VYSOKÉ UČENÍ TECHNICKÉ V BRNĚ
FAKULTA STROJNÍHO INŽENÝRSTVÍ

O JISTÝCH ÚLOHÁCH Z KVALITATIVNÍ TEORIE OBYČEJNÝCH A FUNKCIONÁLNÍCH DIFERENCIÁLNÍCH ROVNIC

ON CERTAIN PROBLEMS OF QUALITATIVE THEORY OF ORDINARY AND FUNCTIONAL DIFFERENTIAL EQUATIONS

HABILITAČNÍ PRÁCE
HABILITATION THESIS

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Contents

Basic notation and definitions

(1) $\mathbb{N}$ is the set of natural numbers.

(2) $\mathbb{R}$ is the set of real numbers.

(3) $\mathbb{R}_+$ is the set of nonnegative real numbers.

(4) For any $x \in \mathbb{R}$, we put

$$[x]_- = \frac{1}{2}(|x| - x), \quad [x]_+ = \frac{1}{2}(|x| + x).$$

(5) $\text{ess inf}\{f(t) : t \geq a\} = \sup\{b \in \mathbb{R} \cup \{-\infty\} : f(t) > b \text{ for a.e. } t \geq a\}$

(6) $C([a,b];\mathbb{R})$ is the Banach space of continuous functions $u : [a,b] \to \mathbb{R}$ equipped with the norm

$$||u||_C = \max\{|u(t)| : t \in [a,b]\}.$$

(7) $AC([a,b];\mathbb{R})$ is the set of absolutely continuous functions $u : [a,b] \to \mathbb{R}$.

(8) $AC'_{loc}(I)$ is the set of functions $u : I \to \mathbb{R}$ which are absolutely continuous with their first derivative on every compact subinterval of I.

(9) $L([a,b];\mathbb{R})$ is the Banach space of Lebesgue integrable functions $f : [a,b] \to \mathbb{R}$ equipped with the norm

$$||f||_L = \int_a^b |f(s)|\, ds.$$

(10) $L_{loc}(I)$ is the set of functions $f : I \to \mathbb{R}$ which are Lebesgue integrable on every compact subinterval of I.

(11) $L([a,b];\mathbb{R}_+) = \big\{f \in L([a,b];\mathbb{R}) : f(t) \geq 0 \text{ for a. e. } t \in [a,b]\big\}$.

(12) $\mathcal{L}_{ab}$ is the set of linear bounded operators $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$.

(13) $\mathcal{F}_{ab}$ is the set of linear bounded functionals $h : C([a,b];\mathbb{R}) \to \mathbb{R}$.

(14) $K([a,b] \times A; B)$, where $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$, is the set of functions $f : [a,b] \times A \to B$ satisfying the Carathéodory conditions, i. e.,

(a) $f(\cdot, x) : [a,b] \to B$ is a measurable function for all $x \in A$,

(b) $f(t, \cdot) : A \to B$ is a continuous function for almost all $t \in [a,b]$,

(c) for every $r > 0$ there exists $q_r \in L([a,b];\mathbb{R}_+)$ such that

$$|f(t,x)| \leq q_r(t) \quad \text{for a. e. } t \in [a,b] \text{ and all } x \in A,\ |x| \leq r.$$

Definition 0.1. An operator $\ell \in \mathcal{L}_{ab}$ is said to be *positive* if the relation

$$\ell(u)(t) \geq 0 \quad \text{for a. e. } t \in [a, b]$$

holds for every function $u \in C([a, b]; \mathbb{R})$ satisfying the condition

$$u(t) \geq 0 \quad \text{for } t \in [a, b].$$

The set of positive operators we denote by $\mathcal{P}_{ab}$.

We say that an operator $\ell \in \mathcal{L}_{ab}$ is *negative* if $-\ell \in \mathcal{P}_{ab}$.

Definition 0.2. A functional $h \in \mathcal{F}_{ab}$ is said to be *positive* if the relation

$$h(u) \geq 0$$

holds for every function $u \in C([a, b]; \mathbb{R})$ satisfying the condition

$$u(t) \geq 0 \quad \text{for } t \in [a, b].$$

The set of positive functionals we denote $\mathcal{PF}_{ab}$.

Definition 0.3. An operator $\mathcal{K} : C([a, b]; \mathbb{R}) \to L([a, b]; \mathbb{R})$ is said to be an a–Volterra operator if for every $t_0 \in]a, b]$ and $v_1, v_2 \in C([a, b]; \mathbb{R})$ satisfying

$$v_1(t) = v_2(t) \quad \text{for } t \in [a, t_0],$$

we have

$$\mathcal{K}(v_1)(t) = \mathcal{K}(v_2)(t) \quad \text{for a. e. } t \in [a, t_0].$$

1 Introduction

The habilitation thesis is based on the author's results obtained in the years 2003–2015 and published in the papers [27–29, 39–45]. Three main topics are discussed: asymptotic theory of delay differential equations, boundary value problems for functional differential equations, and the singular Dirichlet problem.

The text consists of introduction, where a brief review of the discussed topics is given, and three chapters, where the author's results are presented.

Asymptotic properties

Second order nonautonomous ordinary differential equations (ODE) and the asymptotic behaviour of their solutions attracted attention in the early 20th century in connection with the astrophysical investigations by R. Emden, where the equation of the type

$$u'' \pm t^{\sigma} u^{n} = 0$$

appeared. Detailed qualitative investigation of that equation, which subsequently became known as the Emden-Fowler equation, was realized by R. Fowler. The interest in the study of the asymptotic properties of second order nonlinear equations essentialy enhanced after the appearance of the well-known monograph [2] by R. Bellman, where all main results are stated dealing with the Emden-Fowler equations. The current state of this theory is presented in the monograph [15].

On the other hand, at the beginning of the 20th century the interest in studying so-called differential equations with delay arguments (DDE) grew, especially in connection with their extensive applications in mechanics, physics, biology, medicine, and economy. The main reason is that many mathematical models cannot be really described by ordinary differential equations. Indeed, the evolution of the process depends not only on the current value of an unknown function but also on its past or future.

In particular, we mention two typical branches, where the DDEs are widely used. DDE models naturally appear in the control processes. Almost every system including a feedback control involves time delays. This happens because some (finite) time is required to "transport" the information and then react to it (see, e.g., model of gantry crane in Section 2.1). Aftereffects in biology has an important influence on biological systems. These are usually related to such long processes as birth, growth, and death. Therefore, the evolution of these processes depends in an essential way on the whole previous history, and can be modelled by DDEs succesfully (see for example the models of population dynamics in Section 2.2).

The bases of the qualitative theory of the equations with delayed arguments and so-called integro-differential equations were put in the works of A. Myshkis and R. Bellman (see, e.g., [2, 22]) in the second half of the 20th century.

Boundary value problem for functional differential equations

The foundation of the theory of boundary value problems (BVP) for functional differential equations (FDE) was laid in 70's of the 20th century (see, e.g., [1, 11, 22, 48]). This theory has been intensively developed in the last 50 years. During this period, the particular types of FDEs were studied, e.g., equations with delay arguments and equations with Volterra's right-hand sides. Some special types of boundary conditions were also considered, e.g., two-point or periodic type conditions (see $[1, 11, 20]$). However, a quite wide class of BVPs for FDEs was not sufficiently investigated till now. The reasons are obvious: the right-hand side of FDE contains operators, which are in general nonlocal, and therefore, the investigation of FDEs is more complicated than the study of ordinary differential equations (ODE). Other difficulties arise, when boundary conditions are also nonlocal. Such kind of problems can be represented by the following BVP

$$u'(t) = \int_a^b K(t,s)u(s)\,ds + q(t); \quad u(a) = \int_a^b \sigma(s)u(s)\,ds,$$

where $K : [a,b] \times [a,b] \to \mathbb{R}$ and $q, \sigma : [a,b] \to \mathbb{R}$ are suitable functions.

The analysis of simple FDEs shows that unlike ODEs, a theorem on differential inequalities is not valid, in general. It seems that the mentioned "pathological" property is the main reason why the most fruitful technique for ODEs cannot be used for the investigation of FDEs.

Singular Dirichlet problem

Boundary value problems for singular second order ordinary differential equations frequently arise in applications. It is sufficient to mention, e.g., the Bessel equation or the hypergeometric equation. Nowadays, there is a quite complete theory of singular boundary value problems for ordinary differential equations (see, e.g., [4]). In this theory, it is usually assumed that the right-hand side of the equation is integrable with a "linear weight". However, many interesting problems do not satisfy this assumption (for example, the Bessel equation). It is therefore desirable to extend the theory to cover such cases.

The first step in this effort, of course, concerns the linear part of the theory, which includes the Fredholm theory, well-posedness, and eigenvalue problems. These topics are also studied in the given order in the case of nonsingular problems. For example, the treatment of the eigenvalue problem in the regular case is based on the Fredholm's alternative and the continuous dependence on parameters.

In Section 5, we present our results concerning the Fredholm theory and well-posedness of the singular Dirichlet problem. These results could be useful, in particular, for the study of eigenvalue problems and nonlinear singular problems, which are the topics of our further research.

2 Motivation

Functional differential equations, boundary value problems for FDEs as well as ordinary differential equations with singularities arise in many applications in biological models, engineering processes, mechanics, technical problems, medicines, chemistry, economy, etc. (see, e.g., [7, 22, 49]). We introduce, for instance, the following three models as a motivation for the study of qualitative properties of ODEs and FDEs.

2.1 Model of a gantry crane

Gantry cranes are used for transportation of objects within factories, railyards, shipyards, ports, etc. We show a simple one-dimensional model (see Figures 2.1 and 2.2) which was introduced in [7]. In this system, almost all controlling motions are done automatically with some anti-sway control technique. Gantry cranes can transport heavy objects, which weight several tons. Moreover, cable length can be over ten meters. So it is necessary for the crane motion to be smooth, otherwise a subject may start sway and the operator can lose control of the payload. We assume that the crane

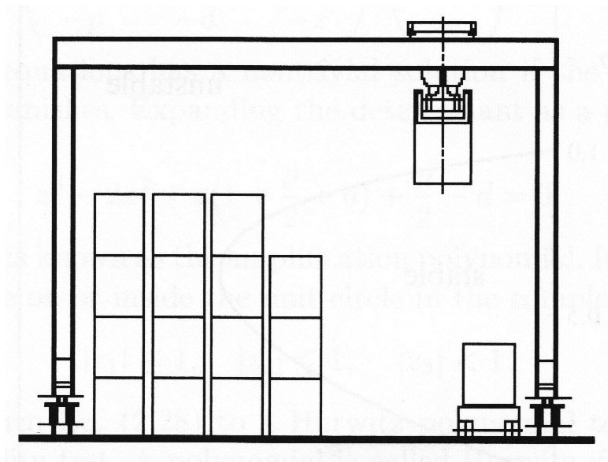


Figure 2.1: Schema of gantry cranes

rides on the frictionless rails, the payload rotates around a frictionless pivot P and the cable is nonelastic. By using the second Newton's law, one can derive from Figure 2.2 the following equations, which describe the motions of the crane and payload

$$\begin{aligned} &Mu'' + m(u'' + l\theta'') = F, \\ &m(u'' + l\theta'')l\cos\theta + mgl\sin\theta = 0. \end{aligned} \tag{2.1}$$

Here, M and m are masses of the trolley and payload, F denotes the force applied to the motor of the trolley and θ is the angle of deviation. By elimination of the function u in (2.1) we obtain

$$\theta'' + \operatorname{tg}\theta + \frac{F(s)}{(M+m)g} = 0, \tag{2.2}$$

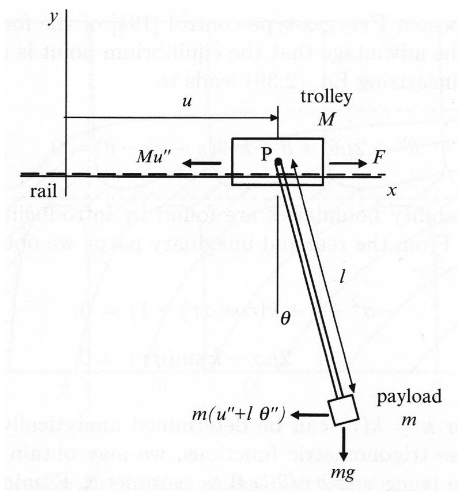


Figure 2.2: Pendulum model for container crane

where the prime denotes the derivative with respect to the dimensionless time $s = \omega t$ and ω is the payload frequency introduced by $\omega = \sqrt{\frac{(M+m)g}{Ml}}$. By using the so-called Pyrygas-type control $\frac{F(s)}{(M+m)g} = k(\theta(s-T) - \theta(s))$, where k is a real parameter and $T > 0$ is a constant delay (see [47]), we get from (2.2)

$$\theta''(s) + \operatorname{tg}\theta(s) + k(\theta(s-T) - \theta(s)) = 0. \tag{2.3}$$

For small value of θ, the equation (2.3) can be linearized and thus, we arrived at the *linear second order differential equation with a constant delay*

$$\theta''(s) + (k-1)\theta(s) + k\theta(s-T) = 0.$$

In Chapter 3, we investigate oscillations of a two-term linear delay differential equation with a non-constant coefficient and a non-constant delay.

2.2 Population dynamics

Functional differential equations appear in mathematical models of many biological processes because, for example, population dynamic is related to long processes as birth, growth, and death or food supply, etc. It is the reason why the evolution of population systems depends on the previous history and can be modelled by FDEs successfully.

We start with a basic population model of single species without "delay". It is assumed that the per-capita growth rate F depends on the size of the population, i.e.,

$$N' = NF(N), \tag{2.4}$$

where $N(t)$ denotes the size (density) of the population at the time t. One of the first and the most known example of the function F is

$$F(N) = r\left(1 - \frac{N}{K}\right), \tag{2.5}$$

where the real constant $r > 0$ is a specific coefficient of the growth and $K > 0$ is a measure of the environment carrying capacity. The model (2.4) with the function F defined by (2.5) was introduced by Verhulst in 1838 and is commonly known as the *logistic equation*.

In the above-mentioned model we assume that the growth rate F at time t depends on the population size (density) N at the same time t. But there are models of population dynamics, where changes in the population size (density) do not correspond to the growth rate instantly. Consequently, using DDEs is much more suitable than using ODEs in these cases. Models with a "time delay" are, for example, those which incorporate gestation and maturation of populations, differences in resource consumption with respect to the age structure, dependence on a food supply, migration and diffusion of populations, etc.

There are two basic concepts how a time delay is represented in biological models. The first one lies in assuming that the growth rate is a function of the population size (density) $N(\tau(t))$ at previous time $\tau(t) \leq t$. These models can be introduced as follows

$$N'(t) = R(N(\tau(t))) - D(N(t)). \tag{2.6}$$

Here, the function R denotes the birth rate and D denotes the death rate of population size. For easier investigation of properties of population models, a so-called "constant delay" is usually used. Consequently, if we put $\tau(t) := t - T$, where $T > 0$ is a real constant, then we obtain from (2.6) the equation

$$N'(t) = R(N(t-T)) - D(N(t)),$$

where T is the time that members of population need to mature (i.e., to have an ability of reproduction) and thus, $N(t-T)$ is the number of adult members. Now one can see that the function of birth rate depends only on the size (density) of adult population. If the death rate of population is zero and the growth rate is given as in formula (2.5), we get the *logistic equation* with a constant delay

$$N'(t) = N(t)r\left(1 - \frac{N(t-T)}{K}\right).$$

The second concept in biological models lies in assuming that a time delay is distributed over the time. Let p be a nonnegative function such that $\int_0^{+\infty} p(s)ds = 1$ and $p(s)\Delta s$ is an approximation of the probability that the delay $\tau(s)$ is between s and $s + \Delta s$. Then we obtain the integro-differential equation

$$N'(t) = N(t)\int_0^{+\infty} F(N(\tau(s)))p(s)ds.$$

Finally, we mention a particular population model, which is commonly known as the "Harvesting of a single population". This model is widely studied not only to investigate biological systems with external influence on population but it also has applications in ecological and economical processes. It is important to develop such strategy for harvesting any renewable resources (fish, animals, plants) in order to maximize the yield, but the species do not die out. We consider the population model (2.6) with some harvesting function H, namely, the equation

$$N'(t) = R(N(\tau(t))) - D(N(t)) - H(t)N(t), \tag{2.7}$$

where function H describes the harvesting rate per-capita and include a harvesting strategy, costs, effort, etc. Together with the equation (2.7) we consider the boundary condition

$$N(a) = N(b), \tag{2.8}$$

which can be interpreted as a periodic behaviour of the population size (density) in the time period $[a, b]$ under a "harvesting pressure". The task is to regulate the harvesting in order to guarantee that the population does not die, respectively, that the population size at the time $t = b$ returns to the beginning value $N(a)$. It means to find the conditions on the function H, which guarantee that the boundary value problem (2.7), (2.8) is solvable. This BVP as well as the above-mentioned DDEs are particular cases of boundary value problems and functional differential equations studied in Chapter 4.

2.3 Model of moving of dislocations in crystals

Most of the technologically important materials are crystals, where atoms are arranged in a periodic lattice of a defined symmetry (cubic, hexagonal, orthorhombic, etc.). Due to the finite rate of solidification, the atoms do not have sufficient time to find their perfect lattice positions which results in the formation of defects. There is a wide variety of such defects but the most important ones from the point of view of mechanical properties are line defects, the so-called *dislocations*. Each dislocation is characterized by so-called Burgers vector $\vec{b}$ and the local orientation of the dislocation specified by the tangential vector $\vec{\tau}$. The Burgers vector is fixed for the whole dislocation but the tangential vector changes from place to place. We distinguish two basic types of dislocation segments: *edge* segment ($\vec{b} \perp \vec{\tau}$), *screw* segment ($\vec{b} \parallel \vec{\tau}$). If none of these conditions is satisfied, we characterize the segment as *mixed*. The motion of dislocations is thermally activated – they move due to the applied load and this motion is aided by thermal fluctuations. The higher the applied load, the lower the thermal energy is needed for its motion and vice versa. We are interested in screw dislocations (see Figure 2.3 and Figure 2.4a).

The task is to calculate the activated shape of the dislocation which minimizes the activation enthalpy that has to be supplied by the thermal fluctuations. The dislocation first moves by the applied stress alone as a straight line from $x = 0$ to $x = x_0$ (see Figure 2.4b), where the latter will be determined later. From this straight shape, the dislocation vibrates due to finite thermal energy until it reaches the activated shape $x = x(z)$ for which the dislocation needs no more energy to move through the lattice (see Figure 2.4c). The task is to calculate this shape, which corresponds to a stationary state of the activation enthalpy.

The activated shape of the dislocation can be mathematically described as a non-constant solution of the boundary value problem

$$x''(z) = \frac{E_p'(x(z))}{E_p(x(z))}\big(1 + x'(z)^2\big) - \frac{\sigma b}{E_p(x(z))}\big(1 + x'(z)^2\big)^{\frac{3}{2}}, \quad z \in]-\infty, +\infty[\,, \tag{2.9}$$

$$\lim_{z\to-\infty} x(z) = x_0, \qquad \lim_{z\to+\infty} x(z) = x_0. \tag{2.10}$$

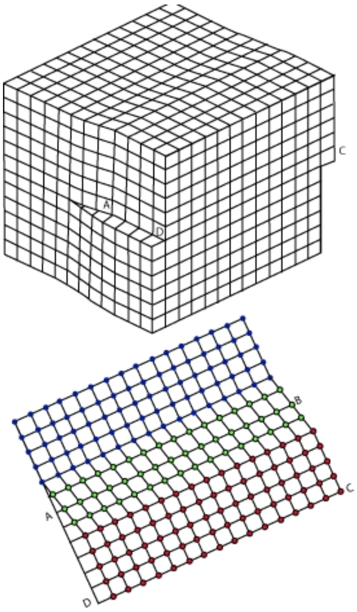


Figure 2.3: Screw dislocation

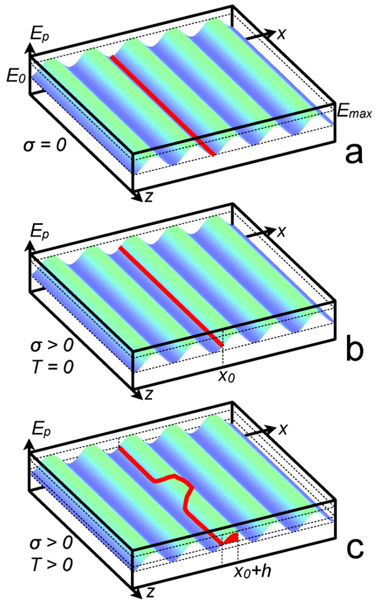


Figure 2.4: Dislocation process

Here, σ is the shear stress, b is the magnitude of the Burgers vector, and E_p is the so-called Peierls barrier representing a lattice friction that acts against moving of the dislocation (see Figure 2.5).

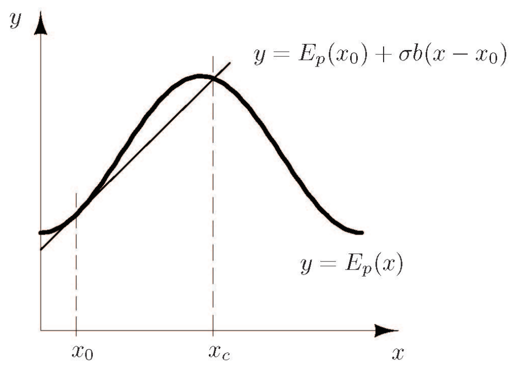


Figure 2.5: Peierls barrier, $E_p'(y_0) = \sigma b$.

One of possibilities how to find a solution of boundary value problem (2.9), (2.10) is to transform it into a suitable problem given on a finite interval and to use available software packages. Introducing the transformation $z = \frac{t}{1-t^2}$, one can show that the boundary value problem (2.9), (2.10) is equivalent to the problem

$$\begin{aligned} u''(t) = {} & \frac{1}{(1-t)(1+t)} \frac{2t(t^2+3)}{1+t^2} u'(t) \\ & + \frac{(1+t^2)^2}{(1-t)^4(1+t)^4} f\Big(u(t), \frac{(1-t^2)^2}{1+t^2} u'(t)\Big), \quad t \in]-1,1[\,, \end{aligned} \tag{2.11}$$

$$\lim_{t\to -1+} u(t) = x_0, \qquad \lim_{t\to 1-} u(t) = x_0,$$

where $f(v_1, v_2) := \frac{1+v_2^2}{E_p(v_1)} \big(E_p'(v_1) - \sigma b\sqrt{1+v_2^2}\big)$. Observe that unlike (2.9), the equation (2.11) contains time singularities for both $t = -1$ and $t = 1$. A standard method how to investigate nonlinear problems is to compare them to suitable linear problems. Precisely, a linear problem with singularities of the same kind as in (2.11) is studied in Chapter 5.

3 Asymptotic properties

3.1 Introduction

In this chapter, we present our contribution in the oscillatory theory of ordinary and functional differential equations obtained in [43–45]. First of all, in Section 3.2, we present Hille-Nehari's type results for DDEs. This type of oscillation criteria concerns the case when the delay is "small enough". Hence, it is natural to expect that the properties of solutions of DDE are close to the properties of solutions of ODE. However, if the delay is "large enough", then qualitative properties of solutions of DDE are not necessarily close to the properties of solutions ODE. Oscillation criteria, specific for DDEs, were suggested for the first time by A. Myshkis (see [37]). In Section 3.3, we present our contribution along this line. Finally, in Section 3.4, we deal with so-called half-linear equations. It is worth mentioning that the criteria stated there generalize results presented in the book [6] by O. Došlý and P. Řehák.

3.2 Hille-Nehari's type criteria for DDE

On the half-line $[0, +\infty[$, we consider the second-order linear delay differential equation

$$u''(t) + p(t)u(\tau(t)) = 0, \tag{3.1}$$

where $p\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a locally Lebesgue integrable function and $\tau\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a measurable function such that

$$\tau(t) \leq t \quad \text{for a. e. } t \geq 0 \tag{3.2}$$

and

$$\lim_{t\to+\infty} \operatorname{ess\,inf}\{\tau(s) : s \geq t\} = +\infty. \tag{3.3}$$

Solutions of equation (3.1) can be defined in various ways. Since we are interested in properties of solutions in a neighbourhood of $+\infty$, we introduce the following commonly used definition.

Definition 3.1. Let $t_0 \in \mathbb{R}_+$ and $a_0 = \operatorname{ess\,inf}\{\tau(t) : t \geq t_0\}$. A continuous function $u\colon [a_0, +\infty[\to \mathbb{R}$ is said to be a *solution of equation* (3.1) *on the interval* $[t_0, +\infty[$ if it is absolutely continuous together with its first derivative on every compact interval contained in $[t_0, +\infty[$ and satisfies equality (3.1) almost everywhere in $[t_0, +\infty[$.

Although equation (3.1) is linear, the presence of the argument deviation τ in this equation causes many peculiar properties, which do not appear in the case of ordinary differential equations. In particular, it may happen that a nontrivial solution of equation (3.1) is identically equal to zero in some neighbourhood of $+\infty$. Indeed, let $t^* \in]3\pi/2, 2\pi[$ be such that

$$\frac{\sin t^*}{(t^* - 3\pi)^2} = -k, \quad \text{where} \quad k = \max\left\{-\frac{\sin t}{(t-3\pi)^2} : t \in [3\pi/2, 2\pi]\right\},$$

and

$$p(t) = \begin{cases} 1 & \text{for } t \in [0, t^*[\cup]3\pi, +\infty[\,, \\ 2k & \text{for } t \in [t^*, 3\pi], \end{cases} \qquad \tau(t) = \begin{cases} t & \text{for } t \in [0, t^*[\cup]3\pi, +\infty[\,, \\ \pi/2 & \text{for } t \in [t^*, 3\pi]. \end{cases}$$

Then

$$u(t) = \begin{cases} -\sin t & \text{for } t \in [0, t^*[\,, \\ k(t - 3\pi)^2 & \text{for } t \in [t^*, 3\pi[\,, \\ 0 & \text{for } t \in [3\pi, +\infty[\end{cases}$$

is a nontrivial solution of equation (3.1) on $\mathbb{R}_+$, which is equal to zero on the interval $[3\pi, +\infty[$. To exclude from our consideration such kind of solutions, we introduce the following definition.

Definition 3.2. A solution u of equation (3.1) on the interval $[t_0, +\infty[$ is called *proper* if the inequality $\sup\big\{|u(s)| : s \geq t\big\} > 0$ holds for $t \geq t_0$.

Now we are in a position to introduce definitions of oscillatory and non-oscillatory solutions of equation (3.1).

Definition 3.3. A proper solution u of equation (3.1) is said to be *oscillatory* if it has a sequence of zeros tending to infinity, and *non-oscillatory* otherwise.

3.2.1 Main results

Oscillation criteria presented in this section guarantee that every proper solution of equation (3.1) is oscillatory. The main results are proved by using lemmas on a priori estimates of non-oscillatory solutions (see Section 3.2.2). To do this, having a proper non-oscillatory solution u of equation (3.1), we need to find a suitable a priori lower bound of the quantity $u(\tau(t))/u(t)$, which is equal to 1 in the case of ordinary differential equations. It is not difficult to verify that

$$\frac{\tau(t)}{t} \leq \frac{u(\tau(t))}{u(t)} \quad \text{for } t \text{ large enough.}$$

However, we succeeded to find a more precise estimate (see Lemma 3.22 below) which allows one to establish more sophisticated results.

We first present a rather simple result which, for ordinary differential equations, follows for example from [13, Theorem 2].

Proposition 3.4 ([44, Prop. 2.1]). *Let*

$$\int_0^{+\infty} sp(s)ds < +\infty.$$

Then equation (3.1) *has a proper non-oscillatory solution.*

Remark 3.5. It follows from the proof of Proposition 3.4 that the assertion of this proposition remains true without assumption (3.2), i. e., equation (3.1) may not be delayed.

Recall that we are interested in oscillation criteria for equation (3.1), i. e., conditions guaranteeing that every proper solution of equation (3.1) is oscillatory. Therefore, in view of Proposition 3.4, we assume in what follows that

$$\int_0^{+\infty} sp(s)ds = +\infty. \tag{3.4}$$

Put

$$G_* = \liminf_{t\to+\infty} \frac{1}{t}\int_0^t s\tau(s)p(s)ds, \qquad G^* = \limsup_{t\to+\infty} \frac{1}{t}\int_0^t s\tau(s)p(s)ds. \tag{3.5}$$

The following statement has been established in [43] in the case, where the function τ is continuous. In [44], the result mentioned is proved in a more general case.

Proposition 3.6 ([44, Prop. 2.3]). *Let condition* (3.4) *hold and*

$$G^* > 1.$$

Then every proper solution of equation (3.1) *is oscillatory.*

In view of Proposition 3.6, it is natural to suppose in the sequel that

$$G_* \le 1. \tag{3.6}$$

Theorem 3.7 ([44, Thm. 2.4]). *Let conditions* (3.4) *and* (3.6) *be fulfilled and let there exist* $\lambda < 1$ *and* $\varepsilon \in [0,1[$ *such that*

$$\int_0^{+\infty} s^\lambda \left(\frac{\tau(s)}{s}\right)^{1-\varepsilon G_*} p(s)ds = +\infty. \tag{3.7}$$

Then every proper solution of equation (3.1) *is oscillatory.*

Remark 3.8. Observe that for any $\varepsilon \in]0,1[$, we have

$$\int_0^{+\infty} s^\lambda \left(\frac{\tau(s)}{s}\right)^{1-\varepsilon G_*} p(s)ds \le \int_0^{+\infty} s^\lambda \left(\frac{\tau(s)}{s}\right)^{1-G_*} p(s)ds.$$

Under some additional assumption imposed on the argument deviation τ, assumption (3.7) in the previous theorem can be replaced by the more convenient assumption

$$\int_0^{+\infty} s^\lambda \left(\frac{\tau(s)}{s}\right)^{1-G_*} p(s)ds = +\infty. \tag{3.8}$$

More precisely, if we assume in Theorem 3.7 that there exist numbers $\alpha > 0$ and $t_0 \ge 0$ such that

$$\frac{\tau(t)}{t} \ge \alpha \quad \text{for a. e. } t \ge t_0, \tag{3.9}$$

then assumption (3.7) is, in fact, equivalent to (3.8). Indeed, for any $\varepsilon \in]0,1[$, the inequality

$$\int_{t_0}^{t} s^{\lambda} \left(\frac{\tau(s)}{s}\right)^{1-\varepsilon G_*} p(s)ds \geq \alpha^{(1-\varepsilon)G_*} \int_{t_0}^{t} s^{\lambda} \left(\frac{\tau(s)}{s}\right)^{1-G_*} p(s)ds \quad \text{for } t \geq t_0$$

holds and thus, equality (3.8) yields the validity of assumption (3.7).

Similarly, we can put $\varepsilon = 1$ in all theorems stated in this section provided that additional assumption (3.9) is satisfied.

Now we provide a criterion which generalizes a result of E. Müller-Pfeiffer proved for ordinary differential equations in [36].

THEOREM 3.9 ([44, Thm. 2.6]). *Let conditions* (3.4) *and* (3.6) *hold and let there exist* $\varepsilon \in [0,1[$ *such that*

$$\limsup_{t\to+\infty} \frac{1}{\ln t} \int_{0}^{t} s \left(\frac{\tau(s)}{s}\right)^{1-\varepsilon G_*} p(s)ds > \frac{1}{4}. \tag{3.10}$$

Then every proper solution of equation (3.1) *is oscillatory.*

In view of Theorem 3.7, we can assume in the sequel that

$$\int_{0}^{+\infty} s^{\lambda} \left(\frac{\tau(s)}{s}\right)^{1-\varepsilon G_*} p(s)ds < +\infty \quad \text{for all } \lambda < 1,\ \varepsilon \in [0,1[. \tag{3.11}$$

It allows us to define, for any $\lambda < 1$ and $\varepsilon \in [0,1[$, the function

$$Q(t;\lambda,\varepsilon) := t^{1-\lambda} \int_{t}^{+\infty} s^{\lambda} \left(\frac{\tau(s)}{s}\right)^{1-\varepsilon G_*} p(s)ds \quad \text{for } t > 0.$$

Moreover, for any $\mu > 1$ and $\varepsilon \in [0,1[$, we put

$$H(t;\mu,\varepsilon) := \frac{1}{t^{\mu-1}} \int_{0}^{t} s^{\mu} \left(\frac{\tau(s)}{s}\right)^{1-\varepsilon G_*} p(s)ds \quad \text{for } t > 0.$$

By using the lower and upper limits

$$\begin{aligned} Q_*(\lambda,\varepsilon) &:= \liminf_{t\to+\infty} Q(t;\lambda,\varepsilon), \qquad Q^*(\lambda,\varepsilon) := \limsup_{t\to+\infty} Q(t;\lambda,\varepsilon), \\ H_*(\mu,\varepsilon) &:= \liminf_{t\to+\infty} H(t;\mu,\varepsilon), \qquad H^*(\mu,\varepsilon) := \limsup_{t\to+\infty} H(t;\mu,\varepsilon), \end{aligned} \tag{3.12}$$

we establish Hille-Nehari's type oscillation criteria, which coincide with the well-known results in the case of ordinary differential equations (see, e.g., [13,16,25,38,46]).

THEOREM 3.10 ([44, Thm. 2.7]). *Let conditions* (3.4) *and* (3.6) *be fulfilled and let there exist* $\lambda < 1$, $\mu > 1$, *and* $\varepsilon \in [0,1[$ *such that*

$$\limsup_{t\to+\infty} \Big(Q(t;\lambda,\varepsilon) + H(t;\mu,\varepsilon)\Big) > \frac{\lambda^2}{4(1-\lambda)} + \frac{\mu^2}{4(\mu-1)}. \tag{3.13}$$

Then every proper solution of equation (3.1) *is oscillatory.*

As corollaries of Theorem 3.10 (with $\mu = 2$ and $\lambda = 0$, respectively) we obtain the following statements, which coincide with the Nehari's classical results (see [38]) in the case of ordinary differential equations.

Corollary 3.11 ([44, Cor. 2.8]). *Let conditions* (3.4) *and* (3.6) *be fulfilled and let there exist* $\lambda < 1$ *and* $\varepsilon \in [0, 1[$ *such that*

$$Q^*(\lambda, \varepsilon) > \frac{(2-\lambda)^2}{4(1-\lambda)} \ .$$

Then every proper solution of equation (3.1) *is oscillatory.*

Corollary 3.12 ([44, Cor. 2.9]). *Let conditions* (3.4) *and* (3.6) *be fulfilled and let there exist* $\mu > 1$ *and* $\varepsilon \in [0, 1[$ *such that*

$$H^*(\mu, \varepsilon) > \frac{\mu^2}{4(\mu-1)} \ .$$

Then every proper solution of equation (3.1) *is oscillatory.*

The next theorem deals with the lower limit of the sum on the left-hand side of inequality (3.13) and thus, it complements Theorem 3.10 in a certain sense.

Theorem 3.13 ([44, Thm. 2.10]). *Let conditions* (3.4) *and* (3.6) *be fulfilled and let there exist* $\lambda < 1$*,* $\mu > 1$*, and* $\varepsilon \in [0, 1[$ *such that*

$$\liminf_{t\to+\infty} \Big(Q(t; \lambda, \varepsilon) + H(t; \mu, \varepsilon) \Big) > \frac{1}{4(1-\lambda)} + \frac{1}{4(\mu-1)} \ .$$

Then every proper solution of equation (3.1) *is oscillatory.*

Theorem 3.13 yields the following corollaries.

Corollary 3.14 ([44, Cor. 2.11]). *Let conditions* (3.4) *and* (3.6) *be fulfilled and let there exist* $\lambda < 1$ *and* $\varepsilon \in [0, 1[$ *such that*

$$Q_*(\lambda, \varepsilon) > \frac{1}{4(1-\lambda)} \ . \tag{3.14}$$

Then every proper solution of equation (3.1) *is oscillatory.*

Corollary 3.15 ([44, Cor. 2.12]). *Let conditions* (3.4) *and* (3.6) *be fulfilled and let there exist* $\mu > 1$ *and* $\varepsilon \in [0, 1[$ *such that*

$$H_*(\mu, \varepsilon) > \frac{1}{4(\mu-1)} \ . \tag{3.15}$$

Then every proper solution of equation (3.1) *is oscillatory.*

Now we provide two statements complementing Corollaries 3.14 and 3.15 in certain sense (see Example 3.20).

Theorem 3.16 ([44, Thm. 2.13]). *Let conditions* (3.4) *and* (3.6) *be fulfilled and let there exist* $\lambda < 1$, $\mu > 1$, *and* $\varepsilon \in [0, 1[$ *such that*

$$\frac{\lambda(2-\lambda)}{4(1-\lambda)} \le Q_*(\lambda, \varepsilon) \le \frac{1}{4(1-\lambda)} \tag{3.16}$$

and

$$H^*(\mu, \varepsilon) > \frac{\mu^2}{4(\mu-1)} - \frac{1}{2}\left(1 - \sqrt{1 - 4(1-\lambda)Q_*(\lambda, \varepsilon)}\right). \tag{3.17}$$

Then every proper solution of equation (3.1) *is oscillatory.*

Theorem 3.17 ([44, Thm. 2.14]). *Let conditions* (3.4) *and* (3.6) *be fulfilled and let there exist* $\lambda < 1$, $\mu > 1$, *and* $\varepsilon \in [0, 1[$ *such that*

$$\frac{\mu(2-\mu)}{4(\mu-1)} \le H_*(\mu, \varepsilon) \le \frac{1}{4(\mu-1)} \tag{3.18}$$

and

$$Q^*(\lambda, \varepsilon) > \frac{\lambda^2}{4(1-\lambda)} + \frac{1}{2}\left(1 + \sqrt{1 - 4(\mu-1)H_*(\mu, \varepsilon)}\right). \tag{3.19}$$

Then every proper solution of equation (3.1) *is oscillatory.*

If both conditions (3.16) and (3.18) are satisfied then oscillation criteria (3.17) and (3.19) can be slightly refined as is presented in the last two statements (see also Example 3.21).

Theorem 3.18 ([44, Thm. 2.15]). *Let conditions* (3.4) *and* (3.6) *be fulfilled and let there exist* $\lambda < 1$, $\mu > 1$, *and* $\varepsilon \in [0, 1[$ *such that inequalities* (3.16) *and* (3.18) *are satisfied. If, moreover,*

$$\begin{aligned}\limsup_{t\to+\infty}\Big(Q(t;\lambda,\varepsilon) + H(t;\mu,\varepsilon)\Big) &> Q_*(\lambda, \varepsilon) + H_*(\mu, \varepsilon) \\ &+\frac{1}{2}\left(\sqrt{1 - 4(1-\lambda)Q_*(\lambda, \varepsilon)} + \sqrt{1 - 4(\mu-1)H_*(\mu, \varepsilon)}\right),\end{aligned} \tag{3.20}$$

then every proper solution of equation (3.1) *is oscillatory.*

Corollary 3.19 ([44, Cor. 2.16]). *Let conditions* (3.4) *and* (3.6) *be fulfilled and let there exist* $\lambda < 1$, $\mu > 1$, *and* $\varepsilon \in [0, 1[$ *such that inequalities* (3.16) *and* (3.18) *are satisfied. Then each of the relations*

$$Q^*(\lambda, \varepsilon) > Q_*(\lambda, \varepsilon) + \frac{1}{2}\left(\sqrt{1 - 4(1-\lambda)Q_*(\lambda, \varepsilon)} + \sqrt{1 - 4(\mu-1)H_*(\mu, \varepsilon)}\right) \tag{3.21}$$

and

$$H^*(\mu, \varepsilon) > H_*(\mu, \varepsilon) + \frac{1}{2}\left(\sqrt{1 - 4(1-\lambda)Q_*(\lambda, \varepsilon)} + \sqrt{1 - 4(\mu-1)H_*(\mu, \varepsilon)}\right) \tag{3.22}$$

guarantees that every proper solution of equation (3.1) *is oscillatory.*

EXAMPLE 3.20. On $\mathbb{R}_+$, we consider the equation with a proportional delay

$$u''(t) + \frac{\cos(\ln(t+1)) + \sin(\ln(t+1)) + 2}{(t+1)^2} u\left(\frac{t}{4}\right) = 0. \tag{3.23}$$

One can easily derive that

$$\begin{aligned} Q(t;0,0) &= t\int_t^{+\infty} \frac{\tau(s)}{s} p(s)ds = \frac{1}{4}t\int_t^{+\infty} \frac{\cos(\ln(s+1)) + \sin(\ln(s+1)) + 2}{(s+1)^2} ds \\ &= \frac{t}{4(t+1)}\left(2 + \cos(\ln(t+1))\right) \quad \text{for } t > 0 \end{aligned}$$

and

$$\begin{aligned} H(t;2,0) &= \frac{1}{t}\int_0^t s^2 \frac{\tau(s)}{s} p(s)ds = \frac{1}{4t}\int_0^t \frac{s^2}{(s+1)^2}(\cos(\ln(s+1)) + \sin(\ln(s+1)) + 2)ds \\ &= \frac{t^2}{4t(t+1)}\left(2 + \sin(\ln(t+1)\right) + \phi(t) \quad \text{for } t > 0, \end{aligned}$$

where $\lim_{t\to+\infty} \phi(t) = 0$. Hence,

$$\begin{aligned} Q_*(0,0) &:= \liminf_{t\to+\infty} Q(t;0,0) = \frac{1}{4}, \qquad Q^*(0,0) := \limsup_{t\to+\infty} Q(t;0,0) = \frac{3}{4}, \\ H_*(2,0) &:= \liminf_{t\to+\infty} H(t;2,0) = \frac{1}{4}, \qquad H^*(2,0) := \limsup_{t\to+\infty} H(t;2,0) = \frac{3}{4}. \end{aligned} \tag{3.24}$$

Moreover,

$$\begin{aligned} \int_0^{+\infty} sp(s)ds &= \int_0^{+\infty} s\frac{\cos(\ln(s+1)) + \sin(\ln(s+1)) + 2}{(s+1)^2} ds \\ &= \lim_{t\to+\infty}\left(\frac{-t(\cos(\ln(t+1)) + 2)}{t+1} + \sin(\ln(t+1)) + 2\ln(t+1)\right) = +\infty \end{aligned}$$

and

$$G_* = \liminf_{t\to+\infty} \frac{1}{t}\int_0^t s\tau(s)p(s)ds = H_*(2,0) = \frac{1}{4} \leq 1,$$

i.e., conditions (3.4) and (3.6) are fulfilled. It is clear that inequality (3.14) with $\lambda = 0$ (resp. (3.15) with $\mu = 2$) is not satisfied. Therefore, Corollary 3.14 with $\lambda = 0$ (resp. Corollary 3.15 with $\mu = 2$) cannot be applied.

However, by virtue of (3.24), one can see that (3.16) and (3.17) (resp. (3.18) and (3.19)) with $\lambda = 0$ and $\mu = 2$. Consequently, according to Theorem 3.16 (resp. Theorem 3.17), every proper solution of equation (3.23) is oscillatory.

EXAMPLE 3.21. On $\mathbb{R}_+$, we consider the equation with proportional delay

$$u''(t) + \frac{\cos(\ln(t+1)) + \sin(\ln(t+1)) + 3}{(t+1)^2} u\left(\frac{t}{8}\right) = 0. \tag{3.25}$$

Analogously as above one can derive that

$$Q(t;0,0) = \frac{t}{8(t+1)}\,(3+\cos(\ln(t+1))) \quad \text{for } t>0$$

and

$$H(t;2,0) = \frac{t^2}{8t(t+1)}\,(3+\cos(\ln(t+1)) + \tilde{\phi}(t) \quad \text{for } t>0,$$

where $\lim_{t\to+\infty}\tilde{\phi}(t)=0$. Hence,

$$\begin{aligned} Q_*(0,0) &:= \liminf_{t\to+\infty} Q(t;0,0) = \frac{1}{4}, \qquad Q^*(0,0) := \limsup_{t\to+\infty} Q(t;0,0) = \frac{1}{2}, \\ H_*(2,0) &:= \liminf_{t\to+\infty} H(t;2,0) = \frac{1}{4}, \qquad H^*(2,0) := \limsup_{t\to+\infty} H(t;2,0) = \frac{1}{2}. \end{aligned} \tag{3.26}$$

Analogously as in the previous example, one can show that (3.4) and (3.6) are fulfilled. Moreover, it is clear that (3.16) with $\lambda=0$ and (3.18) with $\mu=2$ hold, but inequality (3.17) (resp. (3.19)) with $\lambda=0$ and $\mu=2$ is not satisfied. Therefore, neither of Theorems 3.16 and Theorem 3.17 can be applied for $\lambda=0$ and $\mu=2$.

However, by virtue of (3.26), one can see that condition (3.21) (resp. (3.22)) is fulfilled with $\lambda=0$ and $\mu=2$ and thus, according to Corollary 3.19, every proper solution of equation (3.25) is oscillatory.

3.2.2 Auxiliary statements

The following lemma on an a priori estimate of proper non-oscillatory solutions of equation (3.1) plays a crucial role in the proofs of the main results.

LEMMA 3.22. *Let u be a solution of equation* (3.1) *on the interval* $[t_u,+\infty[$ *satisfying the inequality*

$$u(t)>0 \quad \text{for } t\geq t_u. \tag{3.27}$$

Then the inequalities

$$\int_0^{+\infty} \frac{\tau(s)}{s}\,p(s)ds < +\infty \tag{3.28}$$

and

$$\limsup_{t\to+\infty} \frac{1}{t}\int_0^t s\tau(s)p(s)ds \leq 1, \qquad \limsup_{t\to+\infty} t\int_t^{+\infty} \frac{\tau(s)}{s}\,p(s)ds \leq 1 \tag{3.29}$$

are satisfied. Moreover, for any $\zeta\in[0,1[$, there exists $t_0(\zeta)\geq t_u$ such that

$$\left(\frac{T_1}{T_2}\right)^{1-\zeta G_*} \leq \frac{u(T_1)}{u(T_2)} \leq \left(\frac{T_1}{T_2}\right)^{\zeta F_*} \quad \text{for } T_2\geq T_1\geq t_0(\zeta), \tag{3.30}$$

where the number G_* is defined by (3.5) *and*

$$F_* := \liminf_{t\to+\infty} t\int_t^{+\infty} \frac{\tau(s)}{s}\,p(s)ds. \tag{3.31}$$

Proof. It is not difficult to verify that the inequality

$$u'(t) \geq 0$$

holds for sufficiently large t. Since equation (3.1) is homogeneous, we can assume without loss of generality that $u(t) \geq 1$ for t large enough. Consequently, by virtue of assumption (3.3), there exists $t_1 > t_u$ such that

$$u'(t) \geq 0, \quad u(\tau(t)) \geq 1 \quad \text{for } t \geq t_1. \tag{3.32}$$

Obviously, relation (3.1) yields that

$$\big(tu'(t) - u(t)\big)' = -tp(t)u(\tau(t)) \quad \text{for a. e. } t \geq t_u.$$

The integration of the latter equality from t_1 to t leads to

$$\theta(t) = \theta(t_1) - \int_{t_1}^{t} sp(s)u(\tau(s))ds \quad \text{for } t \geq t_1,$$

where $\theta(t) := tu'(t) - u(t)$ for $t \geq t_1$. In view of relations (3.4) and (3.32), we have $\theta(t) \to -\infty$ as $t \to +\infty$ and thus, there exists $t_2 \geq t_1$ such that

$$tu'(t) - u(t) \leq -\int_{t_2}^{t} sp(s)u(\tau(s))ds \leq 0 \quad \text{for } t \geq t_2, \tag{3.33}$$

whence we get

$$\left(\frac{u(t)}{t}\right)' = \frac{1}{t^2}\big(tu'(t) - u(t)\big) \leq 0 \quad \text{for } t \geq t_2. \tag{3.34}$$

By virtue of assumption (3.3), there exists a number $t_3 \geq t_2$ such that

$$\tau(t) \geq t_2 \quad \text{for } t \geq t_3. \tag{3.35}$$

Using inequalities (3.34) and (3.35) in relation (3.33), we get

$$tu'(t) - u(t) \leq -\int_{t_3}^{t} s\tau(s)p(s)\frac{u(\tau(s))}{\tau(s)}\, ds \leq -\frac{u(t)}{t}\int_{t_3}^{t} s\tau(s)p(s)ds \quad \text{for } t \geq t_3.$$

Hence, we have

$$tu'(t) \leq u(t)\left[1 - \frac{1}{t}\int_{t_3}^{t} s\tau(s)p(s)ds\right] \quad \text{for } t \geq t_3. \tag{3.36}$$

In particular, by virtue of relations (3.27) and (3.32), inequality (3.36) yields that

$$\frac{1}{t}\int_{t_3}^{t} s\tau(s)p(s)ds \leq 1 \quad \text{for } t \geq t_3$$

and therefore, the first inequality in (3.29) holds.

On the other hand, integrating of equality (3.1) from t to T, one gets

$$u'(t) - u'(T) = \int_t^T p(s)u(\tau(s))ds \quad \text{for } T \geq t \geq t_3.$$

Using inequalities (3.32), (3.34) and (3.35), from the last equality it follows that

$$\begin{aligned} u'(t) &\geq \int_t^T \tau(s)p(s)\frac{u(\tau(s))}{\tau(s)}ds \geq \int_t^T \tau(s)p(s)\frac{u(s)}{s}ds \\ &\geq u(t)\int_t^T \frac{\tau(s)}{s}\, p(s)ds \quad \text{for } T \geq t \geq t_3. \end{aligned} \tag{3.37}$$

Hence, in view of relations (3.27) and (3.33), we get

$$t\int_t^T \frac{\tau(s)}{s}\, p(s)ds \leq 1 \quad \text{for } T \geq t \geq t_3$$

and therefore, the desired relation (3.28) holds. It is clear that the second inequality in (3.29) is satisfied as well. Moreover, it follows from (3.37) that

$$u'(t) \geq u(t)\int_t^{+\infty} \frac{\tau(s)}{s}\, p(s)ds \quad \text{for } t \geq t_3. \tag{3.38}$$

Now let $\zeta \in [0, 1[$ be arbitrary. According to notation (3.5) and (3.31), there exists a number $t_0(\zeta) \geq t_3$ such that

$$\frac{1}{t}\int_{t_3}^t s\tau(s)p(s)ds \geq \zeta G_*, \quad t\int_t^{+\infty} \frac{\tau(s)}{s}\, p(s)ds \geq \zeta F_* \quad \text{for } t \geq t_0(\zeta).$$

Then, in view of relation (3.27), from inequality (3.36) we get that

$$tu'(t) - u(t) \leq -\zeta G_* u(t) \quad \text{for } t \geq t_0(\zeta)$$

and thus, we have

$$\left(\frac{u(t)}{t}\right)' = \frac{1}{t^2}\big(tu'(t) - u(t)\big) \leq -\frac{\zeta\, G_*}{t}\,\frac{u(t)}{t} \quad \text{for } t \geq t_0(\zeta).$$

Hence, we get that

$$\ln \frac{\frac{u(T_2)}{T_2}}{\frac{u(T_1)}{T1}} \leq -\zeta\, G_* \ln \frac{T_2}{T_1} \quad \text{for } T_2 \geq T_1 \geq t_0(\zeta). \tag{3.39}$$

On the other hand, in view of relation (3.27), from inequality (3.38) we obtain

$$u'(t) \geq \frac{\zeta F_*}{t}\, u(t) \quad \text{for } t \geq t_0(\zeta),$$

whence we get

$$\ln \frac{u(T_2)}{u(T_1)} \geq \zeta F_* \ln \frac{T_2}{T_1} \quad \text{for } T_2 \geq T_1 \geq t_0(\zeta).$$

The latter inequality and relation (3.39) guarantee the validity of the desired estimates (3.30). To conclude the proof we mention only that ζ was arbitrary. $\square$

Now we introduce the following notation. Let u be a solution to equation (3.1) on the interval $[t_u, +\infty[$ satisfying relation (3.27). For any $\lambda < 1$, we put

$$c(t;\lambda,u) := \int_{t_u}^{t} s^{\lambda}\,\frac{u(\tau(s))}{u(s)}\,p(s)ds \quad \text{for } t \geq t_u. \tag{3.40}$$

LEMMA 3.23. *Let u be a solution of equation* (3.1) *on the interval* $[t_u, +\infty[$ *satisfying relation* (3.27). *Then, for any $\lambda < 1$, there exists a finite limit*

$$c_0(\lambda,u) := \lim_{t\to+\infty} c(t;\lambda,u), \tag{3.41}$$

where the function c is defined by formula (3.40).

Proof. Let $\lambda < 1$ be arbitrary and put

$$\varrho(t) := \frac{u'(t)}{u(t)} \quad \text{for } t \geq t_u. \tag{3.42}$$

Then equality (3.1) yields that

$$\varrho'(t) = -p(t)\frac{u(\tau(t))}{u(t)} - \varrho^2(t) \quad \text{for a. e. } t \geq t_u. \tag{3.43}$$

Multiplying both sides of this equality by t^{λ} and integrating them from t_u to t, we get that

$$\begin{aligned} t^{\lambda}\varrho(t) - t_u^{\lambda}\varrho(t_u) - \lambda\int_{t_u}^{t} s^{\lambda-1}\varrho(s)ds = &-\int_{t_u}^{t} s^{\lambda}\,\frac{u(\tau(s))}{u(s)}\,p(s)ds \\ &-\int_{t_u}^{t} s^{\lambda}\,\varrho^2(s)ds \quad \text{for } t \geq t_u, \end{aligned}$$

whence we obtain

$$\begin{aligned} \frac{1}{t^{1-\lambda}}\left[t\varrho(t) - \frac{\lambda}{2}\right] = \delta_0 - \frac{\lambda(2-\lambda)}{4(1-\lambda)}\,\frac{1}{t^{1-\lambda}} &- \int_{t_u}^{t} s^{\lambda}\,\frac{u(\tau(s))}{u(s)}\,p(s)ds \\ -\int_{t_u}^{t} s^{\lambda-2}\left[s\varrho(s) - \frac{\lambda}{2}\right]^2 ds &\quad \text{for } t \geq t_u, \end{aligned} \tag{3.44}$$

where $\delta_0 = t_u^{\lambda}\varrho(t_u) + \frac{\lambda^2}{4(1-\lambda)}\frac{1}{t_u^{1-\lambda}}$.

We first show that

$$\int_{t_u}^{+\infty} s^{\lambda-2}\left[s\varrho(s) - \frac{\lambda}{2}\right]^2 ds < +\infty. \tag{3.45}$$

Assume on the contrary that the integral in (3.45) is divergent. In view of assumption (3.3), there exists a number $t_u^* \geq t_u$ such that

$$\tau(t) \geq t_u \quad \text{for a. e. } t \geq t_u^*$$

and thus,

$$\int_{t_u}^{t} s^{\lambda}\,\frac{u(\tau(s))}{u(s)}\,p(s)ds \geq \int_{t_u}^{t_u^*} s^{\lambda}\,\frac{u(\tau(s))}{u(s)}\,p(s)ds \quad \text{for } t \geq t_u^*.$$

Therefore, it follows from equality (3.44) that for some $a > t_u^*$, the inequality

$$t\varrho(t) - \frac{\lambda}{2} \leq -\frac{1}{2}\,t^{1-\lambda} \int_{t_u}^{t} s^{\lambda-2}\left[s\varrho(s) - \frac{\lambda}{2}\right]^2 ds < 0 \quad \text{for } t \geq a \tag{3.46}$$

holds. Put

$$x(t) := \int_{t_u}^{t} s^{\lambda-2}\left[s\varrho(s) - \frac{\lambda}{2}\right]^2 ds \quad \text{for } t \geq a.$$

Using relation (3.46), we get

$$x'(t) = t^{\lambda-2}\left[t\varrho(t) - \frac{\lambda}{2}\right]^2 \geq \frac{1}{4t^{\lambda}}\,x^2(t) \quad \text{for } t \geq a.$$

Therefore, the integration of the latter inequality from a to t yields that

$$\frac{4(1-\lambda)}{x(a)} + a^{1-\lambda} \geq t^{1-\lambda} \quad \text{for } t \geq a,$$

which is a contradiction. The contradiction obtained proves the validity of inequality (3.45).

Now, in view of notation (3.40), equality (3.44) can be rewritten to the form

$$\begin{aligned} t^{\lambda}\varrho(t) = \delta(\lambda,u) - \frac{\lambda^2}{4(1-\lambda)}\,\frac{1}{t^{1-\lambda}} - c(t;\lambda,u) \\ + \int_{t}^{+\infty} s^{\lambda-2}\left[s\varrho(s) - \frac{\lambda}{2}\right]^2 ds \quad \text{for } t \geq t_u, \end{aligned} \tag{3.47}$$

where

$$\delta(\lambda,u) := t_u^{\lambda}\varrho(t_u) + \frac{\lambda^2}{4(1-\lambda)t_u^{1-\lambda}} - \int_{t_u}^{+\infty} s^{\lambda-2}\left[s\varrho(s) - \frac{\lambda}{2}\right]^2 ds.$$

Consequently, we get

$$-\infty < \lim_{t\to+\infty} c(t;\lambda,u) = \delta(\lambda,u) < +\infty \tag{3.48}$$

because, by virtue of condition (3.33), the inequality $\varrho(t) \leq 1/t$ holds for large t. $\square$

Lemma 3.24. *Let $\lambda < 1$ and u be a solution of equation* (3.1) *on the interval* $[t_u, +\infty[$ *satisfying relation* (3.27). *Then*

$$\limsup_{t\to+\infty} \frac{t^{1-\lambda}}{\ln t}\left(c_0(\lambda,u) - \frac{1-\lambda}{t^{1-\lambda}}\int_{t_u}^{t} \frac{c(s;\lambda,u)}{s^{\lambda}}\,ds\right) \leq \frac{1}{4}\,, \tag{3.49}$$

where the function c is defined by relation (3.40) *and the number $c_0(\lambda,u)$ is given by formula* (3.41).

Proof. It follows from the proof of Lemma 3.23 (see relations (3.45), (3.47), and (3.48)) and notation (3.41) that

$$
\begin{aligned}
c_0(\lambda,u)-c(t;\lambda,u) = \frac{1}{t^{1-\lambda}}\left[t\varrho(t)-\frac{\lambda}{2}\right]+\frac{\lambda(2-\lambda)}{4(1-\lambda)}\,\frac{1}{t^{1-\lambda}} \\
-\int_t^{+\infty} s^{\lambda-2}\left[s\varrho(s)-\frac{\lambda}{2}\right]^2 ds \quad \text{for } t\geq t_u,
\end{aligned}
\tag{3.50}
$$

where the function ϱ is defined by formula (3.42). Multiplying both sides of this equality by $t^{-\lambda}$ and integrating them from a to t, we get

$$
\begin{aligned}
&\frac{t^{1-\lambda}}{1-\lambda}\left(c_0(\lambda,u)-\frac{1-\lambda}{t^{1-\lambda}}\int_a^t \frac{c(s;\lambda,u)}{s^{\lambda}}\,ds\right) \\
&\quad = \frac{a^{1-\lambda}}{1-\lambda}\,c_0(\lambda,u)+\frac{\lambda(2-\lambda)}{4(1-\lambda)}\,\ln\frac{t}{a}+\int_a^t \frac{1}{s}\left[s\varrho(s)-\frac{\lambda}{2}\right] ds \\
&\quad\quad -\int_a^t \frac{1}{s^{\lambda}}\left(\int_s^{+\infty}\xi^{\lambda-2}\left[\xi\varrho(\xi)-\frac{\lambda}{2}\right]^2 d\xi\right) ds \quad \text{for } t\geq a>t_u,
\end{aligned}
$$

whence we obtain

$$
\begin{aligned}
\frac{t^{1-\lambda}}{1-\lambda}\left(c_0(\lambda,u)-\frac{1-\lambda}{t^{1-\lambda}}\int_a^t \frac{c(s;\lambda,u)}{s^{\lambda}}\,ds\right) = \delta(a)+\frac{\lambda(2-\lambda)}{4(1-\lambda)}\,\ln\frac{t}{a} \\
+\frac{1}{1-\lambda}\int_a^t \frac{1}{s}\left[s\varrho(s)-\frac{\lambda}{2}\right]\left(1-\lambda-\left[s\varrho(s)-\frac{\lambda}{2}\right]\right) ds \\
-\frac{t^{1-\lambda}}{1-\lambda}\int_t^{+\infty} s^{\lambda-2}\left[s\varrho(s)-\frac{\lambda}{2}\right]^2 ds \quad \text{for } t\geq a>t_u,
\end{aligned}
\tag{3.51}
$$

where

$$
\delta(a):=\frac{a^{1-\lambda}}{1-\lambda}\,c_0(\lambda,u)+\frac{a^{1-\lambda}}{1-\lambda}\int_a^{+\infty} s^{\lambda-2}\left[s\varrho(s)-\frac{\lambda}{2}\right]^2 ds \quad \text{for } a>t_u.
$$

By using the inequality $4x(1-\lambda-x)\leq(1-\lambda)^2$ for $x\in\mathbb{R}$, it follows from equality (3.51) that

$$
\frac{t^{1-\lambda}}{1-\lambda}\left(c_0(\lambda,u)-\frac{1-\lambda}{t^{1-\lambda}}\int_a^t \frac{c(s;\lambda,u)}{s^{\lambda}} ds\right)\leq\delta(a)+\frac{1}{4(1-\lambda)}\,\ln\frac{t}{a} \quad \text{for } t\geq a>t_u.
$$

However, the latter relation yields that

$$
\limsup_{t\to+\infty}\frac{t^{1-\lambda}}{\ln t}\left(c_0(\lambda,u)-\frac{1-\lambda}{t^{1-\lambda}}\int_a^t \frac{c(s;\lambda,u)}{s^{\lambda}}\,ds\right)\leq\frac{1}{4} \quad \text{for } a>t_u,
$$

which implies the validity of desired inequality (3.49). ☐

Let u be a solution of equation (3.1) on the interval $[t_u, +\infty[$ satisfying relation (3.27). For any $\lambda < 1$ and $\mu > 1$, we put

$$q(t; \lambda, u) := t^{1-\lambda} \int_t^{+\infty} s^\lambda \frac{u(\tau(s))}{u(s)} p(s) ds \quad \text{for } t > t_u \tag{3.52}$$

and

$$h(t; \mu, u) := \frac{1}{t^{\mu-1}} \int_{t_u}^t s^\mu \frac{u(\tau(s))}{u(s)} p(s) ds \quad \text{for } t > t_u. \tag{3.53}$$

Note that the function q is well defined because, in view of Lemma 3.23, we have

$$q(t; \lambda, u) = t^{1-\lambda} \Big(c_0(\lambda, u) - c(t; \lambda, u) \Big) \quad \text{for } t > t_u. \tag{3.54}$$

Moreover, we put

$$q_*(\lambda, u) := \liminf_{t\to+\infty} q(t; \lambda, u), \qquad q^*(\lambda, u) := \limsup_{t\to+\infty} q(t; \lambda, u), \tag{3.55}$$

$$h_*(\mu, u) := \liminf_{t\to+\infty} h(t; \mu, u), \qquad h^*(\mu, u) := \limsup_{t\to+\infty} h(t; \mu, u). \tag{3.56}$$

LEMMA 3.25. *Let u be a solution of equation* (3.1) *on the interval* $[t_u, +\infty[$ *satisfying relation* (3.27). *Then*

$$\limsup_{t\to+\infty} \frac{1}{\ln t} \int_{t_u}^t s \frac{u(\tau(s))}{u(s)} p(s) ds \le \frac{1}{4}. \tag{3.57}$$

Proof. Let $\lambda < 1$ be arbitrary. In view of assumptions (3.3) and (3.27), there exists $t_u^* > \max\{1, t_u\}$ such that

$$u(\tau(t)) > 0 \quad \text{for a. e. } t \ge t_u^*.$$

According to Lemma 3.23, the function c defined by formula (3.40) possesses a finite limit (3.41). Therefore, by using relation (3.54), we get

$$\begin{aligned} \frac{t^{1-\lambda}}{\ln t} \left(c_0(\lambda, u) - \frac{1-\lambda}{t^{1-\lambda}} \int_{t_u}^t \frac{c(s; \lambda, u)}{s^\lambda} ds \right) & \\ = \frac{q(t; \lambda, u)}{\ln t} + \frac{1}{\ln t} \int_{t_u}^t s \frac{u(\tau(s))}{u(s)} p(s) ds & \\ \ge \frac{1}{\ln t} \int_{t_u}^t s \frac{u(\tau(s))}{u(s)} p(s) ds \quad \text{for } t \ge t_u^*, & \end{aligned} \tag{3.58}$$

which, by virtue of Lemma 3.24, guarantees desired estimate (3.57). □

LEMMA 3.26. *Let $\lambda < 1$ and u be a solution of equation* (3.1) *on the interval* $[t_u, +\infty[$ *such that relation* (3.27) *holds and*

$$\frac{\lambda(2-\lambda)}{4(1-\lambda)} \le q_*(\lambda, u) \le \frac{1}{4(1-\lambda)}, \tag{3.59}$$

where the number $q_*(\lambda, u)$ *is defined by formula* (3.55)*. Then either*

$$\liminf_{t\to+\infty} \frac{tu'(t)}{u(t)} = +\infty \tag{3.60}$$

or

$$\begin{aligned} \frac{1}{2}\left(1-\sqrt{1-4(1-\lambda)q_*(\lambda,u)}\right) &\le \liminf_{t\to+\infty} \frac{tu'(t)}{u(t)} \\ &\le \frac{1}{2}\left(1+\sqrt{1-4(1-\lambda)q_*(\lambda,u)}\right). \end{aligned} \tag{3.61}$$

Proof. It follows from the proof of Lemma 3.23 (see relations (3.45), (3.47), and (3.48)) and notation (3.41) that equality (3.50) is satisfied, where the function ϱ is defined by formula (3.42). Therefore, in view of relation (3.54), equality (3.50) leads to

$$\begin{aligned} t\varrho(t) - \frac{\lambda}{2} &= q(t;\lambda,u) - \frac{\lambda(2-\lambda)}{4(1-\lambda)} \\ &\quad + t^{1-\lambda}\int_t^{+\infty} s^{\lambda-2}\left[s\varrho(s)-\frac{\lambda}{2}\right]^2 ds \quad \text{for } t \ge t_u. \end{aligned} \tag{3.62}$$

Now we put

$$m := \liminf_{t\to+\infty}\left[t\varrho(t)-\frac{\lambda}{2}\right]. \tag{3.63}$$

If $m = +\infty$, then condition (3.60) holds. Therefore, assume that $m < +\infty$. Then it follows from relation (3.62) that

$$m \ge q_*(\lambda,u) - \frac{\lambda(2-\lambda)}{4(1-\lambda)}. \tag{3.64}$$

If $q_*(\lambda,u) = \frac{\lambda(2-\lambda)}{4(1-\lambda)}$, then desired estimate (3.61) holds, because relation (3.64) yields that $m \ge 0$. Hence, we suppose in what follows that $q_*(\lambda,u) > \frac{\lambda(2-\lambda)}{4(1-\lambda)}$ and thus, we have $m > 0$ (see relation (3.64)).

Let $\varepsilon \in\,]0,m]$ be arbitrary and choose $t_\varepsilon \ge t_u$ such that

$$t\varrho(t) - \frac{\lambda}{2} \ge m-\varepsilon, \quad q(t;\lambda,u) \ge q_*(\lambda,u)-\varepsilon \quad \text{for } t \ge t_\varepsilon.$$

Then from equality (3.62) we get

$$t\varrho(t) - \frac{\lambda}{2} \ge q_*(\lambda,u) - \varepsilon - \frac{\lambda(2-\lambda)}{4(1-\lambda)} + \frac{(m-\varepsilon)^2}{1-\lambda} \quad \text{for } t \ge t_\varepsilon,$$

which implies that

$$m \ge q_*(\lambda,u) - \varepsilon - \frac{\lambda(2-\lambda)}{4(1-\lambda)} + \frac{(m-\varepsilon)^2}{1-\lambda}\,.$$

Since ε was arbitrary, the latter relation leads to the inequality

$$m^2 - (1-\lambda)m + (1-\lambda)q_*(\lambda,u) - \frac{\lambda(2-\lambda)}{4} \le 0.$$

Consequently, we have

$$\frac{1}{2}\left(1-\lambda-\sqrt{1-4(1-\lambda)q_*(\lambda,u)}\right) \leq m \leq \frac{1}{2}\left(1-\lambda+\sqrt{1-4(1-\lambda)q_*(\lambda,u)}\right)$$

which, in view of notation (3.63), yields estimate (3.61). $\square$

LEMMA 3.27. *Let $\mu > 1$ and u be a solution of equation* (3.1) *on the interval* $[t_u, +\infty[$ *such that relation* (3.27) *holds and*

$$\frac{\mu(2-\mu)}{4(\mu-1)} \leq h_*(\mu,u) \leq \frac{1}{4(\mu-1)}\ , \tag{3.65}$$

where the number $h_*(\mu,u)$ is defined by formula (3.56). *Then*

$$\limsup_{t\to+\infty} \frac{tu'(t)}{u(t)} \leq \frac{1}{2}\left(1+\sqrt{1-4(\mu-1)h_*(\mu,u)}\right). \tag{3.66}$$

Proof. Define the function ϱ by formula (3.42). Then, in view of (3.1), relation (3.42) yields that equality (3.43) is satisfied. Multiplying both sides of equality (3.43) by t^μ and integrating them from a to t, we get

$$\begin{aligned} t^\mu\varrho(t) - a^\mu\varrho(a) - \mu\int_a^t s^{\mu-1}\varrho(s)ds = &-\int_a^t s^\mu\,\frac{u(\tau(s))}{u(s)}\,p(s)ds \\ &-\int_a^t s^\mu\varrho^2(s)ds \quad \text{for } t > a \geq t_u, \end{aligned}$$

whence we obtain

$$\begin{aligned} t\varrho(t) = &\ \frac{\delta(a)}{t^{\mu-1}} - h(t;\mu,u) \\ &+ \frac{1}{t^{\mu-1}}\int_a^t s^{\mu-2}\Big[s\varrho(s)\Big(\mu - s\varrho(s)\Big)\Big]ds \quad \text{for } t > a \geq t_u, \end{aligned} \tag{3.67}$$

where

$$\delta(a) := a^\mu\varrho(a) + \int_{t_u}^a s^\mu\,\frac{u(\tau(s))}{u(s)}\,p(s)ds \quad \text{for } a \geq t_u. \tag{3.68}$$

Now we put

$$M := \limsup_{t\to+\infty} t\varrho(t). \tag{3.69}$$

It is not difficult to verify that inequality $u'(t) \geq 0$ holds for t large enough and thus we have $M \geq 0$. According to the inequality $4x(\mu - x) \leq \mu^2$ for $x \in \mathbb{R}$, it follows from relation (3.67) that

$$t\varrho(t) \leq \frac{\delta(a)}{t^{\mu-1}} - h(t;\mu,u) + \frac{\mu^2}{4(\mu-1)} \quad \text{for } t > a \geq t_u,$$

which implies

$$M \leq -h_*(\mu,u) + \frac{\mu^2}{4(\mu-1)}\ . \tag{3.70}$$

If $h_*(\mu,u) = \frac{\mu(2-\mu)}{4(\mu-1)}$, then desired estimate (3.66) is fulfilled because relation (3.70) yields that $M \le \frac{\mu}{2}$. Hence, we suppose in the sequel that $h_*(\mu,u) > \frac{\mu(2-\mu)}{4(\mu-1)}$ and thus, we have $M < \frac{\mu}{2}$ (see relation (3.70)).

Let $\varepsilon \in]0, \frac{\mu}{2} - M]$ be arbitrary and choose $t_\varepsilon \ge t_u$ such that

$$t\varrho(t) \le M + \varepsilon, \quad h(t;\mu,u) \ge h_*(\mu,u) - \varepsilon \quad \text{for } t \ge t_\varepsilon. \tag{3.71}$$

Since we have $M + \varepsilon \le \frac{\mu}{2}$, it is easy to check that

$$s\varrho(s)\big(\mu - s\varrho(s)\big) \le (M+\varepsilon)(\mu - M - \varepsilon) \quad \text{for } s \ge t_\varepsilon. \tag{3.72}$$

Therefore, by using relations (3.71) and (3.72), from equality (3.67) with $a = t_\varepsilon$ we get

$$t\varrho(t) \le \frac{\delta(t_\varepsilon)}{t^{\mu-1}} - h_*(\mu,u) + \varepsilon + \frac{(M+\varepsilon)(\mu-M-\varepsilon)}{\mu-1} \quad \text{for } t > t_\varepsilon,$$

which yields that

$$M \le -h_*(\mu,u) + \varepsilon + \frac{(M+\varepsilon)(\mu-M-\varepsilon)}{\mu-1}\,.$$

Since ε was arbitrary, the latter relation leads to the inequality

$$M^2 - M + (\mu-1)h_*(\mu,u) \le 0.$$

Consequently, we have

$$M \le \frac{1}{2}\left(1 + \sqrt{1 - 4(\mu-1)h_*(\mu,u)}\right)$$

which, in view of notation (3.69), proves desired estimate (3.66). □

Lemma 3.28. *Let $\lambda < 1$, $\mu > 1$, and u be a solution of equation* (3.1) *on the interval* $[t_u, +\infty[$ *satisfying relation* (3.27). *Then*

$$\limsup_{t\to+\infty}\Big(q(t;\lambda,u) + h(t;\mu,u)\Big) \le \frac{\lambda^2}{4(1-\lambda)} + \frac{\mu^2}{4(\mu-1)}\,, \tag{3.73}$$

where the functions q and h are defined by formulas (3.52) *and*(3.53), *respectively.*

Proof. Analogously to the proofs of Lemmas 3.26 and 3.27 we get equalities (3.62) and (3.67), where $\delta(a)$ is given by formula (3.68), combining of which leads to the relation

$$\begin{aligned} &q(t;\lambda,u) + h(t;\mu,u) \\ &\quad = \frac{\lambda^2}{4(1-\lambda)} + \frac{\delta(a)}{t^{\mu-1}} + \frac{1}{t^{\mu-1}}\int_a^t s^{\mu-2}\Big[s\varrho(s)\Big(\mu - s\varrho(s)\Big)\Big]ds \\ &\quad\quad - t^{1-\lambda}\int_t^{+\infty} s^{\lambda-2}\left[s\varrho(s) - \frac{\lambda}{2}\right]^2 ds \quad \text{for } t > a \ge t_u. \end{aligned} \tag{3.74}$$

Putting $a = t_u$ and using the inequality $4x(\mu - x) \le \mu^2$ for $x \in \mathbb{R}$, from equality (3.74) we get

$$q(t;\lambda,u) + h(t;\mu,u) \le \frac{\lambda^2}{4(1-\lambda)} + \frac{\mu^2}{4(\mu-1)} + \frac{\delta(t_u)}{t^{\mu-1}} \quad \text{for } t > t_u,$$

which yields desired estimate (3.73). □

LEMMA 3.29. *Let* $\lambda < 1$ *and* u *be a solution of equation* (3.1) *on the interval* $[t_u, +\infty[$ *satisfying relation* (3.27)*. Then*

$$q_*(\lambda, u) \leq \frac{1}{4(1-\lambda)}\,, \tag{3.75}$$

where the number $q_*(\lambda, u)$ *is defined by formula* (3.55)*.*

Proof. In view of Lemma 3.23, the function c defined by formula (3.40) possesses a finite limit (3.41).

Assume on the contrary that inequality (3.75) does not hold. Then there exist $\varepsilon > 0$ and $t_\varepsilon > \max\{1, t_u\}$ such that

$$q(t; \lambda, u) \geq \frac{1+\varepsilon}{4(1-\lambda)} \quad \text{for } t \geq t_\varepsilon.$$

By using this relation, for $t \geq t_\varepsilon$ we get

$$\begin{aligned}
\frac{t^{1-\lambda}}{\ln t}&\left(c_0(\lambda, u) - \frac{1-\lambda}{t^{1-\lambda}} \int_{t_u}^{t} \frac{c(s; \lambda, u)}{s^\lambda}\, ds\right) \\
&\qquad\qquad = \frac{t_u^{1-\lambda}}{\ln t}\, c_0(\lambda, u) + \frac{1-\lambda}{\ln t} \int_{t_u}^{t} \frac{q(s; \lambda, u)}{s}\, ds \\
&\qquad\qquad \geq \frac{t_u^{1-\lambda}}{\ln t}\, c_0(\lambda, u) + \frac{1-\lambda}{\ln t} \int_{t_u}^{t_\varepsilon} \frac{q(s; \lambda, u)}{s}\, ds \\
&\qquad\qquad\quad + \frac{1+\varepsilon}{4 \ln t} \ln \frac{t}{t_\varepsilon},
\end{aligned}$$

which yields that

$$\limsup_{t \to +\infty} \frac{t^{1-\lambda}}{\ln t} \left(c_0(\lambda, u) - \frac{1-\lambda}{t^{1-\lambda}} \int_{t_u}^{t} \frac{c(s; \lambda, u)}{s^\lambda}\, ds\right) \geq \frac{1+\varepsilon}{4}\,. \tag{3.76}$$

However, this is in a contradiction with the assertion of Lemma 3.24. ☐

LEMMA 3.30. *Let* $\mu > 1$ *and* u *be a solution of equation* (3.1) *on the interval* $[t_u, +\infty[$ *satisfying relation* (3.27)*. Then*

$$h_*(\mu, u) \leq \frac{1}{4(\mu - 1)}\,, \tag{3.77}$$

where the number $h_*(\mu, u)$ *is defined by formula* (3.56)*.*

Proof. Let $\lambda < 1$ be arbitrary. In view of assumptions (3.3) and (3.27), there exists $t_u^* > \max\{1, t_u\}$ such that

$$u(\tau(t)) > 0 \quad \text{for a. e. } t \geq t_u^*.$$

According to Lemma 3.23, the function c defined by formula (3.40) possesses a finite limit (3.41). Therefore, by using equality (3.54), it is not difficult to verify that relation (3.58) is fulfilled, where the function q is defined by formula (3.52).

Assume on the contrary that inequality (3.77) does not hold. Then there exist $\varepsilon > 0$ and $t_\varepsilon \geq t_u^*$ such that

$$h(t;\mu,u) \geq \frac{1+\varepsilon}{4(\mu-1)} \quad \text{for } t \geq t_\varepsilon.$$

By using this relation, we get

$$\begin{aligned}\int_{t_u}^{t} s\,\frac{u(\tau(s))}{u(s)}\,p(s)ds &= h(t;\mu,u) + (\mu-1)\int_{t_u}^{t}\frac{h(s;\mu,u)}{s}\,ds \\ &\geq \frac{1+\varepsilon}{4(\mu-1)} + (\mu-1)\int_{t_u}^{t_\varepsilon}\frac{h(s;\mu,u)}{s}\,ds + \frac{1+\varepsilon}{4}\ln\frac{t}{t_\varepsilon}\end{aligned}$$

for $t \geq t_\varepsilon$. Consequently, relation (3.58) yields the validity of inequality (3.76), which is in a contradiction with the assertion of Lemma 3.24. $\square$

LEMMA 3.31. *Let $\lambda < 1$, $\mu > 1$, and u be a solution of equation* (3.1) *on the interval* $[t_u, +\infty[$ *such that relation* (3.27) *holds. If, moreover, inequalities* (3.59) *and* (3.65) *are satisfied then*

$$\begin{aligned}\limsup_{t\to+\infty}\Big(q(t;\lambda,u)+h(t;\mu,u)\Big) &\leq q_*(\lambda,u)+h_*(\mu,u) \\ &+\frac{1}{2}\left(\sqrt{1-4(1-\lambda)q_*(\lambda,u)}+\sqrt{1-4(\mu-1)h_*(\mu,u)}\right),\end{aligned} \tag{3.78}$$

where the functions q and h are defined by relations (3.52) *and* (3.53)*, respectively, and the numbers $q_*(\lambda,u)$ and $h_*(\mu,u)$ are given by formulas* (3.55) *and* (3.56)*, respectively.*

Proof. Analogously to the proofs of Lemmas 3.26 and 3.27 we get equalities (3.62) and (3.67), where $\delta(a)$ is given by formula (3.68), combining of which leads to relation (3.74).

Let the numbers m and M be given by formulas

$$m := \frac{1}{2}\left(1-\sqrt{1-4(1-\lambda)q_*(\lambda,u)}\right) \tag{3.79}$$

and

$$M := \frac{1}{2}\left(1+\sqrt{1-4(\mu-1)h_*(\mu,u)}\right), \tag{3.80}$$

respectively. It follows from Lemmas 3.26 that and 3.27

$$\liminf_{t\to+\infty} t\varrho(t) \geq m \quad \text{and} \quad \limsup_{t\to+\infty} t\varrho(t) \leq M,$$

where the function ϱ is defined by relation (3.42). Since we assume that inequalities (3.59) and (3.65) are fulfilled, we have that $m \geq \frac{\lambda}{2}$ and $M \leq \frac{\mu}{2}$.

Suppose that $m > \frac{\lambda}{2}$ and $M < \frac{\mu}{2}$. Let $0 < \varepsilon \le \min\{m - \frac{\lambda}{2}, \frac{\mu}{2} - M\}$ be arbitrary and choose $t_\varepsilon \ge t_u$ such that

$$t\varrho(t) \ge m - \varepsilon, \quad t\varrho(t) \le M + \varepsilon \quad \text{for } t \ge t_\varepsilon$$

hold. Since $M + \varepsilon \le \frac{\mu}{2}$, it is easy to check that

$$s\varrho(s)\big(\mu - s\varrho(s)\big) \le (M+\varepsilon)(\mu - M - \varepsilon) \quad \text{for } s \ge t_\varepsilon\,.$$

Therefore, from equality (3.74) with $a = t_\varepsilon$ we get

$$\begin{aligned} q(t;\lambda,u) + h(t;\mu,u) \le \frac{\lambda^2}{4(1-\lambda)} &+ \frac{(M+\varepsilon)(\mu-M-\varepsilon)}{\mu-1} \\ &- \frac{\big(m - \frac{\lambda}{2} - \varepsilon\big)^2}{1-\lambda} + \frac{\delta(t_\varepsilon)}{t^{\mu-1}} \quad \text{for } t \ge t_\varepsilon. \end{aligned}$$

Since ε was arbitrary, it follows from the latter inequality that

$$\begin{aligned} \limsup_{t\to+\infty}\Big(q(t;\lambda,u) + h(t;\mu,u)\Big) \le &\frac{\lambda^2}{4(1-\lambda)} \\ &+ \frac{M(\mu-M)}{\mu-1} - \frac{\big(m-\frac{\lambda}{2}\big)^2}{1-\lambda}\,. \end{aligned} \tag{3.81}$$

If $m = \frac{\lambda}{2}$ (respectively, $M = \frac{\mu}{2}$), then we prove similarly as above that relation (3.81) holds whereas we use the fact that

$$-t^{1-\lambda}\int_t^{+\infty} s^{\lambda-2}\left[s\varrho(s) - \frac{\lambda}{2}\right]^2 ds \le 0 = -\frac{\big(m-\frac{\lambda}{2}\big)^2}{1-\lambda} \quad \text{for } t \ge t_u$$

$$\begin{aligned} \Bigg(\text{respectively,} \quad &\frac{1}{t^{\mu-1}}\int_a^t s^{\mu-2}\Big[s\varrho(s)\Big(\mu - s\varrho(s)\Big)\Big]ds \\ &\le \frac{\mu^2}{4(\mu-1)} = \frac{M(\mu-M)}{\mu-1} \quad \text{for } t \ge a \ge t_u\Bigg). \end{aligned}$$

Consequently, relation (3.81) and notation (3.79) and (3.80) guarantee the validity of desired estimate (3.78). $\square$

3.2.3 Proofs of main results

Proof of Proposition 3.4. It can be found in [44]. $\square$

Proof of Proposition 3.6. It can be found in [44]. $\square$

Proof of Theorem 3.7. It can be found in [44]. $\square$

Proof of Theorem 3.9. Suppose on the contrary that equation (3.1) has a proper non-oscillatory solution. Let u be a solution of equation (3.1) on the interval $[t_u, +\infty[$ satisfying relation (3.27). According to Lemma 3.22, there exists $t_0(\varepsilon) > \max\{1, t_u\}$ such that inequality (3.30) holds. Let, moreover, $t_0^*(\varepsilon) \geq t_0(\varepsilon)$ be such that $\tau(t) \geq t_0(\varepsilon)$ for a.e. $t \geq t_0^*(\varepsilon)$. Then we have

$$\frac{1}{\ln t}\int_0^t s\left(\frac{\tau(s)}{s}\right)^{1-\varepsilon G_*} p(s)ds \leq \frac{1}{\ln t}\int_0^{t_0^*(\varepsilon)} s\left(\frac{\tau(s)}{s}\right)^{1-\varepsilon G_*} p(s)ds + \frac{1}{\ln t}\int_{t_0*(\varepsilon)}^t s\,\frac{u(\tau(s))}{u(s)}\,p(s)ds \quad \text{for } t \geq t_0^*(\varepsilon).$$

Therefore, by virtue of Lemma 3.25, we get

$$\limsup_{t\to+\infty}\frac{1}{\ln t}\int_0^t s\left(\frac{\tau(s)}{s}\right)^{1-\varepsilon G_*} p(s)ds \leq \frac{1}{4}\,,$$

which is in a contradiction with assumption (3.10). □

Proof of Theorem 3.10. Suppose on the contrary that equation (3.1) has a proper non-oscillatory solution. Let u be a solution of equation (3.1) on the interval $[t_u, +\infty[$ satisfying relation (3.27). Define the functions q and h by formulas (3.52) and (3.53), respectively. According to Lemma 3.22, there exists $t_0(\varepsilon) \geq t_u$ such that inequality (3.30) is fulfilled. Let, moreover, $t_0^*(\varepsilon) \geq t_0(\varepsilon)$ be such that $\tau(t) \geq t_0(\varepsilon)$ for a.e. $t \geq t_0^*(\varepsilon)$. Then we have

$$\begin{aligned} Q(t;\lambda,\varepsilon) + H(t;\mu,\varepsilon) &\leq q(t;\lambda,u) + h(t;\mu,u) \\ &\quad + \frac{1}{t^{\mu-1}}\int_0^{t_0^*(\varepsilon)} s^\mu\left(\frac{\tau(s)}{s}\right)^{1-\varepsilon G_*} p(s)ds \\ &\quad - \frac{1}{t^{\mu-1}}\int_{t_u}^{t_0^*(\varepsilon)} s^\mu\,\frac{u(\tau(s))}{u(s)}\,p(s)ds \quad \text{for } t > t_0^*(\varepsilon). \end{aligned} \tag{3.82}$$

Consequently, by virtue of Lemma 3.28, we get

$$\limsup_{t\to+\infty}\Big(Q(t;\lambda,\varepsilon) + H(t;\mu,\varepsilon)\Big) \leq \frac{\lambda^2}{4(1-\lambda)} + \frac{\mu^2}{4(\mu-1)}\,,$$

which contradicts assumption (3.13). □

Proof of Corollary 3.11. It immediately follows from Theorem 3.10 with $\mu = 2$, because the inequality $H(t;2,\varepsilon) \geq 0$ holds for $t > 0$. □

Proof of Corollary 3.12. Since we have $Q(t;0,\varepsilon) \geq 0$ for $t > 0$, the assertion of the corollary immediately follows from Theorem 3.10 with $\lambda = 0$. □

Proof of Theorem 3.13. It can be found in [44]. □

Proof of Corollary 3.14. According to assumption (3.14), there exists $\mu > 1$ such that

$$Q_*(\lambda, \varepsilon) > \frac{1}{4(1-\lambda)} + \frac{1}{4(\mu-1)} \,.$$

Since we have $H(t; \mu, \varepsilon) \geq 0$ for $t > 0$, the assertion of the corollary immediately follows from Theorem 3.13 □

Proof of Corollary 3.15. By virtue of assumption (3.15), there exists $\lambda < 1$ such that

$$H_*(\mu, \varepsilon) > \frac{1}{4(1-\lambda)} + \frac{1}{4(\mu-1)} \,.$$

Consequently, the assertion of the corollary follows from Theorem 3.13, because the inequality $Q(t; \lambda, \varepsilon) \geq 0$ holds for $t > 0$.

□

Proof of Theorem 3.16. It can be found in [44]. □

Proof of Theorem 3.17. It can be found in [44]. □

Proof of Theorem 3.18. Suppose on the contrary that equation (3.1) has a proper non-oscillatory solution. Let u be a solution of equation (3.1) on the interval $[t_u, +\infty[$ satisfying relation (3.27). Define the functions q and h by formulas (3.52) and (3.53), respectively. According to Lemma 3.22, there exists $t_0(\varepsilon) \geq t_u$ such that inequality (3.30) is fulfilled and thus, in view of (3.3), we easily get inequalities

$$Q_*(\lambda, \varepsilon) \leq q_*(\lambda, u), \qquad H^*(\mu, \varepsilon) \leq h^*(\mu, u),$$

and

$$H_*(\mu, \varepsilon) \leq h_*(\mu, u), \qquad Q^*(\lambda, \varepsilon) \leq q^*(\lambda, u),$$

where the numbers $q_*(\lambda, u)$, $q^*(\lambda, u)$ and $h_*(\mu, u)$, $h^*(\mu, u)$ are defined by formulas (3.55) and (3.56), respectively. Therefore, assumptions (3.16), (3.18) and Lemmas 3.29, 3.30 (see relations (3.75) and (3.77)) immediately yield the validity of inequalities (3.59) and (3.65), i. e., the assumptions of Lemma 3.31 are satisfied. Obviously, inequality (3.82) holds, the function

$$x \mapsto x + \frac{1}{2}\sqrt{1-4(1-\lambda)x} \quad \text{is non-increasing on } \left[\tfrac{\lambda(2-\lambda)}{4(1-\lambda)}, \tfrac{1}{4(1-\lambda)}\right],$$

and the function

$$y \mapsto y + \frac{1}{2}\sqrt{1-4(\mu-1)y} \quad \text{is non-increasing on } \left[\tfrac{\mu(2-\mu)}{4(\mu-1)}, \tfrac{1}{4(\mu-1)}\right].$$

Consequently, by using Lemma 3.31, we obtain

$$
\begin{aligned}
\limsup_{t\to+\infty}\Big(Q(t;\lambda,\varepsilon)+H(t;\mu,\varepsilon)\Big) &\le \limsup_{t\to+\infty}\Big(q(t;\lambda,u)+h(t;\mu,u)\Big)\\
&\le q_*(\lambda,u)+\frac{1}{2}\,\sqrt{1-4(1-\lambda)q_*(\lambda,u)}\\
&\quad + h_*(\mu,u)+\frac{1}{2}\,\sqrt{1-4(\mu-1)h_*(\mu,u)}\\
&\le Q_*(\lambda,\varepsilon)+\frac{1}{2}\,\sqrt{1-4(1-\lambda)Q_*(\lambda,\varepsilon)}\\
&\quad + H_*(\mu,\varepsilon)+\frac{1}{2}\,\sqrt{1-4(\mu-1)H_*(\mu,\varepsilon)}\,,
\end{aligned}
$$

which contradicts assumption (3.20). □

Proof of Corollary 3.19. It is easy to show that

$$
\limsup_{t\to+\infty}\Big(Q(t;\lambda,\varepsilon)+H(t;\mu,\varepsilon)\Big)\ge Q^*(\lambda,\varepsilon)+H_*(\mu,\varepsilon)
$$

and

$$
\limsup_{t\to+\infty}\Big(Q(t;\lambda,\varepsilon)+H(t;\mu,\varepsilon)\Big)\ge H^*(\mu,\varepsilon)+Q_*(\lambda,\varepsilon).
$$

Consequently, in both cases (3.21) and (3.22), inequality (3.20) is satisfied and thus, the assertion of the corollary follows immediately from Theorem 3.18. □

3.3 Myshkis's type criteria for DDE

In this section, we present other type of oscillation criteria for equation (3.1), so-called Myshkis's type oscillation criteria, which generalise known results of R. Koplatadze. Below we assume that $\tau\colon \mathbb{R}_+\to\mathbb{R}_+$ is a continuous function satisfying

$$
\tau(t)\le t \quad \text{for } t\ge 0 \tag{3.83}
$$

and

$$
\lim_{t\to+\infty}\tau(t)=+\infty. \tag{3.84}
$$

The assumption of continuity imposed on τ is motivated by the form of relations (conditions) of type (3.97), (3.101), (3.102), etc. Similar statements can also be formulated for τ that is only measurable. In order to do so, one should use suitable notions of upper and lower limits for measurable functions.

3.3.1 Main results

In the paper [23], R. Koplatadze proved, among other things, the following statements.

Theorem 3.32 ([23, Thm. 1]). *Let there exist a continuous non-decreasing function* $\sigma\colon \mathbb{R}_+ \to \mathbb{R}_+$ *such that the inequalities*

$$\tau(t) \le \sigma(t) \le t \quad \textit{for } t \ge 0 \tag{3.85}$$

and

$$\limsup_{t\to+\infty} \int_{\sigma(t)}^{t} \tau(s)p(s)ds > 1 \tag{3.86}$$

are fulfilled. Then every proper solution of equation (3.1) *is oscillatory.*

Remark 3.33. It is necessary for the validity of assumption (3.86) that

$$\int_0^{+\infty} \tau(s)p(s)ds = +\infty. \tag{3.87}$$

Theorem 3.34 ([23, Thm. 2]). *Let the inequality*

$$\liminf_{t\to+\infty} \int_{\tau(t)}^{t} \tau(s)p(s)ds > \frac{1}{\mathrm{e}} \tag{3.88}$$

hold. Then every proper solution of equation (3.1) *is oscillatory.*

Remark 3.35. If assumption (3.88) is fulfilled then condition (3.87) necessarily holds. In Theorem 3.34, the constant $\frac{1}{\mathrm{e}}$ is optimal and can not be in general improved. A counterexample is constructed in [23] for equation (3.1) with a proportional delay.

Remark 3.36. Oscillation criteria (3.86) and (3.88) are usually called Myshkis's type oscillation criteria because results of that kind were firstly achieved for first-order linear delay differential equations by famous mathematician A. D. Myshkis (see, e. g., [37]).

In what follows, we show that condition (3.87), necessary in statements of R. Koplatadze, can be relaxed. Moreover, under some natural additional assumptions, we improve constants on the right-hand side of inequalities (3.86) and (3.88).

Let the number G_* be defined by (3.5). By virtue of Theorem 3.7, Proposition 3.6, and Corollary 3.11, we assume in the sequel that conditions (3.11) and (3.29) are fulfilled, because otherwise every proper solution of equation (3.1) is oscillatory without any additional assumption.

Define the number F_* by (3.31). In view of assumption (3.11), the number F_* is well defined. Moreover, assumptions (3.29) yield that

$$G_* \le 1, \qquad F_* \le 1.$$

Furthermore, Corollaries 3.14 and 3.15 claim that every proper solution of equation (3.1) is oscillatory provided that either

$$Q_*(\lambda, \varepsilon) > \frac{1}{4(1-\lambda)} \quad \text{for some } \lambda < 1,\ \varepsilon \in [0,1[\,,$$

or

$$H_*(\mu,\varepsilon) > \frac{1}{4(\mu-1)} \quad \text{for some } \mu < 1,\ \varepsilon \in [0,1[\,,$$

where the numbers $Q_*(\lambda,\varepsilon)$ and $H_*(\mu,\varepsilon)$ are defined by (3.12). Therefore, it is natural to restrict ourself to the case, where

$$Q_*(\lambda,\varepsilon) \le \frac{1}{4(1-\lambda)}, \quad H_*(\mu,\varepsilon) \le \frac{1}{4(\mu-1)} \quad \text{for all } \lambda < 1,\ \mu < 1,\ \varepsilon \in [0,1[\,.$$

Under these assumptions, we can improve Theorems 3.32 and 3.34 as follows.

THEOREM 3.37 ([45, Thm. 3]). *Let there exist numbers* $\lambda < 1$, $\mu > 1$, $\varepsilon, \delta \in [0,1[$ *and continuous functions* $\nu, \sigma\colon \mathbb{R}_+ \to \mathbb{R}_+$ *such that* σ *is non-decreasing,*

$$\tau(t) \le \nu(t) \le \sigma(t) \le t \quad \textit{for } t \ge 0, \tag{3.89}$$

$$\frac{\lambda(2-\lambda)}{4(1-\lambda)} \le Q_*(\lambda,\varepsilon) \le \frac{1}{4(1-\lambda)}\,, \quad \frac{\mu(2-\mu)}{4(\mu-1)} \le H_*(\mu,\varepsilon) \le \frac{1}{4(\mu-1)}\,, \tag{3.90}$$

and

$$\limsup_{t\to+\infty} \left(\frac{\nu(t)}{\sigma(t)}\right)^{1-\delta F_*} \int_{\nu(t)}^{t} \tau(s)p(s)\left(\frac{\sigma(s)}{\tau(s)}\right)^{\varepsilon G_*} ds > R_0 - \alpha_* r_0, \tag{3.91}$$

where

$$\begin{aligned} r_0 &:= \frac{1}{2}\left(1 - \sqrt{1-4(1-\lambda)Q_*(\lambda,\varepsilon)}\right), \\ R_0 &:= \frac{1}{2}\left(1 + \sqrt{1-4(\mu-1)H_*(\mu,\varepsilon)}\right), \end{aligned} \tag{3.92}$$

and

$$\alpha_* := \liminf_{t\to+\infty} \left(\frac{\nu(t)}{t}\right)^{1-\delta F_*}. \tag{3.93}$$

Then every proper solution of equation (3.1) *is oscillatory.*

REMARK 3.38. Observe that $0 \le \alpha_* \le 1$ and

$$\max\left\{\frac{\lambda}{2}, 0\right\} \le r_0 \le \frac{1}{2} \le R_0 \le \min\left\{\frac{\mu}{2}, 1\right\}. \tag{3.94}$$

REMARK 3.39. For the validity of assumption (3.91) it is necessary that

$$\int_0^{+\infty} \tau(s)p(s)\left(\frac{\sigma(s)}{\tau(s)}\right)^{\varepsilon G_*} ds = +\infty. \tag{3.95}$$

On the other hand, we suppose that (3.11) hold. It worth mentioning that relations (3.11) and (3.95) are not in any contradiction to each other and thus, Theorem 3.37 is meaningful.

REMARK 3.40. The condition (3.87), necessary for the validity of assumption (3.86) in Theorem 3.32, is weakened in Theorem 3.37 to condition (3.95). Consequently, Theorem 3.37 can be applied also in the case, where

$$\int_0^{+\infty} \tau(s)p(s)ds < +\infty. \tag{3.96}$$

If we put $\nu \equiv \sigma$ and $\varepsilon = \delta$ in Theorem 3.37, we obtain

Corollary 3.41 ([45, Cor. 1]). *Let there exist numbers $\lambda < 1$, $\mu > 1$, $\varepsilon \in [0,1[$ and a non-decreasing function $\sigma\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that conditions* (3.85) *and* (3.90) *are fulfilled and*

$$\limsup_{t\to+\infty} \int_{\sigma(t)}^{t} \tau(s)p(s) \left(\frac{\sigma(s)}{\tau(s)}\right)^{\varepsilon G_*} ds > R_0 - \beta_* r_0,$$

where the numbers r_0 and R_0 are given by relations (3.92) *and*

$$\beta_* := \liminf_{t\to+\infty} \left(\frac{\sigma(t)}{t}\right)^{1-\varepsilon F_*}.$$

Then every proper solution of equation (3.1) *is oscillatory.*

Remark 3.42. In view of relations (3.94), it is clear that $R_0 - \beta_* r_0 \leq 1$ and thus, Corollary 3.41 improves (under additional assumptions (3.90)) Theorem 3.32.

Now we show that, under additional assumptions (3.90), we can also improve the constant $\frac{1}{\mathrm{e}}$ in Theorem 3.34. However, to prove Theorem 3.44 below we need the technical assumption

$$\liminf_{t\to+\infty} \frac{t}{\tau(t)} < +\infty. \tag{3.97}$$

Therefore, we first give an oscillation criterion for the case, where condition (3.97) does not hold.

Theorem 3.43 ([45, Thm. 4]). *Let*

$$\lim_{t\to+\infty} \frac{t}{\tau(t)} = +\infty.$$

Then every proper solution of equation (3.1) *is oscillatory provided $G_* > 0$.*

Now we present above-mentioned statements improving Theorem 3.34.

Theorem 3.44 ([45, Thm. 5]). *Let condition* (3.97) *hold and there exist numbers $\lambda < 1$, $\mu > 1$, and $\varepsilon \in [0,1[$ such that inequalities* (3.90) *are satisfied. Let, moreover, there exist a continuous function $\nu\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying*

$$\tau(t) \leq \nu(t) \leq t \quad \text{for } t \geq 0 \tag{3.98}$$

and

$$\liminf_{t\to+\infty} \nu^{\varepsilon G_*}(t) \int_{\nu(t)}^{t} \tau^{1-\varepsilon G_*}(s)p(s)ds > R_0 - \gamma_* r_0, \tag{3.99}$$

where the numbers r_0 and R_0 are given by relations (3.92) *and*

$$\gamma_* := \liminf_{t\to+\infty} \left(\frac{\nu(t)}{t}\right)^{\varepsilon G_*}. \tag{3.100}$$

Then every proper solution of equation (3.1) *is oscillatory.*

REMARK 3.45. If $\varepsilon G_* > 0$ then inequality (3.99) can be fulfilled even when condition (3.96) hold and thus, condition (3.87), necessary for the validity of assumption (3.88) in Theorem 3.34, is relaxed in Theorem 3.44.

If we put $\nu \equiv \tau$ and $\sigma \equiv \mathrm{id}_{\mathbb{R}_+}$ in Theorems 3.37 and 3.44 then we get

COROLLARY 3.46 ([45, Cor. 2]). *Let there exist numbers $\lambda < 1$, $\mu > 1$ and ε, $\delta \in [0, 1[$ such that inequalities* (3.90) *are satisfied and either*

$$\limsup_{t\to+\infty} \left(\frac{\tau(t)}{t}\right)^{1-\delta F_*} \int_{\tau(t)}^{t} \tau(s)p(s) \left(\frac{s}{\tau(s)}\right)^{\varepsilon G_*} ds > R_0 - \eta_* r_0, \tag{3.101}$$

or condition (3.97) *holds and*

$$\liminf_{t\to+\infty} \tau^{\varepsilon G_*}(t) \int_{\tau(t)}^{t} \tau^{1-\varepsilon G_*}(s)p(s)ds > R_0 - \xi_* r_0, \tag{3.102}$$

where the numbers r_0 and R_0 are given by relations (3.92) *and*

$$\eta_* := \liminf_{t\to+\infty} \left(\frac{\tau(t)}{t}\right)^{1-\delta F_*}, \quad \xi_* := \liminf_{t\to+\infty} \left(\frac{\tau(t)}{t}\right)^{\varepsilon G_*}. \tag{3.103}$$

Then every proper solution of equation (3.1) *is oscillatory.*

Observe that $0 \le \xi_* \le 1$ and the numbers r_0 and R_0 given by relations (3.92) satisfy

$$R_0 - r_0 = \frac{1}{2}\left(\sqrt{1 - 4(1-\lambda)Q_*(\lambda, \varepsilon)} + \sqrt{1 - 4(\mu - 1)H_*(\mu, \varepsilon)}\right)$$

and thus, the difference $R_0 - r_0$ converges to zero if $Q_*(\lambda, \varepsilon) \to \frac{1}{4(1-\lambda)}$ and $H_*(\mu, \varepsilon) \to \frac{1}{4(\mu-1)}$. Consequently, it may happen that $R_0 - \xi_* r_0 < \frac{1}{\mathrm{e}}$ in which case Corollary 3.46 improves Theorem 3.34 (see Example 3.50.)

In the last two statements we ensure that the number ξ_* given by formula (3.103) is equal to 1. At first we put $\varepsilon = 0$ in Corollary 3.46 and we obtain

COROLLARY 3.47 ([45, Cor. 3]). *Let condition* (3.97) *be fulfilled and there exist numbers $\lambda < 1$ and $\mu > 1$ such that*

$$\frac{\lambda(2-\lambda)}{4(1-\lambda)} \le Q_*(\lambda, 0) \le \frac{1}{4(1-\lambda)}, \quad \frac{\mu(2-\mu)}{4(\mu-1)} \le H_*(\mu, 0) \le \frac{1}{4(\mu-1)}.$$

If, moreover,

$$\liminf_{t\to+\infty} \int_{\tau(t)}^{t} \tau(s)p(s)ds > \frac{1}{2}\left(\sqrt{1 - 4(1-\lambda)Q_*(\lambda, 0)} + \sqrt{1 - 4(\mu - 1)H_*(\mu, 0)}\right)$$

then every proper solution to equation (3.1) *is oscillatory.*

It is clear that if

$$\liminf_{t\to+\infty} \frac{\tau(t)}{t} > 0$$

then, in view of assumption (3.11), we have

$$\int_0^{+\infty} s^{\lambda} p(s)ds < +\infty \quad \text{for all } \lambda < 1.$$

It allows one to define in this case for every $\lambda < 1$ and $\mu > 1$, the numbers

$$\widetilde{Q}_*(\lambda) := \liminf_{t\to+\infty} t^{1-\lambda} \int_t^{+\infty} s^{\lambda} p(s)ds, \quad \widetilde{H}_*(\mu) := \liminf_{t\to+\infty} \frac{1}{t^{\mu-1}} \int_0^t s^{\mu} p(s)ds.$$

Therefore, from Corollary 3.46 we derive

Corollary 3.48 ([45, Cor. 4]). *Let*

$$\lim_{t\to+\infty} \frac{\tau(t)}{t} = 1 \tag{3.104}$$

and there exist numbers $\lambda < 1$ *and* $\mu > 1$ *such that the inequalities*

$$\frac{\lambda(2-\lambda)}{4(1-\lambda)} \le \widetilde{Q}_*(\lambda) \le \frac{1}{4(1-\lambda)}, \quad \frac{\mu(2-\mu)}{4(\mu-1)} \le \widetilde{H}_*(\mu) \le \frac{1}{4(\mu-1)}$$

are satisfied. If, moreover,

$$\limsup_{t\to+\infty} \int_{\tau(t)}^t sp(s)ds > \frac{1}{2}\left(\sqrt{1-4(1-\lambda)\widetilde{Q}_*(\lambda)} + \sqrt{1-4(\mu-1)\widetilde{H}_*(\mu)}\right)$$

then every proper solution of equation (3.1) *is oscillatory.*

Remark 3.49. Under the additional assumption (3.104), not only the constant $\frac{1}{\mathrm{e}}$ on the right-hand side of inequality (3.88) in Theorem 3.34 can be improved, but it is also possible to replace the lower limit in (3.88) by the upper limit.

We mention that assumption (3.104) is meaningful because it holds for a wide class of delay differential equations, namely, for differential equations with a bounded delay frequently studied in the literature. On the other hand, we can easily find an example of an unbounded delay for which equality (3.104) is satisfied, as well (e. g., if $\tau(t) = t - \sqrt{t}$ for t large enough).

Example 3.50. On $\mathbb{R}_+$, we consider the equation with a proportional delay

$$u''(t) + \min\left\{\frac{1}{2}, \frac{1}{2t^2}\right\} u\left(\frac{t}{2}\right) = 0. \tag{3.105}$$

One can easily see that condition (3.88) is not fulfilled, because

$$\liminf_{t\to+\infty} \int_{\tau(t)}^t \tau(s)p(s)ds = \liminf_{t\to+\infty} \int_{\frac{t}{2}}^t \frac{s}{2}\frac{1}{2s^2}ds = \frac{\ln 2}{4} \not> \frac{1}{e}. \tag{3.106}$$

Therefore, Theorem 3.34 cannot be applied. On the other hand, condition (3.90) with $\lambda = 0$ and $\mu = 2$ is satisfied, since

$$Q_*(0,0) = \liminf_{t\to+\infty} t\int_t^{+\infty} \frac{\tau(s)}{s}\,p(s)ds = \liminf_{t\to+\infty} t\int_t^{+\infty} \frac{s}{2s}\,\frac{1}{2s^2}\,ds = \frac{1}{4},$$

$$H_*(2,0) = \liminf_{t\to+\infty} \frac{1}{t}\int_0^t s^2\frac{\tau(s)}{s}\,p(s)ds = \liminf_{t\to+\infty} \frac{1}{t}\int_0^t s^2\frac{s}{2s}\,\frac{1}{2s^2}\,ds = \frac{1}{4}$$

and, moreover, $r_0 = 0$, $R_0 = 0$, where numbers r_0 and R_0 are defined by (3.92). By virtue of this and (3.106), one can see that (3.102) holds for $\varepsilon = 0$. Finally, it is clear that assumption (3.97) is satisfied and thus, according to Corollary 3.46, every proper solution of equation (3.105) is oscillatory.

3.3.2 Proofs of main results

Proof of Theorem 3.37. Suppose on the contrary that equation (3.1) has a proper non-oscillatory solution. Let u be a solution of equation (3.1) on the interval $[t_u, +\infty[$ satisfying relation (3.27). Analogously to the proof of Lemma 3.22 one can show that there exist numbers $t_2 \geq t_1 > t_u$ such that relations (3.32) and (3.34) hold.

For given ε and δ, let $t_0(\varepsilon)$ and $t_0(\delta)$ be numbers appearing in Lemma 3.22 with $\zeta = \varepsilon$ and $\zeta = \delta$, respectively. In view of assumption (3.84), there exists $t_3 \geq t_2$ such that

$$\tau(t) \geq \max\left\{t_2, t_0(\varepsilon), t_0(\delta)\right\} \quad \text{for } t \geq t_3. \tag{3.107}$$

Furthermore,there exists a number $t_4 \geq t_3$ such that

$$\nu(t) \geq t_3 \quad \text{for } t \geq t_4. \tag{3.108}$$

The integration of equality (3.1) from $\nu(t)$ to t leads to the equality

$$u'(\nu(t)) - u'(t) = \int_{\nu(t)}^t p(s)u(\tau(s))ds \quad \text{for } t \geq t_4.$$

Therefore, using relations (3.89), (3.30), (3.34), (3.107), (3.108) and the assumption that the function σ is non-decreasing, one gets

$$\begin{aligned}
u'(\nu(t)) - u'(t) &= \int_{\nu(t)}^t p(s)\frac{u(\tau(s))}{u(\sigma(s))}\,u(\sigma(s))ds \\
&\geq \int_{\nu(t)}^t p(s)\left(\frac{\tau(s)}{\sigma(s)}\right)^{1-\varepsilon G_*} u(\sigma(s))ds \\
&= \int_{\nu(t)}^t \tau(s)p(s)\left(\frac{\sigma(s)}{\tau(s)}\right)^{\varepsilon G_*} \frac{u(\sigma(s))}{\sigma(s)}\,ds \\
&\geq \frac{u(\sigma(t))}{\sigma(t)}\int_{\nu(t)}^t \tau(s)p(s)\left(\frac{\sigma(s)}{\tau(s)}\right)^{\varepsilon G_*} ds \quad \text{for } t \geq t_4,
\end{aligned}$$

whence we obtain

$$\nu(t)\varrho(\nu(t)) \geq t\varrho(t)\frac{\nu(t)}{t}\frac{u(t)}{u(\nu(t))} + \frac{\nu(t)}{\sigma(t)}\frac{u(\sigma(t))}{u(\nu(t))}\int_{\nu(t)}^{t}\tau(s)p(s)\left(\frac{\sigma(s)}{\tau(s)}\right)^{\varepsilon G_*} ds \quad \text{for } t \geq t_4,$$

where

$$\varrho(t) := \frac{u'(t)}{u(t)} \quad \text{for } t \geq t_u. \tag{3.109}$$

Hence, by virtue of estimates (3.30), we obtain

$$\nu(t)\varrho(\nu(t)) \geq t\varrho(t)\left(\frac{\nu(t)}{t}\right)^{1-\delta F_*} + \left(\frac{\nu(t)}{\sigma(t)}\right)^{1-\delta F_*}\int_{\nu(t)}^{t}\tau(s)p(s)\left(\frac{\sigma(s)}{\tau(s)}\right)^{\varepsilon G_*} ds \quad \text{for } t \geq t_4. \tag{3.110}$$

Now we put

$$r := \liminf_{t\to+\infty} t\varrho(t), \qquad R := \limsup_{t\to+\infty} t\varrho(t), \tag{3.111}$$

and

$$A(t) := \left(\frac{\nu(t)}{\sigma(t)}\right)^{1-\delta F_*}\int_{\nu(t)}^{t}\tau(s)p(s)\left(\frac{\sigma(s)}{\tau(s)}\right)^{\varepsilon G_*} ds \quad \text{for } t \geq t_4. \tag{3.112}$$

It follows from Lemmas 3.29 and 3.30 that

$$q_*(\lambda, u) \leq \frac{1}{4(1-\lambda)}, \quad h_*(\mu, u) \leq \frac{1}{4(\mu-1)}, \tag{3.113}$$

where the numbers $q_*(\lambda, u)$ and $h_*(\mu, u)$ are defined by (3.55) and (3.56). Moreover, using assumption (3.83) and estimates (3.30) we easily get the inequalities

$$q_*(\lambda, u) \geq Q_*(\lambda, \varepsilon), \qquad h_*(\mu, u) \geq H_*(\mu, \varepsilon). \tag{3.114}$$

Therefore, conditions (3.113), (3.114) and assumption (3.90) yield the validity of inequalities (3.59) and (3.65) and thus, it follows from Lemmas 3.26 and 3.27 that

$$r \geq \frac{1}{2}\left(1-\sqrt{1-4(1-\lambda)q_*(\lambda, u)}\right) \geq \frac{1}{2}\left(1-\sqrt{1-4(1-\lambda)Q_*(\lambda, \varepsilon)}\right) \geq 0$$

and

$$R \leq \frac{1}{2}\left(1+\sqrt{1-4(\mu-1)h_*(\mu, u)}\right) \leq \frac{1}{2}\left(1+\sqrt{1-4(\mu-1)H_*(\mu, \varepsilon)}\right). \tag{3.115}$$

Let $\varepsilon_0 > 0$ be arbitrary and choose $t_{\varepsilon_0} \geq t_4$ such that

$$t\varrho(t) \geq r - \varepsilon_0, \quad \nu(t)\varrho(\nu(t)) \leq R + \varepsilon_0 \quad \text{for } t \geq t_{\varepsilon_0}. \tag{3.116}$$

Then, using assumption (3.89), from inequality (3.110) we get

$$A(t) \le R + \varepsilon_0 - r\left(\frac{\nu(t)}{t}\right)^{1-\delta F_*} + \varepsilon_0 \quad \text{for } t \ge t_{\varepsilon_0}$$

which, in view of notation (3.93), implies that

$$\limsup_{t\to+\infty} A(t) \le R - r\alpha_*, \tag{3.117}$$

because the number ε_0 was arbitrary. Hence, by virtue of notation (3.92), inequality (3.117) guarantees that

$$\limsup_{t\to+\infty} A(t) \le R_0 - \alpha_* r_0$$

which, in view of notation (3.112), contradicts assumption (3.91). □

Proof of Corollary 3.41. It immediately follows from Theorem 3.37 with $\nu \equiv \sigma$ and $\varepsilon = \delta$. □

Proof of Theorem 3.43. It can be found in [45]. □

Proof of Theorem 3.44. Suppose on the contrary that equation (3.1) has a proper non-oscillatory solution. Let u be a solution of equation (3.1) on the interval $[t_u, +\infty[$ satisfying relation (3.27). It is not difficult to verify that there exists $t_1 > t_u$ such that

$$u'(t) \ge 0 \quad \text{for } t \ge t_1.$$

For given ε, let $t_0(\varepsilon)$ be the number appearing in Lemma 3.22 with $\zeta = \varepsilon$. In view of assumption (3.84), there exists $t_2 \ge t_1$ such that

$$\tau(t) > t_0(\varepsilon) \quad \text{for } t \ge t_2. \tag{3.118}$$

Furthermore, there exists a number $t_3 \ge t_2$ such that

$$\nu(t) \ge t_2 \quad \text{for } t \ge t_3. \tag{3.119}$$

The integration of equation (3.1) from $\nu(t)$ to t leads to the equality

$$u'(\nu(t)) - u'(t) = \int_{\nu(t)}^{t} p(s)u(\tau(s))ds \quad \text{for } t \ge t_3.$$

Therefore, using relations (3.83), (3.98), (3.30), (3.118), and (3.119), one gets

$$\begin{aligned}
u'(\nu(t)) - u'(t) &= \int_{\nu(t)}^{t} p(s)s^{1-\varepsilon G_*}\frac{u(\tau(s))}{u(s)}\,\frac{u(s)}{s^{1-\varepsilon G_*}}\,ds \\
&\ge \frac{u(t)}{t^{1-\varepsilon G_*}}\int_{\nu(t)}^{t} p(s)s^{1-\varepsilon G_*}\frac{u(\tau(s))}{u(s)}\,ds \\
&= \frac{u(t)}{t^{1-\varepsilon G_*}}\int_{\nu(t)}^{t} \tau^{1-\varepsilon G_*}(s)p(s)\varphi(s)ds \quad \text{for } t \ge t_3,
\end{aligned}$$

where

$$\varphi(t) := \frac{u(\tau(t))}{u(t)} \left(\frac{t}{\tau(t)}\right)^{1-\varepsilon G_*} \quad \text{for } t \geq t_2. \tag{3.120}$$

Hence, by virtue of estimates (3.30), for $t \geq t_3$ we obtain

$$\begin{aligned} \nu(t)\varrho(\nu(t))\varphi(t) &\geq t\varrho(t) \left(\frac{\nu(t)}{t}\right)^{\varepsilon G_*} \frac{u(\tau(t))}{u(\nu(t))} \left(\frac{\nu(t)}{\tau(t)}\right)^{1-\varepsilon G_*} \\ &\quad + \frac{u(\tau(t))}{u(\nu(t))} \left(\frac{\nu(t)}{\tau(t)}\right)^{1-\varepsilon G_*} \nu^{\varepsilon G_*}(t) \int_{\nu(t)}^{t} \tau^{1-\varepsilon G_*}(s)p(s)\varphi(s)ds \qquad (3.121) \\ &\geq t\varrho(t) \left(\frac{\nu(t)}{t}\right)^{\varepsilon G_*} + \nu^{\varepsilon G_*}(t) \int_{\nu(t)}^{t} \tau^{1-\varepsilon G_*}(s)p(s)\varphi(s)ds, \end{aligned}$$

where the function ϱ is given by formula (3.109).

Define the numbers r and R by relations (3.111) and put

$$B(t) := \nu^{\varepsilon G_*}(t) \int_{\nu(t)}^{t} \tau^{1-\varepsilon G_*}(s)p(s)ds \quad \text{for } t \geq t_3 \tag{3.122}$$

and

$$\varphi_* := \liminf_{t\to+\infty} \varphi(t), \tag{3.123}$$

where the function φ is defined by (3.120). Observe that, in view of assumptions (3.83) and (3.97), estimates (3.30) yield that

$$1 \leq \varphi_* < +\infty.$$

On the other hand, it follows from Lemmas 3.26 and 3.27 that inequalities (3.113) hold, where the numbers $q_*(\lambda, u)$ and $h_*(\mu, u)$ are defined by (3.55) and (3.56). Moreover, using assumption (3.83) and estimates (3.30) we easily get inequalities (3.114). Therefore, conditions (3.113), (3.114) and assumption (3.90) yield the validity of inequalities (3.59) and (3.65) and thus, it follows from Lemmas 3.26 and 3.27 that relation (3.115) holds and

$$0 \leq \frac{1}{2}\left(1 - \sqrt{1 - 4(1-\lambda)q_*(\lambda, u)}\right) \leq r \leq \frac{1}{2}\left(1 + \sqrt{1 - 4(1-\lambda)q_*(\lambda, u)}\right). \tag{3.124}$$

Let $\varepsilon_0 \in]0, 1]$ be arbitrary and choose $t_{\varepsilon_0} \geq t_3$ such that relations (3.116) hold and

$$\varphi(t) \geq \varphi_* - \varepsilon_0 \quad \text{for } t \geq t_{\varepsilon_0}.$$

Furthermore, we find a number $t^*_{\varepsilon_0} \geq t_{\varepsilon_0}$ such that

$$\nu(t) \geq t_{\varepsilon_0} \quad \text{for } t \geq t^*_{\varepsilon_0}.$$

Then, using assumption (3.98), from inequalities (3.121) we get

$$\begin{aligned} (R + \varepsilon_0)\varphi(t) &\geq (r - \varepsilon_0) \left(\frac{\nu(t)}{t}\right)^{\varepsilon G_*} + (\varphi_* - \varepsilon_0)B(t) \\ &\geq r \left(\frac{\nu(t)}{t}\right)^{\varepsilon G_*} - \varepsilon_0 + (\varphi_* - \varepsilon_0)B(t) \quad \text{for } t \geq t^*_{\varepsilon_0} \end{aligned}$$

which, in view of notation (3.100) and (3.123), implies that

$$\liminf_{t\to+\infty} B(t) \le R - \frac{r}{\varphi_*}\,\gamma_*, \tag{3.125}$$

because the number ε_0 was arbitrary.

Observe that inequalities (3.124) can be rewritten equivalently to the form

$$r^2 - r + (1-\lambda)q_*(\lambda, u) \le 0. \tag{3.126}$$

Moreover, according to notation (3.123), we get

$$q_*(\lambda, u) \ge \varphi_* Q_*(\lambda, \varepsilon).$$

Hence, it follows immediately from inequality (3.126) that

$$r^2 - r + (1-\lambda)\varphi_* Q_*(\lambda, \varepsilon) \le 0,$$

whence we get

$$\left(\frac{r}{\varphi_*}\right)^2 - \frac{r}{\varphi_*} + (1-\lambda)Q_*(\lambda, \varepsilon) \le 0,$$

because $\varphi_* \ge 1$. Consequently, we have

$$\frac{r}{\varphi_*} \ge \frac{1}{2}\left(1 - \sqrt{1 - 4(1-\lambda)Q_*(\lambda, \varepsilon)}\right).$$

Finally, by virtue of relation (3.115) and notation (3.92), inequality (3.125) guarantees that

$$\liminf_{t\to+\infty} B(t) \le R_0 - \gamma_* r_0$$

which, in view of notation (3.122), contradicts assumption (3.99). □

Proof of Corollary 3.46. The assertion of the Corollary follows from Theorems 3.37 and 3.44 with $\nu \equiv \tau$ and $\sigma \equiv \mathrm{id}_{\mathbb{R}_+}$. □

Proof of Corollary 3.47. It immediately follows from Corollary 3.46 with $\varepsilon = 0$. □

Proof of Corollary 3.48. Using assumptions (3.83), (3.84), and (3.104), we easily get

$$Q_*(\lambda, 0) = \widetilde{Q}_*(\lambda), \qquad H_*(\mu, 0) = \widetilde{H}_*(\mu),$$

and

$$\limsup_{t\to+\infty} \frac{\tau(t)}{t} \int_{\tau(t)}^{t} \tau(s)p(s)ds = \limsup_{t\to+\infty} \int_{\tau(t)}^{t} sp(s)ds.$$

Consequently, the assertion of the corollary follows from Corollary 3.46 with $\varepsilon = \delta = 0$. □

3.4 Nonlinear system

3.4.1 Introduction

On the half-line $[0, +\infty[$, we consider the two-dimensional system of nonlinear ordinary differential equations

$$\begin{aligned} u' &= g(t)|v|^{\frac{1}{\alpha}} \operatorname{sgn} v, \\ v' &= -p(t)|u|^{\alpha} \operatorname{sgn} u, \end{aligned} \tag{3.127}$$

where $\alpha > 0$ and p, $g : [0, +\infty[\to \mathbb{R}$ are locally Lebesgue integrable functions.

By a solution of system (3.127) on the interval $J \subseteq [0, +\infty[$ we understand a pair (u, v) of functions $u, v : J \to \mathbb{R}$, which are absolutely continuous on every compact interval contained in J and satisfy equalities (3.127) almost everywhere in J.

It was proved by Mirzov in [35] that all non-extendable solutions of system (3.127) are defined on the whole interval $[0, +\infty[$. Therefore, when we are speaking about a solution of system (3.127), we assume that it is defined on $[0, +\infty[$.

Definition 3.51. A solution (u, v) of system (3.127) is called *non-trivial* if $u \not\equiv 0$ on any neighborhood of $+\infty$. We say that a non-trivial solution (u, v) of system (3.127) is *oscillatory* if the function u has a sequence of zeros tending to infinity, and *non-oscillatory* otherwise.

In [35, Theorem 1.1], it is shown that a certain analogue of Sturm's theorem holds for system (3.127), if the additional assumption

$$g(t) \geq 0 \quad \text{for a. e. } t \geq 0 \tag{3.128}$$

is satisfied. Especially, under assumption (3.128), if system (3.127) has an oscillatory solution, then any other its non-trivial solution is also oscillatory.

On the other hand, it is clear that if $g \equiv 0$ on some neighborhood of $+\infty$, then all non-trivial solutions of system (3.127) are non-oscillatory. That is why it is natural to assume that inequality (3.128) is satisfied and

$$\operatorname{meas}\{\tau \geq t : g(\tau) > 0\} > 0 \quad \text{for } t \geq 0. \tag{3.129}$$

Definition 3.52. We say that system (3.127) is *oscillatory* if all its non-trivial solutions are oscillatory.

Oscillation theory for ordinary differential equations and their systems is a widely studied and well-developed topic of the qualitative theory of differential equations. As for the results which are closely related to those of this section, we should mention [6, 13, 14, 16, 24–26, 34, 38, 46]. Some criteria established in these papers for the second order linear differential equations or for two-dimensional systems of linear differential equations are generalized to the considered system (3.127) below.

Many results (see, e.g., survey given in [6]) have been obtained in oscillation theory of the so-called "half-linear" equation

$$\big(r(t)|u'|^{q-1}\operatorname{sgn} u'\big)' + p(t)|u|^{q-1}\operatorname{sgn} u = 0 \tag{3.130}$$

(alternatively this equation is referred as "equation with the scalar q-Laplacian"). Equation (3.130) is usually considered under the assumptions $q > 1$, $p, r : [0, +\infty[\to \mathbb{R}$ are continuous and r is positive. One can see that equation (3.130) is a particular case of system (3.127). Indeed, if the function u, with properties $u \in C^1$ and $r|u'|^{q-1}\text{sgn}\, u' \in C^1$, is a solution of equation (3.130), then the vector function $(u, r|u'|^{q-1}\text{sgn}\, u')$ is a solution of system (3.127) with $g(t) := r^{\frac{1}{1-q}}(t)$ for $t \geq 0$ and $\alpha := q - 1$.

Moreover, the equation

$$u'' + \frac{1}{\alpha} p(t)|u|^{\alpha}|u'|^{1-\alpha}\text{sgn} u = 0 \tag{3.131}$$

is also studied in the existing literature under the assumptions $\alpha \in]0, 1]$ and $p : \mathbb{R}_+ \to \mathbb{R}$ is a locally integrable function. It is mentioned in [16] that if u is a so-called proper solution of (3.131) then it is also a solution of system (3.127) with $g \equiv 1$ and vice versa. Some oscillation and non-oscillation criteria for equation (3.131) can be found, e.g., in [16, 24].

Finally, we mention the paper [5], where a certain analogy of Hartman-Wintner's theorem is established (origin one can find in [12, 50]), which allows us to derive oscillation criteria of Hille-Nehari's type for system (3.127).

In what follows, we assume that the coefficient g is non-integrable on $[0, +\infty[$, i.e.,

$$\int_0^{+\infty} g(s)ds = +\infty. \tag{3.132}$$

Let

$$f(t) := \int_0^t g(t)ds \quad \text{for } t \geq 0.$$

In view of assumptions (3.128), (3.129), and (3.132), we have

$$\lim_{t\to+\infty} f(t) = +\infty \tag{3.133}$$

and there exists $t_g \geq 0$ such that $f(t) > 0$ for $t > t_g$ and $f(t_g) = 0$. We can assume without loss of generality that $t_g = 0$, since we are interested in behaviour of solutions in the neighbourhood of $+\infty$, i.e., we have

$$f(t) > 0 \quad \text{for } t > 0. \tag{3.134}$$

For any $\lambda \in [0, \alpha[$, we put

$$c_\alpha(t; \lambda) := \frac{\alpha - \lambda}{f^{\alpha-\lambda}(t)} \int_0^t \frac{g(s)}{f^{\lambda-\alpha+1}(s)} \left(\int_0^s f^{\lambda}(\xi)p(\xi)d\xi \right) ds \quad \text{for } t > 0.$$

Now, we formulate an analogue (in a suitable form for us) of the Hartman-Wintner's theorem for the system (3.127) established in [5].

THEOREM 3.53 ([5, Corollary 2.5 (with $\nu = 1 - \alpha + \lambda$)]). *Let conditions* (3.128), (3.129), *and* (3.132) *hold,* $\lambda < \alpha$, *and either*

$$\lim_{t\to+\infty} c_\alpha(t; \lambda) = +\infty,$$

or

$$-\infty < \liminf_{t\to+\infty} c_\alpha(t;\lambda) < \limsup_{t\to+\infty} c_\alpha(t;\lambda).$$

Then system (3.127) *is oscillatory.*

Clearly, two cases are not covered by Theorem 3.53, namely, $\liminf_{t\to+\infty} c_\alpha(t;\lambda) = -\infty$ and the function $c_\alpha(t;\lambda)$ has a finite limit. The aim of this section is to find oscillation criteria for system (3.127) in the second mentioned case. Consequently, in what follows, we assume that

$$\lim_{t\to+\infty} c_\alpha(t;\lambda) =: c^*_\alpha(\lambda) \in \mathbb{R}. \tag{3.135}$$

3.4.2 Main results

In this section, we formulate main results and theirs corollaries.

THEOREM 3.54. *Let* $\lambda \in [0,\alpha[$ *and* (3.135) *hold. Let, moreover, the inequality*

$$\limsup_{t\to+\infty} \frac{f^{\alpha-\lambda}(t)}{\ln f(t)} \left(c^*_\alpha(\lambda) - c_\alpha(t;\lambda)\right) > \left(\frac{\alpha}{1+\alpha}\right)^{1+\alpha} \tag{3.136}$$

be satisfied. Then system (3.127) *is oscillatory.*

We introduce the following notations. For any $\lambda \in [0,\alpha[$ and $\mu \in]\alpha,+\infty[$, we put

$$Q(t;\alpha,\lambda) := f^{\alpha-\lambda}(t)\left(c^*_\alpha(\lambda) - \int_0^t p(s)f^\lambda(s)ds\right) \quad \text{for } t>0,$$
$$H(t;\alpha,\mu) := \frac{1}{f^{\mu-\alpha}(t)}\left(\int_0^t p(s)f^\mu(s)ds\right) \quad \text{for } t>0,$$

where the number $c^*_\alpha(\lambda)$ is given by (3.135). Moreover, we denote lower and upper limits of the functions $Q(\cdot;\alpha,\lambda)$ and $H(\cdot;\alpha,\mu)$ as follows

$$Q_*(\alpha,\lambda) := \liminf_{t\to+\infty} Q(t;\alpha,\lambda), \qquad H_*(\alpha,\mu) := \liminf_{t\to+\infty} H(t;\alpha,\mu),$$
$$Q^*(\alpha,\lambda) := \limsup_{t\to+\infty} Q(t;\alpha,\lambda), \qquad H^*(\alpha,\mu) := \limsup_{t\to+\infty} H(t;\alpha,\mu).$$

Now we formulate two corollaries of Theorem 3.54.

COROLLARY 3.55. *Let* $\lambda \in [0,\alpha[$, $\mu \in]\alpha,+\infty[$, *and* (3.135) *hold. Let, moreover,*

$$\liminf_{t\to+\infty} \left(Q(t;\alpha,\lambda) + H(t;\alpha,\mu)\right) > \frac{\mu-\lambda}{(\alpha-\lambda)(\mu-\alpha)}\left(\frac{\alpha}{1+\alpha}\right)^{1+\alpha}. \tag{3.137}$$

Then system (3.127) *is oscillatory.*

Corollary 3.56. *$\lambda \in [0,\alpha[\,,\ \mu \in]\alpha,+\infty[\,,$ and* (3.135) *hold. Let, moreover, either*

$$Q_*(\alpha,\lambda) > \frac{1}{\alpha-\lambda}\left(\frac{\alpha}{1+\alpha}\right)^{1+\alpha}, \tag{3.138}$$

or

$$H_*(\alpha,\mu) > \frac{1}{\mu-\alpha}\left(\frac{\alpha}{1+\alpha}\right)^{1+\alpha}. \tag{3.139}$$

Then system (3.127) *is oscillatory.*

Remark 3.57. Oscillation criteria (3.138) and (3.139) coincide with the well-known Hille-Nehari's results for the second order linear differential equations established in [13, 38].

Theorem 3.58. *Let $\lambda \in [0,\alpha[\,,\ \mu \in]\alpha,+\infty[\,,$ and* (3.135) *hold. Let, moreover,*

$$\limsup_{t\to+\infty}\left(Q(t;\alpha,\lambda)+H(t;\alpha,\mu)\right) > \frac{1}{\alpha-\lambda}\left(\frac{\lambda}{1+\alpha}\right)^{1+\alpha}+\frac{1}{\mu-\alpha}\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha}. \tag{3.140}$$

Then system (3.127) *is oscillatory.*

Now we give two statements complementing Corollary 3.56 in a certain sense.

Theorem 3.59. *Let $\lambda \in [0,\alpha[\,,\ \mu \in]\alpha,+\infty[\,,$ and* (3.135) *hold. Let, moreover, inequalities*

$$\frac{\alpha}{\alpha-\lambda}\left(\gamma-\gamma^{\frac{1+\alpha}{\alpha}}\right) \le Q_*(\alpha,\lambda) \le \frac{1}{\alpha-\lambda}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \tag{3.141}$$

and

$$H^*(\alpha,\mu) > \frac{1}{\mu-\alpha}\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha} - \gamma - A(\alpha,\lambda) \tag{3.142}$$

be satisfied, where

$$\gamma := \left(\frac{\lambda}{1+\alpha}\right)^{\alpha} \tag{3.143}$$

and $A(\alpha,\lambda)$ is the smallest root of the equation

$$\alpha|x+\gamma|^{\frac{1+\alpha}{\alpha}} - \alpha x + (\alpha-\lambda)Q_*(\alpha,\lambda) - \alpha\gamma = 0. \tag{3.144}$$

Then system (3.127) *is oscillatory.*

Theorem 3.60. *Let $\lambda \in [0,\alpha[\,,\ \mu \in]\alpha,+\infty[\,,$ and* (3.135) *hold. Let, moreover, inequalities*

$$\left(\frac{\mu}{1+\alpha}\right)^{\alpha}\frac{\alpha(1+\alpha-\mu)}{(\mu-\alpha)(1+\alpha)} \le H_*(\alpha,\mu) \le \frac{1}{\mu-\alpha}\left(\frac{\alpha}{1+\alpha}\right)^{1+\alpha} \tag{3.145}$$

and

$$Q^*(\alpha,\lambda) > B(\alpha,\mu) + \frac{1}{\alpha-\lambda}\left(\frac{\lambda}{1+\alpha}\right)^{1+\alpha} \tag{3.146}$$

be satisfied, where $B(\alpha,\mu)$ is the greatest root of the equation

$$\alpha|x|^{\frac{1+\alpha}{\alpha}} - \alpha x + (\mu-\alpha)H_*(\alpha,\mu) = 0. \tag{3.147}$$

Then system (3.127) *is oscillatory.*

Finally, we formulate an assertion for the case, when both conditions (3.141) and (3.145) are fulfilled. In this case we can obtain better results than in Theorems 3.59 and 3.60.

THEOREM 3.61. *Let* $\lambda \in [0, \alpha[$, $\mu \in]\alpha, +\infty[$, *and* (3.135) *hold. Let, moreover, conditions* (3.141) *and* (3.145) *be satisfied and*

$$\begin{aligned}\limsup_{t\to+\infty} \left(Q(t;\alpha,\lambda) + H(t;\alpha,\mu)\right) > B(\alpha,\mu) - A(\alpha,\lambda) \\ + Q_*(\alpha,\lambda) + H_*(\alpha,\mu) - \gamma,\end{aligned} \tag{3.148}$$

where the number γ *is defined by* (3.143), $A(\alpha,\lambda)$ *is the smallest root of equation* (3.144), *and* $B(\alpha,\mu)$ *is the greatest root of equation* (3.147). *Then system* (3.127) *is oscillatory.*

REMARK 3.62. Presented statements generalize results stated in [6,13,14,16,24–26,34,38,46] concerning system (3.127) as well as equations (3.130) and (3.131). In particular, if we put $\alpha = 1$, $\lambda = 0$, and $\mu = 2$, then we obtain oscillation criteria for linear system of differential equations presented in [46]. Moreover, the results of [16] obtained for equation (3.131) are in a compliance with those above, where we put $g \equiv 1$, $\lambda = 0$, and $\mu = 1 + \alpha$. Observe also that Corollary 3.56 and Theorems 3.59 and 3.60 extend oscillation criteria for equation (3.131) stated in [24], where the coefficient p is supposed to be nonnegative. In the monograph [6], it is noted that the assumption $p(t) \geq 0$ for t large enough can be easily relaxed to $\int_0^t p(s)ds > 0$ for large t. It is worth mentioning here that we do not require any assumption of this kind.

Finally we provide an example, where we cannot apply oscillation criteria from the above-mentioned papers, but we can use Theorem 3.54 successfully.

EXAMPLE 3.63. Let $\alpha = 2$ and

$$g(t) := 1, \quad p(t) := t\cos\left(\frac{t^2}{2}\right) + \frac{1}{(t+1)^3} \quad \text{for } t \geq 0.$$

It is clear that the function p and its integral

$$\int_0^t p(s)ds = \sin\left(\frac{t^2}{2}\right) - \frac{1}{2(t+1)^2} + \frac{1}{2} \quad \text{for } t \geq 0$$

change their signs in any neighbourhood of $+\infty$. Therefore, neither of results mentioned in Remark 3.62 can be applied.

On the other hand, we have

$$\begin{aligned}c_2(t;0) &= \frac{2}{t^2}\int_0^t s\left(\int_0^s \left(\xi\cos\frac{\xi^2}{2} + \frac{1}{(\xi+1)^3}\right)d\xi\right)ds \\ &= \frac{1}{2} - \frac{2\cos\frac{t^2}{2}}{t^2} + \frac{3}{t^2} - \frac{\ln(t+1)}{t^2} - \frac{1}{t^2(t+1)} \quad \text{for } t > 0\end{aligned}$$

and thus, the function $c_2(\cdot, 0)$ has the finite limit

$$c_\alpha^*(0) = \lim_{t\to+\infty} c_2(t;0) = \frac{1}{2}.$$

Moreover,

$$\limsup_{t\to+\infty} \frac{t^2}{\ln t}\left(c_\alpha^*(0) - c_2(t;0)\right) = \limsup_{t\to+\infty}\left(\frac{2\cos\frac{t^2}{2} - 3}{\ln t} + \frac{\ln(t+1)}{\ln t} + \frac{1}{(t+1)\ln t}\right) = 1.$$

Consequently, according to Theorem 3.54 with $\lambda = 0$, system (3.127) is oscillatory.

3.4.3 Auxiliary lemmas

We first formulate two lemmas established in [5], which we use in this section.

LEMMA 3.64 ([5, Lemma 3.1]). *Let $\alpha > 0$ and $\omega \geq 0$. Then the inequality*

$$\omega x - \alpha |x|^{\frac{1+\alpha}{\alpha}} \leq \left(\frac{\omega}{1+\alpha}\right)^{1+\alpha}$$

is satisfied for all $x \in \mathbb{R}$.

LEMMA 3.65 ([5, Lemma 3.2]). *Let $\alpha > 0$. Then*

$$\alpha|x+y|^{\frac{1+\alpha}{\alpha}} \geq \alpha|y|^{\frac{1+\alpha}{\alpha}} + (1+\alpha)x|y|^{\frac{1}{\alpha}}\mathrm{sgn} y \quad \text{for } x, y \in \mathbb{R}.$$

REMARK 3.66. One can easily verify (see the proofs of Lemma 4.2 and Corollary 2.5 in [5]) that if (u, v) is a solution of system (3.127) satisfying

$$u(t) \neq 0 \quad \text{for } t \geq t_u \tag{3.149}$$

with $t_u > 0$ and the function $c_\alpha(\cdot;\lambda)$ has a finite limit (3.135), then

$$\begin{aligned} c_\alpha^*(\lambda) = f^\lambda(t_u)\rho(t_u) + \int_0^{t_u} f^\lambda(s)p(s)ds + \frac{\alpha(\gamma - \gamma^{\frac{1+\alpha}{\alpha}})}{\alpha - \lambda} \cdot \frac{1}{f^{\alpha-\lambda}(t_u)} \\ - \int_{t_u}^{+\infty} g(s)f^{\lambda-1-\alpha}(s)h(s)ds, \end{aligned} \tag{3.150}$$

where the number γ is defined by (3.143),

$$h(t) := \alpha|f^\alpha(t)\rho(t) + \gamma|^{\frac{1+\alpha}{\alpha}} - (1+\alpha)f^\alpha(t)\rho(t)\gamma^{\frac{1}{\alpha}} - \alpha\gamma^{\frac{1+\alpha}{\alpha}} \quad \text{for } t \geq t_u, \tag{3.151}$$

and

$$\rho(t) := \frac{v(t)}{|u(t)|^\alpha \mathrm{sgn}\, u(t)} - \frac{1}{f^\alpha(t)}\left(\frac{\lambda}{1+\alpha}\right)^\alpha \quad \text{for } t \geq t_u. \tag{3.152}$$

Moreover, acording to Lemma 3.65, we have

$$h(t) \geq 0 \quad \text{for } t \geq t_u \tag{3.153}$$

and one can show (see Lemma 4.1 and the proof of Corollary 2.5 in [5]) that

$$\int_{t_u}^{+\infty} g(s)f^{\lambda-1-\alpha}(s)h(s)ds < +\infty. \tag{3.154}$$

LEMMA 3.67. *Let $\lambda \in [0,\alpha[$, (3.135) and (3.141) hold, where the number γ is defined by (3.143). Then every non-oscillatory solution (u,v) of system (3.127) satisfies*

$$\liminf_{t\to+\infty}\left(\frac{f^{\alpha}(t)v(t)}{|u(t)|^{\alpha}\operatorname{sgn}u(t)}-\gamma\right)\geq A(\alpha,\lambda), \tag{3.155}$$

where $A(\alpha,\lambda)$ denotes the smallest root of equation (3.144) .

Proof. Let (u,v) be a non-oscillatory solution of system (3.127). Then there exists $t_u > 0$ such that (3.149) holds. Define the function ρ by (3.152). Then we obtain from (3.127) that

$$\rho'(t) = -p(t) - \alpha g(t)\left|\rho(t) + \frac{\gamma}{f^{\alpha}(t)}\right|^{\frac{1+\alpha}{\alpha}} + \alpha\gamma\frac{g(t)}{f^{1+\alpha}(t)} \quad \text{for a. e. } t \geq t_u. \tag{3.156}$$

Multiplaying the last equality by $f^{\lambda}(t)$ and integrating it from t_u to t, we get

$$\begin{aligned}\int_{t_u}^{t} f^{\lambda}(s)\rho'(s)ds = -\alpha\int_{t_u}^{t} g(s)f^{\lambda-1-\alpha}(s)\,|\rho(s)f^{\alpha}(s)+\gamma|^{\frac{1+\alpha}{\alpha}}\,ds\\ +\alpha\gamma\int_{t_u}^{t} g(s)f^{\lambda-1-\alpha}(s)ds - \int_{t_u}^{t} f^{\lambda}(s)p(s)ds \quad \text{for } t\geq t_u.\end{aligned} \tag{3.157}$$

Integrating the left-hand side of (3.157) by parts, we obtain

$$\begin{aligned}f^{\lambda}(t)\rho(t) = \left(\alpha\gamma - \alpha\gamma^{\frac{1+\alpha}{\alpha}}\right)\int_{t_u}^{t} g(s)f^{\lambda-1-\alpha}(s)ds - \int_{t_u}^{t} f^{\lambda}(s)p(s)ds\\ + f^{\lambda}(t_u)\rho(t_u) - \int_{t_u}^{t} g(s)f^{\lambda-1-\alpha}(s)h(s)ds \quad \text{for } t\geq t_u,\end{aligned}$$

where the function h is defined in (3.151). Hence,

$$\begin{aligned}f^{\lambda}(t)\rho(t) = \delta(t_u) - \int_{0}^{t} f^{\lambda}(s)p(s)ds - \int_{t_u}^{t} g(s)f^{\lambda-1-\alpha}(s)h(s)ds\\ -\frac{\alpha\left(\gamma-\gamma^{\frac{1+\alpha}{\alpha}}\right)}{\alpha-\lambda}\frac{1}{f^{\alpha-\lambda}(t)} \quad \text{for } t\geq t_u,\end{aligned} \tag{3.158}$$

where

$$\delta(t_u) := f^{\lambda}(t_u)\rho(t_u) + \int_{0}^{t_u} f^{\lambda}(s)p(s)ds + \frac{\alpha\left(\gamma-\gamma^{\frac{1+\alpha}{\alpha}}\right)}{\alpha-\lambda}\frac{1}{f^{\alpha-\lambda}(t_u)}.$$

Therefore, in view of relations (3.150) and (3.154), it follows from (3.158) that

$$\begin{aligned}f^{\lambda}(t)\rho(t) = c_{\alpha}^{*}(\lambda) - \int_{0}^{t} f^{\lambda}(s)p(s)ds + \int_{t}^{+\infty} g(s)f^{\lambda-1-\alpha}(s)h(s)ds\\ -\frac{\alpha\left(\gamma-\gamma^{\frac{1+\alpha}{\alpha}}\right)}{\alpha-\lambda}\frac{1}{f^{\alpha-\lambda}(t)} \quad \text{for } t\geq t_u.\end{aligned} \tag{3.159}$$

Hence,

$$f^{\alpha}(t)\rho(t) = Q(t;\alpha,\lambda) + f^{\alpha-\lambda}(t)\int_t^{+\infty} g(s)f^{\lambda-1-\alpha}(s)h(s)ds \\ - \frac{\alpha\left(\gamma - \gamma^{\frac{1+\alpha}{\alpha}}\right)}{\alpha-\lambda} \quad \text{for } t \geq t_u. \tag{3.160}$$

Put

$$m := \liminf_{t\to+\infty} f^{\alpha}(t)\rho(t). \tag{3.161}$$

It is clear that if $m = +\infty$, then (3.155) holds. Therefore, we suppose that

$$m < +\infty.$$

In view of (3.141), (3.153), and (3.161), relation (3.160) yields that

$$m \geq Q_*(\alpha,\lambda) - \frac{\alpha}{\alpha-\lambda}\left(\gamma - \gamma^{\frac{1+\alpha}{\alpha}}\right) \geq 0. \tag{3.162}$$

If $Q_*(\alpha,\lambda) = \frac{\alpha}{\alpha-\lambda}(\gamma - \gamma^{\frac{1+\alpha}{\alpha}})$, then 0 is a root of equation (3.144). Moreover, in view of Lemma 3.65 and the assumption $\lambda < \alpha$, we see that the function $x \mapsto \alpha|x+\gamma|^{\frac{1+\alpha}{\alpha}} - \alpha x - \alpha\gamma^{\frac{1+\alpha}{\alpha}}$ is positive on $]-\infty, 0[$. Consequently, by virtue of notations (3.152), (3.161) and relation (3.162), desired estimate (3.155) holds.

Now suppose that $Q_*(\alpha,\lambda) > \frac{\alpha}{\alpha-\lambda}(\gamma - \gamma^{\frac{1+\alpha}{\alpha}})$. Let $\varepsilon \in]0, Q_*(\alpha,\lambda) - \frac{\alpha}{\alpha-\lambda}(\gamma - \gamma^{\frac{1+\alpha}{\alpha}})[$ be arbitrary. According to (3.162), it is clear that

$$m > \varepsilon. \tag{3.163}$$

Choose $t_\varepsilon \geq t_u$ such that

$$f^{\alpha}(t)\rho(t) \geq m - \varepsilon \quad \text{and} \quad Q(t;\alpha,\lambda) \geq Q_*(\alpha,\lambda) - \varepsilon \quad \text{for } t \geq t_\varepsilon. \tag{3.164}$$

Then it follows from (3.160) that

$$f^{\alpha}(t)\rho(t) \geq Q_*(\alpha,\lambda) - \varepsilon + f^{\alpha-\lambda}(t)\int_t^{+\infty} g(s)f^{\lambda-1-\alpha}(s)h(s)ds \\ - \frac{\alpha\big(\gamma - \gamma^{\frac{1+\alpha}{\alpha}}\big)}{\alpha-\lambda} \quad \text{for } t \geq t_\varepsilon. \tag{3.165}$$

On the other hand, the function $x \mapsto \alpha|x+\gamma|^{\frac{1+\alpha}{\alpha}} - (1+\alpha)x\gamma^{\frac{1}{\alpha}} - \alpha\gamma^{\frac{1+\alpha}{\alpha}}$ is non-decreasing on $[0,+\infty[$. Therefore, by virtue of (3.153), (3.163), and (3.164), one gets from (3.165) that

$$f^{\alpha}(t)\rho(t) \geq Q_*(\alpha,\lambda) - \varepsilon + \frac{\alpha|(m-\varepsilon)+\gamma|^{\frac{1+\alpha}{\alpha}} - \alpha\gamma - \lambda(m-\varepsilon)}{\alpha-\lambda} \quad \text{for } t \geq t_\varepsilon,$$

which implies

$$m \geq Q_*(\alpha,\lambda) - \varepsilon + \frac{\alpha|(m-\varepsilon)+\gamma|^{\frac{1+\alpha}{\alpha}} - \alpha\gamma - \lambda(m-\varepsilon)}{\alpha-\lambda}.$$

Since ε was arbitrary, the latter relation leads to the inequality

$$\alpha|m+\gamma|^{\frac{1+\alpha}{\alpha}} - \alpha m + Q_*(\alpha,\lambda)(\alpha-\lambda) - \alpha\gamma \le 0. \tag{3.166}$$

One can easily derive that the function $y : x \mapsto \alpha|x+\gamma|^{\frac{1+\alpha}{\alpha}} - \alpha x + Q_*(\alpha,\lambda)(\alpha-\lambda) - \alpha\gamma$ is decreasing on $]-\infty, (\frac{\alpha}{1+\alpha})^\alpha - \gamma]$ and increasing on $[(\frac{\alpha}{1+\alpha})^\alpha - \gamma, +\infty[$. Therefore, in view of assumption (3.141), the function y is non-positive at the point $\left(\frac{\alpha}{1+\alpha}\right)^\alpha - \gamma$, which together with (3.152), (3.161), and (3.166) implies desired estimate (3.155). $\square$

LEMMA 3.68. *Let $\mu \in]\alpha, +\infty[$ and* (3.145) *hold. Then every non-oscillatory solution* (u, v) *of system* (3.127) *satisfies*

$$\limsup_{t\to+\infty} \frac{f^\alpha(t)v(t)}{|u(t)|^\alpha \operatorname{sgn} u(t)} \le B(\alpha,\mu), \tag{3.167}$$

where $B(\alpha,\mu)$ is the greatest root of equation (3.147).

Proof. Let (u, v) be a non-oscillatory solution of system (3.127). Then there exists $t_u > 0$ such that (3.149) holds. Define the function ρ by (3.152). Then from (3.127) we obtain the equality (3.156), where the number γ is defined by (3.143).

Multiplaying (3.156) by $f^\mu(t)$ and integrating it from t_u to t, we obtain

$$\int_{t_u}^{t} f^\mu(s)\rho'(s)ds = -\int_{t_u}^{t} f^\mu(s)p(s)ds - \alpha\int_{t_u}^{t} g(s)f^{\mu-\alpha-1}(s)|\rho(s)f^\alpha(s)+\gamma|^{\frac{1+\alpha}{\alpha}}ds$$
$$+\alpha\gamma\int_{t_u}^{t} g(s)f^{\mu-\alpha-1}(s)ds \quad \text{for } t \ge t_u.$$

Integrating the left-hand side of the last equality by parts, we get

$$\begin{aligned} f^\alpha(t)\rho(t) = f^{\alpha-\mu}(t)\int_{t_u}^{t} g(s)f^{\mu-\alpha-1}(s)\left[\mu f^\alpha(s)\rho(s) - \alpha|\rho(s)f^\alpha(s)+\gamma|^{\frac{1+\alpha}{\alpha}}\right]ds \\ + \delta(t_u)f^{\alpha-\mu}(t) - H(t;\alpha,\mu) + \frac{\alpha\gamma}{\mu-\alpha} \quad \text{for } t \ge t_u, \end{aligned} \tag{3.168}$$

where

$$\delta(t_u) := f^\mu(t_u)\rho(t_u) + \int_0^{t_u} f^\mu(s)p(s)ds - \frac{\alpha\gamma}{\mu-\alpha}f^{\mu-\alpha}(t_u). \tag{3.169}$$

According to Lemma 3.64, it follows from (3.168) that

$$f^\alpha(t)\rho(t) \le \delta_1(t_u)f^{\alpha-\mu}(t) - H(t;\alpha,\mu) + \frac{1}{\mu-\alpha}\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha} - \gamma \quad \text{for } t \ge t_u, \tag{3.170}$$

where

$$\delta_1(t_u) := \delta(t_u) - \frac{f^{\mu-\alpha}(t_u)}{\mu-\alpha}\left(\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha} - \mu\gamma\right). \tag{3.171}$$

Put
$$M := \limsup_{t\to+\infty} \left(f^{\alpha}(t)\rho(t) + \gamma\right). \tag{3.172}$$
Obviously, if $M = -\infty$ then (3.167) holds. Therefore, suppose that
$$M > -\infty.$$
By virtue of (3.133), inequality (3.170) yields
$$M \le -H_*(\alpha, \mu) + \frac{1}{\mu-\alpha}\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha}. \tag{3.173}$$

If $H_*(\alpha,\mu) = \left(\frac{\mu}{1+\alpha}\right)^{\alpha} \frac{\alpha(1+\alpha-\mu)}{(\mu-\alpha)(1+\alpha)}$, then it is not difficult to verify that $(\frac{\mu}{1+\alpha})^{\alpha}$ is a root of the equation (3.147) and the function $x \mapsto \alpha|x|^{\frac{1+\alpha}{\alpha}} - \alpha x + (\mu-\alpha)H_*(\alpha,\mu)$ is positive on $](\frac{\mu}{1+\alpha})^{\alpha}, +\infty[$. Consequently, it follows from (3.172) and (3.173) that (3.167) is satisfied.

Now suppose that
$$H_*(\alpha,\mu) > \left(\frac{\mu}{1+\alpha}\right)^{\alpha} \frac{\alpha(1+\alpha-\mu)}{(\mu-\alpha)(1+\alpha)}.$$
Using the latter inequality in (3.173), we get
$$M < \left(\frac{\mu}{1+\alpha}\right)^{\alpha}.$$
Let $\varepsilon \in]0, \left(\frac{\mu}{1+\alpha}\right)^{\alpha} - M[$ be arbitrary and choose $t_\varepsilon \ge t_u$ such that
$$\gamma + f^{\alpha}(t)\rho(t) \le M + \varepsilon, \quad H(t;\alpha,\mu) \ge H_*(\alpha,\mu) - \varepsilon \quad \text{for } t \ge t_\varepsilon. \tag{3.174}$$
Observe that the function $x \mapsto \mu x - \alpha|x|^{\frac{1+\alpha}{\alpha}}$ is non-decrasing on $]-\infty, \left(\frac{\mu}{1+\alpha}\right)^{\alpha}]$ and thus, using relations (3.174) and $M + \varepsilon < \left(\frac{\mu}{1+\alpha}\right)^{\alpha}$, from (3.168) we get
$$\begin{aligned} f^{\alpha}(t)\rho(t) \le{}& \delta_2(t_u) f^{\alpha-\mu}(t) - H_*(\alpha,\mu) + \varepsilon + \frac{\alpha\gamma}{\mu-\alpha} - \frac{\mu\gamma}{\mu-\alpha} \\ &+ f^{\alpha-\mu}(t)\int_{t_u}^{t} g(s) f^{\mu-\alpha-1}(s)\left[\mu\left(M+\varepsilon\right) - \alpha|M+\varepsilon|^{\frac{1+\alpha}{\alpha}}\right] ds \quad \text{for } t \ge t_\varepsilon, \end{aligned}$$
where
$$\delta_2(t) := f^{\mu}(t_u)\rho(t_u) + \int_0^{t_u} f^{\mu}(s)p(s)ds + \gamma f^{\mu-\alpha}(t_u).$$
Consequently,
$$\begin{aligned} f^{\alpha}(t)\rho(t) + \gamma \le{}& \delta_3(t_u) f^{\alpha-\mu}(t) - H_*(\alpha,\mu) + \varepsilon \\ &+ \frac{\mu\left(M+\varepsilon\right) - \alpha|M+\varepsilon|^{\frac{1+\alpha}{\alpha}}}{\mu-\alpha} \quad \text{for } t \ge t_\varepsilon, \end{aligned}$$

where

$$\delta_3(t_u) := \delta_2(t_u) - \frac{\mu\,(M+\varepsilon) - \alpha|M+\varepsilon|^{\frac{1+\alpha}{\alpha}}}{\mu-\alpha} f^{\mu-\alpha}(t_u),$$

which, by virtue of the assumption $\alpha < \mu$ and conditiond (3.133) and (3.172), yields that

$$M \leq -H_*(\alpha,\mu) + \varepsilon + \frac{\mu\,(M+\varepsilon) - \alpha|M+\varepsilon|^{\frac{1+\alpha}{\alpha}}}{\mu-\alpha}.$$

Since ε was arbitrary, the latter inequality leads to

$$\alpha|M|^{\frac{1+\alpha}{\alpha}} - \alpha M + (\mu-\alpha)H_*(\alpha,\mu) \leq 0. \tag{3.175}$$

One can easily derive that the function $y : x \mapsto \alpha|x|^{\frac{1+\alpha}{\alpha}} - \alpha x + H_*(\alpha,\mu)(\mu-\alpha)$ is decreasing on $]-\infty, (\frac{\alpha}{1+\alpha})^{\alpha}]$ and increasing on $[(\frac{\alpha}{1+\alpha})^{\alpha}, +\infty[$. Therefore, in view of assumption (3.145), the function y is non-positive at the point $\left(\frac{\alpha}{1+\alpha}\right)^{\alpha}$, which together with (3.152), (3.172), and (3.175) implies desired estimate (3.167). □

3.4.4 Proofs of main results

Proof of Theorem 3.54. Assume on the contrary that system (3.127) is not oscillatory, i.e., there exists a solution (u,v) of system (3.127) satisfying relation (3.149) with $t_u > 0$. Analogously to the proof of Lemma 3.67 we show that equality (3.159) holds, where the functions h, ρ and the number γ are defined by (3.151), (3.152), and (3.143). Moreover, conditions (3.153) and (3.154) are satisfied.

Multiplaying of (3.159) by $g(t)f^{\alpha-1-\lambda}(t)$ and integrating it from t_u to t, one gets

$$\begin{aligned}\int_{t_u}^{t} g(s)f^{\alpha-1}(s)\rho(s)ds = c_\alpha^*(\lambda)&\int_{t_u}^{t}\frac{g(s)}{f^{1+\lambda-\alpha}(s)}ds \\ &- \int_{t_u}^{t}\frac{g(s)}{f^{1+\lambda-\alpha}(s)}\left(\int_0^s f^{\lambda}(\xi)p(\xi)d\xi\right)ds \\ &+ \int_{t_u}^{t}\frac{g(s)}{f^{1+\lambda-\alpha}(s)}\left(\int_s^{+\infty} g(\xi)f^{\lambda-1-\alpha}(\xi)h(\xi)d\xi\right)ds \\ &- \frac{\alpha}{\alpha-\lambda}\left(\gamma-\gamma^{\frac{1+\alpha}{\alpha}}\right)\int_{t_u}^{t}\frac{g(s)}{f(s)}ds \quad \text{for } t \geq t_u,\end{aligned} \tag{3.176}$$

Observe that

$$\begin{aligned}&\int_{t_u}^{t}\frac{g(s)}{f^{1+\lambda-\alpha}(s)}\left(\int_s^{+\infty} g(\xi)f^{\lambda-1-\alpha}(\xi)h(\xi)d\xi\right)ds \\ &\qquad = -\frac{f^{\alpha-\lambda}(t)}{\alpha-\lambda}\int_t^{+\infty} g(s)f^{\lambda-1-\alpha}(s)h(s)ds + \frac{1}{\alpha-\lambda}\int_{t_u}^{t}\frac{g(s)}{f(s)}h(s)ds \\ &\qquad\quad - \frac{f^{\alpha-\lambda}(t_u)}{\alpha-\lambda}\int_{t_u}^{+\infty} g(s)f^{\lambda-1-\alpha}(s)h(s)ds \quad \text{for } t \geq t_u.\end{aligned}$$

Hence, it follows from (3.176) that

$$
\begin{aligned}
&f^{\alpha-\lambda}(t)\left(c_\alpha^*(\lambda)-c_\alpha(t;\lambda)\right)\\
&\qquad=\int_{t_u}^{t}\frac{g(s)}{f(s)}\left[(\alpha-\lambda)f^{\alpha}(s)\rho(s)-h(s)+\alpha\left(\gamma-\gamma^{\frac{1+\alpha}{\alpha}}\right)\right]ds\\
&\qquad\quad+f^{\alpha-\lambda}(t_u)\left[c_\alpha^*(\lambda)-c_\alpha(t_u;\lambda)+\int_{t_u}^{+\infty}g(s)f^{\lambda-1-\alpha}(s)h(s)ds\right]\\
&\qquad\quad-f^{\alpha-\lambda}(t)\int_{t}^{+\infty}g(s)f^{\lambda-1-\alpha}(s)h(s)ds\quad\text{for } t\geq t_u.
\end{aligned}
\tag{3.177}
$$

On the other hand, according to (3.143), (3.151), and Lemma 3.64 with $\omega:=\alpha$, the estimate

$$
\begin{aligned}
&(\alpha-\lambda)f^{\alpha}(s)\rho(s)-h(s)+\alpha\left(\gamma-\gamma^{\frac{1+\alpha}{\alpha}}\right)\\
&\qquad=\alpha\left(f^{\alpha}(s)\rho(s)+\gamma\right)-\alpha|f^{\alpha}(s)\rho(s)+\gamma|^{\frac{1+\alpha}{\alpha}}\leq\left(\frac{\alpha}{1+\alpha}\right)^{\frac{1+\alpha}{\alpha}}
\end{aligned}
\tag{3.178}
$$

holds for $s\geq t_u$. Moreover, in view of (3.128), (3.134), and (3.153), it is clear that

$$
f^{\alpha-\lambda}(t)\int_{t}^{+\infty}g(s)f^{\lambda-1-\alpha}(s)h(s)ds\geq 0\quad\text{for } t\geq t_u.
$$

Consequently, by virtue of the last inequality and (3.178), it follows from (3.177) that

$$
\begin{aligned}
f^{\alpha-\lambda}(t)\left[c_\alpha^*(\lambda)-c_\alpha(t;\lambda)\right]&\leq\left(\frac{\alpha}{1+\alpha}\right)^{\frac{1+\alpha}{\alpha}}\ln\frac{f(t)}{f(t_u)}\\
&+f^{\alpha-\lambda}(t_u)\left[c_\alpha^*(\lambda)-c_\alpha(t_u;\lambda)+\int_{t_u}^{+\infty}g(s)f^{\lambda-1-\alpha}(s)h(s)ds\right]\quad\text{for } t\geq t_u.
\end{aligned}
$$

Hence, in view of (3.133), we get

$$
\limsup_{t\to+\infty}\frac{f^{\alpha-\lambda}(t)}{\ln f(t)}\left[c_\alpha^*(\lambda)-c_\alpha(t;\lambda)\right]\leq\left(\frac{\alpha}{1+\alpha}\right)^{\frac{1+\alpha}{\alpha}},
$$

which contradicts (3.136). □

Proof of Corollary 3.55. Observe that for $t>0$, we have

$$
\frac{f^{\alpha-\lambda}(t)}{\ln f(t)}\left(c_\alpha^*(\lambda)-c_\alpha(t;\lambda)\right)=\frac{\alpha-\lambda}{\ln f(t)}\int_0^t\frac{g(s)}{f(s)}Q(s;\alpha,\lambda)ds
\tag{3.179}
$$

and

$$Q(t;\alpha,\lambda)+H(t;\alpha,\mu)=(\mu-\lambda)f^{\alpha-\mu}(t)\int\limits_0^t g(s)f^{\mu-\alpha-1}(s)Q(s;\alpha,\lambda)ds. \tag{3.180}$$

Moreover, it is easy to show that

$$\begin{aligned}\int\limits_0^t \frac{g(s)}{f(s)}Q(s;\alpha,\lambda)ds &= f^{\alpha-\mu}(t)\int\limits_0^t g(s)f^{\mu-\alpha-1}(s)Q(s;\alpha,\lambda)ds\\ &+(\mu-\alpha)\int\limits_0^t g(s)f^{\alpha-\mu-1}(s)\left(\int\limits_0^s g(\xi)f^{\mu-\alpha-1}(\xi)Q(\xi;\alpha,\lambda)d\xi\right)ds \quad \text{for } t>0.\end{aligned} \tag{3.181}$$

On the other hand, by virtue of (3.137), from relation (3.180) one gets

$$\liminf_{t\to+\infty} f^{\alpha-\mu}(t)\int\limits_0^t g(s)f^{\mu-\alpha-1}(s)Q(s;\alpha,\lambda)ds>\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{(\alpha-\lambda)(\mu-\alpha)}.$$

Therefore, in view of relation (3.133), it follows from (3.181) that

$$\liminf_{t\to+\infty}\frac{1}{\ln f(t)}\int\limits_0^t \frac{g(s)}{f(s)}Q(s;\alpha,\lambda)ds>\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{\alpha-\lambda}. \tag{3.182}$$

Now, equality (3.179) and inequality (3.182) guarantee the validity of condition (3.136) and thus, the assertion of the corollary follows from Theorem 3.54. □

Proof of Corollary 3.56. If assumption (3.138) holds, then it follows from (3.179) that condition (3.136) is satisfied and thus, the assertion of the corollary follows from Theorem 3.54.

Let now assumption (3.139) be fulfilled. Observe that

$$\int\limits_0^t f^{\alpha}(s)p(s)ds=H(t;\alpha,\mu)+(\mu-\alpha)\int_0^t \frac{g(s)}{f(s)}H(s;\alpha,\mu)ds \quad \text{for } t>0.$$

Therefore, in view of (3.139), we obtain

$$\liminf_{t\to+\infty}\frac{1}{\ln f(t)}\int_0^t f^{\alpha}(s)p(s)ds>\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}. \tag{3.183}$$

On the other hand, it is clear that

$$\begin{aligned}c'_\alpha(t;\lambda)&=\frac{-(\alpha-\lambda)^2 g(t)}{f^{1+\alpha-\lambda}}\int_0^t g(s)f^{\alpha-\lambda-1}(s)\left(\int_0^s f^{\lambda}(\xi)p(\xi)d\xi\right)ds\\ &\quad+\frac{(\alpha-\lambda)g(t)}{f(t)}\int_0^t f^{\lambda}(s)p(s)ds\\ &=\frac{(\alpha-\lambda)g(t)}{f^{\alpha-\lambda+1}(t)}\int_0^t f^{\alpha}(s)p(s)ds \quad \text{for } t>0.\end{aligned}$$

Hence, we have

$$c_\alpha(\tau;\lambda) - c_\alpha(t;\lambda) = (\alpha-\lambda)\int_t^\tau \frac{g(s)}{f^{\alpha-\lambda+1}(s)}\left(\int_0^s f^\alpha(\xi)p(\xi)d\xi\right)ds \quad \tau \ge t > 0$$

and consequently, by virtue of assumption (3.135) and condition (3.183), we get

$$c_\alpha^*(\lambda) - c_\alpha(t;\lambda) = \\ (\alpha-\lambda)\int_t^{+\infty} \frac{g(s)\ln f(s)}{f^{\alpha-\lambda+1}(s)}\left(\frac{1}{\ln f(s)}\int_0^s f^\alpha(\xi)p(\xi)d\xi\right)ds \quad \text{for } t>0. \tag{3.184}$$

In view of (3.183), there exist $\varepsilon > 0$ and $t_\varepsilon > 0$ such that

$$\frac{1}{\ln f(t)}\int_0^t f^\alpha(s)p(s)ds \ge \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} + \varepsilon \quad \text{for} \quad t \ge t_\varepsilon.$$

Hence, it follows from (3.184) that

$$c_\alpha^*(\lambda) - c_\alpha(t;\lambda) \ge (\alpha-\lambda)\left(\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} + \varepsilon\right)\int_t^{+\infty}\frac{g(s)\ln f(s)}{f^{\alpha-\lambda+1}(s)} \quad \text{for} \quad t \ge t_\varepsilon.$$

Since $\varepsilon > 0$, by virtue of (3.133), from the last relation we derive inequality (3.136). Therefore, the assertion of the corollary follows from Theorem 3.54. □

Proof of Theorem 3.58. Assume on the contrary that system (3.127) is not oscillatory, i.e., there exists a solution (u,v) of system (3.127) satisfying relation (3.149) with $t_u > 0$. Analogously to the proofs of Lemmas 3.67 and 3.68 we derive equalities (3.159) and (3.168), where the numbers γ, $\delta(t_u)$ and the functions h, ρ are given by (3.143), (3.169) and (3.151), (3.152).

It follows from (3.159) and (3.168) that

$$\begin{aligned} Q(t;\alpha,\lambda) + H(t;\alpha,\mu) = &-f^{\alpha-\lambda}(t)\int_t^{+\infty} g(s)f^{\lambda-1-\alpha}(s)h(s)ds \\ &+ \frac{\alpha}{\alpha-\lambda}\left(\gamma - \gamma^{\frac{1+\alpha}{\alpha}}\right) + \frac{\alpha\gamma}{\mu-\alpha} + \delta(t_u)f^{\alpha-\mu}(t) \\ &+ f^{\alpha-\mu}(t)\int_{t_u}^{t} g(s)f^{\mu-\alpha-1}(s)\left[\mu f^\alpha(s)\rho(s) - \alpha|\rho(s)f^\alpha(s)+\gamma|^{\frac{1+\alpha}{\alpha}}\right]ds \end{aligned} \tag{3.185}$$

is satisfied for $t \ge t_u$. Moreover, according to Lemma 3.64 with $\omega := \mu$, it is clear that

$$\mu\left(f^\alpha(t)\rho(t)+\gamma\right) - \alpha|\rho(t)f^\alpha(t)+\gamma|^{\frac{1+\alpha}{\alpha}} \le \left(\frac{\mu}{1+\alpha}\right)^{1+\alpha} \quad \text{for } t \ge t_u. \tag{3.186}$$

Therefore, using (3.143), (3.153), and (3.186) in relation (3.185), we get

$$\begin{aligned} Q(t;\alpha,\lambda) + H(t;\alpha,\mu) \le \frac{1}{\alpha-\lambda}\left(\frac{\lambda}{1+\alpha}\right)^{1+\alpha} + \frac{1}{\mu-\alpha}\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha} \\ +\widetilde{\delta}(t_u)f^{\alpha-\mu}(t) \quad \text{for } t \ge t_u, \end{aligned} \tag{3.187}$$

where

$$\widetilde{\delta}(t_u) := \delta(t_u) - \left[\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha} - \mu\gamma\right]\frac{f^{\mu-\alpha}(t_u)}{\mu-\alpha}.$$

Consequently, by virtue of (3.133), relation (3.187) leads to a contradiction with assumption (3.140). □

Proof of Theorem 3.59. Suppose on the contrary that system (3.127) is not oscillatory. Then there exists a solution (u, v) of system (3.127) satisfying relation (3.149) with $t_u > 0$. Analogously to the proof of Lemma 3.68 one can show that relation (3.170) holds, where the numbers γ, $\delta_1(t_u)$ and the function ρ are given by (3.143), (3.171), and (3.152). On the other hand, according to Lemma 3.67, estimate (3.155) is fulfilled, where $A(\alpha, \lambda)$ is the smallest root of equation (3.144).

Let $\varepsilon > 0$ be arbitrary. Then there exists $t_\varepsilon \geq t_u$ such that

$$f^\alpha(t)\rho(t) \geq A(\alpha, \lambda) - \varepsilon \quad \text{for } t \geq t_\varepsilon.$$

Hence, it follows from (3.170) that

$$H(t; \alpha, \mu) \leq \delta_1(t_u) f^{\alpha-\mu}(t) - A(\alpha, \lambda) + \varepsilon + \frac{1}{\mu-\alpha}\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha} - \gamma \quad \text{for } t \geq t_\varepsilon.$$

Since ε was arbitrary, in view of (3.133), from the latter inequality we get

$$H^*(\alpha, \mu) \leq \frac{1}{\mu-\alpha}\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha} - \gamma - A(\alpha, \lambda, \gamma),$$

which contradicts assumption (3.142). □

Proof of Theorem 3.60. Assume on the contrary that system (3.127) is not oscillatory, i.e., there exists a solution (u, v) of system (3.127) satisfying relation (3.149) with $t_u > 0$. Analogously to the proof of Lemma 3.67 we show that equality (3.160) holds, where the number γ and the functions h, ρ are defined by (3.143), (3.151), and (3.152).

On the other hand, according to Lemma 3.68, estimate (3.167) is fulfilled, where $B(\alpha, \mu)$ is the greatest root of equation (3.147). Let $\varepsilon > 0$ be arbitrary. Then there exists $t_\varepsilon \geq t_u$ such that

$$f^\alpha(t)\rho(t) + \gamma \leq B(\alpha, \mu) + \varepsilon \quad \text{for } t \geq t_\varepsilon.$$

In view of the last inequality, (3.128), (3.134) and (3.153), it follows from (3.160) that

$$Q(t; \alpha, \lambda) \leq B(\alpha, \mu) + \varepsilon - \gamma + \frac{\alpha}{\alpha-\lambda}\left(\gamma - \gamma^{\frac{1+\alpha}{\alpha}}\right) \quad \text{for } t \geq t_\varepsilon.$$

Since ε was arbitrary, we get

$$Q^*(\alpha, \lambda) \leq B(\alpha, \mu) + \frac{\gamma^{\frac{1+\alpha}{\alpha}}}{\alpha-\lambda},$$

which contradicts (3.146). □

Proof of Theorem 3.61. Suppose on the contrary that system (3.127) is not oscillatory. Then there exists a solution (u, v) of system (3.127) satisfying relation (3.149) with $t_u > 0$. Put

$$m := A(\alpha, \lambda), \quad M := B(\alpha, \mu), \tag{3.188}$$

i.e., m denotes the smallest root of equation (3.144) and M is the greatest root of equation (3.147). According to Lemmas 3.67 and 3.68, we have

$$\liminf_{t\to+\infty} f^\alpha(t)\rho(t) \geq m, \quad \limsup_{t\to+\infty} \left(f^\alpha(t)\rho(t) + \gamma\right) \leq M, \tag{3.189}$$

where the function ρ and the number γ are defined in (3.152) and (3.143).

Analogously to the proof of Theorem 3.58 we show that relation (3.185) holds for $t \geq t_u$, where the number $\delta(t_u)$ and the function h are defined by (3.169) and (3.151).

In view of (3.141), one can easily show that the function $y : x \mapsto \alpha|x+\gamma|^{\frac{1+\alpha}{\alpha}} - \alpha x + Q_*(\alpha, \lambda)(\alpha - \lambda) - \alpha\gamma$ is positive on $]-\infty, 0[$ and there exists $\bar{x} \in [0, +\infty[$ such that $y(\bar{x}) \leq 0$, which yields that $m \geq 0$.

On the other hand, in view of (3.145), one can easily verify that the function $z : x \mapsto \alpha|x|^{\frac{1+\alpha}{\alpha}} - \alpha x + (\mu - \alpha)H_*(\alpha, \mu)$ is positive on $\left]\left(\frac{\mu}{1+\alpha}\right)^\alpha, +\infty\right[$ and there exists $\tilde{x} \leq \left(\frac{\mu}{1+\alpha}\right)^\alpha$ such that $z(\tilde{x}) \leq 0$. Consequently, we have $M \leq \left(\frac{\mu}{1+\alpha}\right)^\alpha$.

We first assume that $m > 0$ and $M < \left(\frac{\mu}{1+\alpha}\right)^\alpha$. Let $\varepsilon \in \left]0, \min\left\{m, \left(\frac{\mu}{1+\alpha}\right)^\alpha - M\right\}\right[$ be arbitrary. Then, by virtue of (3.189), there exists $t_\varepsilon \geq t_u$ such that

$$f^\alpha(t)\rho(t) \geq m - \varepsilon, \quad f^\alpha(t)\rho(t) + \gamma \leq M + \varepsilon \quad \text{for } t \geq t_\varepsilon. \tag{3.190}$$

The function $x \mapsto \alpha|x+\gamma|^{\frac{1+\alpha}{\alpha}} - (1+\alpha)x\gamma^{\frac{1}{\alpha}}$ is non-decrasing on $[0, +\infty[$. Therefore, in view of (3.151) and (3.190), we get

$$f^{\alpha-\lambda}(t)\int_t^{+\infty} g(s)f^{\lambda-1-\alpha}(s)h(s)ds \geq \frac{\alpha|m-\varepsilon+\gamma|^{\frac{1+\alpha}{\alpha}} - \lambda(m-\varepsilon) - \alpha\gamma^{\frac{1+\alpha}{\alpha}}}{\alpha - \lambda} \tag{3.191}$$

for $t \geq t_\varepsilon$. Moreover, the function $x \mapsto \mu x - \alpha|x|^{\frac{1+\alpha}{\alpha}}$ is non-decrasing on $\left]-\infty, \left(\frac{\mu}{1+\alpha}\right)^\alpha\right[$ and thus, in view of (3.190), we obtain

$$\begin{aligned} f^{\alpha-\mu}(t)\int_{t_\varepsilon}^{t} g(s)f^{\mu-\alpha-1}(s)\left[\mu f^\alpha(s)\rho(s) - \alpha|\rho(s)f^\alpha(s)+\gamma|^{\frac{1+\alpha}{\alpha}}\right]ds& \\ \leq \frac{\mu(M+\varepsilon) - \alpha|M+\varepsilon|^{\frac{1+\alpha}{\alpha}} - \mu\gamma}{\mu - \alpha} \quad \text{for} \quad t \geq t_\varepsilon.& \end{aligned} \tag{3.192}$$

Now it follows from (3.185), (3.191), and (3.192) that

$$\begin{aligned} Q(t; \alpha, \lambda) + H(t; \alpha, \mu) \leq{}& M + \varepsilon + H_*(\alpha, \mu) - (m - \varepsilon) + Q_*(\alpha, \lambda) - \gamma \\ &+ \frac{\alpha(M+\varepsilon) - \alpha|M+\varepsilon|^{\frac{1+\alpha}{\alpha}} - (\mu-\alpha)H_*(\alpha,\mu)}{\mu - \alpha} \\ &- \frac{\alpha|m-\varepsilon+\gamma|^{\frac{1+\alpha}{\alpha}} - \alpha(m-\varepsilon) + (\alpha-\lambda)Q_*(\alpha,\lambda) - \alpha\gamma}{\alpha - \lambda} \\ &+ \delta(t_\varepsilon)f^{\alpha-\mu}(t) \quad \text{for } t \geq t_\varepsilon, \end{aligned} \tag{3.193}$$

where

$$\delta(t_\varepsilon) := \delta(t_u) + \int_{t_u}^{t_\varepsilon} g(s) f^{\mu-\alpha-1}(s) \left[\mu f^\alpha(s)\rho(s) - \alpha |\rho(s) f^\alpha(s) + \gamma|^{\frac{1+\alpha}{\alpha}} \right] ds.$$

Since ε was arbitrary, in view of (3.133) and (3.188), inequality (3.193) yields that

$$\begin{aligned} \limsup_{t\to+\infty} \left(Q(t;\alpha,\lambda) + H(t;\alpha,\mu) \right) \le B(\alpha,\mu) - A(\alpha,\lambda,\gamma) \\ + Q_*(\alpha,\lambda) + H_*(\alpha,\mu) - \gamma, \end{aligned} \tag{3.194}$$

which contradicts assumption (3.148).

If $m = 0$ then, in view of (3.153), it is clear that

$$-f^{\alpha-\lambda}(t) \int_t^{+\infty} g(s) f^{\lambda-1-\alpha}(s) h(s) ds \le 0 = -\frac{\alpha |m+\gamma|^{\frac{1+\alpha}{\alpha}} - \lambda m - \alpha \gamma^{\frac{1+\alpha}{\alpha}}}{\alpha - \lambda} \tag{3.195}$$

for $t \ge t_u$. On the other hand, if $M = \left(\frac{\mu}{1+\alpha}\right)^\alpha$ then, using Lemma 3.64 with $\omega := \mu$, one can show that

$$\begin{aligned} & f^{\alpha-\mu}(t) \int_{t_u}^{t} g(s) f^{\mu-\alpha-1}(s) \left[\mu f^\alpha(s)\rho(s) - \alpha |\rho(s) f^\alpha(s) + \gamma|^{\frac{1+\alpha}{\alpha}} \right] ds \\ & \le \frac{\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha} - \mu\gamma}{\mu - \alpha} - \frac{f^{\mu-\alpha}(t_u)}{f^{\mu-\alpha}(t)} \left(\frac{\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha} - \mu\gamma}{\mu-\alpha} \right) \\ & = \frac{\mu M - \alpha |M|^{\frac{1+\alpha}{\alpha}} - \mu\gamma}{\mu - \alpha} - \frac{f^{\mu-\alpha}(t_u)}{f^{\mu-\alpha}(t)} \left(\frac{\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha} - \mu\gamma}{\mu-\alpha} \right) \quad \text{for} \quad t \ge t_u. \end{aligned} \tag{3.196}$$

Consequently, if $m = 0$ (resp. $M = \left(\frac{\mu}{1+\alpha}\right)^\alpha$), then we derive from (3.185), the inequality (3.194) similarly as above, but we use (3.195) instead of (3.191) (resp. (3.196) instead of (3.192)). $\square$

4 Boundary value problems for functional differential equations

4.1 Introduction

On the interval $[a, b]$, we consider the functional differential equation

$$u' = F(u)(t), \tag{4.1}$$

where $F : C([a, b]; \mathbb{R}) \to L([a, b]; \mathbb{R})$ is a continuous (in general) nonlinear operator. As usually, by a solution of this equation we understand an absolutely continuous function $u : [a, b] \to \mathbb{R}$ satisfying equality (4.1) almost everywhere on $[a, b]$. Along with equation (4.1), we consider the nonlocal boundary condition

$$h(u) = \varphi(u), \tag{4.2}$$

where $h : C([a, b]; \mathbb{R}) \to \mathbb{R}$ is a (non-zero) linear bounded functional and $\varphi : C([a, b]; \mathbb{R}) \to \mathbb{R}$ is a continuous (in general) nonlinear functional.

Firstly, in Sections 4.2 and 4.3, we study the question on the unique solvability of problem (4.1), (4.2) in a linear case, i.e., in the case where equation (4.1) is linear and $\varphi \equiv const$. Conditions guaranteeing the solvability and unique solvability of problem (4.1), (4.2), when equation (4.1) is nonlinear, are provided in Section 4.4.

In this chapter we present our result stated in [27, 40, 42]. Besides boundary value problems presented here, we also dealt with the following ones. In the papers [31–33,41], we studied the question of the existence and uniqueness of a solution of the linear problem

$$u'(t) = \ell(u)(t) + q(t), \tag{4.3}$$

$$u(a) = h(u) + c, \tag{4.4}$$

where $\ell\colon C([a, b]; \mathbb{R}) \to L([a, b]; \mathbb{R})$ is a linear bounded operator, $q \in L([a, b]; \mathbb{R})$, $h\colon C([a, b]; \mathbb{R}) \to \mathbb{R}$ is a linear functional, and $c \in \mathbb{R}$.

Particularly, in [31, 41], we considered the boundary condition (4.4) in the form

$$u(a) = \lambda u(b) + h_0(u) - h_1(u) + c, \tag{4.5}$$

where $h_0, h_1\colon C([a, b]; \mathbb{R}) \to \mathbb{R}$ are positive functionals and $\lambda \geq 0$. It is clear that the periodic condition is a particular case of (4.5). Presented results are concretized for boundary value problems with delay differential equations such as

$$u'(t) = p(t)u(\tau(t)) + q(t), \qquad \int_a^b u(s)\, d\sigma(s) = c,$$

where $p, q \in L([a, b]; \mathbb{R})$, $\tau : [a, b] \to [a, b]$ is a measurable function, $\sigma : [a, b] \to \mathbb{R}$ is an absolutely continuous function, $\sigma(a) > 0$, $\sigma(b) > 0$, and $c \in \mathbb{R}$ (see [31]). In [32], we established conditions guaranteeing the unique solvability of problem (4.3), (4.4) as well as nonpositivity of its solution. Obtained general statements are applied to the case, where the operator ℓ is the operator with argument deviations defined by

$$\ell(u)(t) := p(t)u(\tau(t)) - g(t)u(\mu(t)),$$

where $p, g \in L([a,b];\mathbb{R}_+)$ and $\tau, \mu : [a,b] \to [a,b]$ are measurable functions.

Problem (4.1), (4.2) in a full generality, i.e., if F is a continuous nonlinear operator and φ is a continuous nonlinear functional, has been studied in [39].

4.2 Linear problem - nonnegative solutions

In this section, we assume that equation (4.1) is linear and the functional φ in the boundary condition (4.2) is constant, i.e., we consider the boundary value problem

$$u'(t) = \ell(u)(t) + q(t), \tag{4.6}$$

$$u(a) = h(u) + c, \tag{4.7}$$

where $\ell\colon C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ is a linear bounded operator, $h\colon C([a,b];\mathbb{R}) \to \mathbb{R}$ is a linear bounded functional, $q \in L([a,b];\mathbb{R})$, and $c \in \mathbb{R}$. It is natural to assume that

$$\widetilde{h} \not\equiv 0, \quad \text{where} \quad \widetilde{h}(v) := v(a) - h(v).$$

We establish conditions sufficient for the unique solvability of the considered problem. Moreover, if the function q and the number c are nonnegative, then these conditions guarantee also nonnegativity of a solution. Presented results are concretized for differential equations with argument deviations.

Recall that by a solution of problem (4.6), (4.7) we understand a function $u \in AC([a,b];\mathbb{R})$ satisfying equality (4.6) almost everywhere in $[a,b]$ and condition (4.7). Along with problem (4.6), (4.7) we consider the corresponding homogeneous problem

$$u'(t) = \ell(u)(t), \tag{4.6$_0$}$$

$$u(a) = h(u). \tag{4.7$_0$}$$

The following result is well known from the general theory of BVPs for FDEs (see, e.g., [1, 3, 9, 20, 48])

Theorem 4.1. *Problem* (4.6), (4.7) *is uniquely solvable iff the corresponding homogeneous problem* (4.6$_0$), (4.7$_0$) *has only the trivial solution.*

Introduce the definition.

Definition 4.1. We say that an operator $\ell \in \mathcal{L}_{ab}$ belongs to the set $\widetilde{V}^+_{ab}(h)$ if every function $u \in AC([a,b];\mathbb{R})$ satisfying

$$u'(t) \geq \ell(u)(t) \quad \text{for a. e. } t \in [a,b], \tag{4.8}$$

$$u(a) \geq h(u) \tag{4.9}$$

is nonnegative on $[a,b]$.

Remark 4.2. Assume that $\ell \in \widetilde{V}_{ab}^{+}(h)$. Then it is clear that problem (4.6_0), (4.7_0) has only the trivial solution. Therefore, according to Theorem 4.1 problem (4.6), (4.7) is uniquely solvable for any $c \in \mathbb{R}$ and $q \in L([a,b];\mathbb{R})$. If, moreover, $c \in \mathbb{R}_+$ and $q \in L([a,b];\mathbb{R}_+)$, then the unique solution of problem (4.6), (4.7) is nonnegative.

Let us mention some properties of the set $\widetilde{V}_{ab}^{+}(h)$ in the case when $h \in \mathcal{PF}_{ab}$.

Remark 4.3. Let $h \in \mathcal{PF}_{ab}$. It is not difficult to verify that $\mathcal{P}_{ab} \cap \widetilde{V}_{ab}^{+}(h) \neq \varnothing$ if and only if

$$h(1) < 1. \tag{4.10}$$

Indeed, assume that $\ell \in \mathcal{P}_{ab} \cap \widetilde{V}_{ab}^{+}(h)$. Then, according to Remark 4.2, the problem

$$\begin{aligned} u'(t) &= \ell(u)(t) \\ u(a) &= h(u) + 1 \end{aligned} \tag{4.11}$$

has a unique solution u and

$$u(t) \geq 0 \quad \text{for } t \in [a,b]. \tag{4.12}$$

By virtue of (4.12) and the assumption $\ell \in \mathcal{P}_{ab}$, we have

$$u'(t) \geq 0 \quad \text{for a. e. } t \in [a,b]. \tag{4.13}$$

Hence,

$$u(t) \geq u(a) \quad \text{for } t \in [a,b]. \tag{4.14}$$

Now (4.14) and the assumption $h \in \mathcal{PF}_{ab}$ imply that

$$h(u) \geq u(a)h(1), \tag{4.15}$$

whence, together with (4.11), we obtain

$$u(a)(1 - h(1)) \geq 1.$$

Therefore, inequality (4.10) holds.

Assume now that (4.10) is fulfilled. We show that $0 \in \widetilde{V}_{ab}^{+}(h)$. Let the function $u \in AC([a,b];\mathbb{R})$ satisfy (4.9) and (4.13). Clearly, (4.14) holds, as well. Hence, on account of the assumption $h \in \mathcal{PF}_{ab}$, inequality (4.15) is satisfied. By virtue of (4.9) and (4.15), we get

$$u(a)(1 - h(1)) \geq 0,$$

which together with (4.10) implies $u(a) \geq 0$. Taking now into account (4.14), we get (4.12). Therefore, $0 \in \widetilde{V}_{ab}^{+}(h)$.

4.2.1 Main results

In this section we present optimal (nonimprovable in a certain sense) sufficient conditions guaranteeing the inclusion $\ell \in \widetilde{V}_{ab}^{+}(h)$.

THEOREM 4.4 ([27, Thm. 2.1]). *Let $h \in \mathcal{PF}_{ab}$ and $\ell \in \mathcal{P}_{ab}$. Then $\ell \in \widetilde{V}_{ab}^{+}(h)$ if and only if there exists a function $\gamma \in AC([a,b];]0,+\infty[)$ satisfying the inequalities*

$$\gamma'(t) \geq \ell(\gamma)(t) \quad \text{for a. e. } t \in [a,b], \tag{4.16}$$

$$\gamma(a) > h(\gamma). \tag{4.17}$$

In the case, when ℓ is an a-Volterra operator, Theorem 4.4 yields the following statement.

COROLLARY 4.5 ([27, Cor. 2.1]). *Let $h \in \mathcal{PF}_{ab}$, $\ell \in \mathcal{P}_{ab}$ be an a-Volterra operator, and*

$$h(\gamma) < 1, \tag{4.18}$$

where

$$\gamma(t) := \exp\left[\int_a^t \ell(1)(s)\,ds\right] \quad \text{for} \quad t \in [a,b].$$

Then $\ell \in \widetilde{V}_{ab}^{+}(h)$.

REMARK 4.6. Inequality (4.18) is optimal and cannot be replaced by the inequality $h(\gamma) \leq 1$. Indeed, let $\gamma(t) := \exp\left[\int_a^t p(s)\,ds\right]$ for $t \in [a,b]$, where $p \in L([a,b];\mathbb{R}_+)$ is such that $h(\gamma) = 1$. Clearly, the function γ is a nontrivial solution of problem (4.6_0), (4.7_0) with $\ell(v)(t) := p(t)v(t)$. Therefore, according to Remark 4.2, $\ell \notin \widetilde{V}_{ab}^{+}(h)$.

COROLLARY 4.7 ([27, Cor. 2.2]). *Let $h \in \mathcal{PF}_{ab}$, $\ell \in \mathcal{P}_{ab}$, $h(1) < 1$, and let there exist $m, k \in \mathbb{N}$ and a constant $\alpha \in]0,1[$ such that $m > k$ and*

$$\rho_m(t) \leq \alpha \rho_k(t) \quad \text{for} \quad t \in [a,b], \tag{4.19}$$

where

$$\begin{aligned} \rho_1 \equiv 1, \quad \rho_{i+1}(t) &:= \frac{1}{1-h(1)} h(\varphi_i) + \varphi_i(t) \quad \text{for} \quad t \in [a,b],\ i \in \mathbb{N}, \\ \varphi_i(t) &:= \int_a^t \ell(\rho_i)(s)\,ds \quad \text{for} \quad t \in [a,b],\ i \in \mathbb{N}. \end{aligned} \tag{4.20}$$

Then $\ell \in \widetilde{V}_{ab}^{+}(h)$.

In the case, when the operator ℓ is negative, necessary and sufficient condition for the inclusion $\ell \in \widetilde{V}_{ab}^{+}(h)$ is presented in the next theorem.

Theorem 4.8 ([27, Thm. 2.3]). *Let $h \in \mathcal{PF}_{ab}$, $-\ell \in \mathcal{P}_{ab}$ be an a-Vollterа operator, and* (4.10) *hold. Then $\ell \in \widetilde{V}_{ab}^{+}(h)$ if and only if $\ell \in \widetilde{V}_{ab}^{+}(0)$.*

Theorem 4.8 yields the following corollaries.

Corollary 4.9 ([27, Cor. 2.4]). *Let $h \in \mathcal{PF}_{ab}$, $-\ell \in \mathcal{P}_{ab}$ be an a-Volterra operator, and* (4.10) *hold. Let, moreover, there exist a function $\gamma \in AC([a,b];\mathbb{R}_+)$ satisfying*

$$\gamma(t) > 0 \quad \textit{for} \qquad t \in [a,b[\,, \tag{4.21}$$

$$\gamma'(t) \le \ell(\gamma)(t) \quad \textit{for a. e. } t \in [a,b]. \tag{4.22}$$

Then $\ell \in \widetilde{V}_{ab}^{+}(h)$.

Remark 4.10. Corollary 4.9 is nonimprovable in a certain sense. More precisely, condition (4.21) cannot be replaced by the condition

$$\gamma(t) > 0 \quad \text{for} \quad t \in [a,b_1[\,, \tag{4.23}$$

where $b_1 \in]a,b[$ .Indeed, it is shown in [8, Example 4.3] that conditions (4.22) and (4.23) do not guarantee the inclusion $\ell \in \widetilde{V}_{ab}^{+}(0)$. Consequently, by virtue of Theorem 4.8, Corollary 4.9 is nonimprovable in the above-mentioned sense.

Corollary 4.11 ([27, Cor. 2.5]). *Let $h \in \mathcal{PF}_{ab}$, $-\ell \in \mathcal{P}_{ab}$ be an a-Volterra operator, and let* (4.10) *hold. If, moreover,*

$$\int_a^b |\ell(1)(s)|\,ds \le 1, \tag{4.24}$$

then $\ell \in \widetilde{V}_{ab}^{+}(h)$.

Remark 4.12. Corollary 4.11 is nonimprovable in the sense that the inequality (4.24) cannot be replaced by the inequality

$$\int_a^b |\ell(1)(s)|\,ds \le 1 + \varepsilon,$$

no matter how small $\varepsilon > 0$ is (see Theorem 4.8 and [8, Example 4.4]).

The following theorem deals with the case, when the operator ℓ is not monotone.

Theorem 4.13 ([27, Thm. 2.4]). *Let the operator $\ell \in \mathcal{L}_{ab}$ admit the representation $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and*

$$\ell_0 \in \widetilde{V}_{ab}^{+}(h), \qquad -\ell_1 \in \widetilde{V}_{ab}^{+}(h). \tag{4.25}$$

Then $\ell \in \widetilde{V}_{ab}^{+}(h)$.

REMARK 4.14. Assumption (4.25) is nonimprovable in the sense that it can be replaced neither by the assumption

$$(1-\varepsilon)\ell_0 \in \widetilde{V}_{ab}^{+}(h), \quad -\ell_1 \in \widetilde{V}_{ab}^{+}(h), \tag{4.26}$$

nor by

$$\ell_0 \in \widetilde{V}_{ab}^{+}(h), \quad -(1-\varepsilon)\ell_1 \in \widetilde{V}_{ab}^{+}(h), \tag{4.27}$$

no matter how small $\varepsilon > 0$ is (see Examples 4.17 and 4.18).

Now we concretize obtained results for the differential equations with argument deviations. Put

$$\ell(v)(t) := p(t)v(\tau(t)), \tag{4.28}$$

$$\ell(v)(t) := -\,g(t)v(\mu(t)), \tag{4.29}$$

where $p, g \in L([a,b];\mathbb{R}_+)$ and $\tau, \mu : [a,b] \to [a,b]$ are measurable functions.

In the case, when the operator ℓ is defined by (4.28), the following statement follows immediately from Corollary 4.5.

THEOREM 4.15. *Let $\tau(t) \le t$ for a.e. $t \in [a,b]$, $h(1) < 1$, and the inequality*

$$\int_a^b p(s)\,ds < \ln\frac{1}{h(1)}$$

hold. Then the operator ℓ defined by (4.28) *belongs to the set $\widetilde{V}_{ab}^{+}(h)$.*

For the operator ℓ is defined by (4.29), the next statement is true.

THEOREM 4.16 ([27, Thm. 4.4]). *Let $\mu(t) \le t$ for a.e. $t \in [a,b]$, $h(1) < 1$, and at least one of the following conditions be fulfilled:*

a)

$$\int_a^b g(s)\,ds \le 1;$$

b)

$$\int_a^b g(s)\left(\int_{\mu(s)}^{s} g(\xi)\exp\left[\int_{\mu(\xi)}^{s} g(\eta)\,d\eta\right]d\xi\right)ds \le 1;$$

c) $g \not\equiv 0$ and

$$\operatorname{ess\,sup}\left\{\int_{\mu(t)}^{t} g(s)\,ds : t \in [a,b]\right\} < \omega^*,$$

where

$$\omega^* = \sup\left\{\frac{1}{x}\ln\left[x + x\left(\exp\left[x\int_a^b g(s)\,ds\right] - 1\right)^{-1}\right] : x > 0\right\}.$$

Then the operator ℓ defined by (4.29) *belongs to the set $\widetilde{V}_{ab}^{+}(h)$.*

Now we present two examples justifying that assumption (4.25) in Theorem 4.13 is nonimprovable in a certain sense.

EXAMPLE 4.17. Let $\varepsilon \in]0,1[$, $t_0 \in]a,b]$, and the functions $p, g \in L([a,b];\mathbb{R}_+)$ be such that

$$\int_a^{t_0} p(s)ds = 1 + \widetilde{\varepsilon}, \qquad \int_{t_0}^{b} p(s)ds = \varepsilon - \widetilde{\varepsilon}, \tag{4.30}$$

$$\int_a^{t_0} g(s)ds = \frac{\varepsilon}{2}, \qquad \int_{t_0}^{b} g(s)ds = \frac{\varepsilon}{2}, \tag{4.31}$$

where

$$\widetilde{\varepsilon} := \begin{cases} \frac{\varepsilon}{2} & \text{if } t_0 \neq b \\ \varepsilon & \text{if } t_0 = b. \end{cases} \tag{4.32}$$

Consider the boundary value problem

$$u'(t) = p(t)u(t_0) - g(t)u(a), \tag{4.33}$$

$$u(a) = \frac{\varepsilon^2 u(t_0)}{2} + 1, \tag{4.34}$$

i.e., problem (4.6_0), (4.7) with $\ell = \ell_0 - \ell_1$, where

$$\ell_0(v)(t) := p(t)v(\tau(t)), \quad \ell_1(v)(t) := g(t)v(\mu(t)), \tag{4.35}$$

$$h(v) := \frac{\varepsilon^2 v(t_0)}{2}, \tag{4.36}$$

$\tau \equiv t_0$, $\mu \equiv a$, and $c = 1$.

One can show that $(1-\varepsilon)\ell_0 \in \widetilde{V}_{ab}^{+}(h)$ and $-\ell_1 \in \widetilde{V}_{ab}^{+}(h)$. Indeed, by virtue of (4.30), we have

$$\begin{aligned}\rho_2(t) =& \frac{1}{1-h(1)}\, h(\varphi_1) + \varphi_1(t) = (1-\varepsilon)\left(\frac{1}{1-h(1)}\, h\left(\int_a^t \ell_0(1)(s)ds\right) + \int_a^t \ell_0(1)(s)ds\right) \\ =& (1-\varepsilon)\left(\frac{1}{1-h(1)}\, h\left(\int_a^t p(s)ds\right) + \int_a^t p(s)ds\right) \leq (1-\varepsilon)(1+\varepsilon)\frac{2}{2-\varepsilon^2} < 1,\end{aligned}$$

where the functions ρ_2 and φ_1 are defined by (4.20). Hence, (4.19) is satisfied with $\alpha = \frac{2-2\varepsilon^2}{2-\varepsilon^2}$. Consequently, according to Corollary 4.7 (with $k=1, m=2$), we obtain $(1-\varepsilon)\ell_0 \in \widetilde{V}_{ab}^{+}(h)$. On the other hand, by virtue of (4.31) and Corollary 4.11, it is clear that $-\ell_1 \in \widetilde{V}_{ab}^{+}(h)$.

Note that equation (4.33) has only the trivial solution satisfying

$$u(a) = \frac{\varepsilon^2 u(t_0)}{2}. \tag{4.37}$$

Indeed, let u be a solution of problem (4.33), (4.37). Integrating (4.33) from a to t_0 and taking into account (4.37), one gets

$$0 = u(t_0)\left(\widetilde{\varepsilon} + \frac{\varepsilon^2}{2}\left(1 - \frac{\varepsilon}{2}\right)\right).$$

Hence, by virue of (4.32) and (4.37), we obtain $u(t_0) = 0$ and $u(a) = 0$. Consequently, in view of (4.33), we have $u'(t) = 0$, which leads to $u \equiv 0$. Therefore problem (4.33), (4.34) has a unique solution u. Integrating (4.33) from a to t_0 and taking into account (4.34), we get

$$\frac{\varepsilon}{2} - 1 = u(t_0)\left(\widetilde{\varepsilon} + \frac{\varepsilon^2}{2}\left(1 - \frac{\varepsilon}{2}\right)\right).$$

Hence, by virtue of (4.32), we have $u(t_0) < 0$.

Therefore, $\ell_0 - \ell_1 \notin \widetilde{V}_{ab}^+(h)$ and thus, assumption (4.25) of Theorem 4.13 cannot be relaxed to (4.26) no matter how small $\varepsilon > 0$ is.

EXAMPLE 4.18. Let $\varepsilon \in]0,1[$, $t_0 \in]a,b]$, and the functions $p, g \in L([a,b];\mathbb{R}_+)$ be such that

$$\int_a^{t_0} p(s)ds = \frac{\varepsilon}{4}, \qquad \int_{t_0}^b p(s)ds = \frac{\varepsilon}{4}, \tag{4.38}$$

$$\int_a^{t_0} g(s)ds = 1 + \widetilde{\varepsilon}, \qquad \int_{t_0}^b g(s)ds = \varepsilon - \widetilde{\varepsilon}, \tag{4.39}$$

where $\widetilde{\varepsilon}$ is given by (4.32). Define the operators ℓ_0 and ℓ_1 by (4.35).

One can show that $\ell_0 \in \widetilde{V}_{ab}^+(h)$ and $-(1-\varepsilon)\ell_1 \in \widetilde{V}_{ab}^+(h)$. Indeed, by virtue of (4.36) and (4.38), analogously to Example 4.17 we derive

$$\rho_2(t) = \frac{1}{1 - h(1)} h(\varphi_1) + \varphi_1(t) \le \frac{\varepsilon}{2 - \varepsilon^2} < 1,$$

where the functions ρ_2 and φ_1 are defined by (4.20). Hence, according to Corollary 4.7 (with $k = 1, m = 2, \alpha = \frac{\varepsilon}{2-\varepsilon^2}$), we have $\ell_0 \in \widetilde{V}_{ab}^+(h)$. On the other hand, by virtue of (4.39) and Corollary 4.11, it is clear that $-(1-\varepsilon)\ell_1 \in \widetilde{V}_{ab}^+(h)$.

Analogously to Example 4.17, one can show that the homogeneous problem (4.33), (4.37) has only the trivial solution. Therefore, problem (4.33), (4.34) has a unique solution u. Integrating (4.33) from a to t_0 and taking into account (4.34), we get

$$-\widetilde{\varepsilon} = u(t_0)\left(\widetilde{\varepsilon}\frac{\varepsilon^2}{2} + 1 - \frac{\varepsilon}{2}\right).$$

Hence, by virtue of (4.32), we have $u(t_0) < 0$.

Therefore, $\ell_0 - \ell_1 \notin \widetilde{V}_{ab}^+(h)$ and thus, assumption (4.25) of Theorem 4.13 cannot be relaxed to (4.27), no matter how small $\varepsilon > 0$ is.

4.2.2 Proofs of the main results

To prove the main results we need the following auxiliary lemma.

LEMMA 4.19. *Let $\ell \in \mathcal{P}_{ab}$, inequality* (4.10) *be fulfilled, and let there exist no nontrivial function $v \in AC([a,b];\mathbb{R}_+)$ satisfying*

$$v'(t) \le \ell(v)(t) \quad \text{for a. e. } t \in [a,b], \quad v(a) = h(v). \tag{4.40}$$

Then $\ell \in \widetilde{V}_{ab}^+(h)$.

Proof. Let $u \in AC([a,b];\mathbb{R})$ satisfy (4.8) and (4.9). Obviously, (4.6) and (4.7) hold, where

$$q(t) := u'(t) - \ell(u)(t) \quad \text{for a. e. } t \in [a,b], \quad c := u(a) - h(u).$$

It is clear that

$$q(t) \geq 0 \quad \text{for a. e. } t \in [a,b], \quad c \geq 0. \tag{4.41}$$

Taking into account (4.6), (4.7), (4.41), and the assumption $\ell \in \mathcal{P}_{ab}$, we easily get

$$\begin{aligned} [u(t)]'_- = \frac{1}{2}\Big(u'(t)\operatorname{sgn} u(t) - u'(t)\Big) = \frac{1}{2}\Big(\ell(u)(t)\operatorname{sgn} u(t) - \ell(u)(t)\Big) \\ + \frac{1}{2}\, q(t)\Big(\operatorname{sgn} u(t) - 1\Big) \leq \ell([u]_-)(t) \quad \text{for a. e. } t \in [a,b], \end{aligned} \tag{4.42}$$

and

$$[u(a)]_- = \frac{1}{2}\Big(h(u)\operatorname{sgn} u(a) - h(u)\Big) + \frac{1}{2}\, c\Big(\operatorname{sgn} u(a) - 1\Big) \leq h([u]_-). \tag{4.43}$$

Put

$$\begin{aligned} c_0 := \Big(1 - h(1)\Big)^{-1}\Big(h([u]_-) - [u(a)]_-\Big), \\ v(t) := [u(t)]_- + c_0 \quad \text{for} \quad t \in [a,b]. \end{aligned} \tag{4.44}$$

On account of (4.10) and (4.43), we have

$$c_0 \geq 0. \tag{4.45}$$

On the other hand, by virtue of (4.42), (4.44), and (4.45) v is a nonnegative function satisfying (4.40). Therefore $v \equiv 0$ which, in view of (4.45), yields that $[u]_- \equiv 0$. Consequently, $u(t) \geq 0$ for $t \in [a,b]$ and thus, $\ell \in \widetilde{V}^+_{ab}(h)$. ☐

Proof of Theorem 4.4. Let $\ell \in \widetilde{V}^+_{ab}(h)$. Then, according to Remark 4.2, the problem

$$\gamma'(t) = \ell(\gamma)(t), \tag{4.46}$$

$$\gamma(a) = h(\gamma) + 1 \tag{4.47}$$

has a unique solution γ and

$$\gamma(t) \geq 0 \quad \text{for} \quad t \in [a,b]. \tag{4.48}$$

By virtue of (4.48) and the assumption $h \in \mathcal{P}F_{ab}$, it follows from (4.47) that

$$\gamma(a) > 1. \tag{4.49}$$

Now, on account of (4.48), (4.49), and the assumption $\ell \in \mathcal{P}_{ab}$, equality (4.46) yields

$$\gamma(t) = \gamma(a) + \int_a^t \ell(\gamma)(s)\, ds \geq \gamma(a) > 0 \quad \text{for} \quad t \in [a,b].$$

Therefore, $\gamma \in AC([a,b];]0,+\infty[)$. Clearly, (4.16) and (4.17) hold, as well.

Assume now that there exists a function $\gamma \in AC([a,b];]0,+\infty[)$ satisfying (4.16) and (4.17). According to Lemma 4.19, it is sufficient to show that there exist no nontrivial function $v \in AC([a,b];\mathbb{R}_+)$ satisfying (4.40). Assume on the contrary that $v \in AC([a,b];\mathbb{R}_+)$ is a nontrivial function satisfying (4.40).

Put

$$w(t) := \lambda\gamma(t) - v(t) \quad \text{for} \quad t \in [a,b],$$

where

$$\lambda = \max\left\{\frac{v(t)}{\gamma(t)} : t \in [a,b]\right\}. \tag{4.50}$$

Obviously,

$$\lambda > 0. \tag{4.51}$$

It is also evident that

$$w(t) \geq 0 \quad \text{for} \;\; t \in [a,b]. \tag{4.52}$$

On account of (4.17), (4.51), (4.52), and the assumption $h \in \mathcal{PF}_{ab}$, we have

$$w(a) = \lambda\gamma(a) - v(a) > h(w) \geq 0. \tag{4.53}$$

From (4.50) and (4.53) it follows that there exists $t_0 \in]a,b]$ such that

$$w(t_0) = 0. \tag{4.54}$$

On the other hand, by virtue of (4.16), (4.40), (4.51), (4.52), and the assumption $\ell \in \mathcal{P}_{ab}$, we get

$$w'(t) \geq \ell(w)(t) \geq 0 \quad \text{for} \;\; t \in [a,b],$$

which together with (4.53) contradicts (4.54). □

Proof of Corollary 4.5. It is clear that that

$$\gamma(a) = 1 \tag{4.55}$$

and

$$\gamma'(t) = \ell(1)(t)\gamma(t) \quad \text{for a. e. } t \in [a,b]. \tag{4.56}$$

Since $\ell \in \mathcal{P}_{ab}$ is an a-Volterra operator, one can show that

$$\ell(\gamma)(t) \leq \ell(1)(t)\gamma(t) \quad \text{for a. e. } t \in [a,b].$$

The latter inequality together with (4.56) yields that (4.16) is fulfilled. On the other hand, it follows from (4.18) and (4.55) that (4.17) holds. Therefore, by virtue of Theorem 4.4, $\ell \in \widetilde{V}^+_{ab}(h)$. □

Proof of Theorem 4.8. It can be found in [27, Thm. 2.3]. □

Proof of Corollary 4.9. It can be found in [27, Cor. 2.4]. □

Proof of Corollary 4.11. It can be found in [27, Cor. 2.5]. □

Proof of Theorem 4.13. Let $u \in AC([a,b];\mathbb{R})$ satisfy (4.8) and (4.9). On account of the assumption $-\ell_1 \in \widetilde{V}^+_{ab}(h)$, it follows from Remark 4.2 that

$$v'(t) = -\ell_1(v)(t) - \ell_0([u]_-)(t), \tag{4.57}$$

$$v(a) = h(v) \tag{4.58}$$

has a unique solution v and

$$v(t) \le 0 \quad \text{for} \quad t \in [a,b]. \tag{4.59}$$

By virtue of (4.8), (4.9), (4.57), (4.58), and the assumption $\ell_0 \in \mathcal{P}_{ab}$, it is easy to show that

$$w'(t) \ge -\ell_1(w)(t) \quad \text{for a. e. } t \in [a,b], \quad w(a) \ge h(w),$$

where

$$w(t) := u(t) - v(t) \quad \text{for} \quad t \in [a,b].$$

Hence, by virtue of the inclusion $-\ell_1 \in \widetilde{V}^+_{ab}(h)$, we have

$$u(t) \ge v(t) \quad \text{for} \quad t \in [a,b].$$

The latter inequality together with (4.59) yields that

$$-[u(t)]_- \ge v(t) \quad \text{for} \quad t \in [a,b]. \tag{4.60}$$

Therefore, on account of (4.59), (4.60), and the condition $\ell_1 \in \mathcal{P}_{ab}$, it follows from (4.57) that

$$v'(t) \ge \ell_0(v)(t) - \ell_1(v)(t) \ge \ell_0(v)(t) \quad \text{for a. e. } t \in [a,b]. \tag{4.61}$$

Now by virtue of the inclusion $\ell_0 \in \widetilde{V}^+_{ab}(h)$, (4.58) and (4.61) yield

$$v(t) \ge 0 \quad \text{for} \quad t \in [a,b].$$

The latter inequality and (4.59) result in $v \equiv 0$. Therefore, it follows from (4.60) that $[u]_- \equiv 0$, which yields the inequality $u(t) \ge 0$ for $t \in [a,b]$. □

Proof of Theorem 4.13. The assertion of theorem immediately follows from Corrolary 4.5, where the operator ℓ is defined by (4.28) . □

Proof of Theorem 4.16. It can be found in [27, Thm. 4.4]. □

4.3 Linear problem - existence and uniqueness of solutions

In this section, we still assume that equation (4.1) is linear and the functional φ in boundary condition (4.2) is constant. More preciously, we consider the problem on the existence and uniqueness of a solution of the equation

$$u'(t) = \ell(u)(t) + q(t) \tag{4.62}$$

satisfying the nonlocal boundary condition

$$h(u) = c, \tag{4.63}$$

where $\ell\colon C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ is a linear bounded operator, $h\colon C([a,b];\mathbb{R}) \to \mathbb{R}$ is a linear bounded functional, $q \in L([a,b];\mathbb{R})$, and $c \in \mathbb{R}$. As above, by a solution of problem (4.62), (4.63) we understand a function $u \in AC([a,b];\mathbb{R})$ satisfying equality (4.62) almost everywhere in $[a,b]$ and condition (4.63).

In theorems stated below, we assume that the operator ℓ admits the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$.

It is clear that the ordinary differential equation

$$u' = p(t)u + q(t), \tag{4.64}$$

where $p, q \in L([a,b];\mathbb{R})$, is a particular case of equation (4.62). Moreover, it is easy to show that problem (4.64), (4.63) is uniquely solvable if and only if the condition

$$h\left(e^{\int_a^{\cdot} p(s)\mathrm{d}s}\right) \neq 0$$

is satisfied. Below, we establish solvability conditions for problem (4.62), (4.63) in terms of norms of the operators appearing in (4.62) and (4.63). Further, we apply the results obtained to the differential equation with an argument deviation

$$u'(t) = p(t)u(\tau(t)) + q(t) \tag{4.65}$$

in which $p, q \in L([a,b];\mathbb{R})$ and $\tau\colon [a,b] \to [a,b]$ is a measurable function.

4.3.1 Main results

We first consider the case, where the boundary condition (4.63) is understood as a nonlocal perturbation of a two-point condition of an anti-periodic type. More precisely, we consider the boundary condition

$$u(a) + \lambda u(b) = h_0(u) - h_1(u) + c, \tag{4.66}$$

where $\lambda \geq 0$, $h_0, h_1 \in \mathcal{PF}_{ab}$, and $c \in \mathbb{R}$. Observe that there is no loss of generality in assuming this, because an arbitrary functional h can be represented in the form

$$h(v) := v(a) + \lambda v(b) - h_0(v) + h_1(v) \quad \text{for } v \in C([a,b];\mathbb{R}),$$

where $\lambda \geq 0$ and $h_0, h_1 \in \mathcal{PF}_{ab}$.

THEOREM 4.20 ([42, Thm. 2.1]). *Let $h_0(1) < 1 + \lambda + h_1(1)$ and $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$. Let, moreover,*

$$\lambda\big(\lambda - h_0(1)\big) \leq \big(1 + h_1(1)\big)^2 \tag{4.67}$$

and either the conditions

$$\|\ell_0\| < 1 - h_0(1) - \big(\lambda + h_1(1)\big)^2,$$
$$\|\ell_1\| < 1 - \lambda - h_1(1) + 2\sqrt{1 - h_0(1) - \|\ell_0\|}\,, \tag{4.68}$$

be satisfied, or the conditions

$$\|\ell_0\| \geq 1 - h_0(1) - \big(\lambda + h_1(1)\big)^2,$$
$$\|\ell_0\| + \big(\lambda + h_1(1)\big)\|\ell_1\| < 1 + \lambda - h_0(1) + h_1(1), \tag{4.69}$$
$$\big(1 + h_1(1)\big)\|\ell_0\| + \lambda\|\ell_1\| < 1 + \lambda - h_0(1) + h_1(1) \tag{4.70}$$

hold. Then problem (4.62), (4.66) *has a unique solution.*

REMARK 4.21. Let $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$. Define the operator $\varphi\colon C([a,b];\mathbb{R}) \to C([a,b];\mathbb{R})$ by setting

$$\varphi(w)(t) := w(a+b-t) \quad \text{for } t \in [a,b],\ w \in C([a,b];\mathbb{R}).$$

For $i = 0, 1$, we put

$$\tilde{\ell}_i(w)(t) := \ell_i(\varphi(w))(a+b-t) \quad \text{for a. e. } t \in [a,b] \text{ and all } w \in C([a,b];\mathbb{R})$$

and

$$\tilde{q}(t) := -q(a+b-t) \quad \text{for a. e. } t \in [a,b],$$
$$\tilde{h}(w) := h(\varphi(w)) \quad \text{for } w \in C([a,b];\mathbb{R}).$$

It is clear that if u is a solution of problem (4.62), (4.63), then the function $v := \varphi(u)$ is a solution of the problem

$$v'(t) = \tilde{\ell}_1(v)(t) - \tilde{\ell}_0(v)(t) + \tilde{q}(t), \qquad \tilde{h}(v) = c, \tag{4.71}$$

and vice versa, if v is a solution of problem (4.71) then the function $u := \varphi(v)$ is a solution of problem (4.62), (4.63).

Using the transformation described in the previous remark, we can immediately derive from Theorem 4.20 the following statement.

THEOREM 4.22 ([42, Thm. 2.2]). *Let* $\lambda > 0$, $h_0(1) < 1 + \lambda + h_1(1)$, *and* $\ell = \ell_0 - \ell_1$, *where* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$. *Let, moreover,*

$$1 - h_0(1) \leq \big(\lambda + h_1(1)\big)^2$$

and either the conditions

$$\|\ell_1\| < 1 - \frac{1}{\lambda}\,h_0(1) - \frac{\big(1 + h_1(1)\big)^2}{\lambda^2}\,,$$
$$\|\ell_0\| < 1 - \frac{1}{\lambda}\big(1 + h_1(1)\big) + 2\sqrt{1 - \frac{1}{\lambda}\,h_0(1) - \|\ell_1\|}$$

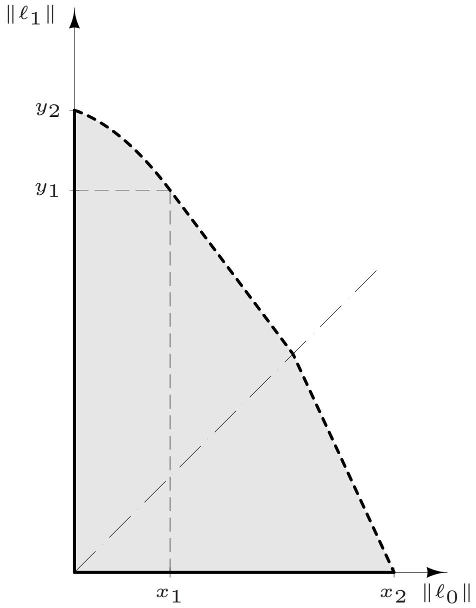
be satisfied, or

$$\|\ell_1\| \geq 1 - \frac{1}{\lambda}\, h_0(1) - \frac{\big(1 + h_1(1)\big)^2}{\lambda^2}$$

and the conditions (4.69) *and* (4.70) *hold. Then problem* (4.62), (4.66) *has a unique solution.*

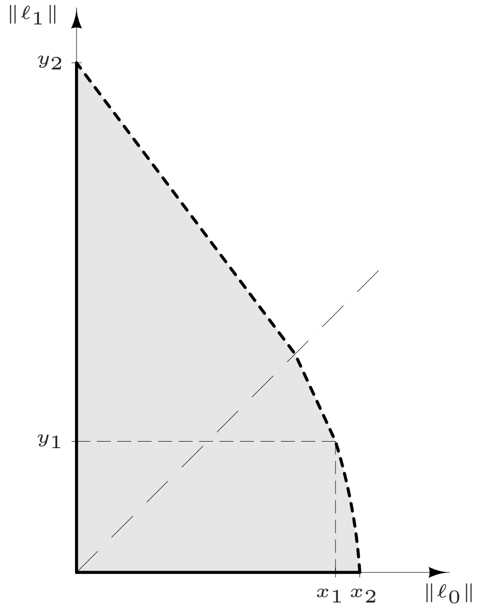
Remark 4.23. Geometrical meaning of the assumptions of Theorems 4.20 and 4.22 is illustrated on Figures 4.1 and 4.2, respectively.

Figure 4.1:



$$x_1 = 1 - h_0(1) - (\lambda + h_1(1))^2$$
$$x_2 = \frac{1 + \lambda - h_0(1) + h_1(1)}{1 + \lambda + h_1(1)}$$
$$y_1 = 1 + \lambda + h_1(1)$$
$$y_2 = 1 - \lambda - h_1(1) + 2\sqrt{1 - h_0(1)}$$

Figure 4.2:



$$x_1 = 1 + \frac{1 + h_1(1)}{\lambda}$$
$$x_2 = 1 - \frac{1 + h_1(1)}{\lambda} + 2\sqrt{1 - \frac{1}{\lambda}\, h_0(1)}$$
$$y_1 = 1 - \frac{1}{\lambda}\, h_0(1) - \frac{(1 + h_1(1)r}{\lambda^2}$$
$$y_2 = \frac{1 + \lambda - h_0(1) + h_1(1)}{\lambda + h_1(1)}$$

If $\lambda = 0$ in (4.66), we arrive at the problem

$$u'(t) = \ell(u)(t) + q(t), \qquad u(a) = h_0(u) - h_1(u) + c \tag{4.72}$$

and from Theorem 4.20 we get the following statement

Corollary 4.24 ([42, Cor. 2.2]). *Let* $h_0(1) < 1 + h_1(1)$ *and* $\ell = \ell_0 - \ell_1$, *where* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$. *Let, moreover, either the conditions*

$$\|\ell_0\| < 1 - h_0(1) - h_1(1)^2,$$
$$\|\ell_1\| < 1 - h_1(1) + 2\sqrt{1 - h_0(1) - \|\ell_0\|}\,,$$

be satisfied, or the conditions

$$1 - h_0(1) - h_1(1)^2 \le \|\ell_0\| < 1 - \frac{h_0(1)}{1 + h_1(1)}\,,$$
$$\|\ell_0\| + h_1(1)\|\ell_1\| < 1 - h_0(1) + h_1(1)$$

hold. Then problem (4.72) *has a unique solution.*

Finally, we give two statements dealing with the unique solvability of problem (4.62), (4.63), where $h = h^+ - h^-$ with $h^+, h^- \in \mathcal{PF}_{ab}$. There is no loss of generality in assuming this, because every linear bounded functional $h\colon C([a,b]) \to \mathbb{R}$ can be expressed in such a form.

THEOREM 4.25 ([42, Thm. 2.3]). *Let* $h(1) > 0$, $h = h^+ - h^-$ *with* $h^+, h^- \in \mathcal{PF}_{ab}$, *and* $\ell = \ell_0 - \ell_1$, *where* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$. *Let, moreover, the conditions*

$$\|\ell_0\| + h^+(1)\|\ell_1\| < h(1)$$

and

$$h^+(1)\|\ell_0\| + \|\ell_1\| < h(1)$$

be fulfilled. Then problem (4.62), (4.63) *has a unique solution.*

THEOREM 4.26 ([42, Thm. 2.4]). *Let* $h(1) < 0$, $h = h^+ - h^-$ *with* $h^+, h^- \in \mathcal{PF}_{ab}$, *and* $\ell = \ell_0 - \ell_1$, *where* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$. *Let, moreover, the conditions*

$$\|\ell_0\| + h^-(1)\|\ell_1\| < |h(1)|$$

and

$$h^-(1)\|\ell_0\| + \|\ell_1\| < |h(1)|$$

be fulfilled. Then problem (4.62), (4.63) *has a unique solution.*

REMARK 4.27. Geometrical meaning of the assumptions of Theorems 4.25 and 4.26 is illustrated on Figures 4.3 and 4.4, respectively.

REMARK 4.28. From Theorems 4.20, 4.22, 4.25 and 4.26 we can immediately obtain conditions guaranteeing the unique solvability of problems (4.65),(4.66) and (4.65), (4.63) when we replace the norms $\|\ell_0\|$ and $\|\ell_1\|$ appearing therein by the integrals $\int_a^b [p(s)]_+ \mathrm{d}s$ and $\int_a^b [p(s)]_- \mathrm{d}s$ (see [42, Thm. 2.5 and 2.6]).

Figure 4.3:

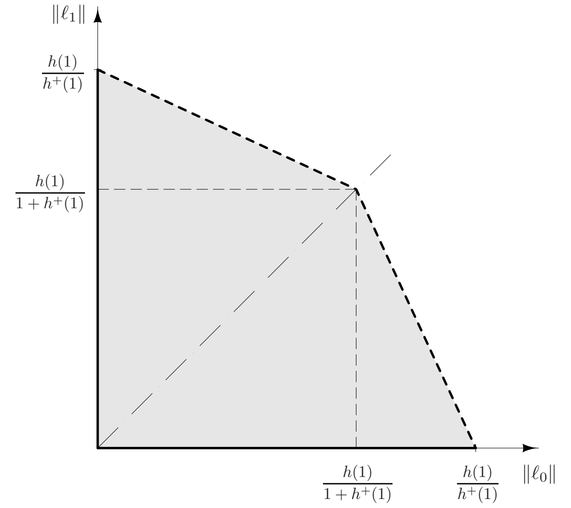
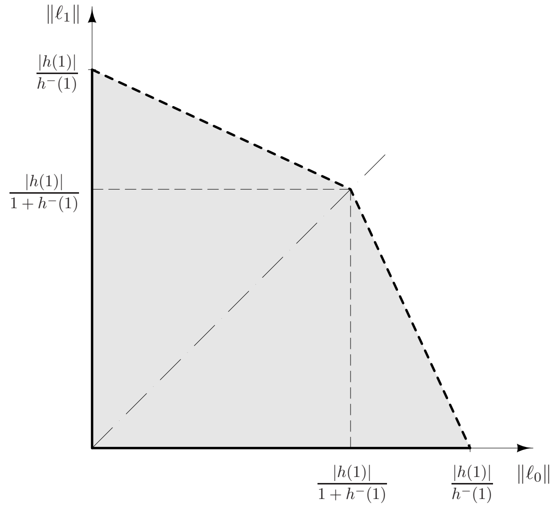


Figure 4.4:



4.3.2 Proofs of the main results

We first recall that linear problem (4.62), (4.63) has Fredholm's property, i. e., the following assertion holds (see, e.g., [1, 3, 9, 20, 48]).

LEMMA 4.29. *The problem* (4.62), (4.63) *has a unique solution for any* $q \in L([a,b];\mathbb{R})$ *and* $c \in \mathbb{R}$ *if and only if the corresponding homogeneous problem*

$$u'(t) = \ell(u)(t), \qquad h(u) = 0$$

has only the trivial solution.

Proof of Theorem 4.20. According to Lemma 4.29, to prove the theorem it is sufficient to show that the homogeneous problem

$$u'(t) = \ell_0(u)(t) - \ell_1(u)(t), \tag{4.73}$$

$$u(a) + \lambda u(b) = h_0(u) - h_1(u) \tag{4.74}$$

has only the trivial solution. Assume on the contrary that, u is a nontrivial solution of problem (4.73), (4.74).

First suppose that u changes its sign. Put

$$M := \max\{u(t) : t \in [a,b]\}, \qquad m := -\min\{u(t) : t \in [a,b]\} \tag{4.75}$$

and choose $t_M, t_m \in [a,b]$ such that

$$u(t_M) = M, \qquad u(t_m) = -m. \tag{4.76}$$

It is clear that

$$M > 0, \qquad m > 0. \tag{4.77}$$

We can assume without loss of generality that $t_M < t_m$. Integrating equality (4.73) from t_M to t_m, from a to t_M, and from t_m to b and taking into account (4.75), (4.76), and the assumption $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, one gets

$$M + m = \int_{t_M}^{t_m} \ell_1(u)(s)\,\mathrm{d}s - \int_{t_M}^{t_m} \ell_0(u)(s)\,\mathrm{d}s \le MB_1 + mA_1, \tag{4.78}$$

$$\begin{aligned} M - u(a) + u(b) + m = \int_a^{t_M} \ell_0(u)(s)\,\mathrm{d}s - \int_a^{t_M} \ell_1(u)(s)\,\mathrm{d}s + \\ + \int_{t_m}^{b} \ell_0(u)(s)\,\mathrm{d}s - \int_{t_m}^{b} \ell_1(u)(s)\,\mathrm{d}s \le MA_2 + mB_2, \end{aligned} \tag{4.79}$$

where

$$A_1 := \int_{t_M}^{t_m} \ell_0(1)(s)\,\mathrm{d}s, \qquad A_2 := \int_a^{t_M} \ell_0(1)(s)\,\mathrm{d}s + \int_{t_m}^{b} \ell_0(1)(s)\,\mathrm{d}s,$$
$$B_1 := \int_{t_M}^{t_m} \ell_1(1)(s)\,\mathrm{d}s, \qquad B_2 := \int_a^{t_M} \ell_1(1)(s)\,\mathrm{d}s + \int_{t_m}^{b} \ell_1(1)(s)\,\mathrm{d}s.$$

On the other hand, in view of relations (4.76), (4.77) and the assumption $h_0, h_1 \in \mathcal{PF}_{ab}$, from the boundary condition (4.74) we obtain

$$u(a) - u(b) = -\big(1+\lambda\big)u(b) + h_0(u) - h_1(u) \le \big(1+\lambda\big)m + Mh_0(1) + mh_1(1)$$

and

$$\begin{aligned} u(a) - u(b) = \Big(1 + \frac{1}{\lambda}\Big)u(a) - \frac{1}{\lambda}h_0(u) + \frac{1}{\lambda}h_1(u) \le \\ \le \Big(1 + \frac{1}{\lambda}\Big)M + m\frac{1}{\lambda}\,h_0(1) + M\frac{1}{\lambda}\,h_1(1). \end{aligned}$$

Hence, it follows from equality (4.79) that

$$M - \lambda m \le MA_2 + mB_2 + Mh_0(1) + mh_1(1) \tag{4.80}$$

and

$$m - \frac{1}{\lambda}\,M \le MA_2 + mB_2 + m\frac{1}{\lambda}\,h_0(1) + M\frac{1}{\lambda}\,h_1(1). \tag{4.81}$$

We first assume that $\|\ell_0\| \ge 1$. Then conditions (4.69) and (4.70) are supposed to be satisfied. It is clear that inequality (4.70) implies $\lambda > 0$ and $\|\ell_1\| < 1 - \frac{1}{\lambda}h_0(1)$ and thus, we have

$$B_1 < 1, \qquad B_2 < 1 - \frac{1}{\lambda}\,h_0(1).$$

Using these inequalities and relations (4.77), from (4.78) and (4.81) we get

$$0 < M(1 - B_1) \le m(A_1 - 1),$$
$$0 < m\Big(1 - \frac{1}{\lambda}\,h_0(1) - B_2\Big) \le M\Big(A_2 + \frac{1}{\lambda}\big(1 + h_1(1)\big)\Big),$$

which yields that

$$(1-B_1)\Big(1-\frac{1}{\lambda}\,h_0(1)-B_2\Big)\le (A_1-1)\Big(A_2+\frac{1}{\lambda}\big(1+h_1(1)\big)\Big). \tag{4.82}$$

Obviously,

$$(1-B_1)\Big(1-\frac{1}{\lambda}h_0(1)-B_2\Big)\ge 1-\frac{1}{\lambda}h_0(1)-\|\ell_1\|. \tag{4.83}$$

On the other hand, by virtue of (4.67), it follows from inequality (4.70) that

$$\|\ell_0\|<1+\frac{\lambda-h_0(1)}{1+h_1(1)}\le 1+\frac{1}{\lambda}\big(1+h_1(1)\big),$$

and thus, we obtain

$$\begin{aligned}(A_1-1)\Big(A_2+\frac{1}{\lambda}\big(1+h_1(1)\big)\Big)&\le (\|\ell_0\|-1)A_2+(A_1-1)\frac{1}{\lambda}\big(1+h_1(1)\big)\le\\ &\le\frac{1}{\lambda}\big(1+h_1(1)\big)(A_1+A_2-1)\le\frac{1}{\lambda}\big(1+h_1(1)\big)(\|\ell_0\|-1).\end{aligned} \tag{4.84}$$

Now from (4.82), (4.83), and (4.84) we get

$$1+\lambda-h_0(1)+h_1(1)\le\big(1+h_1(1)\big)\|\ell_0\|+\lambda\|\ell_1\|,$$

which contradicts inequality (4.70).

Now assume that $\|\ell_0\|<1$. Then, in view of condition (4.77), inequalities (4.78) and (4.80) yield that

$$\begin{gathered}0<m\big(1-A_1\big)\le M\big(B_1-1\big),\\ M\big(1-h_0(1)-A_2\big)\le m\big(B_2+\lambda+h_1(1)\big)\end{gathered}$$

and thus, we get $\|\ell_1\|\ge B_1>1$ and

$$\big(1-A_1\big)\big(1-h_0(1)-A_2\big)\le\big(B_1-1\big)\big(B_2+\lambda+h_1(1)\big). \tag{4.85}$$

Obviously,

$$\big(1-A_1\big)\big(1-h_0(1)-A_2\big)\ge 1-h_0(1)-\|\ell_0\|. \tag{4.86}$$

If $\|\ell_0\|\ge 1-h_0(1)-\big(\lambda+h_1(1)\big)^2$, then conditions (4.69) and (4.70) are supposed to be satisfied. Therefore, we obtain from the inequality (4.69) that $\|\ell_1\|\le 1+\lambda+h_1(1)$ and thus, it is easy to verify that

$$\begin{aligned}\big(B_1-1\big)\big(B_2+\lambda+h_1(1)\big)&\le(\|\ell_1\|-1)B_2+(B_1-1)(\lambda+h_1(1))\le\\ &\le\big(\lambda+h_1(1)\big)\big(B_1+B_2-1\big)\le\big(\lambda+h_1(1)\big)\big(\|\ell_1\|-1\big).\end{aligned} \tag{4.87}$$

Now it follows from (4.85), (4.86), and (4.87) that

$$1+\lambda-h_0(1)+h_1(1)\le\|\ell_0\|+\big(\lambda+h_1(1)\big)\|\ell_1\|,$$

which contradicts inequality (4.69).

If $\|\ell_0\| < 1 - h_0(1) - \big(\lambda + h_1(1)\big)^2$, then taking into account the above-mentioned condition $\|\ell_1\| > 1$ and the obvious inequality

$$\big(B_1 - 1\big)\big(B_2 + \lambda + h_1(1)\big) \leq \frac{1}{4}\big(\|\ell_1\| - 1 + \lambda + h_1(1)\big)^2,$$

from relations (4.85) and (4.86) we get

$$1 - \lambda - h_1(1) + 2\sqrt{1 - h_0(1) - \|\ell_0\|} \leq \|\ell_1\|,$$

which contradicts inequality (4.68).

Now suppose that u does not change its sign. Then, without loss of generality, we can assume that

$$u(t) \geq 0 \quad \text{for } t \in [a,b]. \tag{4.88}$$

Put

$$M_0 := \max\big\{u(t) : t \in [a,b]\big\}, \qquad m_0 := \min\big\{u(t) : t \in [a,b]\big\} \tag{4.89}$$

and choose $t_{M_0}, t_{m_0} \in [a,b]$ such that

$$u(t_{M_0}) = M_0, \qquad u(t_{m_0}) = m_0. \tag{4.90}$$

It is clear that

$$M_0 > 0, \qquad m_0 \geq 0,$$

and either

$$t_{M_0} \geq t_{m_0}, \tag{4.91}$$

or

$$t_{M_0} < t_{m_0}. \tag{4.92}$$

Observe that if the assumptions of the theorem are fulfilled, then both inequalities

$$A + \big(\lambda + h_1(1)\big)B < 1 + \lambda - h_0(1) + h_1(1) \tag{4.93}$$

and

$$\big(1 + h_1(1)\big)A + \lambda B < 1 + \lambda - h_0(1) + h_1(1) \tag{4.94}$$

hold, where $A := \|\ell_0\|$ and $B := \|\ell_1\|$.

Integrating equality (4.73) from a to t_{M_0} and from t_{M_0} to b and taking into account relations (4.88), (4.89), and (4.90), and the assumption $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, one gets

$$M_0 - u(a) = \int_a^{t_{M_0}} \ell_0(u)(s)\,\mathrm{d}s - \int_a^{t_{M_0}} \ell_1(u)(s)\,\mathrm{d}s \leq M_0 A$$

and

$$M_0 - u(b) = \int_{t_{M_0}}^b \ell_1(u)(s)\,\mathrm{d}s - \int_{t_{M_0}}^b \ell_0(u)(s)\,\mathrm{d}s \leq M_0 B.$$

The last two inequalities yield that

$$M_0(1 + \lambda) - u(a) - \lambda u(b) \leq M_0(A + \lambda B)$$

and thus, using (4.74), (4.89), and the assumption $h_0, h_1 \in \mathcal{PF}_{ab}$, we obtain

$$m_0 h_1(1) \leq M_0\big(A + \lambda B + h_0(1) - 1 - \lambda\big). \tag{4.95}$$

First suppose that (4.91) holds. Integrating equality (4.73) from t_{m_0} to t_{M_0} and taking into account (4.88), (4.89), (4.90), and the assumption $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, one gets

$$M_0 - m_0 = \int_{t_{m_0}}^{t_{M_0}} \ell_0(u)(s)\, \mathrm{d}s - \int_{t_{m_0}}^{t_{M_0}} \ell_1(u)(s)\, \mathrm{d}s \leq M_0 A,$$

i. e.,

$$M_0\big(1 - A\big) \leq m_0.$$

It follows from the latter inequality and (4.95) that

$$\big(1 + h_1(1)\big)A + \lambda B \geq 1 + \lambda - h_0(1) + h_1(1),$$

which contradicts inequality (4.94).

Now assume that (4.92) holds. Integrating of equality (4.73) from t_{M_0} to t_{m_0} and taking into account (4.88), (4.89), (4.90), and the assumption $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, we obtain

$$M_0 - m_0 = \int_{t_{M_0}}^{t_{m_0}} \ell_1(u)(s)\, \mathrm{d}s - \int_{t_{M_0}}^{t_{m_0}} \ell_0(u)(s)\, \mathrm{d}s \leq M_0 B,$$

i. e.,

$$M_0(1 - B) \leq m_0.$$

The last inequality, together with (4.95), yields that

$$A + \big(\lambda + h_1(1)\big)B \geq 1 + \lambda - h_0(1) + h_1(1),$$

which contradicts inequality (4.93).

The contradictions obtained prove that the homogeneous problem (4.73), (4.74) has only the trivial solution. □

Proof of Theorem 4.22. Using the transformation described in Remark 4.21, the assertion of the theorem can be derived from Theorem 4.20. □

Proof of Corollary 4.24. It follows from Theorem 4.20 with $\lambda = 0$. □

Proof of Theorem 4.25. Let the functionals h_0 and h_1 be defined by the formulae

$$h_0(v) := v(a) + h^-(v), \quad h_1(v) := h^+(v) \quad \text{for } v \in C([a,b];\mathbb{R}).$$

By virtue of Corollary 4.24, problem (4.62), (4.63) is uniquely solvable under the assumptions

$$\|\ell_0\| < 1 - \frac{1 + h^-(1)}{1 + h^+(1)}, \qquad \|\ell_0\| + h^+(1)\|\ell_1\| < h^+(1) - h^-(1).$$

Moreover, using the transformation described in Remark 4.21, it is not difficult to verify that the problem (4.62), (4.63) is uniquely solvable also under the assumptions

$$\|\ell_1\| < 1 - \frac{1 + h^-(1)}{1 + h^+(1)}, \qquad \|\ell_1\| + h^+(1)\|\ell_0\| < h^+(1) - h^-(1).$$

Combining these two cases, we obtain the desired assertion. □

Proof of Theorem 4.26. The assertion of the theorem follows from Theorem 4.25 and the fact that the problem

$$u'(t) = \ell(u)(t) + q(t), \qquad h(u) = c$$

has a unique solution for every $q \in L([a,b];\mathbb{R})$ and $c \in \mathbb{R}$ if and only if the problem

$$v'(t) = \ell(v)(t) + q(t), \qquad -h(v) = c$$

has a unique solution for every $q \in L([a,b];\mathbb{R})$ and $c \in \mathbb{R}$. □

4.4 Nonlinear problem

In this section, we establish sufficient conditions for the solvability as well as unique solvability of a nonlocal boundary value problem for nonlinear functional differential equations. On the interval $[a,b]$, we consider the functional differential equation

$$u' = F(u)(t), \tag{4.96}$$

where $F : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ is a continuous (in general) nonlinear operator, subjected to the linear nonlocal boundary condition

$$h(u) = c, \tag{4.97}$$

where $h : C([a,b];\mathbb{R}) \to \mathbb{R}$ is a (non-zero) linear bounded functional and c is a real number.

Recall that by a solution of equation (4.96) we understand an absolutely continuous function $u : [a,b] \to \mathbb{R}$ satisfying equality (4.96) almost everywhere on the interval $[a,b]$. A solution of equation (4.96) satisfying the boundary condition (4.97) is said to be a solution of problem (4.96), (4.97).

We assume in theorems below that the functional h in the boundary condition (4.97) admits the representation $h = h_0 - h_1$, where $h_0, h_1 \in \mathcal{PF}_{ab}$. There is no lost of generality in assuming this, because an arbitrary linear bounded functional can be expressed in this form. As fot the operator F in equation (4.96), we introduce the hypothesis:

$$\left.\begin{array}{l} F : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R}) \text{ is a continuous operator such that the relation} \\ \qquad\qquad \sup\big\{|F(v)(\cdot)| : v \in C([a,b];\mathbb{R}),\ \|v\|_C \le r\big\} \in L([a,b];\mathbb{R}_+) \\ \text{is satisfied for every } r > 0. \end{array}\right\} \quad (H)$$

4.4.1 Main results

For any $h_0 \in \mathcal{PF}_{ab}$, $c \in [a,b]$, and $\lambda \geq 0$, we put

$$h_{0,c}^{\lambda}(v) := h_0(v) - \lambda v(c) \quad \text{for } v \in C([a,b];\mathbb{R}), \tag{4.98}$$

Obviously, $h_{0,c}^{0} \in \mathcal{PF}_{ab}$. It allows one to set

$$\lambda_c^* := \sup\{\lambda \geq 0 : h_{0,c}^{\lambda} \in \mathcal{PF}_{ab}\} \tag{4.99}$$

It is clear that $0 \leq \lambda_c^* \leq h_0(1)$ and

$$h_{0,c}^{\lambda_c^*} \in \mathcal{PF}_{ab}. \tag{4.100}$$

Theorem 4.30 ([40, Thm. 2.1]). *Let assumption* (H) *be satisfied, the functional* h *admit the representation* $h = h_0 - h_1$ *with* $h_0, h_1 \in \mathcal{PF}_{ab}$, *and the condition*

$$h_1(1) < \lambda_a^* \tag{4.101}$$

hold, where the number λ_a^* *is defined by formula* (4.99). *Let, moreover, there exist*

$$\ell_0, \ell_1 \in \mathcal{P}_{ab} \tag{4.102}$$

such that for any $v \in C([a,b];\mathbb{R})$, *the inequality*

$$\big[F(v)(t) - \ell_0(v)(t) + \ell_1(v)(t)\big] \operatorname{sgn} v(t) \leq q(t, \|v\|_C) \quad \text{for a. e. } t \in [a,b] \tag{4.103}$$

holds, where the function $q \in K([a,b] \times \mathbb{R}_+;\mathbb{R}_+)$ *satisfies the condition*

$$\lim_{x \to +\infty} \frac{1}{x} \int_a^b q(s,x)\,\mathrm{d}s = 0. \tag{4.104}$$

If, in addition, either

$$\begin{aligned} \|\ell_0\| &< 1 - \frac{1}{\lambda_a^*}\, h_1(1) - \Big(\frac{1}{\lambda_a^*}\Big(h_0(1) - \lambda_a^*\Big)\Big)^2, \\ \|\ell_1\| &< 2\sqrt{1 - \frac{1}{\lambda_a^*}\, h_1(1) - \|\ell_0\|} - \frac{1}{\lambda_a^*}\Big(h_0(1) - \lambda_a^*\Big), \end{aligned} \tag{4.105}$$

or

$$\begin{aligned} \|\ell_0\| &\geq 1 - \frac{1}{\lambda_a^*}\, h_1(1) - \Big(\frac{1}{\lambda_a^*}\Big(h_0(1) - \lambda_a^*\Big)\Big)^2, \\ \lambda_a^* \|\ell_0\| &+ \Big(h_0(1) - \lambda_a^*\Big)\|\ell_1\| < \lambda_a^* - h_1(1), \end{aligned} \tag{4.106}$$

then problem (4.96), (4.97) *has at least one solution.*

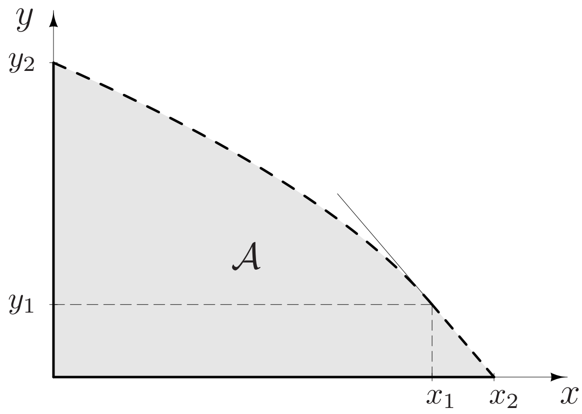


Figure 4.5: The set $\mathcal{A}$: $x_1 = 1 - \frac{1}{\lambda_a^*} h_1(1) - \big(\frac{1}{\lambda_a^*}\big(h_0(1) - \lambda_a^*\big)\big)^2$, $x_2 = 1 - \frac{1}{\lambda_a^*} h_1(1)$, $y_1 = \frac{1}{\lambda_a^*}\big(h_0(1) - \lambda_a^*\big)$, $y_2 = 2\sqrt{1 - \frac{1}{\lambda_a^*} h_1(1)} - \frac{1}{\lambda_a^*}\big(h_0(1) - \lambda_a^*\big)$.

Remark 4.31. Let $h_0, h_1 \in \mathcal{PF}_{ab}$ and $\mathcal{A}$ denote the set of $(x, y) \in \mathbb{R}_+^2$ such that either

$$x < 1 - \frac{1}{\lambda_a^*} h_1(1) - \Big(\frac{1}{\lambda_a^*}\big(h_0(1) - \lambda_a^*\big)\Big)^2, \quad y < 2\sqrt{1 - \frac{1}{\lambda_a^*} h_1(1) - x} - \frac{1}{\lambda_a^*}\big(h_0(1) - \lambda_a^*\big),$$

or

$$x \geq 1 - \frac{1}{\lambda_a^*} h_1(1) - \Big(\frac{1}{\lambda_a^*}\big(h_0(1) - \lambda_a^*\big)\Big)^2, \quad \lambda_a^* x + \big(h_0(1) - \lambda_a^*\big) y < \lambda_a^* - h_1(1),$$

where the number λ_a^* is defined by the formula (4.99) (see Fig. 4.5).

According to Theorem 4.30, if (H) and (4.101) hold, there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ such that inequality (4.103) is satisfied on the set $C([a, b]; \mathbb{R})$, and

$$\big(\|\ell_0\|, \|\ell_1\|\big) \in \mathcal{A},$$

then the problem (4.96), (4.97) with $h = h_0 - h_1$ has at least one solution.

Remark 4.32. If the functional h is defined by the formula

$$h(v) := \alpha v(a) + \beta v(b) \quad \text{for } v \in C([a, b]; \mathbb{R})$$

with $\alpha, \beta > 0$, the assumptions (4.105) and (4.106) of the previous theorem take the form

$$\|\ell_0\| < 1 - \left(\frac{\beta}{\alpha}\right)^2, \qquad \|\ell_1\| < -\frac{\beta}{\alpha} + 2\sqrt{1 - \|\ell_0\|}$$

and

$$\|\ell_0\| \geq 1 - \left(\frac{\beta}{\alpha}\right)^2, \qquad \alpha\|\ell_0\| + \beta\|\ell_1\| < \alpha,$$

respectively. Therefore, in this case, Theorem 4.30 reduces to Theorems 14.1 and 14.6 stated in [10].

Remark 4.33. Let the operator $\omega : C([a,b];\mathbb{R}) \to C([a,b];\mathbb{R})$ be defined by the formula

$$\omega(z)(t) := z(a+b-t) \quad \text{for } t \in [a,b], \ z \in C([a,b];\mathbb{R}).$$

Put

$$\widehat{F}(z)(t) := -F\big(\omega(z)\big)(a+b-t) \quad \text{for a. e. } t \in [a,b] \text{ and all } z \in C([a,b];\mathbb{R})$$

and

$$\widehat{h}(z) := h\big(\omega(z)\big) \quad \text{for } z \in C([a,b];\mathbb{R}).$$

Then u is a solution of problem (4.96), (4.97) if and only if the function $v := \omega(u)$ is a solution of the problem

$$v'(t) = \widehat{F}(v)(t), \qquad \widehat{h}(v) = c.$$

Using the transformation described in the previous remark, we can immediately derive from Theorem 4.30 the following statement.

Theorem 4.34 ([40, Thm. 2.2]). *Let assumption (H) be satisfied, the functional h admit the representation $h = h_0 - h_1$ with $h_0, h_1 \in \mathcal{PF}_{ab}$, and the condition*

$$h_1(1) < \lambda_b^* \tag{4.107}$$

hold, where the number λ_b^* be defined by formula (4.99)*. Let, moreover, there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ such that, for any $v \in C([a,b];\mathbb{R})$, the inequality*

$$\big[F(v)(t) - \ell_0(v)(t) + \ell_1(v)(t)\big] \operatorname{sgn} v(t) \ge -q(t, \|v\|_C) \quad \textit{for a. e. } t \in [a,b]$$

holds, where the function $q \in K([a,b] \times \mathbb{R}_+;\mathbb{R}_+)$ satisfies condition (4.104)*. If, in addition, either*

$$\begin{aligned} \|\ell_1\| &< 1 - \frac{1}{\lambda_b^*} h_1(1) - \Big(\frac{1}{\lambda_b^*}\Big(h_0(1) - \lambda_b^*\Big)\Big)^2, \\ \|\ell_0\| &< 2\sqrt{1 - \frac{1}{\lambda_b^*} h_1(1) - \|\ell_1\|} - \frac{1}{\lambda_b^*}\Big(h_0(1) - \lambda_b^*\Big), \end{aligned} \tag{4.108}$$

or

$$\begin{aligned} \|\ell_1\| &\ge 1 - \frac{1}{\lambda_b^*} h_1(1) - \Big(\frac{1}{\lambda_b^*}\Big(h_0(1) - \lambda_b^*\Big)\Big)^2, \\ \Big(h_0(1) - \lambda_b^*\Big)\|\ell_0\| &+ \lambda_b^*\|\ell_1\| < \lambda_b^* - h_1(1), \end{aligned} \tag{4.109}$$

then problem (4.96), (4.97) *has at least one solution.*

The next theorems deal with the unique solvability of problem (4.96), (4.97).

Theorem 4.35 ([40, Thm. 2.3]). *Let assumption (H) be satisfied, the functional h admit the representation $h = h_0 - h_1$ with $h_0, h_1 \in \mathcal{PF}_{ab}$ and condition* (4.101) *hold,* *where the number λ_a^* be defined by formula* (4.99). *Let, moreover, there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ such that the inequality*

$$\big[F(v)(t) - F(w)(t) - \ell_0(v-w)(t) + \ell_1(v-w)(t)\big]\operatorname{sgn}\big(v(t) - w(t)\big) \le 0 \quad \text{for a. e. } t \in [a,b] \tag{4.110}$$

holds on the space $C([a,b];\mathbb{R})$. If, in addition, either condition (4.105), *or condition* (4.106) *is fulfilled, then problem* (4.96), (4.97) *is uniquely solvable.*

THEOREM 4.36 ([40, Thm. 2.4]). *Let assumption* (H) *be satisfied, the functional h admit the representation $h = h_0 - h_1$ with $h_0, h_1 \in \mathcal{PF}_{ab}$, and condition* (4.107) *hold, where the number λ_b^* be defined by formula* (4.99). *Let, moreover, there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ such that the inequality*

$$\big[F(v)(t) - F(w)(t) - \ell_0(v-w)(t) + \ell_1(v-w)(t)\big]\operatorname{sgn}\big(v(t) - w(t)\big) \ge 0 \quad \text{for a. e. } t \in [a,b]$$

holds on the space $C([a,b];\mathbb{R})$. If, in addition, either condition (4.108), *or condition* (4.109) *is fulfilled, then problem* (4.96), (4.97) *is uniquely solvable.*

4.4.2 Proofs of the main results

The main results are proved using so-called principle of a priory estimate due to Kiguradze and Půža. It can be formulated as follows.

LEMMA 4.37 ([19, Cor. 2]). *Let there exist a positive number ρ and an operator $\ell \in \mathcal{L}_{ab}$ such that homogeneous problem*

$$u' = \ell(u)(t), \qquad h(u) = 0 \tag{4.111}$$

has only the trivial solution and for every $\delta \in\,]0,1[$, an arbitrary function $u \in AC([a,b];\mathbb{R})$ satisfying the relations

$$u' = \ell(u)(t) + \delta[F(u)(t) - \ell(u)(t)] \quad \text{for a. e. } t \in [a,b], \quad h(u) = \delta c \tag{4.112}$$

admits the estimate

$$\|u\|_C \le \rho. \tag{4.113}$$

Then problem (4.96), (4.97) *has at least one solution.*

Now we prove lemma on a apriory estimate suitable for our problem.

LEMMA 4.38. *Let assumption* (H) *be satisfied, the functional h admit the representation $h = h_0 - h_1$ with $h_0, h_1 \in \mathcal{PF}_{ab}$, and condition* (4.101) *hold, where the number λ_a^* be defined by formula* (4.99). *Let moreover, the operators $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ be such that either condition* (4.105), *or condition* (4.106) *is fulfilled. Then there exists $r > 0$ such* *that for any $c^* \in \mathbb{R}_+$ and $q^* \in L([a,b];\mathbb{R}_+)$, an arbitrary function $u \in AC([a,b];\mathbb{R})$ satisfying the inequalities*

$$h(u)\operatorname{sgn} u(a) \le c^*, \tag{4.114}$$

$$\big[u' - \ell_0(u)(t) + \ell_1(u)(t)\big]\operatorname{sgn} u(t) \le q^*(t) \quad \textit{for a. e. } t \in [a,b] \tag{4.115}$$

admits the estimate

$$\|u\|_C \le r(c^* + \|q^*\|_L). \tag{4.116}$$

Proof. Let $c^* \in \mathbb{R}_+$, $q^* \in L([a,b];\mathbb{R}_+)$, and $u \in AC([a,b];\mathbb{R})$ satisfy conditions (4.114) and (4.115). We show that estimate (4.116) holds, where the number r depends only on $\|\ell_0\|$, $\|\ell_1\|$, λ_a^*, $h_0(1)$, and $h_1(1)$. It is clear that

$$u(t)' = \ell_0(u)(t) - \ell_1(u)(t) + \widetilde{q}(t) \quad \text{for a. e. } t \in [a,b], \tag{4.117}$$

where

$$\widetilde{q}(t) := u' - \ell_0(u)(t) + \ell_1(u)(t) \quad \text{for a. e. } t \in [a,b].$$

From condition (4.115) we get

$$\widetilde{q}(t)\operatorname{sgn} u(t) \le q^*(t) \quad \text{for a. e. } t \in [a,b]. \tag{4.118}$$

First suppose that the function u does not change its sign. Put

$$M_0 := \max\{|u(t)| : t \in [a,b]\} \tag{4.119}$$

and choose $t_{M_0} \in [a,b]$ such that

$$|u(t_{M_0})| = M_0.$$

It is clear that $M_0 \ge 0$ and, in view of (4.102), (4.118), and (4.119), from (4.117) we get

$$|u(t)|' \le M_0\,\ell_0(1)(t) + q^*(t) \quad \text{for a. e. } t \in [a,b]. \tag{4.120}$$

By virtue of (4.98), it is clear that

$$h(u)\operatorname{sgn} u(a) = \lambda_a^*|u(a)| - h_1(u)\operatorname{sgn} u(a) + h_{0,a}^{\lambda_a^*}(u)\operatorname{sgn} u(a)$$

and thus, relations (4.100), (4.114), and (4.119) yield

$$|u(a)| \le M\,\frac{h_1(1)}{\lambda_a^*} + c_\alpha^*(\lambda). \tag{4.121}$$

Integrating (4.120) from a to t_{M_0} and taking into account (4.102) and (4.121), one gets

$$M_0 - M_0\,\frac{h_1(1)}{\lambda_a^*} - c_\alpha^*(\lambda) \le M_0\,\|\ell_0\| + \|q^*\|_L.$$

Note that relations (4.105) and (4.106) yield that $\|\ell_0\| < 1 - \frac{h_1(1)}{\lambda_a^*}$. Therefore, from the last inequality we obtain

$$\|u\|_C \le r_0 \max\Big\{\frac{1}{\lambda_a^*}, 1\Big\}\,(c^* + \|q^*\|_L), \quad \text{where} \quad r_0 = \Big(1 - \frac{h_1(1)}{\lambda_a^*} - \|\ell_0\|\Big)^{-1} > 0,$$

and thus, estimate (4.116) holds, where the number r is defined by the formula

$$r := r_0 \max\Big\{\frac{1}{\lambda_a^*}, 1\Big\}.$$

Now suppose that the function u changes its sign. Put

$$M := \max\{u(t) : t \in [a,b]\}, \qquad m := -\min\{u(t) : t \in [a,b]\} \tag{4.122}$$

and choose $t_M, t_m \in [a,b]$ such that

$$u(t_M) = M, \qquad u(t_m) = -m. \tag{4.123}$$

Obviously, $M > 0$, $m > 0$, and either

$$t_m < t_M, \tag{4.124}$$

or

$$t_m > t_M. \tag{4.125}$$

Suppose that relation (4.124) holds. It is clear that there exists $\alpha_2 \in\ ]t_m, t_M[$ such that

$$u(t) > 0 \quad \text{for } \alpha_2 < t \le t_M, \qquad u(\alpha_2) = 0. \tag{4.126}$$

Let

$$\alpha_1 := \inf\{t \in [a, t_m] : u(s) < 0 \text{ for } t \le s \le t_m\}.$$

Obviously,

$$u(t) < 0 \quad \text{for } \alpha_1 < t \le t_m \qquad \text{and} \qquad u(\alpha_1) = 0 \quad \text{if } \alpha_1 > a. \tag{4.127}$$

In view of (4.98), it is clear that

$$\lambda_a^* u(a) = h_1(u) - h_{0,a}^{\lambda_a^*}(u) + h(u)$$

and thus, we obtain from (4.100), (4.114), (4.122), and (4.127) that

$$u(\alpha_1) \ge -m\,\frac{h_1(1)}{\lambda_a^*} - M\,\frac{1}{\lambda_a^*}\Big(h_0(1) - \lambda_a^*\Big) - c_\alpha^*(\lambda). \tag{4.128}$$

Integrating (4.117) from α_1 to t_m and from α_2 to t_M and taking into account (4.102), (4.115), (4.122), (4.123), and (4.126)–(4.128), one gets

$$m - m\,\frac{h_1(1)}{\lambda_a^*} - M\,\frac{1}{\lambda_a^*}\Big(h_0(1) - \lambda_a^*\Big) - c_\alpha^*(\lambda) \le$$
$$\le M\int_{\alpha_1}^{t_m} \ell_1(1)(s)\,\mathrm{d}s + m\int_{\alpha_1}^{t_m} \ell_0(1)(s)\,\mathrm{d}s + \int_{\alpha_1}^{t_m} q^*(s)\,\mathrm{d}s$$

and

$$M \le M \int\limits_{\alpha_2}^{t_M} \ell_0(1)(s)\,\mathrm{d}s + m \int\limits_{\alpha_2}^{t_M} \ell_1(1)(s)\,\mathrm{d}s + \int\limits_{\alpha_2}^{t_M} q^*(s)\,\mathrm{d}s.$$

Hence, we have

$$\begin{aligned} m\Big(1 - \frac{h_1(1)}{\lambda_a^*} - C\Big) &\le M\Big(\frac{1}{\lambda_a^*}\Big(h_0(1) - \lambda_a^*\Big) + A\Big) + \|q^*\|_L + c_\alpha^*(\lambda)\,, \\ M\Big(1 - D\Big) &\le mB + \|q^*\|_L, \end{aligned} \tag{4.129}$$

where

$$A := \int\limits_{\alpha_1}^{t_m} \ell_1(1)(s)\,\mathrm{d}s, \quad B := \int\limits_{\alpha_2}^{t_M} \ell_1(1)(s)\,\mathrm{d}s$$

and

$$C := \int\limits_{\alpha_1}^{t_m} \ell_0(1)(s)\,\mathrm{d}s, \quad D := \int\limits_{\alpha_2}^{t_M} \ell_0(1)(s)\,\mathrm{d}s.$$

In view of relations (4.105) and (4.106), we have $\|\ell_0\| < 1 - \frac{1}{\lambda_a^*} h_1(1)$ and thus, it is clear that $C < 1 - \frac{h_1(1)}{\lambda_a^*}$ and $D < 1$. Consequently, (4.129) implies

$$\begin{aligned} & m\Big(1 - \frac{h_1(1)}{\lambda_a^*} - C\Big)(1 - D) \le \\ & \qquad \le mB\Big(\frac{1}{\lambda_a^*}\Big(h_0(1) - \lambda_a^*\Big) + A\Big) + \Big(\|q^*\|_L + c_\alpha^*(\lambda)\Big)\Big(\frac{h_0(1)}{\lambda_a^*} + A\Big), \\ & M\Big(1 - \frac{h_1(1)}{\lambda_a^*} - C\Big)(1 - D) \le \\ & \qquad \le MB\Big(\frac{h_0(1)}{\lambda_a^*} - 1 + A\Big) + \Big(\|q^*\|_L + c_\alpha^*(\lambda)\Big)(B + 1). \end{aligned} \tag{4.130}$$

Observe that

$$\Big(1 - \frac{h_1(1)}{\lambda_a^*} - C\Big)\Big(1 - D\Big) \ge 1 - \frac{h_1(1)}{\lambda_a^*} - (C + D) \ge 1 - \frac{h_1(1)}{\lambda_a^*} - \|\ell_0\|. \tag{4.131}$$

First suppose that assumption (4.105) holds. Obviously,

$$B\Big(\frac{h_0(1)}{\lambda_a^*} - 1 + A\Big) \le \frac{1}{4}\Big(A + B + \frac{h_0(1)}{\lambda_a^*} - 1\Big)^2 \le \frac{1}{4}\Big(\|\ell_1\| + \frac{h_0(1)}{\lambda_a^*} - 1\Big)^2.$$

By virtue of the last inequality, (4.131) and the second inequality in (4.105), it follows from (4.130) that

$$\begin{aligned} m &\le r_1 \max\Big\{\frac{1}{\lambda_a^*}, 1\Big\}\Big(\|q^*\|_L + c^*\Big)\Big(\frac{h_0(1)}{\lambda_a^*} + \|\ell_1\|\Big), \\ M &\le r_1 \max\Big\{\frac{1}{\lambda_a^*}, 1\Big\}\Big(\|q^*\|_L + c^*\Big)(1 + \|\ell_1\|), \end{aligned}$$

where

$$r_1 := \left[1 - \frac{h_1(1)}{\lambda_a^*} - \|\ell_0\| - \frac{1}{4}\Big(\|\ell_1\| + \frac{h_0(1)}{\lambda_a^*} - 1\Big)^2\right]^{-1}.$$

Consequently, estimate (4.116) holds, where the number r is defined by the formula

$$r := r_1 \max\Big\{\frac{1}{\lambda_a^*}, 1\Big\}\Big(\frac{h_0(1)}{\lambda_a^*} + \|\ell_1\|\Big),$$

because we have $\lambda_a^* \leq h_0(1)$.

Now suppose that the assumption (4.106) holds. Using the relation $\lambda_a^* \leq h_0(1)$, from inequalities (4.106) we get

$$B \leq \|\ell_1\| < \frac{h_0(1)}{\lambda_a^*} - 1$$

and thus,

$$B\Big(\frac{h_0(1)}{\lambda_a^*} - 1 + A\Big) \leq B\Big(\frac{h_0(1)}{\lambda_a^*} - 1\Big) + A\Big(\frac{h_0(1)}{\lambda_a^*} - 1\Big) \leq \|\ell_1\|\Big(\frac{h_0(1)}{\lambda_a^*} - 1\Big).$$

By virtue of the last inequality, (4.131) and the second inequality in (4.106), it follows from (4.130) that

$$\begin{aligned} m &\leq r_2 \max\Big\{\frac{1}{\lambda_a^*}, 1\Big\}\Big(\|q^*\|_L + c^*\Big)\Big(\frac{h_0(1)}{\lambda_a^*} + \|\ell_1\|\Big), \\ M &\leq r_2 \max\Big\{\frac{1}{\lambda_a^*}, 1\Big\}\Big(\|q^*\|_L + c^*\Big)(1 + \|\ell_1\|), \end{aligned}$$

where

$$r_2 := \left[1 - \frac{h_1(1)}{\lambda_a^*} - \|\ell_0\| - \|\ell_1\|\Big(\frac{h_0(1)}{\lambda_a^*} - 1\Big)\right]^{-1}.$$

Consequently, estimate (4.116) holds, where the number r is given by the formula

$$r := r_2 \max\Big\{\frac{1}{\lambda_a^*}, 1\Big\}\Big(\frac{h_0(1)}{\lambda_a^*} + \|\ell_1\|\Big).$$

If relation (4.125) holds, then the validity of estimate (4.116) can be proved analogously. □

Proof of Theorems 4.30. Let $\ell = \ell_0 - \ell_1$. It is clear that $\ell \in \mathcal{L}_{ab}$ and all the assumptions of Lemma 4.38 are satisfied. Let r be the number appearing therein. According to (4.104), there exists $\rho > 2r|c|$ such that

$$\frac{1}{x}\int_a^b q(s,x)\,\mathrm{d}s < \frac{1}{2r} \quad \text{for } x > \rho.$$

First note that the homogeneous problem (4.111) has only the trivial solution. Indeed, if u is a solution of problem (4.111), then the function u satisfies inequalities

(4.114) and (4.115) with $c^* = 0$ and $q^* \equiv 0$. Consequently, Lemma 4.38 guarantees that $u \equiv 0$.

Now let $\delta \in]0,1[$ and $u \in AC([a,b];\mathbb{R})$ be a function satisfying condition (4.112). Then we obtain from (4.103) that inequalities (4.114) and (4.115) are fulfilled with $c^* = |c|$ and $q^* \equiv q(\cdot\,, \|u\|_C)$. Hence, using Lemma 4.38 and the definition of the number ρ, we get estimate (4.113). Indeed, assuming $\|u\|_C > \rho$, from estimate (4.116) we get

$$1 \le \frac{r|c|}{\|u\|_C} + \frac{r}{\|u\|_C} \int\limits_a^b q(s, \|u\|_C)\, \mathrm{d}s < 1,$$

which is a contradiction.

Since ρ depends neither on u nor on δ, it follows from Lemma 4.37 that problem (4.96), (4.97) has at least one solution. $\square$

Proof of Theorem 4.34. Acording to Remark 4.33, the assertion of the theorem follows immediately from Theorem 4.30. $\square$

Proof of Theorem 4.35. First note that the assumptions of Lemma 4.38 are satisfied. It follows from assumption (4.110) that inequality (4.103) is fulfilled on the set $C([a,b];\mathbb{R})$, where $q \equiv |F(0)|$. Consequently, all the assumptions of Theorem 4.30 are satisfied and thus, problem (4.96), (4.97) has at least one solution. It remains to show that problem (4.96), (4.97) has at most one solution.

Let u_1, u_2 be solutions of problem (4.96), (4.97). Put

$$u(t) := u_1(t) - u_2(t) \quad \text{for } t \in [a,b].$$

Then $h(u) = 0$ and, by virtue of inequality (4.110), we have

$$\big[u'(t) - \ell_0(u)(t) + \ell_1(u)(t)\big] \operatorname{sgn} u(t) \le 0 \quad \text{for a.e. } t \in [a,b].$$

Consequently, inequalities (4.114) and (4.115) are satisfied with $c^* = 0$ and $q^* \equiv 0$. Therefore, Lemma 4.38 guarantees that $u \equiv 0$, which yields $u_1 \equiv u_2$. $\square$

Proof of Theorem 4.36. Acording to Remark 4.33, the assertion of the theorem follows immediately from Theorem 4.35. $\square$

5 Singular Dirichlet problem

5.1 Introduction

Consider the boundary value problem

$$u'' = p(t)u + q(t), \tag{5.1}$$

$$u(a) = 0, \quad u(b) = 0, \tag{5.2}$$

where $p, q \in L_{loc}(]a, b[)$. We are mainly interested in the case, when the functions p and q are not (in general) integrable on $[a, b]$. In this case, equation (5.1) as well as problem (5.1), (5.2) are said to be singular. While if $p, q \in L([a, b])$, then equation (5.1) and problem (5.1), (5.2) are referred as regular. Theory of the regular Dirichlet problem is well developed. One of the main part of this theory is the so-called Sturm-Liouville's theory, which in its turn consists of the following three items: Fredholm's theorems, well-posedness and eigenvalue problem. However an analogue of the Sturm-Liouville's theory for the singular problem is far from being complete.

It is well known that for the singular problem (5.1), (5.2), the condition

$$\int_a^b (s-a)(b-s)|p(s)|ds < +\infty \tag{5.3}$$

guarantees the validity of Fredholm's alternative. More precisely, if (5.3) holds then the problem (5.1), (5.2) is uniquely solvable for any q satisfying

$$\int_a^b (s-a)(b-s)|q(s)|ds < +\infty \tag{5.4}$$

iff the corresponding homogeneous equation

$$u'' = p(t)u \tag{5.1$_0$}$$

has no nontrivial solution satisfying (5.2). Above statement plays an important role in the theory of singular problems, however it does not cover many interesting, even rather simple, equations. For example, consider the Dirichlet problem for the Euler equation

$$u'' = \frac{\alpha}{(t-a)^2}\,u + \beta; \qquad u(a) = 0, \quad u(b) = 0, \tag{5.5}$$

where α and β are real constants. By direct calculations one can easily verify that if $\alpha > 0$, then the homogeneous problem

$$u'' = \frac{\alpha}{(t-a)^2}\,u; \qquad u(a) = 0, \quad u(b) = 0,$$

has only the trivial solution, while problem (5.5) is uniquely solvable. However, in this case $p(t) := \frac{\alpha}{(t-a)^2}$ and therefore, condition (5.3) is not satisfied.

In this chapter we show that Fredholm's alternative remains true even in the case, when instead of (5.3) only the condition

$$\int_a^b (s-a)(b-s)[p(s)]_- ds < +\infty \tag{5.6}$$

holds. Moreover, bellow we establish optimal, in a certain sense, conditions for well-posedness of singular problem.

Definition 5.1. Under a solution of equation (5.1) we understand a function $u \in AC'_{loc}(]a,b[)$, which satisfies it almost everywhere in $]a,b[$. A solution of the equation (5.1) satisfying (5.2) is said to be a solution of problem (5.1), (5.2).

We say that a certain property holds in $]\alpha,\beta[$ if it takes place on every closed subinterval of $]\alpha,\beta[$.

5.2 Fredholm's alternative

5.2.1 Main results

Theorem 5.2 ([28, Thm. 1.1]). *Let condition* (5.6) *hold. Then problem* (5.1), (5.2) *is uniquely solvable for any q satisfying* (5.4) *iff the homogeneous problem* (5.1_0), (5.2) *has no nontrivial solution.*

Remark 5.3. In Theorem 5.2, condition (5.4) is essential and cannot be omitted. Indeed, let $p \equiv 0$, $q \in L_{loc}(]a,b[)$, $q(t) \geq 0$ for a.e. $t \in]a,b[$, and

$$\int_a^{\frac{a+b}{2}} (s-a)q(s)ds = +\infty. \tag{5.7}$$

Obviously, (5.6) holds and problem (5.1_0), (5.2) has no nontrivial solution. On the other hand, a general solution of (5.1) is of the form

$$u(t) = \alpha + \beta t + \int_t^{\frac{a+b}{2}} (s-a)q(s)ds - (t-a)\int_t^{\frac{a+b}{2}} q(s)ds \quad \text{for } t \in]a,b[\,.$$

However, for $a < t < x < \frac{a+b}{2}$, we have

$$u(t) \geq \int_x^{\frac{a+b}{2}} (s-a)q(s)ds - (t-a)\int_x^{\frac{a+b}{2}} q(s)ds + \alpha + \beta t.$$

Hence,

$$\liminf_{t\to a+} u(t) \geq \alpha + \beta a + \int_x^{\frac{a+b}{2}} (s-a)q(s)ds.$$

Therefore, in view of (5.7), we get $\lim\limits_{t\to a+} u(t) = +\infty$ and consequently, problem (5.1), (5.2) has no solution.

Remark 5.4. Theorem 5.2 concerns the half-homogeneous problem (5.1), (5.2) and does not remain true for fully nonhomogeneous problem

$$u'' = p(t)u + q(t); \qquad u(a) = c_1, \quad u(b) = c_2. \tag{5.8}$$

Let, for example, $p(t) := \frac{2}{(t-a)^2}$, $q \equiv 0$, $c_1 \neq 0$, and $c_2 = 0$. It is clear that (5.6) holds and the corresponding homogeneous problem (5.1_0), (5.2) has no nontrivial solution. On the other hand, a general solution of (5.1) is of the form $u(t) = \frac{\alpha}{t-a} + \beta(t-a)^2$ for $t \in]a,b[$ and therefore, (5.8) has no solution.

Theorem 5.5 ([28, Thm. 1.2]). *Let* (5.6) *hold and problem* (5.1_0), (5.2) *have no nontrivial solution. Then there exists* $r > 0$ *such that for any* q *satisfying* (5.4), *the solution* u *of problem* (5.1), (5.2) *admits the estimate*

$$|u(t)| + (t-a)(b-t)|u'(t)| \leq r \int_a^b (s-a)(b-s)|q(s)|ds \quad \textit{for } t \in]a,b[\,. \tag{5.9}$$

Consider now a sequence of equations

$$u'' = p(t)u + q_n(t), \tag{5.10_n}$$

where $q_n \in L_{loc}(]a,b[)$ are such that

$$\int_a^b (s-a)(b-s)|q_n(s)|ds < +\infty \quad \text{for } n \in \mathbb{N}. \tag{5.11}$$

Let, moreover, $q \in L_{loc}(]a,b[)$ satisfy (5.4) and

$$\lim_{n\to+\infty} \int_a^b (s-a)(b-s)|q_n(s) - q(s)|ds = 0. \tag{5.12}$$

Corollary 5.6 ([28, Cor. 1.1]). *Let* (5.4), (5.6) *hold and problem* (5.1_0), (5.2) *have no nontrivial solution. Let, moreover,* (5.11) *and* (5.12) *be fulfilled. Then problems* (5.1), (5.2) *and* (5.10_n), (5.2) *have unique solutions* u *and* u_n, *respectively,*

$$\lim_{n\to+\infty} u_n(t) = u(t) \quad \textit{uniformly on } [a,b] \tag{5.13}$$

and

$$\lim_{n\to+\infty} u_n'(t) = u'(t) \quad \textit{uniformly in }]a,b[\,. \tag{5.14}$$

5.2.2 Auxiliary statements

In this section, we consider the equation

$$v'' = h(t)v + q(t),$$

where $h, q \in L_{loc}(]a,b[)$, q satisfies (5.4), and

$$\int_a^b (s-a)(b-s)|h(s)|ds < +\infty. \tag{5.15}$$

Below we state some known results in a suitable for us form.

PROPOSITION 5.7. *Let* (5.15) *hold. Then the problem*

$$v'' = h(t)v + q(t)\,; \qquad v(a) = c_1, \quad v(b) = c_2$$

is uniquely solvable for any $c_1, c_2 \in \mathbb{R}$ *and* q *satisfying* (5.4) *iff the homogeneous problem*

$$v'' = h(t)v\,; \qquad v(a) = 0, \quad v(b) = 0$$

has no nontrivial solution.

Proof. See, e. g., [18, Theorem 3.1] or [21, Theorem 1.1]. □

PROPOSITION 5.8. *Let* (5.15) *hold. Then there exist* $a_0 \in]a, b[$ *and* $b_0 \in]a_0, b[$ *such that for any* $t_1 < t_2$ *satisfying either* $t_1, t_2 \in [a, a_0]$ *or* $t_1, t_2 \in [b_0, b]$*, the homogeneous problem*

$$v'' = h(t)v\,; \qquad v(t_1) = 0, \quad v(t_2) = 0 \tag{5.16}$$

has no nontrivial solution. Moreover, for any $w \in C'_{loc}(]t_1, t_2[)$ *(where* $t_1 < t_2$ *are the same as above) satisfying*

$$w''(t) \geq h(t)w(t) \quad \text{for a. e. } t \in]t_1, t_2[\,, \quad w(t_1) = 0, \quad w(t_2) = 0,$$

the inequality

$$w(t) \leq 0 \quad \text{for } t \in [t_1, t_2]$$

holds.

Proof. In view of (5.15), there exist $a_0 \in]a, b[$ and $b_0 \in]a_0, b[$ such that

$$\int_a^{a_0} (s-a)|h(s)|ds < 1, \qquad \int_{b_0}^{b} (b-s)|h(s)|ds < 1.$$

Hence, the inequalities

$$\int_a^{a_0} (s-a)(a_0-s)|h(s)|ds < a_0 - a, \qquad \int_{b_0}^{b} (s-b_0)(b-s)|h(s)|ds < b - b_0$$

hold, as well. The latter inequalities, by virtue of [21, Lemma 4.1], imply that for any $t_1 < t_2$ satisfying either $t_1, t_2 \in [a, a_0]$ or $t_1, t_2 \in [b_0, b]$, the homogeneous problem (5.16) has no nontrivial solution.

Second part of the proposition follows easily from the above-proved part and [21, Lemma 1.3]. □

PROPOSITION 5.9. *Let* (5.15) *hold. Let, moreover,* $a_0 \in]a, b[$ *and* $b_0 \in]a_0, b[$ *be from the assertion of Proposition 5.8. Then there exists* $\varrho > 0$ *such that for any* $c \in \mathbb{R}$ *and any* q *satisfying* (5.4)*, the solution* v *of the problem*

$$v'' = h(t)v + q(t)\,; \qquad v(a) = 0, \quad v(a_0) = c \tag{5.17}$$

admits the estimate

$$|v(t)| \le \varrho \left(|c|(t-a) + \int_a^t (s-a)|q(s)|ds + (t-a)\int_t^{a_0} |q(s)|ds \right) \tag{5.18}$$

for $t \in]a, a_0]$*, while the solution* v *of the problem*

$$v'' = h(t)v + q(t)\,; \qquad v(b_0) = c, \quad v(b) = 0 \tag{5.19}$$

admits the estimate

$$|v(t)| \le \varrho \left(|c|(b-t) + \int_t^b (b-s)|q(s)|ds + (b-t)\int_{b_0}^{t} |q(s)|ds \right) \tag{5.20}$$

for $t \in [b_0, b[$ *.*

Proof. By virtue of (5.15) and [18, Lemma 2.2], the initial value problems

$$v_1'' = h(t)v_1\,; \qquad v_1(a) = 0, \quad v_1'(a) = 1$$

and

$$v_2'' = h(t)v_2\,; \qquad v_2(a_0) = 0, \quad v_2'(a_0) = -1$$

have unique solutions v_1 and v_2, respectively, and the estimates

$$|v_1(t)| \le \varrho_0(t-a), \quad |v_2(t)| \le \varrho_0(a_0-t) \quad \text{for } t \in [a, a_0] \tag{5.21}$$

are fulfilled, where

$$\varrho_0 := \exp\left(2\int_a^{a_0} (s-a)|h(s)|ds \right).$$

On the other hand, by virtue of Proposition 5.8,

$$v_1(a_0) \ne 0 \qquad \text{and} \qquad v_2(a) \ne 0.$$

In view of Propositions 5.7 and 5.8, problem (5.17) has a unique solution v. By direct calculations one can easily verify that

$$v(t) = \frac{c}{v_1(a_0)}\,v_1(t) - \frac{1}{v_2(a)}\left(v_2(t)\int_a^t v_1(s)q(s)ds + v_1(t)\int_t^{a_0} v_2(s)q(s)ds \right) \tag{5.22}$$

for $t \in [a, a_0]$. Analogously, the (unique) solution v of problem (5.19) is of the form

$$v(t) = \frac{c}{v_4(b_0)}\,v_4(t) - \frac{1}{v_3(b)}\left(v_4(t)\int_{b_0}^t v_3(s)q(s)ds + v_3(t)\int_t^{b} v_4(s)q(s)ds \right) \tag{5.23}$$

for $t \in [b_0, b]$, where v_3 and v_4 are solutions of the problems

$$v_3'' = h(t)v_3\,; \qquad v_3(b_0) = 0, \quad v_3'(b_0) = 1$$

and

$$v_4'' = h(t)v_4\,; \qquad v_4(b) = 0, \quad v_4'(b) = -1,$$

respectively, $v_3(b) \neq 0$, $v_4(b_0) \neq 0$, and the estimates

$$|v_3(t)| \leq \varrho_1(t - b_0), \quad |v_4(t)| \leq \varrho_1(b - t) \quad \text{for } t \in [b_0, b] \tag{5.24}$$

are fulfilled with

$$\varrho_1 := \exp\left(2 \int_{b_0}^{b} (b - s)|h(s)|ds\right).$$

Now, in view of (5.21) and (5.24), it follows from (5.22) and (5.23) that estimates (5.18) and (5.20) hold with

$$\varrho := \frac{\rho_0}{|v_1(a_0)|} + \frac{\rho_1}{|v_4(b_0)|} + \frac{a_0 - a}{|v_2(a)|}\,\varrho_0^2 + \frac{b - b_0}{|v_3(b)|}\,\varrho_1^2.$$

$\square$

5.2.3 Lemmas on a priory estimates

LEMMA 5.10. *Let* (5.4) *and* (5.6) *hold. Then for any* $\alpha \in [a, b[$ *and* $\beta \in]\alpha, b]$, *every solution* u *of equation* (5.1) *satisfying*

$$u(\alpha) = 0, \qquad u(\beta) = 0 \tag{5.25}$$

admits the estimate

$$\begin{aligned}(t - a)(b - t)|u'(t)| \leq \|u\|_{[\alpha,\beta]} &\left(b - a + \int_a^b (s - a)(b - s)[p(s)]_- ds\right) \\ &+ \int_a^b (s - a)(b - s)|q(s)|ds \quad \textit{for } t \in]\alpha, \beta[\,.\end{aligned} \tag{5.26}$$

Proof. Let $t_0 \in]\alpha, \beta[\,$. Then it is clear that either

$$u(t_0)u'(t_0) > 0, \tag{5.27}$$

or

$$u(t_0)u'(t_0) < 0, \tag{5.28}$$

or

$$u(t_0)u'(t_0) = 0. \tag{5.29}$$

Assume that (5.27) (resp., (5.28)) holds. Then, in view of (5.25), there is $t^* \in]t_0, \beta[$ (resp., $t_* \in]\alpha, t_0[$) such that

$$\begin{gathered}u(t) \operatorname{sgn} u'(t_0) > 0 \quad \text{for } t \in [t_0, t^*] \qquad \text{and} \qquad u'(t^*) = 0 \\ \Big(\text{resp.,} \quad u(t) \operatorname{sgn} u'(t_0) < 0 \quad \text{for } t \in [t_*, t_0] \qquad \text{and} \qquad u'(t_*) = 0\Big).\end{gathered} \tag{5.30}$$

Multiplying both sides of (5.1) by $b-t$ (resp., by $t-a$) and integrating it from t_0 to t^* (resp., from t_* to t_0), we get

$$(b-t_0)u'(t_0) = u(t^*) - u(t_0) - \int_{t_0}^{t^*} (b-s)\big(p(s)u(s)+q(s)\big)ds$$

$$\left(\text{resp.,}\quad (t_0-a)u'(t_0) = u(t_0) - u(t_*) + \int_{t_*}^{t_0} (s-a)\big(p(s)u(s)+q(s)\big)ds\right).$$

Hence, in view of (5.30), we obtain

$$(b-t_0)|u'(t_0)| \le \|u\|_{[\alpha,\beta]}\left(1+\int_{t_0}^{b}(b-s)[p(s)]_-ds\right) + \int_{t_0}^{b}(b-s)|q(s)|ds$$

$$\left(\text{resp.,}\quad (t_0-a)|u'(t_0)| \le \|u\|_{[\alpha,\beta]}\left(1+\int_{a}^{t_0}(s-a)[p(s)]_-ds\right) + \int_{a}^{t_0}(s-a)|q(s)|ds\right).$$

Multiplying both parts of the latter inequality by t_0-a (resp., by $b-t_0$), we get

$$\begin{aligned}(t_0-a)(b-t_0)|u'(t_0)| \le \|u\|_{[\alpha,\beta]}&\left(b-a+\int_a^b (s-a)(b-s)[p(s)]_-ds\right)\\ &+\int_a^b (s-a)(b-s)|q(s)|ds.\end{aligned} \tag{5.31}$$

Suppose now that (5.29) holds. Then either there is a $\beta_0 \in\,]t_0,\beta[$ such that

$$u(t)u'(t) = 0 \quad \text{for } t \in [t_0,\beta_0], \tag{5.32}$$

or there is a sequence $\{t_n\}_{n=1}^{+\infty} \subset\,]t_0,\beta[$ such that

$$\lim_{n\to+\infty} t_n = t_0, \tag{5.33}$$

$$u(t_n)u'(t_n) \ne 0 \quad \text{for } n \in \mathbb{N}. \tag{5.34}$$

If (5.32) holds then evidently $u(t) = u(t_0)$ for $t \in [t_0,\beta_0]$ and consequently, (5.31) is fulfilled. On the other hand, if (5.34) holds then, by virtue of the above-proved, the inequalities

$$\begin{aligned}(t_n-a)(b-t_n)|u'(t_n)| \le \|u\|_{[\alpha,\beta]}&\left(b-a+\int_a^b (s-a)(b-s)[p(s)]_-ds\right)\\ &+\int_a^b (s-a)(b-s)|q(s)|ds \quad \text{for } n=1,2,\dots\end{aligned}$$

are fulfilled and therefore, in view of (5.33), inequality (5.31) holds, as well.

Hence, estimate (5.26) is fulfilled. □

LEMMA 5.11. *Let* (5.6) *hold. Then there exist* $a_0 \in\,]a,b[\,$, $b_0 \in\,]a_0,b[\,$, *and* $\varrho > 0$ *such that for any* $\alpha \in [a,a_0[\,$, $\beta \in\,]b_0,b]$, *and any* q *fulfilling* (5.4), *every solution* u *of equation* (5.1) *satisfying*

$$u(\alpha) = 0 \tag{5.35}$$

admits the estimate

$$|u(t)| \leq \varrho\left((t-a)\|u\|_{[\alpha,a_0]} + \int_a^t (s-a)|q(s)|ds + (t-a)\int_t^{a_0} |q(s)|ds\right) \tag{5.36}$$

for $t \in]\alpha, a_0]$, *while every solution* u *of equation* (5.1) *satisfying*

$$u(\beta) = 0 \tag{5.37}$$

admits the estimate

$$|u(t)| \leq \varrho\left((b-t)\|u\|_{[b_0,\beta]} + \int_t^b (b-s)|q(s)|ds + (b-t)\int_{b_0}^t |q(s)|ds\right) \tag{5.38}$$

for $t \in [b_0, \beta[$.

Proof. Let a_0, b_0, and ϱ be from the assertion of Propositions 5.8 and 5.9 with $h(t) := -[p(t)]_-$. Let, moreover, $\alpha \in [a, a_0[$ (resp., $\beta \in]b_0, b]$) and u be a solution of problem (5.1), (5.35) (resp., (5.1), (5.37)). By virtue of Propositions 5.8 and 5.9, the problem

$$v'' = -[p(t)]_- v - |q(t)|, \tag{5.39}$$

$$v(a) = 0, \quad v(a_0) = \|u\|_{[\alpha,a_0]} \qquad \big(\text{resp.,} \quad v(b_0) = \|u\|_{[b_0,\beta]}, \quad v(b) = 0\big)$$

has a unique solution v and, moreover, for any $t \in]a, a_0]$ (resp., $t \in [b_0, b[$), the estimate

$$0 \leq v(t) \leq \varrho\left((t-a)\|u\|_{[\alpha,a_0]} + \int_a^t (s-a)|q(s)|ds + (t-a)\int_t^{a_0} |q(s)|ds\right)$$

$$\left(\text{resp., } 0 \leq v(t) \leq \varrho\left((b-t)\|u\|_{[b_0,\beta]} + \int_t^b (b-s)|q(s)|ds + (b-t)\int_{b_0}^t |q(s)|ds\right)\right) \tag{5.40}$$

holds. We show that

$$|u(t)| \leq v(t) \quad \text{for } t \in [\alpha, a_0] \qquad \big(\text{resp.,} \quad \text{for } t \in [b_0, \beta]\big). \tag{5.41}$$

Assume on the contrary that (5.41) is violated. Put

$$w(t) := |u(t)| - v(t) \quad \text{for } t \in [\alpha, a_0] \qquad \big(\text{resp.,} \quad \text{for } t \in [b_0, \beta]\big).$$

Then there exist $t_1 \in [\alpha, a_0[$ and $t_2 \in]t_1, a_0]$ (resp., $t_1 \in [b_0, \beta[$ and $t_2 \in]t_1, \beta]$) such that

$$w(t) > 0 \quad \text{for } t \in]t_1, t_2[\,, \tag{5.42}$$

$$w(t_1) = 0, \quad w(t_2) = 0. \tag{5.43}$$

In view of (5.1), (5.39), and (5.42), it is clear that $w \in AC'_{loc}(]t_1, t_2[)$ and

$$w''(t) = p(t)|u(t)| + q(t)\operatorname{sgn} u(t) + [p(t)]_- v(t) + |q(t)| \geq -[p(t)]_- w(t) \quad \text{for a. e. } t \in]t_1, t_2[\,.$$

Hence, by virtue of (5.43) and Proposition 5.8, we get $w(t) \leq 0$ for $t \in]t_1, t_2[$, which contradicts (5.42). Therefore, (5.41) is fulfilled. The desired estimate (5.36) (resp., (5.38)) now follows from (5.40) and (5.41). ☐

LEMMA 5.12. *Let* (5.6) *hold and problem* (5.1$_0$), (5.2) *has no nontrivial solution. Then there exist* $\bar{a}_0 \in]a, b[$, $\bar{b}_0 \in]\bar{a}_0, b[$, *and* $r_0 > 0$ *such that for any* $\alpha \in [a, \bar{a}_0]$, $\beta \in [\bar{b}_0, b]$, *and any* q *fulfilling* (5.4), *every solution* u *of equation* (5.1) *satisfying*

$$u(\alpha) = 0, \qquad u(\beta) = 0$$

admits the estimate

$$|u(t)| \le r_0 \int_a^b (s-a)(b-s)|q(s)|ds \quad \textit{for } t \in [\alpha, \beta].$$

Proof. Suppose on the contrary that the lemma is not true. Then there exist the sequences $\{a_n\}_{n=1}^{+\infty} \subset [a, \frac{a+b}{2}[$, $\{b_n\}_{n=1}^{+\infty} \subset]\frac{a+b}{2}, b]$, $\{q_n\}_{n=1}^{+\infty} \subset L_{loc}(]a, b[)$, and $\{u_n\}_{n=1}^{+\infty} \subset AC'_{loc}(]a, b[)$ such that (5.11) holds,

$$\lim_{n \to +\infty} a_n = a, \qquad \lim_{n \to +\infty} b_n = b, \tag{5.44}$$

$$u_n''(t) = p(t)u_n(t) + q_n(t) \quad \text{for a. e. } t \in]a, b[, \quad u_n(a_n) = 0, \quad u_n(b_n) = 0,$$

and

$$\|u_n\|_{[a_n, b_n]} > n \int_a^b (s-a)(b-s)|q_n(s)|ds \quad \text{for } n = 1, 2, \dots . \tag{5.45}$$

Introduce the notation

$$\tilde{u}_n(t) := \frac{1}{\|u_n\|_{[a_n, b_n]}} u_n(t), \qquad \tilde{q}_n(t) := \frac{1}{\|u_n\|_{[a_n, b_n]}} q_n(t).$$

Then it is clear that

$$\|\tilde{u}_n\|_{[a_n, b_n]} = 1 \tag{5.46}$$

and

$$\tilde{u}_n''(t) = p(t)\tilde{u}_n(t) + \tilde{q}_n(t) \quad \text{for a. e. } t \in]a_n, b_n[, \quad \tilde{u}_n(a_n) = 0, \quad \tilde{u}_n(b_n) = 0. \tag{5.47}$$

Moreover, it follows from (5.45) that

$$\lim_{n \to +\infty} \int_a^b (s-a)(b-s)|\tilde{q}_n(s)|ds = 0 \tag{5.48}$$

and consequently,

$$\lim_{n \to +\infty} \int_{\frac{a+b}{2}}^t \left(\int_{\frac{a+b}{2}}^s \tilde{q}_n(\xi)d\xi \right) ds = 0 \quad \text{for } t \in]a, b[. \tag{5.49}$$

By virtue of Lemma 5.11, (5.46), and (5.47), we get

$$(t-a)(b-t)|\tilde{u}_n'(t)| \le b - a + \int_a^b (s-a)(b-s)[p(s)]_- ds +$$
$$+ \int_a^b (s-a)(b-s)|\tilde{q}_n(s)|ds \quad \text{for } t \in]a_n, b_n[.$$

Hence, in view of (5.44) and (5.48), the sequence $\{\tilde{u}_n'\}_{n=1}^{+\infty}$ is uniformly bounded in $]a,b[$ and therefore, the sequence $\{\tilde{u}_n\}_{n=1}^{+\infty}$ is equicontinuous in $]a,b[$. Taking, moreover, into account (5.46), by virtue of Arzelá-Ascoli's lemma, we can assume without loss of generality that

$$\lim_{n\to+\infty} \tilde{u}_n(t) = u_0(t) \quad \text{uniformly in }]a,b[\,, \tag{5.50}$$

where $u_0 \in C(]a,b[)$ and, moreover,

$$\lim_{n\to+\infty} \tilde{u}_n'\left(\frac{a+b}{2}\right) = c_0. \tag{5.51}$$

By direct calculation one can easily verify that

$$\tilde{u}_n(t) = \tilde{u}_n\left(\frac{a+b}{2}\right) + \left(t - \frac{a+b}{2}\right)\tilde{u}_n'\left(\frac{a+b}{2}\right) + \\ + \int_{\frac{a+b}{2}}^{t}\left(\int_{\frac{a+b}{2}}^{s}\big[p(\xi)\tilde{u}_n(\xi) + \tilde{q}_n(\xi)\big]d\xi\right)ds \quad \text{for } t \in]a,b[\,,$$

whence, in view of (5.49)–(5.51), we get

$$u_0(t) = u_0\left(\frac{a+b}{2}\right) + \left(t - \frac{a+b}{2}\right)c_0 + + \int_{\frac{a+b}{2}}^{t}\left(\int_{\frac{a+b}{2}}^{s} p(\xi)u_0(\xi)d\xi\right)ds \quad \text{for } t \in]a,b[\,.$$

Thus, $u_0 \in AC'_{loc}(]a,b[)$ and u_0 is a solution of equation (5.1_0).

Now let $a_0 \in]a,b[$, $b_0 \in]a_0,b[$, and $\varrho > 0$ be from the assertion of Lemma 5.11. Assume without loss of generality that $a_n < a_0$ and $b_n > b_0$ for any natural n. Then, by virtue of Lemma 5.11, (5.46), and (5.47), the estimates

$$|\tilde{u}_n(t)| \le \varrho\left(t - a + \int_a^t (s-a)|\tilde{q}_n(s)|ds + (t-a)\int_t^{a_0}|\tilde{q}_n(s)|ds\right) \quad \text{for } t \in]a_n, a_0],$$
$$|\tilde{u}_n(t)| \le \varrho\left(b - t + \int_t^b (b-s)|\tilde{q}_n(s)|ds + (b-t)\int_{b_0}^{t}|\tilde{q}_n(s)|ds\right) \quad \text{for } t \in [b_0, b_n[\tag{5.52}$$

are fulfilled. Moreover, in view of (5.48), we have

$$\lim_{n\to+\infty}\left(\int_a^t (s-a)|\tilde{q}_n(s)|ds + (t-a)\int_t^{a_0}|\tilde{q}_n(s)|ds\right) = 0 \quad \text{for } t \in]a, a_0]$$

and

$$\lim_{n\to+\infty}\left(\int_t^b (b-s)|\tilde{q}_n(s)|ds + (b-t)\int_{b_0}^{t}|\tilde{q}_n(s)|ds\right) = 0 \quad \text{for } t \in [b_0, b[\,.$$

Taking, moreover, into account (5.50), we get from (5.52) that

$$|u_0(t)| \le \varrho(t-a) \quad \text{for } t \in]a, a_0] \qquad \text{and} \qquad |u_0(t)| \le \varrho(b-t) \quad \text{for } t \in [b_0, b[$$

and thus, u_0 satisfies the conditions

$$u_0(a) = 0, \qquad u_0(b) = 0.$$

On account of (5.44) and (5.48), there exist $\alpha_0 \in]a, a_0[$, $\beta_0 \in]b_0, b[$, and $n_0 \in \mathbb{N}$ such that

$$a_n < \alpha_0, \qquad \varrho\left(\alpha_0 - a + \int_a^{a_0} (s-a)|\tilde{q}_n(s)|ds\right) < 1 \quad \text{for } n > n_0$$

and

$$b_n > \beta_0, \qquad \varrho\left(b - \beta_0 + \int_{b_0}^{b} (b-s)|\tilde{q}_n(s)|ds\right) < 1 \quad \text{for } n > n_0.$$

Then it follows from (5.52) that

$$|\tilde{u}_n(t)| < 1 \quad \text{for } t \in [a_n, \alpha_0] \cup [\beta_0, b_n],\ n > n_0.$$

Hence, in view of (5.46), $\|\tilde{u}_n\|_{[\alpha_0,\beta_0]} = 1$ for $n > n_0$. Taking now into account (5.50), we get $\|u_0\|_{[\alpha_0,\beta_0]} = 1$ and thus, u_0 is a nontrivial solution of problem (5.1_0), (5.2). However, this contradicts an assumption of the lemma. □

5.2.4 Proofs of the main results

Proof of Theorem 5.2. To prove the theorem it is sufficient to show that if problem (5.1_0), (5.2) has no nontrivial solution, then problem (5.1), (5.2) has at least one solution.

Let a_0, b_0, $\bar{a}_0$, $\bar{b}_0$, ϱ, and r_0 be from the assertions of Lemmas 5.11 and 5.12. Let, moreover, the sequences $\{a_n\}_{n=1}^{+\infty} \subset]a, \min\{a_0, \bar{a}_0\}[$ and $\{b_n\}_{n=1}^{+\infty} \subset]\max\{b_0, \bar{b}_0\}, b[$ be such that

$$\lim_{n\to+\infty} a_n = a, \qquad \lim_{n\to+\infty} b_n = b. \tag{5.53}$$

By virtue of Lemma 5.12, the problem

$$u'' = p(t)u\,; \qquad u(a_n) = 0, \quad u(b_n) = 0$$

has no nontrivial solution. Hence, by virtue of Proposition 5.7, the problem

$$\begin{gathered} u_n'' = p(t)u_n + q(t), \\ u_n(a_n) = 0, \quad u_n(b_n) = 0 \end{gathered} \tag{5.54}$$

has a unique solution u_n. Moreover, by virtue of Lemma 5.12, the estimate

$$|u_n(t)| \le r_1 \quad \text{for } t \in [a_n, b_n] \tag{5.55}$$

holds, where

$$r_1 := r_0 \int_a^b (s-a)(b-s)|q(s)|ds.$$

On the other hand, on account of Lemma 5.11 and (5.55), we have

$$(t-a)(b-t)|u_n'(t)| \le r_2 \quad \text{for } t \in [a_n, b_n], \tag{5.56}$$

where

$$r_2 := r_1 \left(b - a + \int_a^b (s-a)(b-s)[p(s)]_- ds \right) + \int_a^b (s-a)(b-s)|q(s)| ds.$$

In view of (5.53), (5.55), and (5.56), the sequence $\{u_n\}_{n=1}^{+\infty}$ is uniformly bounded and equicontinuous in $]a, b[$. Hence, by virtue of Arzelá-Ascoli's lemma, we can suppose without loss of generality that

$$\lim_{n \to +\infty} u_n(t) = u_0(t) \quad \text{uniformly in }]a, b[\,, \tag{5.57}$$

where $u_0 \in C(]a, b[)$ and, moreover,

$$\lim_{n \to +\infty} u_n' \left(\frac{a+b}{2} \right) = c_0. \tag{5.58}$$

Taking into account (5.54), one can easily verify by direct calculation that

$$u_n(t) = u_n \left(\frac{a+b}{2} \right) + \left(t - \frac{a+b}{2} \right) u_n' \left(\frac{a+b}{2} \right) + \\ + \int_{\frac{a+b}{2}}^t \left(\int_{\frac{a+b}{2}}^s \big[p(\xi) u_n(\xi) + q(\xi) \big] d\xi \right) ds \quad \text{for } t \in [a_n, b_n].$$

Hence, in view of (5.57) and (5.58), we get

$$u_0(t) = u_0 \left(\frac{a+b}{2} \right) + \left(t - \frac{a+b}{2} \right) c_0 + \int_{\frac{a+b}{2}}^t \left(\int_{\frac{a+b}{2}}^s \big[p(\xi) u_0(\xi) + q(\xi) \big] d\xi \right) ds$$

for $t \in]a, b[$. Thus, $u_0 \in AC_{loc}'(]a, b[)$ and u_0 is a solution of equation (5.1).

Further, by virtue of Lemma 5.11 and (5.55), the inequalities

$$|u_n(t)| \le \varrho \left(r_1(t-a) + \int_a^t (s-a)|q(s)| ds + (t-a) \int_t^{a_0} |q(s)| ds \right) \quad \text{for } t \in]a_n, a_0]$$

and

$$|u_n(t)| \le \varrho \left(r_1(b-t) + \int_t^b (b-s)|q(s)| ds + (b-t) \int_{b_0}^t |q(s)| ds \right) \quad \text{for } t \in [b_0, b_n[$$

are fulfilled. Hence, on account of (5.57), we get

$$|u_0(t)| \le \varrho \left(r_1(t-a) + \int_a^t (s-a)|q(s)| ds + (t-a) \int_t^{a_0} |q(s)| ds \right) \quad \text{for } t \in]a, a_0],$$
$$|u_0(t)| \le \varrho \left(r_1(b-t) + \int_t^b (b-s)|q(s)| ds + (b-t) \int_{b_0}^t |q(s)| ds \right) \quad \text{for } t \in [b_0, b[\,,$$

and thus, $u_0(a) = 0$ and $u_0(b) = 0$. Consequently, u_0 is a solution of problem (5.1), (5.2). $\square$

Proof of Theorem 5.5. According to Theorem 5.2, problem (5.1), (5.2) has a unique solution u. By virtue of Lemma 5.12, the estimate

$$|u(t)| \le r_0 \int_a^b (s-a)(b-s)|q(s)|ds \quad \text{for } t \in [a,b]$$

holds. On the other hand, it follows from Lemma 5.11 that

$$(t-a)(b-t)|u'(t)| \le \|u\|_{[a,b]} \left(b-a + \int_a^b (s-a)(b-s)[p(s)]_- ds \right) + \int_a^b (s-a)(b-s)|q(s)|ds \quad \text{for } t \in \,]a,b[\,.$$

The latter two inequalities imply (5.9) with

$$r := 1 + r_0 \left(b-a + \int_a^b (s-a)(b-s)[p(s)]_- ds \right).$$

□

Proof of Corollary 5.6. By virtue of Theorem 5.2, problems (5.1), (5.2) and (5.10_n), (5.2) have unique solutions u and u_n, respectively. Let

$$v_n(t) := u_n(t) - u(t) \quad \text{for } t \in [a,b]. \tag{5.59}$$

Then it is clear that

$$v_n''(t) = p(t)v_n(t) + \tilde{q}_n(t) \quad \text{for a. e. } t \in \,]a,b[\,, \quad v_n(a) = 0, \quad v_n(b) = 0,$$

where

$$\tilde{q}_n(t) := q_n(t) - q(t) \quad \text{for } t \in \,]a,b[\,. \tag{5.60}$$

Hence, by virtue of Theorem 5.5, we obtain

$$|v_n(t)| + (t-a)(b-t)|v_n'(t)| \le r \int_a^b (s-a)(b-s)|\tilde{q}_n(s)|ds \quad \text{for } t \in \,]a,b[\,.$$

Taking now into account (5.12), (5.59), and (5.60), we get (5.13) and (5.14). □

5.3 Fredholm's third theorem

5.3.1 Main results

Now we show that, under assumption (5.6), Fredholm's third theorem remains true as well.

THEOREM 5.13 ([29, Thm. 1.2]). *Let* (5.6) *hold. Then the homogeneous problem* (5.1_0), (5.2) *has no more then one, up to a constant multiple, nontrivial solution.*

Remark 5.14. Below we show (see Proposition 5.17) that if (5.6) holds and u_0 is a nontrivial solution of problem (5.1_0), (5.2), then there exists $r_0 > 0$ such that

$$|u_0(t)| \leq r_0(t-a)(b-t) \quad \text{for } t \in [a,b].$$

Theorem 5.15 ([29, Thm. 1.3]). *Let* (5.6) *hold and the homogeneous problem* (5.1_0), (5.2) *have a nontrivial solution* u_0. *Then problem* (5.1), (5.2), *where the function* q *satisfies* (5.4), *is solvable iff the condition*

$$\int_a^b q(s)u_0(s)ds = 0 \tag{5.61}$$

is fulfilled.

Remark 5.16. In view of Remark 5.14 and condition (5.4), the function qu_0 is integrable on $[a,b]$ and therefore, condition (5.61) is meaningfull.

5.3.2 Auxiliary statements

Next proposition immediately follows from Lemma 5.11.

Proposition 5.17. *Let* (5.6) *hold and* u_0 *be a nontrivial solution of the homogeneous problem* (5.1_0), (5.2). *Then there exists* $r_0 > 0$ *such that*

$$|u_0(t)| \leq r_0(t-a)(b-t) \quad \textit{for } t \in [a,b].$$

Proposition 5.18. *Let* (5.6) *hold and* u_0 *be a nontrivial solution of equation* (5.1_0) *satisfying* $u_0(a) = 0$ *(respectively,* $u_0(b) = 0$*). Then there exists* $a_1 \in\,]a,b[$ *(respectively,* $b_1 \in\,]a,b[$*) such that*

$$u_0(t) \neq 0 \quad \textit{for } t \in\,]a,a_1] \quad \Big(\textit{respectively,} \quad u_0(t) \neq 0 \quad \textit{for } t \in [b_1,b[\,\Big). \tag{5.62}$$

Proof. In view of (5.6), there exists $a_0 \in\,]a,b[$ (respectively, $b_0 \in\,]a,b[$) such that

$$\int_a^{a_0}(s-a)[p(s)]_- ds < 1 \quad \left(\text{respectively, } \int_{b_0}^b (b-s)[p(s)]_- ds < 1\right).$$

Hence, the inequality

$$\int_a^{a_0}(s-a)(a_0-s)[p(s)]_- ds < a_0 - a$$

$$\left(\text{respectively,} \quad \int_{b_0}^b (s-b_0)(b-s)[p(s)]_- ds < b - b_0\right)$$

holds, as well. The latter inequality, by virtue of [21, Lemma 4.1], implies that for any $a < t_1 < t_2 < a_0$ (respectively, $b_0 < t_1 < t_2 < b$), the problem

$$u'' = p(t)u; \qquad u(t_1) = 0, \quad u(t_2) = 0$$

has no nontrivial solution.

Now suppose that u_0 is a nontrivial solution of equation (5.1_0) satisfying $u_0(a)=0$ (respectively, $u_0(b)=0$). Then it follows from the above-mentioned that either

$$u_0(t)\neq 0 \quad \text{for } t\in\,]a,a_0] \quad \big(\text{respectively,} \quad u_0(t)\neq 0 \quad \text{for } t\in[b_0,b[\,\big), \tag{5.63}$$

or there is a $t_0\in\,]a,a_0]$ (respectively, $t_0\in[b_0,b[\,$) such that

$$\begin{gathered} u_0(t)\neq 0 \quad \text{for } t\in\,]a,t_0[\,, \quad u_0(t_0)=0 \\ \Big(\text{respectively,} \quad u_0(t)\neq 0 \quad \text{for } t\in\,]t_0,b[\,, \quad u_0(t_0)=0\Big). \end{gathered} \tag{5.64}$$

It is now clear that (5.62) holds with $a_1:=a_0$ (respectively, $b_1:=b_0$) if (5.63) holds, and with $a_1:=\frac{a+t_0}{2}$ $\big($respectively, $b_1:=\frac{t_0+b}{2}\big)$ if (5.64) is satisfied. $\square$

Lemma 5.19. *Let* (5.6) *and* (5.4) *hold. Let, moreover, u be a solution of problem* (5.1), (5.2) *and u_0 be a solution of problem* (5.1_0), (5.2). *Then*

$$\lim_{t\to a+}\big(u'(t)u_0(t)-u(t)u_0'(t)\big)=0, \qquad \lim_{t\to b-}\big(u'(t)u_0(t)-u(t)u_0'(t)\big)=0. \tag{5.65}$$

Proof. It is clear that

$$\big(u'(t)u_0(t)-u(t)u_0'(t)\big)'=q(t)u_0(t) \quad \text{for a. e. } t\in\,]a,b[\,.$$

Hence,

$$u'(t)u_0(t)-u(t)u_0'(t)=\delta-\int_t^c q(s)u_0(s)ds \quad \text{for } t\in\,]a,b[\,, \tag{5.66}$$

where

$$c:=\frac{a+b}{2} \qquad \text{and} \qquad \delta:=u'(c)u_0(c)-u(c)u_0'(c).$$

By virtue of Proposition 5.17 and condition (5.4), the function qu_0 is integrable on $[a,b]$. Thus, it follows from (5.66) that there exists a finite limit

$$\lim_{t\to a+}\big|u'(t)u_0(t)-u(t)u_0'(t)\big|=\varepsilon_0. \tag{5.67}$$

Now we show that $\varepsilon_0=0$. Suppose on the contrary that

$$\varepsilon_0>0. \tag{5.68}$$

Then there is $\alpha\in\,]a,b[$ such that

$$\big|u'(t)u_0(t)-u(t)u_0'(t)\big|>\frac{\varepsilon_0}{2} \quad \text{for } t\in\,]a,\alpha]. \tag{5.69}$$

On account of Proposition 5.18, we can assume without loss of generality that

$$u_0(t)\neq 0 \quad \text{for } t\in\,]a,\alpha].$$

Then it follows from (5.69) that

$$\left|\left(\frac{u(t)}{u_0(t)}\right)'\right| > \frac{\varepsilon_0}{2u_0^2(t)} \quad \text{for } t \in \,]a, \alpha].$$

Hence,

$$\left|\mu u_0(t) - u(t)\right| > \frac{\varepsilon_0}{2}\,|u_0(t)| \int_t^\alpha \frac{ds}{u_0^2(s)} \quad \text{for } t \in \,]a, \alpha], \tag{5.70}$$

where $\mu := \frac{u(\alpha)}{u_0(\alpha)}$.

Taking now into account Proposition 5.17, we get from (5.70) that

$$\left|\mu u_0(t) - u(t)\right| > \varepsilon_1 |u_0(t)| \left(\frac{1}{t-a} - \frac{1}{\alpha - a}\right) \quad \text{for } t \in \,]a, \alpha],$$

where $\varepsilon_1 := \frac{\varepsilon_0}{2r_0^2(b-a)^2}$. The latter inequality, in view of the conditions $u_0(a) = 0$ and $u(a) = 0$, implies that

$$\lim_{t \to a+} \frac{|u_0(t)|}{t-a} = 0. \tag{5.71}$$

On the other hand, by virtue of Lemma 5.11, there is $M > 0$ such that

$$(t-a)|u'(t)| \le M \quad \text{for } t \in \,]a, \alpha]. \tag{5.72}$$

In view of (5.71) and (5.72), we get

$$\lim_{t \to a+} |u'(t)u_0(t)| = \lim_{t \to a+} (t-a)|u'(t)| \frac{|u_0(t)|}{t-a} = 0$$

and therefore, on account of (5.67), we obtain

$$\lim_{t \to a+} |u(t)u_0'(t)| = \varepsilon_0.$$

Now let $\alpha_0 \in \,]a, \alpha[$ be such that

$$|u(t)u_0'(t)| > \frac{\varepsilon_0}{2} \quad \text{for } t \in \,]a, \alpha_0].$$

Then it is clear that

$$\|u\|_{[a,b]} |u_0'(t)| > \frac{\varepsilon_0}{2} \quad \text{for } t \in \,]a, \alpha_0]$$

and consequently,

$$\|u\|_{[a,b]} |u_0(t)| > \frac{\varepsilon_0}{2}\,(t-a) \quad \text{for } t \in \,]a, \alpha_0].$$

However, the latter inequality and (5.71) yield that $\varepsilon_0 \le 0$, which contradicts (5.68). The contradiction obtained proves the first equality in (5.65). By the same arguments one can prove the second equality in (5.65). □

The next lemma we need in the proof of the sufficiency part of Theorem 5.15 and thus, we suppose that Theorem 5.13 and the necessity part of Theorem 5.15 are true.

LEMMA 5.20. *Let (5.6) hold and the homogeneous problem* (5.1_0), *(5.2) have a nontrivial solution* u_0. *Then there exists* $n_0 \in \mathbb{N}$ *and* $r > 0$ *such that for any* q *satisfying (5.4) and (5.61) and every* $n > n_0$, *the solution* u *of the problem*

$$u'' = \left(p(t) + \frac{1}{n}\,[p(t)]_-\right) u + q(t); \qquad u(a) = 0, \quad u(b) = 0$$

admits the estimate

$$|u(t)| \leq r \int_a^b (s-a)(b-s)|q(s)|ds \quad \textit{for } t \in [a,b].$$

Proof. Suppose on the contrary that the assertion of the lemma is violated. Then for any $n \in \mathbb{N}$, there exist $k_n \geq n$, $q_n \in L_{loc}(]a,b[)$, and $u_n \in AC'_{loc}(]a,b[)$ such that

$$\int_a^b (s-a)(b-s)|q_n(s)|ds < +\infty, \qquad \int_a^b q_n(s)u_0(s)ds = 0,$$

$$u_n''(t) = \left(p(t) + \frac{1}{k_n}\,[p(t)]_-\right) u_n(t) + q_n(t) \quad \text{for a. e. } t \in]a,b[\,,$$

$$u_n(a) = 0, \qquad u_n(b) = 0,$$

and

$$\|u_n\|_{[a,b]} > n \int_a^b (s-a)(b-s)|q_n(s)|ds.$$

Introduce the notation

$$\tilde{u}_n(t) := \frac{1}{\|u_n\|_{[a,b]}}\, u_n(t) \quad \text{for } t \in]a,b[\,, \quad \tilde{q}_n(t) := \frac{1}{\|u_n\|_{[a,b]}}\, q_n(t) \quad \text{for a. e. } t \in]a,b[\,.$$

Then it is clear that

$$\tilde{u}_n''(t) = \left(p(t) + \frac{1}{k_n}\,[p(t)]_-\right) \tilde{u}_n(t) + \tilde{q}_n(t) \quad \text{for a. e. } t \in]a,b[\,, \tag{5.73}$$

$$\tilde{u}_n(a) = 0, \qquad \tilde{u}_n(b) = 0,$$

$$\|\tilde{u}_n\|_{[a,b]} = 1, \tag{5.74}$$

$$\int_a^b (s-a)(b-s)|\tilde{q}_n(s)|ds < \frac{1}{n}\,, \tag{5.75}$$

and

$$\int_a^b \tilde{q}_n(s)u_0(s)ds = 0. \tag{5.76}$$

By virtue of Lemma 5.11 (with $q(t) := \frac{1}{k_n}\,[p(t)]_-\tilde{u}_n(t) + \tilde{q}_n(t)$) and (5.74), we have

$$\begin{aligned} (t-a)(b-t)|\tilde{u}_n'(t)| \leq b-a + \frac{n+1}{n} &\int_a^b (s-a)(b-s)[p(s)]_- ds \\ + &\int_a^b (s-a)(b-s)|\tilde{q}_n(s)|ds \quad \text{for } t \in]a,b[\,, \end{aligned} \tag{5.77}$$

while, by virtue of Lemma 5.11 (with $q(t) := \frac{1}{k_n}[p(t)]_-\tilde{u}_n(t) + \tilde{q}_n(t)$), there exist $a_0 \in]a,b[$, $b_0 \in]a_0,b[$, and $\varrho > 0$ such that

$$
\begin{aligned}
|\tilde{u}_n(t)| &\le \varrho \left[t - a + \int_a^{a_0} (s-a) \left| \frac{1}{k_n} [p(s)]_- \tilde{u}_n(s) + \tilde{q}_n(s) \right| ds \right] \quad \text{for } t \in]a, a_0], \\
|\tilde{u}_n(t)| &\le \varrho \left[b - t + \int_t^{b} (b-s) \left| \frac{1}{k_n} [p(s)]_- \tilde{u}_n(s) + \tilde{q}_n(s) \right| ds \right] \quad \text{for } t \in [b_0, b[\,.
\end{aligned}
\tag{5.78}
$$

On account of (5.74) and (5.77), the sequence $\{u_n\}_{n=1}^{+\infty}$ is uniformly bounded and equicontinuous in $]a,b[$. Thus, by virtue of Arzelà-Ascoli's lemma, we can assume without loss of generality that

$$
\lim_{n\to+\infty} \tilde{u}_n(t) = v_0(t) \quad \text{uniformly in }]a,b[\,, \tag{5.79}
$$

where $v_0 \in C(]a,b[)$ and, moreover,

$$
\lim_{n\to+\infty} \tilde{u}_n'\left(\frac{a+b}{2}\right) = c_0. \tag{5.80}
$$

In view of (5.73), it is clear that

$$
\begin{aligned}
\tilde{u}_n(t) &= \tilde{u}_n\left(\frac{a+b}{2}\right) + \left(t - \frac{a+b}{2}\right)\tilde{u}_n'\left(\frac{a+b}{2}\right) \\
&+ \int_{\frac{a+b}{2}}^{t} \left(\int_{\frac{a+b}{2}}^{s} \left[\left(p(\xi) + \frac{1}{k_n}[p(\xi)]_- \right) \tilde{u}_n(\xi) + \tilde{q}_n(\xi) \right] d\xi \right) ds \quad \text{for } t \in]a,b[\,.
\end{aligned}
$$

Hence, on account of (5.74), (5.75), (5.79), and (5.80), we get

$$
v_0(t) = v_0\left(\frac{a+b}{2}\right) + c_0\left(t - \frac{a+b}{2}\right) + \int_{\frac{a+b}{2}}^{t} \left(\int_{\frac{a+b}{2}}^{s} p(\xi) v_0(\xi) d\xi \right) ds \quad \text{for } t \in]a,b[\,.
$$

Therefore, $v_0 \in AC'_{loc}(]a,b[)$ and v_0 is a solution of equation (5.1_0). On the other hand, in view of (5.74), (5.75), and (5.79), it follows from (5.78) that

$$
|v_0(t)| \le \rho(t-a) \quad \text{for } t \in]a,a_0] \qquad \text{and} \qquad |v_0(t)| \le \rho(b-t) \quad \text{for } t \in [b_0,b[\,,
$$

and thus, v_0 is a solution of problem (5.1_0), (5.2).

By virtue of (5.75) and (5.78), it is clear that there are $n_1 \in \mathbb{N}$, $a_1 \in]a,a_0]$, and $b_1 \in [b_0,b[$ such that

$$
\left|\tilde{u}_n(t)\right| < 1 \quad \text{for } t \in [a,a_1] \cup [b_1,b], \; n > n_1.
$$

Therefore, $\|\tilde{u}_n\|_{[a_1,b_1]} = 1$ for $n > n_1$. Taking now into account (5.79), we get $\|v_0\|_{[a_1,b_1]} = 1$ and therefore, v_0 is a nontrivial solution of problem (5.1_0), (5.2).

By virtue of Theorem 5.13, there is $\lambda \neq 0$ such that

$$
v_0(t) = \lambda u_0(t) \quad \text{for } t \in [a,b]. \tag{5.81}
$$

Moreover, in view of the necessity part of Theorem 5.15 (with $q(t) := \frac{1}{k_n}[p(t)]_-\tilde{u}_n(t) + \tilde{q}_n(t))$, (5.73), (5.74), (5.76), and (5.81), we get

$$\int_a^b [p(s)]_-\tilde{u}_n(s)v_0(s)ds = 0. \tag{5.82}$$

Let now $\alpha \in]a,b[$ and $\beta \in]\alpha,b[$ be arbitrary. Then, in view of (5.79), we have

$$\lim_{n\to+\infty}\int_\alpha^\beta [p(s)]_-\tilde{u}_n(s)v_0(s)ds = \int_\alpha^\beta [p(s)]_-v_0^2(s)ds. \tag{5.83}$$

On account of (5.6), (5.81), and Proposition 5.17, the function $[p]_-v_0$ is integrable on $[a,b]$. Taking into account (5.74), we get

$$\left|\int_a^\alpha [p(s)]_-\tilde{u}_n(s)v_0(s)ds\right| \le \int_a^\alpha [p(s)]_-|v_0(s)|ds$$

and

$$\left|\int_\beta^b [p(s)]_-\tilde{u}_n(s)v_0(s)ds\right| \le \int_\beta^b [p(s)]_-|v_0(s)|ds.$$

Hence, (5.82) implies the inequality

$$\int_\alpha^\beta [p(s)]_-\tilde{u}_n(s)v_0(s)ds \le \int_a^\alpha [p(s)]_-|v_0(s)|ds + \int_\beta^b [p(s)]_-|v_0(s)|ds,$$

which, together with (5.83), results in

$$\int_\alpha^\beta [p(s)]_-v_0^2(s)ds \le \int_a^\alpha [p(s)]_-|v_0(s)|ds + \int_\beta^b [p(s)]_-|v_0(s)|ds.$$

Since α and β were arbitrary, we get from the latter inequality that

$$\int_a^b [p(s)]_-v_0^2(s)ds = 0.$$

Taking now into account that $v_0 \not\equiv 0$, we get $[p]_- \equiv 0$, i. e., $p(t) \ge 0$ for a.e. $t \in]a,b[$. However, in this case, problem (5.1_0), (5.2) has no nontrivial solution, which contradicts the assumption of the lemma. □

Proof of Theorem 5.13. Let u_0 and v_0 be nontrivial solutions of (5.1_0). By virtue of Lemma 5.19 (with $u \equiv v_0$ and $q \equiv 0$), we get

$$\lim_{t\to a+}\Big(u_0'(t)v_0(t) - u_0(t)v_0'(t)\Big) = 0.$$

On the other hand, clearly,

$$\big(u_0'(t)v_0(t) - u_0(t)v_0'(t)\big)' = 0 \quad \text{for a. e. } t \in]a,b[$$

and therefore,
$$u_0'(t)v_0(t) - u_0(t)v_0'(t) = 0 \quad \text{for } t \in [a,b]. \tag{5.84}$$
Choose $t_0 \in]a,b[$ such that
$$u_0'(t_0) = 0.$$
It is clear that $u_0(t_0) \neq 0$ since otherwise $u_0 \equiv 0$. Then it follows from (5.84) that
$$v_0'(t_0) = 0$$
and moreover, $v_0(t_0) \neq 0$. Put $\lambda := \frac{u_0(t_0)}{v_0(t_0)}$ and
$$w(t) := u_0(t) - \lambda v_0(t) \quad \text{for } t \in [a,b].$$
Obviously, w is a solution of equation (5.1_0) and $w(t_0) = 0$. However, it follows from (5.84) that $w'(t_0) = 0$. Consequently, $w \equiv 0$ and thus, $u_0 \equiv \lambda v_0$. □

Proof of Theorem 5.15. Let u_0 be a nontrivial solution of problem (5.1_0), (5.2), while u be a solution of problem (5.1), (5.2). Put
$$f(t) := u'(t)u_0(t) - u(t)u_0'(t) \quad \text{for } t \in]a,b[\,.$$
It is clear that
$$f'(t) = q(t)u_0(t) \quad \text{for a. e. } t \in]a,b[\,.$$
Hence,
$$f\left(\frac{a+b}{2}\right) - f(t) = \int_t^{\frac{a+b}{2}} q(s)u_0(s)ds \quad \text{for } t \in]a,b[\,. \tag{5.85}$$
By virtue of Lemma 5.19, Proposition 5.17, and condition (5.4), we get from (5.85) that
$$f\left(\frac{a+b}{2}\right) = \int_a^{\frac{a+b}{2}} q(s)u_0(s)ds \quad \text{and} \quad f\left(\frac{a+b}{2}\right) = -\int_{\frac{a+b}{2}}^b q(s)u_0(s)ds,$$
and therefore, (5.61) is fulfilled.

Let now u_0 be a nontrivial solution of problem (5.1_0), (5.2), $q \in L_{loc}(]a,b[)$ satisfy (5.4), and (5.61) be fulfilled. Let, moreover, $n_0 \in \mathbb{N}$ and $r > 0$ be from the assertion of Lemma 5.20. By virtue of Lemma 5.20, for any $n > n_0$, the problem
$$u'' = \left(p(t) + \frac{1}{n}[p(t)]_-\right)u; \qquad u(a) = 0, \quad u(b) = 0$$
has no nontrivial solution. Since
$$\left[p(t) + \frac{1}{n}[p(t)]_-\right]_- = \frac{n-1}{n}[p(t)]_- \tag{5.86}$$
and (5.6) holds, it follows from Theorem 5.2 that for any $n > n_0$, the problem
$$u'' = \left(p(t) + \frac{1}{n}[p(t)]_-\right)u + q(t); \qquad u(a) = 0, \quad u(b) = 0 \tag{5.87}$$

has a unique solution u_n.

In view of Lemma 5.20, the inequalities

$$|u_n(t)| \le M \quad \text{for } t \in]a,b[\,,\ n > n_0 \tag{5.88}$$

are fulfilled, where

$$M := r \int_a^b (s-a)(b-s)|q(s)|ds.$$

On the other hand, on account of Lemma 5.11, (5.86), and (5.88), we get

$$(t-a)(b-t)|u_n'(t)| \le M_1 \quad \text{for } t \in]a,b[\,,\ n > n_0, \tag{5.89}$$

where

$$M_1 := M\left(b-a+2\int_a^b (s-a)(b-s)[p(s)]_- ds\right) + \int_a^b (s-a)(b-s)|q(s)|ds.$$

It follows from (5.88) and (5.89) that the sequence $\{u_n\}_{n=n_0}^{+\infty}$ is uniformly bounded and equicontinuous in $]a,b[$. Hence, by virtue of Arzelà-Ascoli's lemma, we can assume without loss of generality that

$$\lim_{n\to+\infty} u_n = u(t) \quad \text{uniformly in }]a,b[\,, \tag{5.90}$$

where $u \in C(]a,b[)$ and, moreover,

$$\lim_{n\to+\infty} u_n'\left(\frac{a+b}{2}\right) = c_0. \tag{5.91}$$

In view of (5.87), it is clear that

$$\begin{aligned} u_n(t) = u_n\left(\frac{a+b}{2}\right) + \left(t - \frac{a+b}{2}\right) u_n'\left(\frac{a+b}{2}\right) \\ + \int_{\frac{a+b}{2}}^t \left(\int_{\frac{a+b}{2}}^s \left[\left(p(\xi) + \frac{1}{n}[p(\xi)]_-\right)u_n(\xi) + q(\xi)\right] d\xi\right) ds \quad \text{for } t \in]a,b[\,. \end{aligned}$$

Hence, on account of (5.90) and (5.91), we get

$$\begin{aligned} u(t) = u\left(\frac{a+b}{2}\right) + c_0\left(t - \frac{a+b}{2}\right) \\ + \int_{\frac{a+b}{2}}^t \left(\int_{\frac{a+b}{2}}^s \Big[p(\xi)u(\xi) + q(\xi)\Big] d\xi\right) ds \quad \text{for } t \in]a,b[\,. \end{aligned}$$

Therefore, $u \in AC'_{loc}(]a,b[)$ and u is a solution of equation (5.1).

On the other hand, by virtue of Lemma 5.11 and (5.88), there are $a_0 \in]a,b[$, $b_0 \in]a_0,b[$, and $\varrho > 0$ such that for any $n > n_0$, the inequalities

$$\begin{aligned} |u_n(t)| \le \varrho \Bigg[M(t-a) + \int_a^t (s-a)\left(\frac{M}{n}[p(s)]_- + |q(s)|\right) ds \\ +(t-a)\int_t^{a_0} \left(\frac{M}{n}[p(s)]_- + |q(s)|\right) ds \Bigg] \quad \text{for } t \in]a,a_0] \end{aligned}$$

and

$$|u_n(t)| \le \varrho \left[M(b-t) + \int_t^b (b-s) \left(\frac{M}{n} [p(s)]_- + |q(s)| \right) ds \right.$$
$$\left. +(b-t) \int_{b_0}^t \left(\frac{M}{n} [p(s)]_- + |q(s)| \right) ds \right] \quad \text{for } t \in [b_0, b[$$

are fulfilled. Hence, in view of (5.90), we get

$$|u(t)| \le \varrho \left[M(t-a) + \int_a^t (s-a)|q(s)|ds + (t-a) \int_t^{a_0} |q(s)|ds \right] \quad \text{for } t \in]a, a_0]$$

and

$$|u(t)| \le \varrho \left[M(b-t) + \int_t^b (b-s)|q(s)|ds + (b-t) \int_{b_0}^t |q(s)|ds \right] \quad \text{for } t \in [b_0, b[\,.$$

Consequently, u satisfies (5.2) and thus, u is a solution of problem (5.1), (5.2). □

5.4 Well-posedness of the second order linear singular Dirichlet problem

Consider the problem

$$u'' = p_0(t)u + q_0(t), \tag{5.92}$$
$$u(a) = 0, \quad u(b) = 0, \tag{5.93}$$

where $p_0, q_0 \in L_{loc}(]a, b[)$, and the sequence of equations

$$u'' = p_n(t)u + q_n(t), \tag{5.92$_n$}$$

where $p_n, q_n \in L_{loc}(]a, b[)$, $n \in \mathbb{N}$. Under the well-posedness of the above problem, the statement of the following type is usually understood.

Statement. Let problem (5.92), (5.93) is uniquely solvable and let the functions p_n and q_n converge, in a certain sense, to the functions p_0 and q_0, respectively. Then for any $n \in \mathbb{N}$ large enough, problem (5.92$_n$), (5.93) is uniquely solvable and the sequence of solutions converge, in a certain sense, to the solution of problem (5.92), (5.93).

In the case, when the functions p_n and q_n, $n \in \mathbb{N} \cup \{0\}$, are integrable on $[a, b]$, well-posedness of the problem considered one can deduce from the general theory of linear boundary value problems (see, e. g., [17, Theorem 1.2]). However, we are interested in the case, when the functions p_n and q_n are not, in general, integrable on $[a, b]$, having singularities at $t = a$ and $t = b$. The aim of this section is to establish conditions guaranteeing well-posedness of the singular problem (5.92), (5.93).

Along with (5.92) we consider also the corresponding homogeneous equation

$$u'' = p_0(t)u. \tag{5.92$_0$}$$

It has been proved in Section 5.2 that Fredholm's alternative remains true even in the case, when

$$\int_a^b (s-a)(b-s)[p_0(s)]_- ds < +\infty \tag{5.94}$$

and

$$\int_a^b (s-a)(b-s)|q_0(s)| ds < +\infty. \tag{5.95}$$

Hence, it is natural to assume along with (5.94) and (5.95) that for any $n \in \mathbb{N}$, the conditions

$$\int_a^b (s-a)(b-s)[p_n(s)]_- ds < +\infty \tag{5.96}$$

and

$$\int_a^b (s-a)(b-s)|q_n(s)| ds < +\infty. \tag{5.97}$$

hold as well.

Now introduce the following definitions.

Definition 5.21. We say that the condition (P) is fulfilled if (5.94) and (5.96) hold,

$$\lim_{n\to+\infty} \int_{\frac{a+b}{2}}^t p_n(s)ds = \int_{\frac{a+b}{2}}^t p_0(s)ds \quad \text{uniformly in }]a,b[\,, \tag{5.98}$$

and either there exists $p^* \in L_{loc}(]a,b[)$ such that

$$[p_n(t)]_- \le p^*(t) \quad \text{for } t \in]a,b[\,,\ n \in \mathbb{N}, \text{ and } \int_a^b (s-a)(b-s)p^*(s)ds < +\infty, \tag{5.99}$$

or

$$\lim_{n\to+\infty} \int_a^b (s-a)(b-s)\Big[p_n(s) + [p_0(s)]_-\Big]_- ds = 0. \tag{5.100}$$

Remark 5.22. If the condition (P) is fulfilled, then there exists $M > 0$ such that

$$\int_a^b (s-a)(b-s)[p_n(s)]_- ds \le M \quad \text{for } n \in \mathbb{N}.$$

Definition 5.23. We say that the condition (Q) is fulfilled if (5.95) and (5.97) hold and either

$$\lim_{n\to+\infty} \int_a^b (s-a)(b-s)|q_n(s) - q_0(s)| ds = 0, \tag{5.101}$$

or

$$\lim_{n\to+\infty} \int_{\frac{a+b}{2}}^t q_n(s)ds = \int_{\frac{a+b}{2}}^t q_0(s)ds \quad \text{uniformly in }]a,b[\,, \tag{5.102}$$

and there exists $q^* \in L_{loc}(]a,b[)$ such that

$$|q_n(t)| \le q^*(t) \quad \text{for } t \in]a,b[\,,\ n \in \mathbb{N}, \text{ and } \int_a^b (s-a)(b-s)q^*(s)ds < +\infty. \tag{5.103}$$

Remark 5.24. It is clear that (5.101) implies (5.102). Mention also that if the condition (Q) is fulfilled, then there exists $M > 0$ such that

$$\int_a^b (s-a)(b-s)|q_n(s)|ds \le M \quad \text{for } n \in \mathbb{N}.$$

5.4.1 Main result

Theorem 5.25 ([30, Thm. 1.2]). *Let problem* (5.92_0), (5.93) *have no nontrivial solution and the conditions* (P) *and* (Q) *be fulfilled. Then there exists* $n_0 \in \mathbb{N}$ *such that for any* $n > n_0$, *problem* (5.92_n), (5.93) *possesses a unique solution* u_n,

$$\lim_{n\to+\infty} u_n(t) = u(t) \quad \textit{uniformly on } [a,b], \tag{5.104}$$

and

$$\lim_{n\to+\infty} u_n'(t) = u'(t) \quad \textit{uniformly in }]a,b[\,, \tag{5.105}$$

where u *is a (unique) solution of problem* (5.92), (5.93).

Remark 5.26. As it follows from Definition 5.21, the condition (P) implies condition (5.98). In certain cases, these two conditions are equivalent. For example, if $p_n(t) := p_0(t) + g_n(t)$ with $g_n \in L_{loc}(]a,b[)$, $g_n(t) \ge 0$, $p_0(t) \le 0$ for a.e. $t \in]a,b[$, and

$$\lim_{n\to+\infty} \int_{\frac{a+b}{2}}^t g_n(s)ds = 0 \quad \text{uniformly in }]a,b[\,.$$

However, in Theorem 5.25, the condition (P) cannot be replaced by condition (5.98) (see Example 5.29 bellow). Analogously, the condition (Q) yields condition (5.102). However, Example 5.28 shows that, in Theorem 5.25, the condition (Q) cannot be replaced by condition (5.102). In this sense, Theorem 5.25 is optimal and its assumptions cannot be weakened.

Remark 5.27. In Theorem 5.25, assertion (5.105) cannot be replaced by the stronger assertion

$$\lim_{n\to+\infty} u_n'(t) = u'(t) \quad \text{uniformly on } [a,b]. \tag{5.106}$$

Indeed, let $a = 0$, $b = 1$, $p_0(t) \overset{\text{def}}{=} 0$, $q_0(t) := 0$, $p_n(t) := 0$, and $q_n(t) := \frac{1}{nt(1-t)}$. Then all the assumptions of Theorem 5.25 are fulfilled, $u(t) \equiv 0$, and $u_n(t) = \frac{1}{n}\big(t\ln t + (1-t)\ln(1-t)\big)$. However, (5.106) is violated because $\lim_{n\to+\infty} u_n'(e^{-n}) = -1$.

Example 5.28. Let $a = 0$, $b = 2\pi$, $p_0 \equiv 0$, $q_0 \equiv 0$,

$$p_n(t) := \begin{cases} -n^2 & \text{for } t \in \left[0, \frac{\pi}{2n^2}\right] \cup \left[2\pi - \frac{\pi}{2n^2}, 2\pi\right], \\ 0 & \text{for } t \in \left]\frac{\pi}{2n^2}, 2\pi - \frac{\pi}{2n^2}\right[\,, \end{cases}$$

and

$$q_n(t) := \begin{cases} -n^2\sin(nt) & \text{for } t \in \left]\frac{\pi}{2n^2}, \frac{\pi}{2n}\right[\cup \left]2\pi - \frac{\pi}{2n}, 2\pi - \frac{\pi}{2n^2}\right[, \\ 0 & \text{for } t \in \left[0, \frac{\pi}{2n^2}\right] \cup \left[\frac{\pi}{2n}, 2\pi - \frac{\pi}{2n}\right] \cup \left[2\pi - \frac{\pi}{2n^2}, 2\pi\right] . \end{cases}$$

It is clear that all the assumptions of Theorem 5.25 with (Q) replaced by (5.102) are fulfilled. Problem (5.92), (5.93) has a unique solution $u \equiv 0$, while the function

$$u_n(t) := \begin{cases} \sin(nt) & \text{for } t \in \left[0, \frac{\pi}{2n}\right] \cup \left[2\pi - \frac{\pi}{2n}, 2\pi\right] , \\ 1 & \text{for } t \in \left]\frac{\pi}{2n}, 2\pi - \frac{\pi}{2n}\right[\end{cases} \tag{5.107}$$

is a unique solution of problem (5.92$_n$), (5.93). However, $\|u_n\|_{[a,b]} = 1$ for $n \in \mathbb{N}$ and therefore, (5.104) is violated.

Example 5.29. Let $a = 0$, $b = 2\pi$, $p_0 \equiv 0$, $q_0 \equiv 0$, $q_n(t) := \frac{1}{n}$, and

$$p_n(t) := \begin{cases} -n^2 & \text{for } t \in \left[0, \frac{\pi}{2n}\right] \cup \left[2\pi - \frac{\pi}{2n}, 2\pi\right] , \\ 0 & \text{for } t \in \left]\frac{\pi}{2n}, 2\pi - \frac{\pi}{2n}\right[. \end{cases}$$

It is clear that the all assumptions of Theorem 5.25 are fulfilled with (P) replaced by (5.98). However, for any $n \in \mathbb{N}$, the function u_n defined by (5.107) is a nontrivial solution of the homogeneous problem

$$u'' = p_n(t)u; \qquad u(a) = 0, \quad u(b) = 0.$$

Since $\int_a^b u_n(s)ds \neq 0$ for $n \in N$, by virtue of (classical) Fredholm's third theorem, for any $n \in N$, problem (5.92$_n$), (5.93) has no solution.

5.4.2 Auxiliary statements

First of all, for the sake of convenience of references, we recall some results (in a suitable form for us) established above.

Consider the equation

$$v'' = h(t)v + q(t),$$

where $h, q \in L_{loc}(]a, b[)$,

$$\int_a^b (s-a)(b-s)|h(s)|ds < +\infty, \tag{5.108}$$

and

$$\int_a^b (s-a)(b-s)|q(s)|ds < +\infty. \tag{5.109}$$

In Propositions 5.7–5.9, it is stated that

Proposition 5.30. *Let* (5.108) *hold. Then there exist* $a_0 \in]a, b[$, $b_0 \in]a_0, b[$, *and* $\varrho_0 > 0$ *such that the following assertions hold:*

(1) For any $t_1 < t_2$ satisfying either $t_1, t_2 \in [a, a_0]$ or $t_1, t_2 \in [b_0, b]$ and for any $w \in AC'_{loc}(]t_1, t_2[)$ satisfying

$$w''(t) \geq h(t)w(t) \quad \text{for a. e. } t \in [t_1, t_2], \quad w(t_1) = 0, \quad w(t_2) = 0,$$

the inequality

$$w(t) \leq 0 \quad \text{for } t \in [t_1, t_2]$$

holds.

(2) For any $c \in \mathbb{R}$ and any $q \in L_{loc}(]a, b[)$ satisfying (5.109)*, the problems*

$$v'' = h(t)v + q(t); \qquad v(a) = 0, \quad v(a_0) = c \tag{5.110}$$

and

$$v'' = h(t)v + q(t); \qquad v(b_0) = c, \quad v(b) = 0 \tag{5.111}$$

are uniquely solvable.

(3) The solution v of problem (5.110)*, resp.,* (5.111)*, admits the estimate*

$$|v(t)| \leq \varrho_0 \left(|c|(t-a) + \int_a^t (s-a)|q(s)|ds + (t-a) \int_t^{a_0} |q(s)|ds \right) \quad \text{for } t \in]a, a_0],$$

resp.,

$$|v(t)| \leq \varrho_0 \left(|c|(b-t) + \int_t^b (b-s)|q(s)|ds + (b-t) \int_{b_0}^t |q(s)|ds \right) \quad \text{for } t \in [b_0, b[\,.$$

PROPOSITION 5.31. *Let* (5.94) *hold. Then there exist $a_0 \in]a, b[$, $b_0 \in]a_0, b[$, and $\varrho_0 > 0$ such that for any $p_n, q_n \in L_{loc}(]a, b[)$ satisfying* (5.96) *and* (5.97)*, any solution u_n of problem* (5.92_n)*,* (5.93) *admits the estimate*

$$|u_n(t)| \leq \varrho_0 \Big(\|u_n\|_{[a,b]} P_n(t) + Q_n(t) \Big) \quad \text{for } t \in]a, a_0] \cup [b_0, b[\,, \tag{5.112}$$

where

$$\begin{aligned} P_n(t) &:= t - a + \int_a^t (s-a) \Big[p_n(s) + [p_0(s)]_- \Big]_- ds \\ &\qquad + (t-a) \int_t^{a_0} \Big[p_n(s) + [p_0(s)]_- \Big]_- ds \quad \text{for } t \in]a, a_0], \\ P_n(t) &:= b - t + \int_t^b (b-s) \Big[p_n(s) + [p_0(s)]_- \Big]_- ds \\ &\qquad + (b-t) \int_{b_0}^t \Big[p_n(s) + [p_0(s)]_- \Big]_- ds \quad \text{for } t \in [b_0, b[\,, \\ Q_n(t) &:= \int_a^t (s-a)|q_n(s)|ds + (t-a) \int_t^{a_0} |q_n(s)|ds \quad \text{for } t \in]a, a_0], \\ Q_n(t) &:= \int_t^b (b-s)|q_n(s)|ds + (b-t) \int_{b_0}^t |q_n(s)|ds \quad \text{for } t \in [b_0, b[\,. \end{aligned} \tag{5.113}$$

Proof. Let $a_0 \in]a,b[$, $b_0 \in]a_0,b[$, and $\varrho_0 > 0$ be from the assertion of Proposition 5.30 with $h \equiv -[p_0]_-$. Let, moreover, the functions $p_n, q_n \in L_{loc}(]a,b[)$ satisfy (5.96) and (5.97) and u_n be a solution of problem (5.92$_n$), (5.93).

Consider the problem

$$v'' = -[p_0(t)]_- v - \Big[p_n(t) + [p_0(t)]_-\Big]_- |u_n(t)| - |q_n(t)|,$$
$$v(a) = 0, \qquad v(a_0) = \|u_n\|_{[a,b]}.$$

By virtue of Proposition 5.30(2), this problem has a unique solution v_n. In view of Proposition 5.30(1), we have

$$v_n(t) \geq 0 \quad \text{for } t \in]a, a_0], \tag{5.114}$$

as well. Now we show that

$$|u_n(t)| \leq v_n(t) \quad \text{for } t \in]a, a_0]. \tag{5.115}$$

Suppose on the contrary that (5.115) is violated. Then there exist $t_1 \in [a, a_0[$ and $t_2 \in]t_1, a_0]$ such that

$$w(t) > 0 \quad \text{for } t \in]t_1, t_2[, \tag{5.116}$$
$$w(t_1) = 0, \qquad w(t_2) = 0,$$

where $w(t) := |u_n(t)| - v_n(t)$ for $t \in]a, a_0]$. In view of (5.114) and (5.116), it is clear that $w \in AC'_{loc}(]t_1, t_2[)$. Moreover

$$w''(t) = u_n''(t) \operatorname{sgn} u_n(t) - v_n''(t) \geq -[p_0(t)]_- w(t) \quad \text{for a. e. } t \in]t_1, t_2[.$$

Hence, by virtue of Proposition 5.30(1), we get $w(t) \leq 0$ for $t \in]t_1, t_2[$, which contradicts (5.116).

Analogously one can prove that

$$|u_n(t)| \leq v_n(t) \quad \text{for } t \in [b_0, b[, \tag{5.117}$$

where v_n is a solution of the problem

$$v'' = -[p_0(t)]_- v - \Big[p_n(t) + [p_0(t)]_-\Big]_- |u_n(t)| - |q_n(t)|,$$
$$v(b_0) = \|u_n\|_{[a,b]}, \qquad v(b) = 0.$$

Estimate (5.112) now follows from (5.115), (5.117), and Proposition 5.30(3). $\square$

Remark 5.32. Suppose that the conditions (P) and (Q) hold and the functions P_n and Q_n are defined by (5.113). Then there exist nonnegative functions $\varphi, \psi \in C(]a, a_0]) \cup C([b_0, b[)$ such that

$$\varphi(a) = 0, \qquad \varphi(b) = 0, \quad \psi(a) = 0, \quad \psi(b) = 0, \tag{5.118}$$

and

$$\limsup_{n\to+\infty} P_n(t) \le \varphi(t), \quad \limsup_{n\to+\infty} Q_n(t) \le \psi(t) \quad \text{for } t \in \,]a,a_0] \cup [b_0,b[\,.$$

Indeed, if (5.99), resp., (5.103), holds, then we set

$$\begin{aligned}\varphi(t) &:= t-a+\int_a^t (s-a)\big(p^*(s)+[p_0(s)]_-\big)ds\\ &\qquad + (t-a)\int_t^{a_0}\big(p^*(s)+[p_0(s)]_-\big)ds \quad \text{for } t\in\,]a,a_0],\\ \varphi(t) &:= b-t+\int_t^b (b-s)\big(p^*(s)+[p_0(s)]_-\big)ds\\ &\qquad + (b-t)\int_{b_0}^t\big(p^*(s)+[p_0(s)]_-\big)ds \quad \text{for } t\in[b_0,b[\,,\end{aligned}$$

resp.,

$$\begin{aligned}\psi(t) &:= \int_a^t (s-a)q^*(s)ds + (t-a)\int_t^{a_0} q^*(s)ds \quad \text{for } t\in\,]a,a_0],\\ \psi(t) &:= \int_t^b (b-s)q^*(s)ds + (b-t)\int_{b_0}^t q^*(s)ds \quad \text{for } t\in[b_0,b[\,.\end{aligned}$$

On the other hand, if (5.100), resp., (5.101), is satisfied, then we put

$$\varphi(t) := t-a \quad \text{for } t\in\,]a,a_0], \qquad \varphi(t) := b-t \quad \text{for } t\in[b_0,b[\,,$$

resp.,

$$\begin{aligned}\psi(t) &:= \int_a^t (s-a)|q_0(s)|ds + (t-a)\int_t^{a_0} |q_0(s)|ds \quad \text{for } t\in\,]a,a_0],\\ \psi(t) &:= \int_t^b (b-s)|q_0(s)|ds + (b-t)\int_{b_0}^t |q_0(s)|ds \quad \text{for } t\in[b_0,b[\,.\end{aligned}$$

In particular, for any $\varepsilon>0$, there exist $a_\varepsilon\in\,]a,a_0]$, $b_\varepsilon\in[b_0,b[\,$, and $n_\epsilon\in\mathbb{N}$ such that

$$P_n(t)<\varepsilon, \quad Q_n(t)<\varepsilon \quad \text{for } t\in\,]a,a_\varepsilon]\cup[b_\varepsilon,b[\,,\ n>n_\varepsilon.$$

PROPOSITION 5.33. *Let* (5.98) *and* (5.102) *hold. Let, moreover,* $\{u_n\}_{n=1}^{+\infty}$ *be a uniformly bounded and equicontinuous in* $]a,b[$ *sequence of solutions of problem* (5.92_n), (5.93). *Then there exist a subsequence* $\{u_{n_k}\}_{k=1}^{+\infty}$ *and a solution* u *of equation* (5.92) *such that*

$$\lim_{k\to+\infty} u_{n_k}^{(i)}(t) = u^{(i)}(t) \quad \textit{uniformly in }]a,b[\,,\ i=0,1.$$

Proof. By virtue of Arzelá-Acsoli's lemma, we can assume without loss of generality that

$$\lim_{n\to+\infty} u_n(t) = u(t) \quad \text{uniformly in }]a,b[\,, \tag{5.119}$$

where $u \in C(]a,b[)$. Let $a_0 \in]a,b[$ and $b_0 \in]a_0,b[$ be fixed. Then for any $n \in \mathbb{N}$, there exists $t_n \in [a_0,b_0]$ such that

$$u_n(b_0) - u_n(a_0) = (b_0 - a_0)u_n'(t_n).$$

Hence, there is a subsequence $\{t_{n_k}\}_{k=1}^{+\infty} \subset [a_0,b_0]$ and $t_0 \in [a_0,b_0]$ such that

$$\lim_{k\to+\infty} t_{n_k} = t_0 \tag{5.120}$$

and

$$\lim_{k\to+\infty} u_{n_k}'(t_{n_k}) = c. \tag{5.121}$$

In view of (5.102), (5.119), and (5.120), it is clear that

$$\begin{aligned}\lim_{k\to+\infty} \int_{t_{n_k}}^t & \big(p_0(s)u_{n_k}(s) + q_{n_k}(s)\big)ds \\ &= \int_{t_0}^t \big(p_0(s)u(s) + q_0(s)\big)ds \quad \text{uniformly in }]a,b[\,. \end{aligned}\tag{5.122}$$

We first show that

$$\lim_{k\to+\infty} \int_{t_{n_k}}^t p_{n_k}(s)u_{n_k}(s)ds = \int_{t_0}^t p_0(s)u(s)ds \quad \text{uniformly in }]a,b[\,. \tag{5.123}$$

Observe that

$$u_{n_k}'(t) = u_{n_k}'(t_{n_k}) + \int_{t_{n_k}}^t \big(p_{n_k}(s)u_{n_k}(s) + q_{n_k}(s)\big)ds \quad \text{for } t \in]a,b[\,,\ k \in \mathbb{N}. \tag{5.124}$$

Introduce the notations

$$\Phi_k(t) := \int_{t_{n_k}}^t \big(p_{n_k}(s) - p_0(s)\big)u_{n_k}(s)ds, \quad F_k(t) := \int_{t_{n_k}}^t \big(p_{n_k}(s) - p_0(s)\big)ds,$$

$$H_k(t) := \int_{t_{n_k}}^t \big(p_0(s)u_{n_k}(s) + q_{n_k}(s)\big)ds \quad \text{for } t \in]a,b[\,,\ k \in \mathbb{N}.$$

One can easily verify that

$$\Phi_k(t) = u_{n_k}(t)F_k(t) - \int_{t_{n_k}}^t u_{n_k}'(s)F_k(s)ds \quad \text{for } t \in]a,b[\,,\ k \in \mathbb{N}.$$

Taking now into account (5.124), we get

$$\begin{aligned}\Phi_k(t) = u_{n_k}(t)F_k(t) - u_{n_k}'(t_{n_k}) \int_{t_{n_k}}^t F_k(s)ds - \int_{t_{n_k}}^t F_k(s)\Phi_k(s)ds \\ - \int_{t_{n_k}}^t F_k(s)H_k(s)ds \quad \text{for } t \in]a,b[\,,\ k \in \mathbb{N}. \end{aligned}\tag{5.125}$$

It follows from (5.125) that for any $\alpha \in]a, a_0[$ and $\beta \in]b_0, b[$, the inequality

$$\|\Phi_k\|_{[\alpha,\beta]} \leq \|F_k\|_{[\alpha,\beta]}\Big(|u'_{n_k}(t_{n_k})| + \|u_{n_k}\|_{[\alpha,\beta]} + (b-a)\big(\|\Phi_k\|_{[\alpha,\beta]} + \|H_k\|_{[\alpha,\beta]}\big)\Big)$$

holds for $k \in \mathbb{N}$. Hence, on account of (5.98), (5.121), and (5.122), we conclude that for any $\alpha \in]a, a_0[$ and $\beta \in]b_0, b[$,

$$\lim_{k\to+\infty} \|\Phi_k\|_{[\alpha,\beta]} = 0$$

and consequently, (5.123) holds.

It follows from (5.124) that

$$\begin{aligned} u_{n_k}(t) &= u_{n_k}(t_{n_k}) + (t - t_{n_k})u'_{n_k}(t_{n_k}) \\ &\quad + \int_{t_{n_k}}^{t} \left(\int_{t_{n_k}}^{s} \Big(p_{n_k}(\xi)u_{n_k}(\xi) + q_{n_k}(\xi)\Big)d\xi \right) ds \quad \text{for } t \in]a, b[\,,\ k \in \mathbb{N}. \end{aligned}$$

Hence, in view of (5.119), (5.120), (5.121), (5.123), and (5.102), we get

$$u(t) = u(t_0) + c(t - t_0) + \int_{t_0}^{t} \left(\int_{t_0}^{s} \Big(p_0(\xi)u(\xi) + q_0(\xi)\Big)d\xi \right) ds \quad \text{for } t \in]a, b[\,.$$

Consequently, $u \in AC'_{loc}(]a, b[)$ and the function u is a solution of equation (5.92). In particular,

$$u'(t) = c + \int_{t_0}^{t} \Big(p_0(s)u(s) + q_0(s)\Big)ds \quad \text{for } t \in]a, b[\,. \tag{5.126}$$

On the other hand, in view of (5.102), (5.120), and (5.123), it follows from (5.124) that

$$\lim_{k\to+\infty} u'_{n_k}(t) = c + \int_{t_0}^{t} \Big(p_0(s)u(s) + q_0(s)\Big)ds \quad \text{uniformly in }]a, b[$$

and consequently, by virtue of (5.126), we get

$$\lim_{k\to+\infty} u'_{n_k}(t) = u'(t) \quad \text{uniformly in }]a, b[\,.$$

$\square$

LEMMA 5.34. *Let* (5.94) *hold and problem* (5.92$_0$), (5.93) *have no nontrivial solution. Let, moreover, the condition* (P) *be fulfilled. Then there exist* $r > 0$ *and* $n_0 \in \mathbb{N}$ *such that for any* $q \in L_{loc}(]a, b[)$ *satisfying* (5.109) *and any* $n > n_0$*, any solution* u_n *of the problem*

$$u'' = p_n(t)u + q(t); \qquad u(a) = 0, \quad u(b) = 0$$

admits the estimate

$$\|u_n\|_{[a,b]} \leq r \int_a^b (s-a)(b-s)|q(s)|ds.$$

Proof. Suppose on the contrary that the assertion of the lemma is violated. Then there exist $\{\widetilde{q}_{n_k}\}_{k=1}^{+\infty} \subset L_{loc}(]a,b[)$ and $\{\widetilde{u}_{n_k}\}_{k=1}^{+\infty} \subset AC'_{loc}(]a,b[)$ such that for any $k \in \mathbb{N}$,

$$\widetilde{u}_{n_k}(t)'' = p_{n_k}(t)\widetilde{u}_{n_k}(t) + \widetilde{q}_{n_k}(t) \quad \text{for a. e. } t \in]a,b[\,, \quad \widetilde{u}_{n_k}(a) = 0, \quad \widetilde{u}_{n_k}(b) = 0,$$

and

$$\|\widetilde{u}_{n_k}\|_{[a,b]} > k \int_a^b (s-a)(b-s)|\widetilde{q}_{n_k}(s)|ds.$$

Introduce the notation

$$u_{n_k}(t) := \frac{\widetilde{u}_{n_k}(t)}{\|\widetilde{u}_{n_k}\|_{[a,b]}}, \quad q_{n_k}(t) := \frac{\widetilde{q}_{n_k}(t)}{\|\widetilde{u}_{n_k}\|_{[a,b]}} \quad \text{for } k \in \mathbb{N}.$$

It is clear that for any $k \in \mathbb{N}$, we have

$$u''_{n_k}(t) = p_{n_k}(t)u_{n_k}(t) + q_{n_k}(t) \quad \text{for a. e. } t \in]a,b[\,, \quad u_{n_k}(a) = 0, \quad u_{n_k}(b) = 0,$$
$$\|u_{n_k}\|_{[a,b]} = 1, \tag{5.127}$$

and

$$\lim_{k \to +\infty} \int_a^b (s-a)(b-s)|q_{n_k}(s)|ds = 0. \tag{5.128}$$

In view of (5.128), we get

$$\lim_{k \to +\infty} \int_{\frac{a+b}{2}}^t q_{n_k}(s)ds = 0 \quad \text{uniformly in }]a,b[\,, \tag{5.129}$$

as well. By virtue of Proposition 5.10 and (5.127), the inequality

$$(t-a)(b-t)|u'_{n_k}(t)| \le b-a + \int_a^b (s-a)(b-s)[p_{n_k}(s)]_- ds$$
$$+ \int_a^b (s-a)(b-s)|q_{n_k}(s)|ds \quad \text{for } t \in]a,b[\,,\ k \in \mathbb{N}$$

holds. Hence, in view of (5.127), (5.128), and Remark 5.22, the sequence $\{u_{n_k}\}_{k=1}^{+\infty}$ is uniformly bounded and equicontinuous in $]a,b[$. Taking now into account (5.98), (5.129), and Proposition 5.33, we can assume without loss of generality that

$$\lim_{k \to +\infty} u_{n_k}^{(i)}(t) = u^{(i)}(t) \quad \text{uniformly in }]a,b[\,,\ i = 0,1, \tag{5.130}$$

where u is a solution of equation (5.92_0).

Let now $a_0 \in]a,b[$, $b_0 \in]a_0,b[$, and $\varrho_0 > 0$ be from the assertion of Proposition 5.31. Then, in view of (5.127) and Proposition 5.31, we get

$$|u_{n_k}(t)| \le \varrho_0 \Big(P_{n_k}(t) + Q_{n_k}(t)\Big) \quad \text{for } t \in]a,a_0] \cup [b_0,b[\,,\ k \in \mathbb{N}, \tag{5.131}$$

where P_{n_k} and Q_{n_k} are defined by (5.113). Hence, in view of Remark 5.32, there exist $a_1 \in]a,a_0]$, $b_1 \in [b_0,b[$, and $k_1 \in \mathbb{N}$ such that

$$|u_{n_k}(t)| < 1 \quad \text{for } t \in]a,a_1] \cup [b_1,b[\,,\ k > k_1,$$

whence, on account of (5.127), we get $\|u_{n_k}\|_{[a_1,b_1]} = 1$ for $k > k_1$. Therefore, in view of (5.130), we have

$$u \not\equiv 0. \tag{5.132}$$

On the other hand, on account of (5.128), (5.130), and Remark 5.32, we get from (5.131) that

$$|u(t)| \le \rho_0 \varphi(t) \quad \text{for } t \in \,]a, a_0] \cup [b_0, b[\,,$$

where $\varphi \in C(]a, a_0] \cup [b_0, b[)$ and $\varphi(a) = 0$, $\varphi(b) = 0$. Hence, the function u satisfies (5.93) and therefore, in view of (5.132), u is a nontrivial solution of problem (5.92_0), (5.93). However, this contradicts the assumption of the lemma. $\square$

5.4.3 Proof of the main result

Proof of Theorem 5.25. According to Theorem 5.2, problem (5.92), (5.93) has a unique solution u. Let $r > 0$ and $n_0 \in \mathbb{N}$ be from the assertion of Lemma 5.34. Then it follows from Lemma 5.34 that for any $n > n_0$, the problem

$$u'' = p_n(t)u; \qquad u(a) = 0, \quad u(b) = 0$$

has no nontrivial solution. Therefore, by virtue of Theorem 5.2, for any $n > n_0$, the problem (5.92_n), (5.93) possesses a unique solution u_n.

To prove (5.104) and (5.105) it is sufficient to show that for any unbounded set $J \subset \{n_0, n_0 + 1, \dots\}$, the sequence $\{u_n\}_{n \in J}$ contains a subsequence $\{u_{n_k}\}_{k=1}^{+\infty}$ such that

$$\lim_{k \to +\infty} u_{n_k}(t) = u(t) \quad \text{uniformly on } [a, b] \tag{5.133}$$

and

$$\lim_{k \to +\infty} u'_{n_k}(t) = u'(t) \quad \text{uniformly in } \,]a, b[\,.$$

Let $J \subset \{n_0, n_0 + 1, \dots\}$ be an arbitrary unbounded set. By virtue of Lemma 5.34 and Proposition 5.10, for any $n > n_0$, the estimates

$$\|u_n\|_{[a,b]} \le r \int_a^b (s - a)(b - s)|q_n(s)| ds \tag{5.134}$$

and

$$(t - a)(b - t)|u_n'(t)| \le \|u_n\|_{[a,b]} \left(b - a + \int_a^b (s - a)(b - s)[p_n(s)]_- ds \right) + \\ + \int_a^b (s - a)(b - s)|q_n(s)| ds \quad \text{for } t \in \,]a, b[\tag{5.135}$$

hold. On account of Remarks 5.22 and 5.24, there exists $M > 0$ such that

$$\int_a^b (s - a)(b - s)[p_n(s)]_- ds \le M \quad \text{for } n \in \mathbb{N} \tag{5.136}$$

and

$$\int_a^b (s-a)(b-s)|q_n(s)|ds \le M \quad \text{for } n \in \mathbb{N}. \tag{5.137}$$

In view of (5.136) and (5.137), inequalities (5.134) and (5.135) imply that the sequence $\{u_n\}_{n\in J}$ is uniformly bounded and equicontinuous in $]a,b[$. By virtue of Proposition 5.33, there exits a subsequence $\{u_{n_k}\}_{k=1}^{+\infty}$ such that

$$\lim_{k\to+\infty} u_{n_k}^{(i)}(t) = u_0^{(i)}(t) \quad \text{uniformly in }]a,b[\,,\ i=0,1, \tag{5.138}$$

where u_0 is a solution of equation (5.92).

Let now $a_0 \in]a,b[$, $b_0 \in]a_0,b[$, and $\varrho_0 > 0$ be from the assertion of Proposition 5.31. On account of (5.134), (5.137), and Proposition 5.31, we have

$$|u_{n_k}(t)| \le \varrho_0\Big(rMP_{n_k}(t) + Q_{n_k}(t)\Big) \quad \text{for } t \in]a,a_0] \cup [b_0,b[\,,\ k \in \mathbb{N}, \tag{5.139}$$

where P_{n_k} and Q_{n_k} are defined by (5.113). Hence, in view of (5.138) (with $i=0$) and Remark 5.32, we get

$$|u_0(t)| \le \varrho_0\Big(rM\varphi(t) + \psi(t)\Big) \quad \text{for } t \in]a,a_0] \cup [b_0,b[\,,$$

where $\varphi,\psi \in C(]a,a_0]) \cup C([b_0,b[)$ satisfy (5.118). Consequently, $u_0(a)=0$ and $u_0(b)=0$. Therefore, u_0 is a solution of problem (5.92), (5.93). However, this problem has a unique solution and thus, $u_0 \equiv u$. Taking now into account (5.138), we have

$$\lim_{k\to+\infty} u_{n_k}^{(i)}(t) = u^{(i)}(t) \quad \text{uniformly in }]a,b[\,,\ i=0,1. \tag{5.140}$$

It remains to show that (5.133) holds. Let $\varepsilon > 0$ be arbitrary. Then, by virtue of Remark 5.32 and (5.139), there exist $a_\varepsilon \in]a,a_0]$, $b_\varepsilon \in [b_0,b[$, and $k_0 \in \mathbb{N}$ such that

$$|u_{n_k}(t)| \le \varepsilon \quad \text{for } t \in]a,a_\varepsilon] \cup [b_\varepsilon,b[\,,\ k > k_0.$$

Hence, in view of (5.140), we get

$$|u(t)| \le \varepsilon \quad \text{for } t \in]a,a_\varepsilon] \cup [b_\varepsilon,b[\,.$$

Therefore,

$$\|u_{n_k} - u\|_{[a,a_\varepsilon]} \le 2\varepsilon, \quad \|u_{n_k} - u\|_{[b_\varepsilon,b]} \le 2\varepsilon \quad \text{for } k > k_0,$$

which, together with (5.140), imply desired relation (5.133). □

5.4.4 Example

On the interval $]0,1[$, we consider the problem

$$u'' = \frac{1}{t^2}u + \frac{1-\ln t}{t}; \quad u(0)=0,\ u(1)=0 \tag{5.141}$$

and the sequence of the problems

$$u'' = \frac{1+\cos(nt)}{t^2}\,u + \frac{1+\cos(nt)-\ln t}{t}\,; \quad u(0)=0,\; u(1)=0. \tag{5.141$_n$}$$

Put $p_0(t) := \frac{1}{t^2}$, $q_0(t) := \frac{1-\ln t}{t}$, $p_n(t) := p_0(t) + \frac{\cos(nt)}{t^2}$, $q_n(t) := q_0(t) + \frac{\cos(nt)}{t}$, $a=0$ and $b=1$. One can show that (5.94)–(5.97) hold as well as the conditions (P) and (Q) are satisfied. On the other hand, the functions $u_1(t) = t^{\frac{1-\sqrt{5}}{2}}$ and $u_2(t) = t^{\frac{1+\sqrt{5}}{2}}$ are linearly independent solutions of equation (5.92$_0$) and consequently, problem (5.92$_0$), (5.93) has no nontrivial solution. By direct calculation, we easily conclude that the function $u(t) = t\ln t$ is a unique solution of problem (5.141). Therefore, all the assumptions of Theorem 5.25 hold and consequently, solutions u_n of (5.141$_n$) satisfy

$$\lim_{n\to+\infty} u_n(t) = t\ln t \quad \text{uniformly on } [0,1]$$

and

$$\lim_{n\to+\infty} u_n'(t) = 1+\ln t \quad \text{uniformly in }]0,1[\,.$$

6 Summary and further research

The habilitation thesis deals with certain problems of qualitative theory of ordinary and functional differential equations. After a brief introduction, Chapter 2 is devoted to motivating models for the study of the considered problems. The main results are provided in the next three chapters.

We studied asymptotic properties of the second order delay differential equations and the two-dimensional nonlinear systems in Chapter 3. Two types of the oscillation criteria for DDEs with respect to considered delay are presented. If the delay $\tau(t)$ is "close" enough to the argument t, then we obtain criteria which correspond to well-known results from the oscillation theory of ODEs, namely Hille's and Nehari's criteria established in [13, 38]. On the other hand, if the deviation $\tau(t)$ is "large" with respect to the argument t in a certain sense, then we formulated the oscillation criteria of so-called Myshkis's type. Presented statements generalize and improve (under additional assumptions) results stated in [23]. Furthermore, the oscillation of the two-dimensional systems of nonlinear differential equations is discussed in Section 3.4. Obtained results generalize those, which were presented, e.g., in [6, 13, 14, 16, 24, 38].

Chapter 4 deals with certain boundary value problems for functional differential equations. We established new conditions guaranteeing the solvability and unique solvability of linear as well as of nonlinear problems. Boundary conditions were assumed in general form, which includes, e.g., initial, periodic, and antiperiodic condition. Moreover, nonnegativity of solutions was investigated for considered BVPs. General results were applied to the delay differential equations.

In Section 5 we formulated statements concerning the Fredholm theory and well-posedness of the singular Dirichlet problem. In particular, we found conditions, which provided that Fredholm's third theorem remains true for the considered singular problem. We also established optimal (in certain sense) conditions guaranteeing well-posedness of the studied problem.

The natural and interesting direction of a future research is to study more general second order DDEs than those introduced in Section 3.2. It would be useful to investigate the asymptotic properties of the three-term DDE

$$u''(t) + u'(t)q(t) + F(u(t), u(\tau(t))) = 0 \tag{6.1}$$

because this equation appears in mathematical models in control engineering, hereditary phenomena in physics, machine tool vibrations, etc. Specially, modelling of the milling process can lead to equation (6.1), where the additional term $u'(t)q(t)$ describes some damping in the process.

There is also a possibility for further investigation of the two-dimensional system established in Section 3.4. We assumed there that the coefficient g is not integrable on $[0, +\infty[$. One can find that there is very few results in the existing literature in the contrary case, i.e., if the coefficient g is integrable. Consequently, the asymptotic properties (oscillation and non-oscillation) of the considered system can be completed in this sense.

Another direction of a research is continuation in the study of the singular Dirichlet problem. Quite complete theory of singular boundary problems for ordinary differential equations is built up, but many interesting problems are not covered there, e.g., the Dirichlet problem for the Bessel equations. It is usual to study the linear part of the theory first, which includes Fredholm's alternative, well-posedness, and the eigenvalue problem. The first two were investigated in Section 5, so it is natural to continue with studying of the eigenvalue problem, which is (in the regular case) based on Fredholm's alternative and the continuous dependence on parameters. Obtained results will also be very useful for the investigation of nonlinear singular problems, which arise in many applications (see, for example, the model introduced in Section 2.3).

References

[1] N. V. Azbelev, V. P. Maksimov, L. F. Rakhmatulina: *Introduction to the theory of functional differential equations*, Nauka, Moscow, 1991.

[2] R. Bellman: *Stability theory of differential equations*, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953.

[3] E. Bravyi: *A note on the Fredholm property of boundary value problems for linear functional differential equations*, Mem. Differential Equations Math. Phys. **20** (2000), 133-135.

[4] C. De Coster, P. Habets: *Two-point boundary value problems: lower and upper solutions*, Mathematics in Science and Engineering, vol. 205, Elsevier Science B.V., Amsterdam, 2006.

[5] M. Dosoudilová, A. Lomtatidze, J. Šremr: *Oscillatory properties of solutions to certain two-dimensional systems of non-linear ordinary differential equations*, Nonlinear Analysis **120** (2015), 57-75.

[6] O.Došlý, P. Řehák: *Half-linear Differential Equations*, North-Holland Mathematics Studies, vol. 202, Elsevier Science B.V., Amsterdam, 2005.

[7] T. Erneux: *Applied Delay Differential Equations*, Springer, New York, 2009.

[8] R. Hakl, A. Lomtatidze, B. Půža: *On nonnegative solutions of first order scalar functional differential equations*, Mem. Differential Equations Math. Phys. **23** (2001), 51–84.

[9] R. Hakl, A. Lomtatidze, I. P. Stavroulakis: *On a boundary value problem for scalar functional differential equations*, Abstr. Appl. Anal. **2004** (2004), No. 1, 45–67.

[10] R. Hakl, A. Lomtatidze, J. Šremr: *Some boundary value problems for first order scalar functional differential equations*, Folia Facult. Scien. Natur. Univ. Masar. Brunensis, Brno, 2002.

[11] J. Hale: *Theory of functional differential equations*, Springer-Verlag, New York-Heidelberg-Berlin, 1977.

[12] P. Hartman: *Ordinary differential equations*, John Wiley, New York, 1964.

[13] E. Hille: *Non-oscillation theorems*, Trans. Amer. Math. Soc. **64** (1948), No. 2, 234–252.

[14] T.Chantladze, N. Kandelaki, A. Lomtatidze: *Oscillation and nonoscillation criteria for a second order linear equation*, Georgian Math. J. **6** (1999), 401-414.

[15] T. Chanturia, I. Kiguradze: *Asymptotic properties of solutions of nonautonomous ordinary differential equations*, Mathematics and its Applications (Soviet Series), 89, Kluwer Academic Publishers Group, Dordrecht, 1993.

[16] N. Kandelaki, A. Lomtatidze, D. Ugulava: *On oscillation and nonoscillation of a second order half-linear equation*, Georgian Math. J. **7** (2000, No. 2), 329–346.

[17] I. Kiguradze: *Boundary value problems for systems of ordinary differential equations*, J. Sov. Math. **43** (1988), No. 2, 2259–2339.

[18] I. T. Kiguradze, A. G. Lomtatidze: *On certain boundary–value problems for second–order linear ordinary differential equations with singularities*, J. Math. Anal. Appl. **101** (1984), No. 2, 325–347.

[19] I. Kiguradze, B. Půža: *On boundary value problems for functional differential equations*, Mem. Differential Equations Math. Phys. **12** (1997), 106–113.

[20] I. Kiguradze, B. Půža: *Boundary value problems for systems of linear functional differential equations*, Folia Facul. Sci. Natur. Univ. Masar. Brun., Mathematica 12, Brno: Masaryk University, 2003. .

[21] I. T. Kiguradze, B. L. Shekhter: *Singular boundary value problems for second order ordinary differential equations*, J. Sov. Math. **43** (1988), No. 2, 2340–2417.

[22] V. Kolmanovskii, A. Myshkis: *Introduction to the theory and applications of functional differential equations*, Kluwer Academic Publishers, 1999.

[23] R. Koplatadze: *Oscillation criteria for solutions of second order differential inequalities and equations with retarded argument*, Tr. Inst. Prikl. Mat. Im. I. N. Vekua **17** (1986), 104–121.

[24] A. Lomtatidze: *Oscillation and nonoscillation of Emden-Fowler type equation of second-order*, Arch. Math. (Brno) **32** (1996), No. 3, 181–193.

[25] A. Lomtatidze: *Oscillation and nonoscillation criteria for second-order linear differential equations*, Georgian Math. J. **4** (1997), No. 2, 129–138.

[26] A. Lomtatidze, N. Partsvania: *Oscillation and nonoscillation criteria for two-dimensional system of first order linear ordinary differential equations*, Georgian Math. J. **6** (1999), No. 3, 285–298.

[27] A. Lomtatidze, Z. Opluštil: *On nonnegative solutions of a certain boundary value problem for first order linear functional differential equations*, Electron. J. Qual. Theory Differ. Equ., Proc. 7th Coll. QTDE **2004** (2004), No. 16, 1–21.

[28] A. Lomtatidze, Z. Opluštil: *Fredholm alternative for the second-order singular Dirichlet problem*, Bound. Value Probl. **2014** (2014), No. 13, 1-15.

[29] A. Lomtatidze, Z. Opluštil: *Fredholm's third theorem for second-order singular Dirichlet problem*, Bound. Value Probl. **2014** (2014), No. 59, 1–12.

[30] A. Lomtatidze, Z. Opluštil: *Well-posedness of the second-order linear singular Dirichlet problem*, Georgian Math. J. **22** (2015), No. 3, 409–419.

[31] A. Lomtatidze, Z. Opluštil, J. Šremr: *On a nonlocal boundary value problem for first order linear functional differential equations*, Mem. Differential Equations Math. Phys. **47** (2007), 69–85.

[32] A. Lomtatidze, Z. Opluštil, J. Šremr: *Solvability conditions for a nonlocal boundary value problem for linear functional differential equations*, Fasc. Math. **41** (2009), 81–96.

[33] A. Lomtatidze, Z. Opluštil, J. Šremr: *Nonpositive solutions of one functional differential inequality*, Nonlinear Oscil. **12** (2009), 474–509.

[34] J. D. Mirzov: *Asymptotic properties of solutions of systems of nonlinear nonautonomous ordinary differential equations*, Folia Facul. Sci. Natur. Univ. Masar. Brun., Mathematica 14, Brno: Masaryk University, 2004.

[35] J. D. Mirzov: *On some analogs of Sturm's and Kneser's theorems for nonlinear systems*, J. Math. Anal. Appl. **53** (1976), No. 2, 418–425.

[36] E. Müller-Pfeiffer: *Oscillation criteria of Nehari-type for Schrodinger equation*, Math. Nachr. **96** (1980), 185–194.

[37] A. D. Myshkis: *Linear differential equations with retarded argument*, Izdat. Nauka, Moscow, 1972 (in Russian).

[38] Z. Nehari: *Oscillation criteria for second-order linear differential equations*, Trans. Amer. Math. Soc. **85** (1957), No. 2, 428–445.

[39] Z. Opluštil: *New solvability conditions for a non-local boundary value problem for nonlinear functional differential equations*, Nonlinear Oscillations **11** (2008), No. 3, 384 - 404.

[40] Z. Opluštil: *Solvability of a linear non-local boundary value problem for nonlinear functional differential equations*, Tatra Mountains Mathematical Publications **43** (2009), 189–200.

[41] Z. Opluštil: *Solvability of a nonlocal boundary value problem for linear functional differential equations*, Advances in Difference Equations **2013** (2013), 1-19.

[42] Z. Opluštil, J. Šremr: *On a non-local boundary value problem for linear functional differential equations*, Electron. J. Qual. Theory Differ. Equ. **2009** (2009), No. 36, 1–13.

[43] Z. Opluštil, J. Šremr: *Some oscillation criteria for the second-order linear delay differential equation*, Mathematica Bohemica **136** (2011), No. 2, 195–2046.

[44] Z. Opluštil, J. Šremr: *On oscillations of solutions to second-order linear delay differential equations*, Georgian Math. J. **20** (2013), No. 1, 65–94.

[45] Z. Opluštil, J. Šremr: *Myshkis type oscillation criteria for second-order linear delay differential equations*, Monatshefte fur Mathematik **178** (2015), No. 1, 143–161.

[46] L. Polák: *Oscillation and nonoscillation criteria for two–dimensional systems of linear ordinary differential equations*, Georgian Math. J. **11** (2004), No. 1, 137–154.

[47] K. Pyragas: *Control of chaos vie extended delay feedback*, Phys. Lett. **206** (1996), 323–330.

[48] Š. Schwabik, M. Tvrdý, O. Vejvoda: *Differential and integral equations: boundary value problems and adjoints*, Academia, Praha, 1979.

[49] P.Torres: *Mathematical Models with Singularities*, Atlantis Press, Amsterdam - Paris, 2015.

[50] A. Wintner: *On the non-existence of conjugate points*, Amer. J. Math. **73** (1951), No. 2, 368–380.