a lui Garett Birkhoff, *Bul. Şt. Secţ Şti. Mat. Fiz*, **6** (1954), 703–739. MR 17, p.341.937

[22] J. Beránek, J. Chvalina, From groups of linear functions to noncommutative transposition hypergroups, *Univ. of South Bohemia Dept. Math. Rep. Series*, **7**(1999), 1–11.

[23] J. Beránek, J. Chvalina, On a certain group of linear second-order differential operators of the Hill-type, *Acta Math. (Nitra)*, **17** (2014).

[24] J. Beránek, J. Chvalina, On a non-commutative transposition hypergroup of linear spaces of smooth functions of the dimension two, *Acta Math. (Nitra)*, **11** (2008), 17–23. (in Czech)

[25] G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. 1973.

[26] G. Birkhoff, J. D. Lipson, Heterogeneous Algebras, *J. Comb. Theory*, **8** (1970), 115–133.

[27] J. Birman, *Braids, links and mapping class groups*, Annals of Math. Studies **82**, Princeton Univ. Press, 1975.

[28] W. R. Bloom, H. Heyer, *Harmonic Analysis of Probability Measures on Hypergroups*, Walter de Gruyter, Berlin – New York, 1995.

[29] T. S. Blyth, *Lattices and Ordered Algebraic Structures*, Springer Verlag, London, 2005.

[30] M. Bolurian, A. Hasankhani, Hyper BCK–algebra, *Ratio Mathematica*, **12** (1997), 97–106.

[31] H. Bordbar, S. S. Ahn, M. M. Zahedi, Y. B. Jun, Semiring structures based on meet and plus ideals in lower BCK–semilattices, *J. Comput. Anal. Appl.*, **23**(5) (2017), 945–954.

[32] O. Borůvka, *Linear Differential Transformations of the Second Order*, English Univ. Press, London, 1971.

[33] O. Borůvka, *Algebraic spaces and their realizations by differential equations I, II.*, Text of "Seminar on differential equations", PřF UJEP, Brno, 1988. (in Czech)

[34] R. A. Borzooei, A. Radfar, S. Niazian, Relationship between hyper MV–algebras and hyperlattices, *Anal. Univ. de Vest, Timişoara, Seria Matem. – Inform.*, **LIV**(2) (2016), 75–94.

[35] R. A. Borzooei, H. R. Varasteh, A. Hasankhani, $\mathcal{F}$–Multiautomata on Join Spaces Induced by Differential Operators, *Applied Mathematics*, **5** (2014), 1386–1391.

[36] S. Buchholz, *A Theory of Neural Computation with Clifford Algebras*, Technical Report Number 0504, Christian-Albrechts-Universität zu Kiel, Institut für Informatik und Praktische Mathematik, 2005.

[37] G. Călugăreanu, V. Leoreanu, Hypergroups associated with lattices, *Ital. J. Pure Appl. Math.*, **9** (2001), 165–173.

[38] I. Chajda, Š. Hošková, A factorization of quasiorder hypergroups, *Comment. Math. Univ. Carolin.*, **45**(4) (2004), 573–581.

[39] C. C. Chang, Algebraic analysis of many valued logics, *Trans. Amer. Math. Soc.*, **88** (1958), 467-490.

[40] T. Changphas, B. Davvaz, Bi-hyperideals and quasi-hyperideals in ordered semihypergroups, *Ital. J. Pure Appl. Math.*, **35** (2015), 493–508.

[41] S. Chaopraknoi, Y. Kemprasit, Linear transformation semigroups admitting the structure of a semihyperring with zero. *Ital. J. Pure Appl. Math.*, **17** (2005), 213–222.

[42] S. Chattopadhyay, A semihypergroup associated with a Γ–semigroup, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi, Ser. Noua – Mat.*, **LVI**(1) (2010), 209–225.

[43] J. Chvalina, Commutative hypergroups in the sense of Marty and ordered sets, in: *General Algebra and Ordered Sets, Proceedings of the International Conference Olomouc*, (1994), 19–30.

[44] J. Chvalina, *Functional Graphs, Quasi-ordered Sets and Commutative Hypergroups*, Masaryk University, Brno, 1995. (in Czech)[9]

[45] J. Chvalina, Infinite multiautomata with phase hypergroups of various operators, in: *Proc. 10th International Congress on Algebraic Hyperstructures and Applications*, University of Defence, Brno, 2009.

[46] J. Chvalina, The Iséki-type characterization of certain regular ordered semigroups, *Czech. Math. J.* **47**(2) (1997), 261–276.

[47] J. Chvalina, Parametrizable relations on hypergroups determined by quasi-orders, *Univ. S. Boh. Dept. Math. Rep. Ser.*, **8** (1999), 1–12.

[9]Book, in which the concept of "Ends lemma" was introduced.

[48] J. Chvalina, Relational product of join spaces determined by quasi-orders, in: *Proceedings of the 6th International Congress on AHA and Appl.*, Democritus University of Thrace Press, Xanthi, 1996.

[49] J. Chvalina, L. Chvalinová, Action of join spaces of continuous functions on the underlying hypergroups of 2-dimensional linear spaces of functions, *APLIMAT – J. Appl. Math.*, **2**(1) (2009), 23–33.

[50] J. Chvalina, L. Chvalinová, Join spaces of linear ordinary differential operators of the second order, *Folia FSN Universitatis Masarykianae Brunensis, Mathematica* **13**, CDDE – Proc. Colloquium on Differential and Difference Equations, Brno, (2002), 77–86.

[51] J. Chvalina, L. Chvalinová, Modelling of join spaces by n–th order linear ordinary differential operators, in: *4th International Conf. APLIMAT.* FME Slovak University of Technology, Bratislava, 2005, 279–284.

[52] J. Chvalina, L. Chvalinová, Multistructures determined by differential rings, *Arch. Math. (Brno)* **36** (2000) (CDDE 2000 Issue), 429–434.

[53] J. Chvalina, L. Chvalinová, State hypergroups of automata, *Acta Math. et Inform. Univ. Ostraviensis*, **4**(1) (1996), 105–120.

[54] J. Chvalina, L. Chvalinová, Transfer between join spaces of certain integral operators and translation operators on rings of a continuous functions, in: *5th International Conf. APLIMAT.* FME Slovak University of Technology, Bratislava, 2006, 607–612.

[55] J. Chvalina, L. Chvalinová, Transposition hypergroups formed by transformation operators on rings of differentiable functions, *Ital. J. Pure Appl. Math.*, **15** (2004), 93–106.

[56] J. Chvalina, L. Chvalinová, J. Moučka, Solvability of certain group of the third-order linear differential operators, in: *XXX Internat. Colloq. Brno, Proceedings*, University of Defence, Brno, 2012. [CD-ROM].

[57] J. Chvalina, L. Chvalinová, M. Novák, On a certain tree of two-dimensional function spaces growing from a linear space of continuous functions, in: *XXVII Internat. Colloq. Brno, Proceedings*, University of Defence, Brno, 2009. [CD-ROM].

[58] J. Chvalina, Š. Hošková-Mayerová, On certain proximities and preorderings on the transposition hypergroups of linear first-order partial differential operators, *An. Şt. Univ. Ovidius Constanţa*, **22**(1) (2014), 85–103.

[59] J. Chvalina, Š. Hošková-Mayerová, A. Dehghan Nezhad, General actions of hypergroups and some applications, *An. Şt. Univ. Ovidius Constanţa*, **21**(1) (2013), 59–82.

[60] J. Chvalina, Š. Křehlík, Normal subhypergroups of hypergroups of ordinary linear second-order differential operators, *South Bohemia Math. Letters*, **20**(1) (2012), 1–9.

[61] J. Chvalina, Š. Křehlík, M. Novák, Cartesian composition and the problem of generalising the MAC condition to quasi-multiautomata. *An. Şt. Univ. Ovidius Constanţa*, **24**(3) (2016), 79–100.

[62] J. Chvalina, Š. Křehlík, M. Novák, Homomorphisms of *EL*–hyperstructures based on a certain classical transformation, *Int. J. of Algebraic Hyperstructures and Its Applications*, **2**(1) (2015), 101–112.

[63] J. Chvalina, J. Moučka, When certain basic hyperoperations on lattices coincide, in: *5. Matematický workshop s mezinárodní účastí*, FAST VUT, Brno, 2006. [CD-ROM].

[64] J. Chvalina, J. Moučka, M. Novák, Actions of centralizer semihypergroups of certain transformation operators on rings of functions of a complex variable, in: *XXV Internat. Colloq. Brno, Proceedings*, University of Defence, Brno, 2007. [CD-ROM].

[65] J. Chvalina, J. Moučka, M. Novák, Binary multistructures of preference relations, in: *ICSC – 6^{th} Internat. conf. on soft computing applied in computer and economic environments*, Kunovice, EPI Kunovice, 2008.

[66] J. Chvalina, J. Moučka, M. Novák, Multialgebras on vector spaces of smooth functions, in: *XXVIII Internat. Colloq. Brno, Proceedings*, University of Defence, Brno, 2010. [CD-ROM].

[67] J. Chvalina, J. Moučka, R. Vémolová, Functorial passage from quasi-automata to multi-automata, in: *XXIV. Internat. Colloquium on the Acquisition Proces Management, Proc. of Contributions*, University of Defence, Brno, 2006.

[68] J. Chvalina, M. Novák, Direct heterogeneous and homogeneous products of infinite collections of multiautomata, in: *XXIII Internat. Colloq. Brno, Proceedings*, University of Defence, Brno, 2005. [CD-ROM].

[69] J. Chvalina, M. Novák, Hyperstructures of preference relations, in: 10^{th} *International Congress of Algebraic Hyperstructures and Applications, Proceedings of AHA 2008*, University of Defence, Brno, 2009, 131–140.

[70] J. Chvalina, M. Novák, Laplace-type transformation of a centralizer semihypergroup of Volterra integral operators with translation kernel, in: *XXIV Internat. Colloq. Brno, Proceedings*, University of Defence, Brno, 2006.

[71] J. Chvalina, M. Novák, Minimal countable extension of a certain cascade with the ordered phase-set and strongly isotone endomorphisms, in: 6. *konference o matematice a fyzice na vysokých školách technických*, Univerzita obrany, Brno, 2009. [CD-ROM].

[72] J. Chvalina, M. Novák, On isomorphism blocks of multiautomata determined by certain quadratic functions, in: 4. *konference o matematice a fyzice na vysokých školách technických*, Univerzita obrany, Brno, 2005. [CD-ROM].

[73] J. Chvalina, M. Novák, On Levine's decomposition of the power phase set of a semihypergroup act, in: *5th International Conf. APLIMAT.* FME Slovak University of Technology, Bratislava, 2006, 89–94.

[74] J. Chvalina, M. Novák, Semicascades with bitopological phase spaces formed by solution spaces of second-order linear homogeneous differential equations, in: 7. *konference o matematice a fyzice na vysokých školách technických*, Univerzita obrany, Brno, 2011. [CD-ROM].

[75] J. Chvalina, M. Novák, Transitivity of semihypergroups of certain transformation operators on function spaces, in: 5. *konference o matematice a fyzice na vysokých školách technických*, Univerzita obrany, Brno, 2007. [CD-ROM].

[76] J. Chvalina, P. Račková, Join spaces of smooth functions and their actions on transposition hypergroups of second order linear differential operators, *APLIMAT – J. Appl. Math.*, **1**(1) (2008), 55–64.

[77] J. Chvalina, B. Smetana, Gropus and hypergroups of artificial neurons, in: 17^{th} *Conference on Applied Mathematics (APLIMAT 2018), Proceedings.* In press.

[78] A. H. Clifford, G. B. Preston, *The Algebraic Theory of Semigroups*, Amer. Math. Soc., 1961.

[79] D. W. Coble, Y. J. Lee, Use of a generalized sigmoid growth function to predict site index for unmanaged loblolly and slash pine plantations in East Texas, *Gen. Tech. Rep. SRS-92.* Asheville, NC: U.S. Department of Agriculture, Forest Service, Southern Research Station, 291-295.

[80] S. D. Comer, Application-oriented hyperstructures, in: *New Frontiers Hypersructures*, T. Vougiouklis ed., Hadronic Press, 1996, 213–218.

[81] S. D. Comer, Combinatorial aspects of relation, *Algebr. Universalis*, **18** (1984), 77–94.

[82] S. D. Comer, Hyperstructures associated with character algebras and color schemes, in: Th. Vougiouklis ed., *New Frontiers in Hyperstructures*, Hadronic Press, Palm Harbor, 1996.

[83] S. D. Comer, Integral relation algebras via pseudogroups, *Notices AMS* **23** (1976), A-659.

[84] S. D. Comer, Some problems on hypergroups, in: *Algebraic Hyperstructures and Applications. Proceedings of the Fourth International*

Congress, Xanthi, Greece, 1990, Th. Vougiouklis ed., World Scientific, 1991, 67–74.

[85] S. D. Comer, Multi–valued algebras and their graphical representations, *Math. Comp. Sci. Dep. the Citadel. Charleston, South Carolina*, 29409, July 1986.

[86] A. Connes, C. Consani, The hyperring of adéle classes, *J. Number Theory*, **131**(2) (2011), 159–194.

[87] P. Corsini, A new connection between hypergroups and fuzzy sets, *Southeast Asian Bull. Math.*, **27** (2003), 221–229.

[88] P. Corsini, Binary relations and hypergroupoids, *Ital. J. Pure Appl. Math.*, **7** (2000), 11–18.

[89] P. Corsini, Hypergraphs and hypergroups, *Algebr. Universalis*, **36** (1996), 548–555.

[90] P. Corsini, Hyperstructures associated with ordered sets, *Bull. Greek Math. Soc.*, **48** (2003), 7–18.

[91] P. Corsini, Join spaces, power sets, fuzzy sets, in: *Proc. Fifth Internat. Congress on A.H.A., 1993, Iaşi, Romania*. Hadronic Press, 1994, 45–52.

[92] P. Corsini, *Prolegomena of Hypergroup Theory*, Aviani Editore, Tricesimo, 1993.

[93] P. Corsini, Sur les semi-hypergroupes, *Atti Soc. Pelor. Sc. Mat. Fis. Nat. Messina*, **26** (1980), 363–372.

[94] P. Corsini, Sur les semi-hypergroupes completes et les groupoids, *Atti Soc. Pelor. Sc. Mat. Fis. Nat. Messina*, **26** (1980), 391–396.

[95] P. Corsini, V. Leoreanu, *Applications of Hyperstructure Theory*, Kluwer Academic Publishers, Dodrecht – Boston – London, 2003.

[96] P. Corsini, T. Vougiouklis, From groupoids to groups through hypergroups, *Rendiconti di Mat.*, Serie **VII**(9) Roma (1989), 173–181.

[97] I. Cristea, Several aspects on the hypergroups associated with n-ary relations, *An. Şt. Univ. Ovidius Constanta*, **17**(3) (2009), 99–110.

[98] I. Cristea, S. Jančić-Rašović, Composition hyperrings, *An. Şt. Univ. Ovidius Constanţa*, **21**(2) (2013), 81–94.

[99] I. Cristea, M. Ştefănescu, Hypergroups and n-ary relations, *European J. Combin.*, **31** (2010), 780–789.

[100] I. Cristea, M. Ştefănescu, C. Angheluţă, About the fundamental relations defined on the hypergroupoids associated with binary relations, *European J. Combin.*, **32** (2011), 72–81.

[101] V. Dašić, Hypernear-rings, in: *Algebraic Hyperstructures and Applications. Proceedings of the Fourth International Congress, Xanthi, Greece, 1990*, Th. Vougiouklis ed., World Scientific, 1991, 75–85.

[102] A. J. Davidson et al., MonoSLAM: Real-Time Single Camera SLAM, *IEEE Transactions on Pattern Analysis and Machine Intelligence*, **29**(6) (2007), 1052–1067.

[103] B. Davvaz, Fuzzy hyperideals in ternary semihyperrings. *Iran. J. Fuzzy Syst.*, **6**(4) (2009), 21–36.

[104] B. Davvaz, Rings derived from semihyper-rings. *Algebras Groups Geom.*, **20** (2003), 245–252.

[105] B. Davvaz, *Semihypergroup Theory*, Academic Press, 2016.

[106] B. Davvaz, P. Corsini, Fuzzy n–ary hypergroups, *J. Intell. Fuzzy Syst.*, **18**(4) (2007), 377–382.

[107] B. Davvaz, P. Corsini, T. Changphas, Relationship between ordered semihypergroups and ordered semigroups by using peudoorder, *European J. Combin.*, **44** (2015), 208–217.

[108] B. Davvaz, I. Cristea, *Fuzzy Algebraic Hyperstructures: An Introduction*, Studies in Fuzziness and Soft Computing, Springer, 2015.

[109] B. Davvaz, A. Dehghan Nezhad, A. Benvidi, Chain reactions as experimental examples of ternary algebraic hyperstructures, *MATCH Commun. Math. Comput. Chem.*, **65** (2011), 491–499.

[110] B. Davvaz, A. Dehghan Nezhad, M. M. Heidari, Inheritance examples of algebraic hyperstructures, *Inf. Sci.*, **224** (2013), 180–187.

[111] B. Davvaz, V. Leoreanu Fotea, *Applications of Hyperring Theory*, International Academic Press, Palm Harbor, 2007.

[112] B. Davvaz, S. Omidi, Basic notions and properties of ordered semihyperrings, *Categories and General Algebraic Structures with Applications*, **4**(1) (2016), 43–62.

[113] B. Davvaz, A. Salasi, A realization of hyperrings, *Comm. Algebra*, **34** (2006), 4389–4400.

[114] B. Davvaz, N. Rakhsh-Khorshid, K. P. Shum, Construction of composition (m, n, k)–hyperrings, *An. Şt. Univ. Ovidius Constanţa*, **24**(1) (2016), 177–188.

[115] B. Davvaz, T. Vougiouklis, n–ary hypergroups, *Iran. J. Sci. Technol. Trans. A-Sci.*, **30**(A2) (2006), 165–174.

[116] A. Dehghan Nezhad, B. Davvaz, An Introduction to the Theory of H_v–Semilattices, *Bull. Malays. Math. Sci. Soc.*, **32**(3) (2009), 375–390.

[117] A. Dehghan Nezhad, B. Davvaz, Universal hyperdynamical systems, *Bull. Korean Math. Soc.*, **47**(3) (2010), 513–526.

[118] A. Dehghan Nezhad, S. M. Moosavi Nejad, M. Nadjafikhah, B. Davvaz, A physical example of algebraic hyperstructures: leptons, *Indian J. Phys.*, **86**(11) (2012), 1027–1032.

[119] M. De Salvo, I K_H–ipergruppi, *Atti Sem. Mat. Fis. Univ. Modena* **31** (1982).

[120] M. De Salvo, G. Lo Faro, A new class of hypergroupoids associated to binary relations, *J. Mult.-Valued Log. Soft. Comput.*, **9** (2003), 361–375.

[121] M. De Salvo, G. Lo Faro, Hypergroups and binary relations, *Multi. Val. Logic*, **8**(5–6) (2002), 645–657.

[122] M. C. Domingo, R. Prior, A distributed clustering scheme for underwater wireless sensor networks, in: *The 18th Annual IEEE International Symposium on Personal, Indoor and Mobile Radio Communications (PIMRC'07)*, 2007, Article number 4394038.

[123] W. Dörfler, The cartesian composition of automata, *Math. System Theory*, **11** (1978), 239–257.

[124] W. Dörfler, The direct product of automata and quasi-automata, in: A. Mazurkiewicz ed., *Mathematical foundations of Computer Science: 5^{th} Symposium, Gdansk, Sept. 6-10, 1976*, Springer Verlag, 1976.

[125] A. Dramalidis, Dual H_v–rings, *Riv. Mat. Pura Appl.*, **17** (1996), 55–62.

[126] A. Dramalidis, On H_v–structures derived from K_H–hypergroups, *International Journal of Modern Sciences and Engineering Technology*, **2**(8) (2015), 22–29.

[127] A. Dramalidis, Visualization of algebraic properties of special H_v–structures, *Ratio Mathematica*, **24** (2013), 41–52.

[128] M. Dresher, O. Ore, Theory of Multigroups, *Amer. J. Math.*, **60** (1938), 705–733.

[129] W. A. Dudek, On some old and new problems in n–ary groups, *Quasigroups and Related Systems*, **8** (2001), 15–36.

[130] W. A. Dudek, Remarks to Glazek's results on n–ary groups, *Discussiones Mathematicae General Algebra and Applications*, **27** (2007), 199–233.

[131] Ch. Dunkl, The measure algebra of a locally compact hypergroup, *Trans. Amer. Mathem. Soc.*, **179** (1973), 331–348.

[132] J. E. Eaton, Associative multiplicative systems, *Amer. J. Math.*, **62**(1) (1940), 222-232.

[133] L. Fuchs, *Partially Ordered Algebraic Systems*, Dover Publications, Mineola, NY, 2011.

[134] C. Flaut, BCK–algebras arising from block codes, *J. Intell. Fuzzy Syst.*, **28**(4) (2015), 1829–1833.

[135] S. H. Ghazavi, S. M. Anvariyeh, EL–hyperstructures associated to n-ary relations, *Soft Comput.*, **21**(19) (2017), 5841–5850.

[136] S. H. Ghazavi, S. M. Anvariyeh and S. Mirvakili, EL^2–hyperstructures derived from (partially) quasi-ordered hyperstructures. *Iran. J. Math. Sci. Inform.*, **10**(2) (2015), 99–114.

[137] S. H. Ghazavi, S. M. Anvariyeh, S. Mirvakili, Ideals in EL–semihypergroups associated to ordered semigroups. *Journal of Algebraic Systems*, **3**(2) (2016), 109–125.

[138] S. Ghorbani, A. Hasankhani, E. Eslami, Hyper MV–algebras, *Set-Valued Mathematics and Applications*, **1** (2008), 205–222.

[139] A. Ginzburg, *Algebraic Theory of Automata*, New York, Academic Press, 1968.

[140] F. Gécseg, I. Peák, *Algebraic Theory of Automata*, Budapest, Akadémia Kiadó, 1972.

[141] A. M. W. Glass, *Partially Ordered Groups*, World Scientific, 1999.

[142] A. C. de Jesus Golzio, *Non-deterministic Matrices: Theory and Applicataions to Algebraic Semantics*, Universidade Estadual de Campinas, 2017. (Ph.D. Thesis)

[143] U. Hebisch, H. J. Weinert, *Semirings – Algebraic Theory and Applications in Computer Science*. World Scientific Publishing, 1998.

[144] H. Hedayati, R. Ameri, Construction of k–Hyperideals by P–Hyperoperations, *Ratio Mathematica* **15** (2005), 75–89.

[145] H. Hedayati, S. Azizpour, B. Davvaz, Prime (semiprime) bi-hyperideals of semihypergroups based on intuitionistic fuzzy points, *U.P.B. Sci. Bull., Series A*, **75**(3) (2013), 45–58.

[146] D. Heidari, B. Davvaz, On ordered hyperstructures, *U.P.B. Sci. Bull. Series A*, **73**(2) (2011), 85–96.

[147] K. Hila, J. Dine, On hyperideals in left almost semihypergroups, *ISRN Algebra* (2011), 8 pp.

[148] D. Hort, J. Chvalina, On certain compatibility types of binary relations on hypergroups determined by quasi-orders, in: *Contributions to General Algebra*, **11**, *Proceedings of the Olomouc Conference and the Summer School 1998*, Verlag Johannes Heyn, Klagenfurt, 1999.

[149] Š. Hošková, *Binary hyperstructures determined by relational and transformation systems. Habilitation Thesis.* Faculty of Sciences, University of Ostrava, 2008.

[150] Š. Hošková, J. Chvalina, A survey of investigations of the Brno research group, in: 10^{th} *International Congress of Algebraic Hyperstructures and Applications, Proceedings of AHA 2008*, University of Defence, Brno, 2009, 71–83.

[151] Š. Hošková, J. Chvalina, Discrete transformation hypergroups and transformation hypergroups with phase tolerance space, *Discret. Math.*, **308**(18) (2008), 4133–4143.

[152] Š. Hošková, J. Chvalina, P. Račková, Hypergroups of integral operators in connections with transformation structures, *AiMT*, **1**(2) (2006), 105–117.

[153] Š. Hošková, J. Chvalina, P. Račková, Noncommutative join spaces of integral operators and related hyperstructures, *AiMT*, **1**(1) (2006), 7–24.

[154] Š. Hošková, J. Chvalina, Transformation Tolerance Hypergroups, *Thai J. Math.*, **4**(1) (2006), 63–72.

[155] Š. Hošková, J. Chvalina, P. Račková, Transposition hypergroups of Fredholm integral operators and related hyperstructures. Part I, *J. Basic Science*, **4**(1) (2008), 43–54.

[156] C. J. Huang et al., A clustering head selection algorith for underwater sensor networks, in: *2008 Second International Conference on Future Generation Communication and Networking (FGCN 2008)*, 2008, Article number 4734050, 21–24.

[157] C. J. Huang et al., A self-healing clustering algorithm for underwater sensor networks, *Cluster Comput.*, **14** (2011), 91–99.

[158] Y. Huang, *BCI–algebra*, Science Press, Beijing, 2006.

[159] K. Iséki, An algebra related with a propositional calculus, *Proc. Japan. Acad.*, **42** (1966), 26–29.

[160] K. Iséki, An introduction to the theory of *BCK*–algebras, *Math. Japon.*, **23** (1978), 1–26.

[161] K. Iwasava, On linearly ordered groups, *J. Math. Soc.* **1**(1) (1948), 1–9.

[162] H. M. Jafarabadi, N. H. Sarmin, M. R. Molaei, Completely simple and regular semi hypergroups, *Bull. Malays. Math. Sci. Soc.*, **35**(2) (2012), 335–343.

[163] H. M. Jafarabadi, N. H. Sarmin, M. R. Molaei, Simple semi hypergroups, *Australian Journal of Basic and Applied Sciences*, **5**(4) (2011), 51–55.

[164] J. Jan, *Digital Signal Filtering, Analysis and Restoration*, IEEE Publishing, London, 2000.

[165] J. Jan, D. Janová, Complex approach to surface reconstruction of microscopic samples from bimodal image stereo data, *Machine Graphic and Vision*, **10** (2001), 261–288.

[166] S. Jančić-Rašović, About the hyperring of polynomials, *Ital. J. Pure Appl. Math.*, **28** (2007), 223–234.

[167] S. Jančić-Rašović, On hyperrings associated with binary relations on semihypergroup, *Ital. J. Pure Appl. Math.*, **30** (2013), 279–288.

[168] J. Jantosciak, Classical geometries as hypergroups, in *Convegno su: Ipergruppi, altre strutture multivoche e loro applicazioni*, P. Corsini Ed., Udine, Italy, 1985, 93–104.

[169] J. Jantosciak, Transposition Hypergroups: Noncommutative Join Spaces, *J. Algebra*, **187** (1997), 97–119.

[170] J. Jantosciak, Ch. G. Massouros, Strong identities and fortification in transposition hypergroups, *J. of. Disc. Math. Sc. & Crypt.*, **6**(2-3) (2003), 169–193.

[171] Y. B. Jun, M. S. Kang, H. S. Kim, New types of hyper MV-deductive systems in hyper MV–algebras, *Mathematical Logic Quarterly*, **56**(4) (2010), 400–405.

[172] Y. B. Jun, S. Z. Song, Codes based on BCK–algebras, *Inf. Sci.*, **181**(22) (2011), 5102–5109.

[173] A. Kalampakas et al., The picture hyperstructure, *Proceedings of 12th International AHA Conference. Xanthi, 2014* (accepted)

[174] A. Kalampakas, S. Spartalis, Path hypergroupoids: Commutativity and graph connectivity, *European J. Combin.*, **44**(Part B) (2015), 257–264.

[175] A. Kalampakas, S. Spartalis, A. Tsigkas, The path hyperoperation, *An. Şt. Univ. Ovidius Constanţa*, **22**(1) (2014), 141–153.

[176] E. Kamke, *Differentialgleichungen Lösungsmethoden und Lösungen: gewöhnliche Differentialgleichungen*, 2. Auflage, Leipzig 1959. Translated into Russian, Nauka, Moscow, 1965.

[177] A. Kehagias, M. Konstantinidou, Lattice ordered join space: an applications-oriented example, *Ital. J. Pure Appl. Math.*, **9** (2000).

[178] L. Konguetsof, T. Vougiouklis, M. Kessoglides, S. Spartalis, On cyclic hypergroups with period, *Acta Univ. Carolinae, Math. Phys.*, **28**(1) (1987), 3–7.

[179] M. Konstantinidou, On the hyperlattices–ordered groupoids, *Bol. Un. Mat. Ital.*, **6**(2-A) (1983).

[180] M. Konstantinidou-Seramifidou, Distributive and complemented hyperlattices, *Math. Balcanica*, **13** (1981), 339–360.

[181] M. Konstantinidou-Seramifidou, Modular hyperlattices, *Math. Balcanica*, **11** (1978), 202–218.

[182] M. Konstantinidou, J. Mittas, An introduction to the theory of hyperlattices, *Math. Balcanica*, **7** (1977), 187–193.

[183] M. Koskas, Groupoids, demi-hypergroupes et hypergroupes, *J. Math. Pures Appl.*, **49**(9) (1970), 155–192.

[184] M. Kovár, A. Chernikava, On the proof of the existence of undominated strategies in normal form games, *Amer. Math. Monthly*, **121**(4) (2014), 332–337.

[185] M. Krasner, A class of hyperrings and hyperfields, *Int. J. Math. Sci.*, **6**(2) (1983), 307–312.

[186] M. Krasner, Approximation des corps values complets de characteristique $p \neq 0$ par ceux de characteristique 0, in: *Colloque d'Algebre Superieure (Bruxelles, Decembre 1956)*, CBRM, Bruxelles, 1957.

[187] Š. Křehlík, M. Novák, From lattices to H_v–matrices, *An. Şt. Univ. Ovidius Constanţa*, **24**(3) (2016), 209–222.

[188] Š. Křehlík, Properties of quasi-multiautomata created by a hypergroup of linear differential operators in the Jacobi form, *South Bohemia Mathematical Letters*, **21**(1) (2013), 38–46. (in Czech)

[189] Š. Křehlík, Relations between hypergroups of linear differential operators in the Jacobi form, in *EEICT, Proc. 20th Conf.*, Brno University of Technology, 2014.

[190] V. Leoreanu, Canonical n–ary hypergroups, *Ital. J. Pure Appl. Math.*, **24** (2008).

[191] V. Leoreanu-Fotea, P. Corsini, Isomorphisms of hypergroups and of n–hypergroups with applications, *Soft. Comput.*, **13** (2009), 985–994.

[192] V. Leoreanu-Fotea, B. Davvaz, Join n–space and lattice, *J. Mult.-Valued Log. Soft Comput.*, **15** (2009), 421–432.

[193] V. Leoreanu-Fotea, B. Davvaz, n–hypergroups and binary relations, *European J. Combin.*, **29** (2008), 1207–1218.

[194] V. Leoreanu-Fotea, B. Davvaz, F. Feng, C. Chiper, Join spaces, soft join spaces and lattices, *An. Şt. Univ. Ovidius Constanţa*, **22**(1) (2014), 155–167.

[195] V. Leoreanu-Fotea, I. G. Rosenberg, Hypergroupoids determined by lattices, *European J. Combin.*, **31** (2010), 925–931.

[196] V. Leoreanu-Fotea, I. G. Rosenberg, Join spaces determined by lattices, *J. Mult.-Valued Log. Soft Comput.*, **16**(1-2) (2009), 421–432.

[197] V. Leoreanu, L. Leoreanu, Hypergroups associated with hypergraphs, *Ital. J. Pure Appl. Math.*, **4** (1998), 119–126.

[198] N. Li et al., A survey on underwater acoustic sensor network routing protocols, *Sensors* **16**(3)(2016), 414, 28pp.

[199] D. S. Malik, J. N. Mordenson, M. K. Sen, The Cartesian composition of fuzzy finite state machines, *Kybernetics*, **24** (1995), 98–110.

[200] J. Martínez et al., Generalizations of lattices via non-deterministic operators, *Discrete Math.*, **295** (2005), 107–141.

[201] F. Marty, Sur une généralisation de la notion de groupe, *IV Congrès des Mathématiciens Scandinaves*, Stockholm (1934), 45–49.[10]

[202] Ch. G. Massouros, *Algebraic Structures and Hyperoperations*, Doctoral Thesis, University of Patras, Patras, 1986.

[203] Ch. G. Massouros, Hypergroups and geometry, *Mem. Academia Romana, Mathematics*, special issue, Seria IV, **XIX** (1996), 185-191 and *Scripta Scientiarum Mathematicarum*, **I** (1997), 143-152

[204] Ch. G. Massouros, Looking at certain hypergroups and their properties, in: *Proc. of the European Conference on Intelligent Technologies 2000, Iaşi 2000*, Advances in Generalized Structures, Approximate Reasonong and Applications, publ. by the cooperation of University of Teramo, Romanian Society for Fuzzy Systems and BMFSA Japan, 2001, 31–43.

[205] Ch. G. Massouros, Methods of constructing hyperfields, *Internat. J. Math. & Math. Sci.*, **8**(4) (1985), 725–728.

[206] Ch. G. Massouros, On path hypercompositions in graphs and automata, *MATEC Web of Conferences*, **41**, 05003 (2016).

[207] Ch. G. Massouros, On the theory of hyperrings and hyperfields, *Algebra i Logika*, **24**(6) (1985), 728–742.

[208] Ch. G. Massouros, Quasicanonical hypergroups, *Algebraic Hyperstructures and Applications. Proceedings of the Fourth International Congress, Xanthi, Greece, 1990*, Th. Vougiouklis ed., World Scientific, 1991, 129–136.

[209] Ch. G. Massouros, G. G. Massouros, Homomorphic relations on hyperringoids and join hyperrings. *Ratio Mathematica*, **13** (1999), 61–70.

[210] Ch. G. Massouros, G. G. Massouros, Identitites in multivalued algebraic structures, in: *CP1281, ICNAAM, Numerical Analysis and Applied Mathematics, International Conference 2010*, T. E. Simos, G. Psihoyios and Ch. Tsitouras ed., American Institute of Physics, 2010, 2065–2068.

[10] The very first paper on hyperstructure theory.

[211] Ch. G. Massouros, G. G. Massouros, On join hyperrings, in: *Proceedings of the 10th International Congress on Algebraic Hyperstructures and Applications, Brno, Czech Republic, 2009*, 203–215.

[212] Ch. G. Massouros, Ch. Tsitouras, Enumeration of hypercompositional structures defined by binary relations, *Ital. J. Pure Appl. Math.*, **28** (2011), 43–54.

[213] G. G. Massouros, Hypercompositional structures in the theory of languages and automata, *An. Şt. Univ. A.I. Çuza Iaşi, Sect. Inform.*, **III** (1994), 65–73.

[214] G. G. Massouros, The hyperringoid, *Multi. Val. Logic*, **3** (1998), 217–234.

[215] G. G. Massouros, The subhypergroups of the fortified join hypergroup, *Ital. J. Pure Appl. Math.*, **2** (1997), 97–119.

[216] G. G. Massouros, Ch. G. Massouros, J. D. Mittas, Fortified join hypergroups, *Ann. Math. Blaise Pascal*, **3**(2) (1996), 155–169.

[217] Ch. G. Massouros, G. G. Massouros, The transposition axiom in hypercompositional structures, *Ratio Mathematica*, **21** (2011), 75–90.

[218] G. G. Massouros, J. D. Mittas, Languages, Automata and Hypercompositional Structures, in: *Algebraic Hyperstructures and Applications. Proceedings of the Fourth International Congress, Xanthi, Greece, 1990*, Th. Vougiouklis ed., World Scientific, 1991, 137–147.

[219] J. Meng, Y. B. Jun, *BCK–algebras*, Kyong Moon Sa Co., Seoul, 1994.

[220] F. P. Miller, A. F. Vandsome, J. McBrewster, *Iterated Binary Operation*, Alphascript Publishing, 2010.

[221] S. Mirvakili, B. Davvaz, Applications of the a^*–relation to Krasner hyperrings, *J. Algebra*, **362** (2012), 145–156.

[222] S. Mirvakili, B. Davvaz, Relations on Krasner (m, n)–hyperrings, *European J. Combin.*, **31** (2010), 790–802.

[223] J. D. Mittas, Hyperanneaux et certaines de leurs propriétés, *C. R. Acad. Sci.*, **A269** (1969), 623–626.

[224] J. D. Mittas, Sur les hyperanneaux et les hypercorps, *Math. Balcanica*, **3** (1973), 368–382.

[225] J. D. Mittas, Hypergroups canoniques, *Math. Balcanica*, **2** (1972), 165–179.

[226] J. D. Mittas, Sur certaines classes de structures hypercompositionnelles, *Proceedings of the Academy of Athens*, **48** (1973), 298–318.

[227] J. D. Mittas, Ch. G. Massouros, Hypergroups defined from a linear space, *Bull. Greek Math. Soc.*, **30** (1989), 64–78.

[228] J. N. Mordenson, D. S. Malik, *Fuzzy Automata and Languages – Theory and Applications*, Chapman & Hall CRC Press, 2002.

[229] T. Nakano, Rings and partly ordered systems, *Math. Zeitschr.*, **99** (1967), 355-376.

[230] A. Nakasis, Recent results in hyperring and hyperfield theory, *Internat. J. of Math. & Math. Sci.*, **11**(2) (1988), 209–220.

[231] F. Neuman, From local to global investigations of linear differential equations of the n–th order. Jahrbuch Überblicke Mathematik, *Math. Surveys (Mannheim)* **17** (1984), 55–80.

[232] F. Neuman, *Global Properties of Linear Ordinary Differential Equations*, Mathematics and its Applications, East European Series **52**, Kluwer Academic Publishers (with Academia Praha), Dordrecht–Boston–London, 1991.

[233] F. Neuman, Global theory of ordinary linear homogeneous differential equations in the real domain, I *Aequ. Math.* **33** (1987), 123-149, II ibid. **34** (1987), 1–22.

[234] F. Neuman, Transformation theory of linear ordinary differential equations – from local to global investigations, *Arch. Math. (Brno)*, **33**(1-2) (1997), 65–74.

[235] F. Neuman, Algebraic aspects of transformations with an application to differential equations, *Nonlinear Anal.* **40** (2000), 505-511.

[236] J. von Neumann, On regular rings, *Proc. Nat. Acad. Sci. USA*, **22** (1936), 707–713.

[237] J. Nieminen, Join space graphs, *J. of Geometry*, **33** (1988), 99–103.

[238] J. Nieminen, Chordal graphs and join spaces, *J. of Geometry*, **34** (1989), 146–151.

[239] M. Nobusawa, On a generalization of the ring theory, *Osaka J. Math*, **1** (1964), 81–89.

[240] M. Novák, n–ary hyperstructures constructed from binary quasi–ordered semigroups, *An. Şt. Univ. Ovidius Constanţa*, **22**(3) (2014), 147–168.

[241] M. Novák, Important elements of EL–hyperstructures, *Aplimat – J. of Appl. Math.*, **4**(II) (2011), 105–113.

[242] M. Novák, On EL–semihypergroups, *European J. Combin.*, **44**(Part B), (2015), 274–286.

[243] M. Novák, Potential of the "Ends lemma" to create ring-like hyperstructures from quasi-ordered (semi)groups, *South Bohemia Mathematical Letters* **17**(1) (2009), 39–50.

[244] M. Novák, Some basic properties of EL–hyperstructures, *European J. Combin.*, **34** (2013), 446–459.

[245] M. Novák, Some properties of hyperstructures of transformation operators $T(\lambda, F, \varphi)$, in: *XXVIII International Colloquium on the Management of Educational Process aimed at current issues in science, education and creative thinking development*, Universita Obrany, Brno, 2010, 94–102.

[246] M. Novák, The notion of subhyperstructure of "Ends lemma"–based hyperstructures, *Aplimat – J. of Appl. Math.*, **3**(II) (2010), 237–247.

[247] M. Novák, I. Cristea, Composition in EL–hyperstructures, *Hacet. J. Math. Stat.* (accepted).

[248] M. Novák, I. Cristea, Š. Křehlík, A class of hyperlattices induced by quasi-ordered semigroups, in: *Proceedings,* 16^{th} *Conference on Applied Mathematics Aplimat 2017*, Vydavatelstvo Spektrum STU, Bratislava, 2017.

[249] M. Novák, Š. Křehlík, EL–hyperstructures revisited, *Soft Comput.* (2017) (in press – online ready).

[250] M. Novák, Š. Křehlík, Several aspects of generalizing one construction of hyperstructures from quasi-ordered semigroups, *Int. J. of Algebraic Hyperstructures and Its Applications*, **2**(1) (2015), 113-124.

[251] M. Novák, K. Ovaliadis, Š. Křehlík, A hyperstructure model of Underwater Wireless Sensor Network (UWSN) design, in: *Proceedings ICNAAM 2017* (accepted).

[252] V. Novák, On representation of ternary structures, *Math. Slovaca*, **45**(5) (1995), 469–480.

[253] V. Novák, M. Novotný, Pseudodimension of relational structures, *Czech. Math. J.*, **49**(124) (1999), 547–560.

[254] M. Novotný, Ternary structures and groupoids, *Czech. Math. J.*, **41**(116) (1991), 90–98.

[255] S. Omidi, B. Davvaz, Basic notions and properties of ordered semihyperrings, *Categ. General Alg. Structures Appl.*, **4**(1) (2016), 43–62.

[256] D. M. Olson, V. K. Ward, A note on multiplicative hyperrings, *Ital. J. Pure and Appl. Math.* **1** (1997), 77–84.

[257] S. Omidi, B. Davvaz, Contribution to study of special kinds of hyperideals in ordered semihyperrings, *J. Taibah Univ. Sci.* (2016) – in press.

[258] K. Ovaliadis, N. Savage, Cluster protocols in underwater sensor networks: a research review, *Journal of Engineering Science and Technology Review*, **7**(3)(2014), 171-175.

[259] H. E. Pickett, Homomorphisms and subalgebras of multialgebras, *Pac. J. Math.*, **21**(2) (1967), 327–342.

[260] A. C. Pipkin, *A Course on Integral Equations*, New York – Berlin, Springer Verlag, 1991.

[261] W. Phanthawimol, Y. Kemprasit, Homomorphisms and epimorhisms of some hypergroups, *Ital. J. Pure Appl. Math*, **27** (2010), 305–312.

[262] N. Polat, On bipartite graphs whose interval space is a closed join space, *J. Geom.*, **108** (2017), 719–741.

[263] W. Prenowitz, Projective geometries as multigroups, *Amer. J. Math.*, **65** (1943), 235–256.

[264] W. Prenowitz, J. Jantosciak, Geometries and join spaces, *J. Reine Angew. Math.*, **257** (1972), 100–128.

[265] W. Prenowitz, J. Jantosciak, *Join geometries*, UTM, Springer-Verlag, New York, 1979.

[266] T. Rault, B. Abdelmadjid, C. Yacine, Energy efficiency in wireless sensor networks: A top-down survey, *Comput. Netw.* **67**(2014), 104-122.

[267] P. Račková, Closed, reflexive, invertible and normal subhypergroups of special hypergroups, *Ratio Mathematica*, **23** (2012), 81–86.

[268] P. Račková, Hypergroups of symmetric matrices, in: 10^{th} *International Congress of Algebraic Hyperstructures and Applications, Proceedings of AHA 2008*, University of Defence, Brno, 2009, 267–272.

[269] P. Račková, Properties of generalized multiautomata, *Proceedings of 12th International AHA Conference. Xanthi, 2014* (accepted)

[270] S. Rasouli, B. Davvaz, Lattices derived from hyperlattices, *Commun. Algebr.*, **38** (2010), 2720–2737.

[271] I. G. Rosenberg, Hypergroups and join spaces determined by relations, *Ital. J. Pure Appl. Math.*, **4** (1998), 93–101.

[272] I. G. Rosenberg, Hypergroups induced by paths of a directed graph, *Ital. J. Pure Appl. Math.*, **4**(1998), 133–142.

[273] K. A. Ross et. al, *Harmonic Analysis and Hypergroups*, Trends in Mathematics, Birkhäuser Basel, 1998.

[274] R. Rota, Congruenze sugli iperanelli moltiplicativi, *Rend. Di Mat., Series VII*, **3**(1) (1983), 17–31.

[275] R. Rota, Strongly distributive multiplicative hyperrings, *J. Geom.* **39** (1990), 130–138.

[276] R. Rota, Sugli iperanelli moltiplicativi, *Rend. Di Mat., Series VII*, **2** (1982), 711–724.

[277] R. Rota, Sula categoria degli iperanelli moltiplicativi, *Rend. Di Mat., Series VII*, **4**(1) (1984), 75–84.

[278] S. Rudeanu, D. Vaida, Multilattices in informatics: introductory examples, *Rev. de Logica*, ISSN 2065-1449, Nr. citiri: 677.

[279] A. E. Saeid et al. Some connections between BCK–algebras and n–ary block codes, *Soft Comput.* (2017) (online ready).

[280] M. K. Sen, On Γ–semigroups, *Algebra and its applications* (New Delhi, 1981), 301–308, Lecture Notes in Pure and Appl. Math. **91**, Dekker, New York, 1984.

[281] M. K. Sen, N. K. Saha, On Γ–semigroups I, *Bull. Calcutta Math. Soc.*, **78**, (1986), 180–186.

[282] J. Šlapal, On exponentiation of universal hyperalgebras, *Algebr. Universalis*, **44** (2000), 187–193.

[283] A. Soltani Lashkenari, B. Davvaz, Complete join hyperlattices, *Indian J. Pure Appl. Math.*, **46**(5) (2015), 633–645.

[284] S. Spartalis, A class of hyperrings, *Riv. Mat. Pura Appl.*, **4** (1989), 55–64.

[285] S. Spartalis, M. Konstantinidou-Serafimidou, A. Taouktsoglou, C-hypergroupoids obtained by special binary relations, *Comput. Math. Appl.*, **59**(8) (2010), 2628–2635.

[286] S. Spartalis, Hypergroupoids obtained from groupoids with binary relations, *Ital. J. Pure Appl. Math.*, **16** (2004), 201–210.

[287] S. Spartalis, C. Mamaloukas, Hyperstructures associated with binary relations, *Comput. Math. Appl.*, **51** (2006), 41–50.

[288] Srivastava, N. et al., Dropout: a simple way to prevent neural networks from overfitting, *J. Machine Learning Res.*, **15** (2014), 1929–1958.

[289] M. Ştefănescu, I. Cristea, On the fuzzy grade of hypergroups, *Fuzzy Sets Syst.*, **159**(9) (2008), 1097–1106.

[290] D. Stratigopoulos, Hyperanneaux non commutatifs: hyperanneaux, hypercorps, hypermodules, hyperspaces vectoriels et leurs propriétés elementaires, *C. R. Acad. Sci.*, **A269** (1969), 489–492.

[291] S. Subrmaniyan, M. Rajasekar, Cartesian composition in bipolar fuzzy finite state machines, *International Journal of Computer Applications (IJCA)*, **92**(11) (2014), 1–7.

[292] J. L. Thiffeault, M. D. Finn, Topology, braids and mixing in fluids, *Philos. Trans. R. Soc. A-Math. Phys. Eng. Sci.*, **364**(1849) (2006), 3251–3266.

[293] K. Trimèche, *Generalized Wavelets and Hypergroups*, Gordon and Breach Science Publishers, Amsterdam, 1997.

[294] Y. Utumi, On hypergroups of group right cosets, *Osaka Math. J.* **1**(1) (1949), 73–80.

[295] J. V. Varlet, Remarks on distributive lattices, *Bull. de l'Acad. Polonnaise des Sciences, Serie des Sciences Math., Astr. et Phys.*, **23** (1975), 1143–1147.

[296] T. Vougiouklis, Attach, remove, absorb and merge elements, in: T. Vougiouklis ed., 8^{th} *Int. Congress on AHA (2002)*, Spanidis Press, 2003.

[297] T. Vougiouklis, Cyclicity in a special class of hypergroups, *Acta Univ. Carolinae. Math. Phys.*, **22**(1) (1981), 3–6.

[298] T. Vougiouklis, Generalization of P–hypergroups, *Rend. Circ. Mat. Palermo*, **36**(II) (1987), 114–121.

[299] T. Vougiouklis, Hypermatrix representations: The problem and open problems, in: 10^{th} *International Congress of Algebraic Hyperstructures and Applications, Proceedings of AHA 2008*, University of Defence, Brno, 2009, 17–31.

[300] T. Vougiouklis, *Hyperstructures and their Representations*, Monographs in Mathematics, Hadronic Press, 1994.

[301] T. Vougiouklis, On H_v–rings and H_v–representations, *Discrete Math.*, **208/209** (1999), 615–620.

[302] T. Vougiouklis, On some representations of hypergroups, *Annales scientifiques de l'Université de Clermont-Ferrand 2, tome 95, série Mathematiques*, **26**(1990), 21–29.

[303] T. Vougiouklis, Representations of hypergroups by generalized permutations, *Algebr. Universalis*, **29** (1992), 172–183.

[304] T. Vougiouklis, The fundamental relation in hyperrings, The general hyperfield, in: *Algebraic Hyperstructures and Applications. Proceedings of the Fourth International Congress, Xanthi, Greece, 1990*, Th. Vougiouklis ed., World Scientific, 1991, 203–211.

[305] T. D. Vu, O. Aycard, N. Appenrodt, Online Localization and Mapping with Moving Object Tracking in Dynamic Outdoor Environments, in: *Proc. 2007 IEEE Intelligent Vehicles Symposium Istanbul, Turkey, June 13–15, 2007*, 190–195.

[306] A. Wahid, K. Dongkyun, Analyzing Routing Protocols for Underwater Wireless Sensor Networks , *IJCNIS* **2**(3) (2010), 253–261.

[307] H. S. Wall, Hypergroups, *Amer. J. Math.*, **59**(1) (1934), 77–98.

[308] M. W. Warner, Semi-group, Group Quotient and Homogeneous Automata, *Information and Control*, cont. as *Inf. Comput*, **47** (1980), 59–66.

[309] Y. Xiao, B. Zhao, Hypersemilattices and their ideals, *J. Shaanxi Normal Univ. Nat. Sci. Ed.*, **33**(1) (2005), 7–10.

[310] M. Yoeli, Generalized cascade decomposition of automata, *J. Assoc. Comp. Machinery*, **12** (1965), 587–592.

[311] J. Zhan, S. Sh. Mousavi, J. Jafarpour, On hyperactions of hypergroups. *U.P.B. Sci. Bull., Series A*, **73**(1) (2011), 117–128.

Index