Brno University of Technology
Faculty of Civil Engineering
Institute of Mathematics and Descriptive Geometry

Mgr. Irena Hinterleitner, Ph.D.

Speciální zobrazení variet

Special Mappings of Manifolds

KOMENTÁŘ K SOUBORU PUBLIKOVANÝCH PRACÍ
PŘEDLOŽENÝCH K HABILITACI

OBOR

Aplikovaná matematika

CONTENTS

Introduction

The present habilitation thesis contains the set of publications [H1] – [H18].

The history of the topic begins in the 18^{th} and 19^{th} centuries, when G. Monge and C.F. Gauss studied the geometry of surfaces in Euclidean space. These studies were stimulated by the practical application of surfaces in technology and in cartography.

In the second half of the 19^{th} century the theory of surfaces was extended by B. Riemann to n-dimensional spaces. In 1854 in his habilitation thesis he introduced the metric form which generalized Gauss' first quadratic form of surfaces. Spaces with a metric are called Riemannian spaces, they generalize Euclidian geometry in a natural way [12, 22, 27, 28, 67, 70, 71, 89, 102, 108, 111, 118, 148, 155, 156].

The theory of Riemannian spaces and their generalizations found many applications in mechanics and physics, e.g. in theoretical mechanics, electrodynamics and thermodynamics [21, 30, 58, 63, 76, 77, 108].

A. Einstein applied pseudo-Riemannian spaces in the General Theory of Relativity [26]. Groundbreaking work in this field was done by E. Cartan and H. Weyl [12, 23, 29, 34, 63, 68, 72, 76, 89, 102, 105, 108, 118, 146, 151].

Today the application in physics is very wide. Above all it provides the mathematical foundation of General Relativity, more recently it was applied for example in gauge field theory and σ models, popular in string theory [69, 115].

In more detail, in original Riemannian geometry, as it was developed by Gauss, Riemann, Christoffel, Ricci, Bianchi, Levi-Civita, Einstein and Weyl, vectors and tensors are expressed in terms of components in relation to a coordinate system, in modern terms in the so-called "natural"or "holonomic" basis of the tangent bundle of a manifold. Sometimes, however, orthonormal bases are more convenient. When spinors are involved, or in theories of gauge fields, other kinds of bundles than tangent bundles are appropriate.

In the course of their evolution, differential geometric notions were at a certain stage formulated without restriction to holomorphic bases by Cartan, Schouten and others and later generalized by Chevalley, Koszul, Nomizu and others in a coordinate - and basis - free way.

As a detail in this process, general relativity was first formulated in Riemannian space with its Levi-Civita connection, constructed in a natural and unique way from the metric. An important step in the genesis of gauge theories was the separation of the notation of affine connections as independent structures, from the metric. This leads, for example, to a slight generalization of general relativity, the Einstein-Cartan theory, with includes torsion and finds application, when gravity is coupled to fermionic matter. Connections, both in form of metric-derived Levi-Civita connections, and of independent affine connections, as well as curvature, derived from them, play a central role in a large part of the work presented here. Differential geometry on manifolds facilitates the formulation of mappings and can be written in coordinate free form, widely used in contemporary mathematics. In physics, nevertheless, calculations are mostly carried out in local coordinates.

An interesting actual field of differential geometry is the study of diffeomorphisms and automorphisms of different types of geometric structures on smooth manifolds. In geometry the term morphism denotes a mapping between manifolds which preserves some characteristic properties. Important structures in differential geometry are affine and special Riemannian connections, the latter ones expressed by Christoffel symbols. These connections are very important and useful in physics. A generalization of Riemannian geometry is Finsler geometry with Berwald connection [89, 114]. In Finsler geometry the metric depends not only on the position on a manifolds, but also on directions.

These issues have a main meaning in mathematics as well as also in its applications.

The habilitation thesis is devoted to the following problems of differential geometry of (pseudo-) Riemannian manifolds and manifolds with affine connection:

1. Geodesics [H1], [H2],
2. Geodesic mappings [H3 – H8],
3. Equidistant spaces and special mappings [H9 – H12],
4. F-planar and similar mappings [H13 – H18].

The above results were used in the monographs [89, 102], where I am a coauthor. Now I present these topics in detail.

The mathematical apparatus employed here is tensor calculus, which is used for global and local relations on n-dimensional manifolds with affine connection, denoted in the following by A_n, and Riemannian manifolds, denoted by V_n. The signature of the metric of V_n can be indefinite, so under the notion of Riemannian manifolds we understand also pseudo-Riemannian manifolds, irrespectively to the signature of their metric, as for example in the books [27, 28, 87–89, 102, 107, 108, 111, 112, 118, 120, 155, 156].

1 Geodesics

In Riemannian spaces the natural generalization of straight lines are geodesics. This is illustrated by their role in General Relativity: geodesics are trajectories of freely falling particles in curved space-time, replacing the rectilinear motion of free particles in flat space (Euclidian). Today the theory of geodesics has reached the stadium of technical application in GPS, but their physical and mathematical significance is well known since the time of Bernoulli, Euler, Lagrange, and Gauss.

1.1 Variational problem of geodesics

In the paper [H1] I studied generalizations of the variational problem of geodesics in generalized Finsler and (pseudo-) Riemannian manifolds.

In 1696 Johann Bernoulli formulated the brachistochrone problem, this was the first variational problem. The second variational problem was determining the shortest curve on a surface. This problem was solved by Johann Bernoulli in 1698 but it was published in a textbook by L. Euler in 1728.

During this calculation Leonhard Euler developed new methods which have later in 1766 been called the calculus of variations. Afterwards Joseph-Louis Lagrange found results in modern variational calculus: Trajectories of point particles in classical mechanics are derived by variation of the integral over the Lagrange function, which is the difference between the kinetic and the potential energy. Nowadays these methods are still the subject of active research, [72].

Bernoulli solved the problem of the shortest lines on a surface. In contemporary notation the Lagrange function of the corresponding variational problem $I[l] = \int_A^B L(t, x(t), \dot{x}(t))dt$ is $L = \sqrt{|g_{ij}(x)\, \dot{x}^i \dot{x}^j|}$, where $x = x(t)$, $\dot{x} = dx(t)/dt$, $t \in \mathbb{R}$ is a parameter of a curve ℓ, $g_{ij}(x)$ are components of the metric tensor in (pseudo-) Riemannian manifolds.

A possible generalization is Finsler geometry, where the components of the metric tensor g_{ij} depend also on $\dot{x}$, that is $L = \sqrt{|g_{ij}(x, \dot{x})\, \dot{x}^i \dot{x}^j|}$.

Geodesics are often defined as the extremals with respect to

$L = g_{ij}(x,\dot{x})\,\dot{x}^i\dot{x}^j$. In this case L plays a role as a "generalized kinetic energy"and the parameter t is necessarily canonical.

In the paper [H1] I studied the variational problems for functions $L = f(g_{ij}(x,\dot{x})\,\dot{x}^i\dot{x}^j)$, $f' \neq 0$ in (pseudo-) Riemannian and generalized Finslerian spaces. The extremals are geodesics.

1.2 On the existence of pre-geodesic coordinates

Closely associated with geodesics are special coordinates: geodesic, semigeodesic and pre-semigeodesic coordinates. These special coordinates play an important role in calculations.

Geodesic coordinates at a point p and *along a curve* ℓ (*Fermi coordinates*) are characterized by vanishing Christoffel symbols (or components of the affine connection) at the point p and along the curve ℓ, respectively.

Advantages of *semigeodesic coordinates* are known since C.F. Gauss (*Geodätische Parallelkoordinaten*, [71, p. 201]), and *geodesic polar coordinates*: they can be also interpreted as a "limit case" of semigeodesic coordinates: all geodesic coordinate lines $\phi = x^2 = \text{const}$ pass through one point called the pole, corresponding to $r = x^1 = 0$, and lines $r = x^1 = \text{const}$ are geodesic circles (*Geodätische Polarkoordinaten*, [71, pp. 197-204]).

Well known semigeodesic coordinate systems on surfaces and (pseudo-) Riemannian manifolds are generalized in the following way (Mikeš, Vanžurová, Hinterleitner, [102, p. 43]): Coordinates (U,x) in A_n are called *pre-semigeodesic coordinates* if one system of coordinate lines is geodesic and their natural parameter is just the first coordinate.

The following is true (Mikeš, Vanžurová, Hinterleitner, [102, p. 43]): *The conditions* $\Gamma^h_{11}(x) = 0$, $h = 1,\dots,n$, *are satisfied in coordinates* (U,x) *if and only if* (U,x) *is pre-semigeodesic. Here and below* $\Gamma^h_{ij}(x)$, $i = 1,2,\dots,n$, *are components of the connection* ∇ *on* (U,x).

This was observed by Z. Dušek and O. Kowalski [24] who precisely proved the existence of pre-semigeodesic charts in the case when the components of the affine connection are real analytic functions.

It was proved [H2] that *pre-semigeodesic charts exist in the case when the components of the affine connection are twice differentiable functions.*

From the example of the Fermi coordinates we can see that this special system of coordinates plays an important role in physics. Because pre-semigeodesic coordinates on manifolds with affine connection are related with geodesic coordinates which can have a physical meaning, the existence of pre-semigeodesic coordinates under more general condition makes them potentially applicable in more general situations.

2 Geodesic mappings

Already in Beltrami's lifetime geodesic-preserving morphisms were studied – *geodesic mappings*. T. Levi-Civita [77], who laid the foundations of this theory in tensor form, studied it from the point of view of modelling dynamical processes in mechanics. Presently, for example E. Ferapontov [30] and G. Hall and D. Lonie [34–36, 40] continue working on this subject. See also [79, 138].

To the theory of geodesic mappings and transformations were devoted many papers, results are formulated in a large number of research articles and monographs: T. Levi-Civita [77], H. Weyl [152], T. Thomas [139], P.A. Shirokov [116], L.P. Eisenhart [27, 28], A.Z. Petrov [108], N.S. Sinyukov [118, 119], A.S. Solodovnikov [124], A.V. Aminova [4], J. Mikeš [87, 89, 94, 102, 112], etc.

2.1 General dependence of geodesic mappings

In the papers [H5], [H6] and [H8] I studied the general dependence of geodesic mappings of manifolds with affine and projective connection onto (pseudo-) Riemannian manifolds in dependence on the smoothness class of these geometric objects. We presented well known facts, which were proved by H. Weyl [152], T. Thomas [139], L.P. Eisenhart [27, 28], V. Berezovski and J. Mikeš [90], see [27, 28, 87, 94, 102, 105, 112, 118, 156].

In these results no details about the smoothness class of the metric, resp. connection, were stressed. They were formulated "for sufficiently smooth" geometric objects.

We study fundamental equations of geodesic mappings of manifolds with affine and projective connection onto (pseudo-) Riemannian manifolds with respect to the smoothness class of these geometric objects [H8]:

We prove that the natural smoothness class of these problems is preserved.

Similar tasks also were solved for geodesic mappings between (pseudo-) Riemannian manifolds [H5] and [H6].

In [H8] it was proved that an arbitrary manifold with projective connection admits a global geodesic mapping onto a manifold with equiaffine connection. These results in local form were obtained by L.P. Eisenhart [28, p. 105].

From our results follows the validity of the fundamental equation of geodesic mappings onto (pseudo-) Riemannian manifolds, which was obtained by J. Mikeš and V. Berezovski [90].

2.2 Geodesic mappings of Einstein spaces

To geodesic mappings of special manifolds are devoted many papers beginning with E. Beltrami, who studied geodesic mappings of spaces with constant curvature, well known as special cases of Einstein spaces.

Results about geodesic mappings of Einstein spaces until 2006 are summarized in the paper [H3], moreover the paper contains result by A.Z. Petrov [108] and J. Mikeš [81], see [89, 102]. The metrics of Einstein spaces, which admit geodesic mappings, are in the paper by S. Formella and J. Mikeš [31], see [102], [89, pp. 321-326]. The above works were carried out for Einstein spaces $V_n \in C^3$ onto $\bar{V}_n \in C^3$.

In the case $\bar{V}_n \in C^2$ these results were obtained in [64].

From our results (Theorem 7.8 [89, p. 283]) follows the validity of the above results for the case of Einstein spaces $V_n \in C^3$ (besides, for Einstein spaces V_n there exist always coordinates of the real analytic class C^ω, see [20]) and geodesic equivalent spaces $\bar{V}_n \in C^1$. Thus we have $\bar{V}_n \in C^\omega$ and $\bar{V}_n$ is also an Einstein space, see [H6] and [89, p. 320].

2.3 Geodesic mappings of Kähler spaces

Kählerian spaces play an important role in theoretical physics, especially in the theory of σ-models. They are characterized by a symmetric metric tensor and an antisymmetric symplectic form. Geodesic mappings of Kähler spaces were studied by Coburn, Yano, Westlake, Nagano, who proved the non-existence of non-trivial geodesic mappings with further conditions. Their existence for Kähler spaces was found by Mikeš and Starko, see [89, pp. 340–344].

In the paper [H7] fundamental equations of geodesic mappings onto Kähler spaces of the first kind were found. These spaces are generalizations of Riemannian and Kähler spaces in the sense of non-symmetric metrics introduced by Einstein. These results were quoted in [103].

2.4 Geodesic mappings on compact Riemannian manifolds with conditions on the sectional curvature

Many papers about geodesic mappings deal with global problems. A complete overview of these papers is found in [84–87, 93, 102], [89, pp. 345–365].

In [H4] we clarify many results of other authors (N.S. Sinyukov, E.N. Sinyukova [121, 123], S.E. Stepanov [132–134], J. Mikeš and H. Chudá [91, 92]); the results in [H4] are a continuation of the paper [1] by M. Afwat and A. Švec:

A compact Riemannian manifold (M, g) *without boundary of dimension* $n \geq 2$*, where at any point* $x \in M$ *the sectional curvature is non-positive for any two-direction from all the principal orthonormal basis, does not admitt non-trivial geodesic mappings.*

3 Equidistant spaces and special mappings

Geometric properties of Riemannian manifolds are studied with respect to the existence of certain vector fields.

The subject of [H9]-[H12] are selected examples of Riemannian spaces with special symmetry properties, namely equidistant spaces and generalizations thereof, and several kinds of diffeomorphisms which preserve certain geometric structures.

A major part is devoted to mappings between Riemannian spaces of a special kind, so-called equidistant spaces. Equidistant spaces are characterized by the existence of certain vector fields, called concircular (see 2.8). In physics these spaces occur as spatially homogenous and isotropic cosmological models (Friedmann-Robertson-Walker-Lemaître models).

3.1 Concircular mappings and equidistant spaces

Under a geodesic circle we understand a curve for which the first curvature is constant and the second curvature is zero. K. Yano [154] introduced a conformal mapping of (pseudo-) Riemannian spaces which preserves geodesic circles and is called *concircular*, see also H.L. Vries [149]. These mappings are connected with the existence of manifolds with concircular vector fields. In 1925 these vector fields were studied by H.W. Brinkmann [11] besides conformal mappings onto Einstein space. N.S. Sinyukov [117–119] found geometrical properties of spaces which admit a concircular vector field and called them *equidistant spaces*.

In the paper [H11] we show results connected with basic notations under the conditions of minimal differentiability of metrics and geometric objects which define concircular mappings and also concircular vector fields. *We prove that the smoothness class under concircular mappings is preserved.*

In [H12] general dependence of equitorsion concircular tensors on generalized Riemannian spaces are studied. These spaces are generalizations of Riemannian and pseudo-Riemannian manifolds and they were introduced by A. Einstein [26] as possibilities of a generalization of General relativity.

3.2 Special mappings between equidistant spaces

In the paper [H9] we consider special mappings between equidistant spaces in special coordinate systems $ds^2 = a(x^1)\,(dx^1)^2 + b(x^1)\,d\tilde{s}^2$, especially conformal, concircular, affine, geodesic, harmonic, conformally-projective harmonic and equivolume mappings, see also [83, 89, 98, 135, 153].

The composition of conformal and geodesic (projective) mappings in the case when they are harmonic is called *conformally-projective harmonic* [H13]. Finally we consider *equivolume mappings*, which were defined and studied by T.V. Zudina and S.E. Stepanov [160]. The above mentioned mappings are studied in many applications in theoretical physics, see [136].

3.3 On global geodesic mappings of an ellipsoid

In [H10] I considered two aspects of geodesic mappings of ellipsoids. I describe the geodesic deformations in E_3. An interesting property is that on a sphere as a special case of an ellipsoid these transformations act as identity, whereas they act highly nontrivially on general ellipsoids. In the limit of large transformation parameters the transformed surfaces approach a sphere as limiting surface. The second aspect concerns geodesic transformations of the metric on a manifold homeomorphic to the sphere, in accordance with [147], where it is shown by application of a classical theorem by U. Dini [21] that there is (up to a homothety) a one-parameter family of geodesically equivalent metrics on surfaces.

My result can be summarized [H10]: *Rotational ellipsoids admit global nontrivial geodesic deformations under which they remain rotational surfaces. The resulting surfaces are not ellipsoids.*

4 F-planar and similar mappings

The force exerted by an electro-magnetic field, described by the antisymmetric electro-magnetic tensor F, on a particle with electric charge e and mass m on a trajectory $x^i(s)$ in four-dimensional Minkowski space is given by $eF^i{}_k v^k(s)$, where $v^k = \dot{x}^k(s)$ is the tangent vector to $x^i(s)$, such that the equation of motion has the following form

$$m\,\frac{dv^i}{ds} = eF^i{}_k v^k. \tag{4.1}$$

The tangent vector $v^i(s)$ has the meaning of the four-velocity of the particle, parametrized by the particle's proper time s, which is a canonical parameter. The derivative dv^i/ds is the space-time acceleration, the right-hand side is the Lorentz force in four dimensions.

By generalization of the equation (4.1) A.Z. Petrov [109] introduced the notion of *quasigeodesic curves* and *mappings*, which were used for modeling processes in theoretical physics. These notations were defined for 4-dimensional pseudo-Riemannian spaces with Lorentz signature $(+,-,-,-)$.

If the tangent vector is denoted by λ and ∇ denotes the covariant derivative w.r. to the Levi-Civita connection, the equation of this kind of curves reads

$$\nabla_{\lambda(t)}\,\lambda(t) = \varrho_1(t)\,\lambda(t) + \varrho_2(t)\,F\lambda(t), \tag{4.2}$$

where ϱ_1 and ϱ_2 are some functions of the arbitrary parameter t.

Beside this, the tangent vector λ is orthogonal to $F\lambda$. The component of $F\lambda$ in (4.2) is important as "electro-magnetic force" acting on physical particles.

In a further step of generalization Mikeš and Sinyukov [97] introduced the notion of *F-planar curves*, which are given by equation (4.2) on spaces with affine connection of arbitrary dimension. A metric need not necessarily be defined. In analogy to the conditions of geodesic and quasigeodesic mappings, Mikeš and Sinyukov gave the conditions for *F-planar mappings* that map F-planar curves onto F-planar curves.

Much work is spent further on isometric, homothetic and conformal mappings, also on various generalizations of geodesic mappings, among them for example holomorphic-projective, quasi-geodesic, semi-geodesic, F-planar, 4-planar mappings, transformations and deformations. Work on related questions can be found in many monographs, reports and theses: [3, 6–8, 15, 16, 27–29, 41–43, 46, 49, 74, 75, 78, 82, 83, 87–89, 95–97, 102, 106–109, 116, 118–120, 146, 154–156].

An interesting continuation and generalization of these topics is found in the papers [50–53, 55–57] by J. Hrdina, J. Slovák and P. Vašík.

The papers [H12–H18] are dedicated to these problems.

4.1 On F-planar mappings of spaces with affine connections

In the papers [H14-H18] we study F-planar curves and mappings. A mapping is called *F-planar*, if it maps F-planar curves to F-planar curves. Under an F-planar curve we understand a curve the tangent vector λ of which lies in the 2-dimensional distribution spanned by itself and $F\lambda$, where F is a tensor of type $(1,1)$. This condition can be written in the form (4.2), see [97].

In [H14] general properties of F-planar mappings are studied. There we specified the fundamental equation of F-planar mappings.

Results in [H11] were used in [H15] and in our paper [49]. In this paper, as a special case, F_2^ε-planar mappings were introduced and studied. We also proved that PQ^ε-projectivity, see [78, 138], is a special case of F_2-planar mappings [88].

In the paper [H15] infinitesimal F-planar transformations were studied. These papers were quoted frequently [6, 131, 140, 141, 143, 159]. In the paper [6] our results about the fundamental equation, which were derived in the paper [H14], were used.

4.2 Holomorphically-projective mappings

Kähler manifolds K_n are manifolds with a symmetric metric g and, in addition, a covariantly constant tensor F_i^h with the property that $F^2 = -Id$. Such a structure is called *a complex structure* [7, 89, 102, 118, 155] and plays a role in the construction of symplectic structures in quantum field theory and in σ-models [69, 115].

More generally, hyperbolic (or para-) Kähler spaces were characterized by the condition $F^2 = Id$, see [2, 89, 102, 119, 120].

A special case of F-planar mappings are the previously studied holomorphically projective mappings of (pseudo-) Kähler manifolds, see [89, 102].

In [H16,H18] we study fundamental equations of holomorphically projective mappings of (pseudo-) Kähler manifolds with respect to the smoothness class of metrics C^r, $r \geq 1$. We show that holomorphically projective mappings preserve the smoothness class of metrics.

In previous work on this subject by Domashev, Kurbatova, Mikeš, Prvanović, Otsuki, Tashiro etc., see [7,88,89,102,106,118,120,155] no details about the smoothness class of the metric were emphasized. Results were formulated "for sufficiently smooth" geometric objects there.

It [H14] fundamental equations of holomorphically projective mappings for the conditions of minimal differentiability of metrics were found. These clarify results obtained in the case $K_n, \bar{K}_n \in C^3$, see [74,82,120]. These results were used in [93].

4.3 4-planar mappings of quaternionic Kähler manifolds

4-planar and 4-quasiplanar mappings of almost quaternionic spaces have been studied in [75, 95, 96]. These mappings generalize the geodesic, quasigeodesic and holomorphically projective mappings of Riemannian and Kählerian spaces. Almost quaternionic structures were studied by many authors, for example [3, 59, 60]. Generalisations of the above introduced mappings were studied by J. Hrdina and J. Slovák [51, 52, 55], and M. Stanković, Lj. Velimirović [127, 128, 131].

In [H17] I study the general dependence of 4-planar mappings of almost quaternionic manifolds in dependence on the smoothness class of the metric. Some results were obtained by Kurbatova, see [75], without stress on details about the smoothness class of the metric. In [H17] I make this issue more precise.

In [H17] I proved the following theorem: *If $K_n \in C^r$ $(r > 2)$ admits 4-planar mappings onto $\bar{K}_n \in C^2$, then $\bar{K}_n \in C^r$.*

4.4 Conformally-geodesic mappings

In [H14] we study compositions of conformal and geodesic diffeomorphisms, which are at the same time harmonic mappings (conformally-projective harmonic mappings). The equations of conformally-projective harmonic mappings are shown. We obtained the fundamental equations of these mappings in form of a system of differential equations of Cauchy type. Solutions of this system depend on at most $1/2\,(n+1)(n+2)-(n-2)$ independent parameters.

Conformally-projective harmonic diffeomorphisms of equidistant manifolds are shown in [H13].

A continuation of these topics can be found in the papers [15, 16, 91, 93], in which also paper [H14] is cited.

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ONE REMARK ON VARIATIONAL PROPERTIES OF GEODESICS IN PSEUDORIEMANNIAN AND GENERALIZED FINSLER SPACES

JOSEF MIKEŠ[†], IRENA HINTERLEITNER[‡] and ALENA VANŽUROVÁ[†]

[†]*Department of Algebra and Geometry, Faculty of Science, Palacký University 779 00 Olomouc, Czech Republic*

[‡]*Faculty of Mechanical Engineering, Brno University of Technology 616 69 Brno, Czech Republic*

Abstract. A new variational property of geodesics in (pseudo-)Riemannian and Finsler spaces has been found.

1. Introduction

Let us assume an n-dimensional **Finsler space** F_n with local coordinates $x \equiv (x^1, \ldots, x^n)$ on the underlying manifold M_n, and a (positive definite) metric form with local expression

$$\mathrm{d}s^2 = g_{ij}(x, \dot{x})\mathrm{d}x^i\mathrm{d}x^j \, . \tag{1}$$

Here $g_{ij}(x^1, \ldots, x^n, \dot{x}^1, \ldots, \dot{x}^n)$ are components of the metric tensor, and $(x, \dot{x})$ denote adapted local coordinates on the tangent bundle TM, i.e., $(\dot{x}^1, \ldots, \dot{x}^n)$ are coordinates of the "tangent vector" $\dot{x}$ at x. Metric depends on "positions" and "velocities" in general.

In the Finsler space F_n there exists a (fundamental) function $F(x, \dot{x})$ which is homogeneous of the second degree in $\dot{x}^i$ and satisfies

$$g_{ij}(x, \dot{x}) = \frac{\partial^2 F(x, \dot{x})}{\partial \dot{x}^i \partial \dot{x}^j} \, \cdot$$

Particularly, the equality

$$F(x, \dot{x}) = g_{ij}(x, \dot{x})\mathrm{d}x^i\mathrm{d}x^j$$

holds [3]. As it is well known, in the particular case when components of the metric tensor depend only on position coordinates (i.e., are independent of "velocity coordinates" $\dot{x}$) the Finsler space F_n turns out to be a **Riemannian space** V_n.

2. Pseudo-Riemannian and (Generalized) Finslerian Spaces

In what follows, the signature of the (non-degenerate) metric form is supposed to be arbitrary (we no more restrict ourselves onto positive definite metrics only) so that we can write

$$\mathrm{d}s^2 = e g_{ij}(x,\dot{x})\,\mathrm{d}x^i \mathrm{d}x^j, \qquad e = \pm 1 \tag{2}$$

and the sign is determined in such a way that $\mathrm{d}s^2 \geq 0$.

In short, we will call such metrics and spaces **Finslerian metrics** and **Finsler spaces** again, or *Riemannian*, respectively (more usually, they are called pseudo-Riemannian, or semi-Riemannian).

The arc length of a curve γ, given by parametrization $x^i = x^i(t)$, is given in a Finsler or Riemannian space (in our sense) by the integral

$$s = \int_{t_0}^{t_1} \sqrt{e g_{ij}(x(t),\dot{x}(t))\dot{x}^i(t)\dot{x}^j(t)}\,\mathrm{d}t, \qquad \dot{x}^i(t) = \frac{\mathrm{d}x^i(t)}{\mathrm{d}t}\,\cdot \tag{3}$$

It is well known [3], that this integral is stationary in a Finsler space if and only if its extremals are **geodesic curves** determined by the equations

$$\ddot{x}^h + 2G^h(x,\dot{x}) = \varrho(t)\dot{x}^h \tag{4}$$

where $\varrho(t)$ is a function, g^{ij} are components of the matrix inverse to (g_{ij}), and

$$G^h = \frac{1}{2} g^{ij} \left(\frac{\partial^2 F(x,\dot{x})}{\partial \dot{x}^j \partial x^k} \dot{x}^k - \frac{\partial F(x,\dot{x})}{\partial \dot{x}^j} \right)$$

are components of the Berwald connection. Let us emphasize that extremals of the integral of length are independent of reparametrization of geodesics. In Riemannian spaces, [2, 3], the components read

$$G^h = \frac{1}{2} \Gamma^h_{ij}(x)\,\dot{x}^i \dot{x}^j$$

where Γ^h_{ij} are the Christoffels of second type.

Many authors define a **geodesic** in V_n as an extremal curve of the integral

$$I = \int_{t_0}^{t_1} g_{ij}(x)\dot{x}^i \dot{x}^j\,\mathrm{d}t. \tag{5}$$

Extremals of this variational problem are those geodesics which satisfy the equations (4) with $\varrho(t) \equiv 0$.

Analogous situation is in Finsler spaces (in our generalized sense). Extremal curves of the integral (5) are determined together with their parameter, which is used to be called *canonical*. Note that particularly, arc length in V_n or F_n, respectively, is always canonical.

3. Generalized Variational Problem of Geodesics

In a Riemannian or in a Finsler space (in a more general sense explained above) consider the following more general **variational problem**

$$I = \int_{t_0}^{t_1} f(e\, g_{ij}(x, \dot{x})\dot{x}^i \dot{x}^j)\, \mathrm{d}\tau \tag{6}$$

where e takes the values ± 1, and $f(\tau)$ is a differentiable real-valued function (at least of class two) defined on some open domain $D \subset \mathbb{R}$ which is regular on D in the sense that $f'(\tau) \neq 0$ for all $\tau \in D$.

As an immediate consequence of the Euler-Lagrange equations for the Lagrange function $\mathcal{L} = f(e\, g_{ij}\dot{x}^i \dot{x}^j)$, it can be checked that the extremals satisfy the equations

$$\ddot{x}^h + 2G^h(x, \dot{x}) = -\frac{\mathrm{d}}{\mathrm{d}t}(\ln |f'(e g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta)|)\dot{x}^h. \tag{7}$$

We can prove the following theorem.

Theorem 1. *In* (*generalized*) *Finsler or Riemannian spaces, respectively, geodesic lines parameterized by a canonical parameter, which satisfy the condition*

$$e g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = k \in D$$

are extremals of the integral (6).

Theorem 2. *Consider* (*all*) *extremals of the integral* (6) *in a Finsler space* (*or in a Riemannian space, respectively*). *All curves arising under all possible regular reparameterizations of extremal curves belong to extremals, too, if and only if the function f takes the form $f(x) \equiv \alpha\sqrt{x}$ where α is some non-zero constant.*

Theorem 3. *All possible extremals of the integral* (6) *are just those geodesics which figure in Theorem 1 and Theorem 2. More precisely, in the particular case $f(x) \equiv \alpha\sqrt{x}$, $0 \neq \alpha = \mathrm{const}$, they are represented by all unparameterized geodesics (i.e., geodesics under all possible regular reparameterizations), while for all other functions f, satisfying the above assumptions of the problem* (6), *extremals are represented just by canonically parameterized geodesics only.*

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On the existence of pre-semigeodesic coordinates

Irena Hinterleitner, Josef Mikeš

Abstract

In the present paper we consider the problem of the existence of pre-semigeodesic coordinates on manifolds with affine connection. We proved that pre-semigeodesic coordinates exist in the case when the components of the affine connection are twice differentiable functions.

Keywords: Geodesic, pre-semigeodesic coordinates, manifold with affine connection. [1] [2] [3]

1 Introduction

Geodesics are fundamental objects of differential geometry, analogous to straight lines in Euclidean space. A geodesic is a curve whose tangent vectors in all of its points are parallel. Some properties of geodesic lines in mechanics: a point mass without external influences moves on a geodesic line, another example of geodesics is an ideal elastic ribbon without friction between two points on a curved surface [11, 12]. Geodesics are of particular importance in general relativity. Timelike geodesics in general relativity describe the motion of inertial test particles.

Let $A_n = (M, \nabla)$ be an n-dimensional manifold M with affine connection ∇. A curve ℓ in A_n is a *geodesic* when its tangent vector field remains in the tangent distribution of ℓ during parallel transport along the curve or, equivalently if and only if the covariant derivative of its tangent vector, i.e. $\lambda(t) = \dot{\ell}(t)$ is proportional to the tangent vector $\nabla_\lambda \lambda = \rho(t)\,\lambda$, where ϱ is some function of the parameter t of the curve ℓ.

When the parameter t of the geodesic is chosen so that $\varrho(t) \equiv 0$, then this parameter is called *natural* or *affine*. A natural parameter is usually denoted by τ.

With geodesics some special coordinates are closely associated:

geodesic, semigeodesic and pre-semigeodesic coordinates.

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[2]I. Hinterleitner, Dept. of Mathematics, Faculty of Civil Engineering, Brno University of Technology, Žižkova 17, 602 00 Brno, Czech Republic, *e-mail:* hinterleitner.i@fce.vutbr.cz

[3]J. Mikeš, Dept. of Algebra and Geometry, Palacky University, 17. listopadu 12, 77146 Olomouc, Czech Republic, *e-mail:* josef.mikes@upol.cz

Geodesic coordinates at a point p and *along a curve ℓ* (*Fermi coordinates*) are characterized by vanishing Christoffel symbols (or components of the affine connection) at the point p and along the curve ℓ, respectively.

Let us consider a non-isotropic coordinate hypersurface Σ: $x^1 = c$ in (pseudo-) Riemannian space V_n. Let us fix some point $(c, x^2, \ldots, x^n)$ on Σ and construct the geodesic γ passing through the point and tangent to the unit normal of Σ; γ is an x^1-curve, it is parametrized by $\gamma(x^1) = (x^1 + c, x^2, \ldots, x^n)$ and x^1 is the arc length on the geodesic. Coordinates introduced in this way are called *semigeodesic coordinates* in V_n.

It is well known that the metric of V_n in semigeodesic coordinates has the following form: $ds^2 = e\,(dx^1)^2 + g_{ab}(x)\,dx^a\,dx^b,\ a,b > 1,\ e = \pm 1$. On the other hand this coordinate form of the metric is a sufficient condition for the coordinate system to be semigeodesic. In this case for the Christoffel symbols of the second type follows $\Gamma^h_{11} = 0$, $h = 1, \ldots, n$.

Advantages of such coordinates are known since C.F. Gauss (*Geodätische Parallelkoordinaten*, [9, p. 201]),

Geodesic polar coordinates: can be also interpreted as a "limit case" of semigeodesic coordinates: all geodesic coordinate lines $\varphi = x^2 =$ const. pass through one point called the pole, corresponding to $r = x^1 = 0$, and lines $r = x^1 =$ const are geodesic circles (*Geodätische Polarkoordinaten*, [9, pp. 197-204]).

Let $A_n = (M, \nabla)$ be an n-dimensional manifolds M with the affine connection ∇, dimension $n \geq 2$, and let $U \subset M$ be a coordinate neighbourhood at the point $x_0 \in U$. A couple (U, x) is a coordinate map on A_n.

Semigeodesic coordinate systems on surfaces and (pseudo-) Riemannian manifolds are generalized in the following way (Mikeš, Vanžurová, Hinterleitner [14, p. 43]):

Definition 1 Coordinates (U, x) in A_n are called *pre-semigeodesic coordinates* if one system of coordinate lines is geodesic and the coordinate is just the natural parameter.

In a paper by J. Mikeš and A. Vanžurová [15] these coordinates were called *general Fermi coordinates*, and the reconstruction of components of the affine connection in these coordinates is shown, if we known a certain number of components of the curvature tensor.

In [14, p. 43], [15] the following theorems were proved.

Theorem 1 *The conditions* $\Gamma^h_{11}(x) = 0$, $h = 1, \ldots, n$, *are satisfied in* (U, x) *if and only if* (U, x) *is pre-semigeodesic.*

Theorem 2 *The conditions* $\Gamma^h_{11}(x) = 0$, $h = 1, \ldots, n$, *are satisfied in a coordinate map* (U, x) *if and only if the parametrized curves*

$$\ell\colon I \to U,\ \ell(\tau) = (\tau, a_2, \ldots, a_n), \tau \in I,\ a_i \in R,\ i = 2, \ldots, n,$$

are canonically parametrized geodesics of $\nabla_{|U}$, *I is some interval,* a_k *are suitable constants chosen so that* $\ell(I) \subset U$, Γ^h_{ij} *are components of the affine connection* ∇, *the subset* $U \subset M$ *is a coordinate neighbourhood of* $A_n = (M, \nabla)$.

We thought that the existence of this chart is trivial. This problem is obviously more difficult than we supposed. This was observed in [1, 2] where precisely the existence of pre-semigeodesic charts was proved in the case when the components of the affine connection are real analytic functions. In the proof S. Kowalewsky's Theorem [8] was used.

We proved that the pre-semigeodesic charts exist in the case when the components of the affine connection are twice differentiable functions. The following is true

Theorem 3 *For any affine connection determined by $\Gamma^h_{ij}(x) \in C^r(U)$, $r \geq 2$, there exists a local transformation of coordinates determined by $x' = f(x) \in C^r$ such that the connection in the new coordinates (U', x'), $U' \subset U$, satisfies $\Gamma'^h_{11}(x') = 0$, $h = 1, \ldots, n$, i.e. the coordinates (U', x') are pre-semigeodesic and the components $\Gamma'^h_{ij}(x') \in C^{r-2}(U')$.*

The differentiability class r is equal to $0, 1, 2, \ldots, \infty, \omega$, where 0, ∞ and ω denotes continuous, infinitely differentiable, and real analytic functions, respectively.

It therefore follows that the existence of a pre-semigeodesic chart is guaranteed in the case when the components of the affine connection ∇ are twice differentiable. The existence of this chart is not excluded in the case when the components are only continuous.

The components $\Gamma'^h_{ij}(x')$ can have better differentiability than $C^{r-2}(U')$. On the other hand, if the transformation $x' = f(x) \in C^{r^*}$, $2 \leq r^* \leq r$, leads to pre-semigeodesic coordinates (which is possible), then it guarantees that $\Gamma'^h_{ij}(x') \in C^{r^*-2}$.

The affine connection ∇ is defined in general coordinates by n^3 components $\Gamma^h_{ij}(x)$ which are functions of n variables, and ∇ without torsion is defined by $n^2(n+1)/2$ components.

Theorem 3 implies that in pre-semigeodesic coordinates the number of independent functions, which are defined by ∇, is reduced by n functions. It follows that all affine connections ∇ in dimension n depend locally only on $n(n^2-1)$ arbitrary functions of n variables, and all affine connections without torsion depend only on $n(n-1)^2/2$ arbitrary functions of n variables.

A manifold A_n with a symmetric affine connection is called an *equiaffine manifold* if the Ricci tensor is symmetric, or equivalently, in any local coordinates x there exists a function $f(x)$ satisfying [13, 14, 17, 20]:

$$\Gamma^\alpha_{i\alpha} = \partial_i f(x).$$

It is clear to see that for equiaffine connections the number of these functions is reduced by further $(n-1)$ functions.

2 Special coordinates generated by vector fields

Let X be a vector field which is defined in the neighbourhood of the point p on an n-dimensional manifold M_n in the coordinate system $x = (x^1, x^2, \ldots, x^n)$ by the components $\xi^h(x)$; $\xi^h(p) \not\equiv 0$.

It is known, see [3, 4, 13, 14, 16, 18–21], that it is possible to find a coordinate system $x' = (x'^1, \ldots, x'^n)$ such that

$$\xi'^h(x') = \delta_1^h, \tag{1}$$

where δ_i^h is the Kronecker symbol.

The coordinate transformation from x^h to x'^h has the form

$$x'^h = x'^h(x^1, x^2, \ldots, x^n) \tag{2}$$

for which the law of change of the components of contravariant vectors holds:

$$\xi'^h(x') = \xi^\alpha(x) \cdot \partial_\alpha x'^h(x). \tag{3}$$

This task is solved by finding solutions $f(x)$ and $F(x)$ of the linear partial differential equations

$$\xi^\alpha(x) \cdot \partial_\alpha f(x) = 0, \tag{4}$$

and

$$\xi^\alpha(x) \cdot \partial_\alpha F(x) = 1, \tag{5}$$

It is known that equation (4) has $(n-1)$ functionally independent solutions

$$f^2(x), f^3(x), \ldots, f^n(x), \tag{6}$$

which are the first integrals of the system of ordinary differential equations

$$\frac{dx^h(t)}{dt} = \xi^h(x^1(t), x^2(t), \ldots, x^n(t)), \quad h = 1, 2, \ldots, n. \tag{7}$$

Equation (5) is solved in the same way, its solution is denoted by $f^1(x)$.

Then the searched transformation (2) has the following form

$$x'^h = f^h(x). \tag{8}$$

The above solution was found for $\xi^h(x) \in C^1$, see [3, 4, 16, 18, 20, 21].

By detailed analysis, based on the *Theorem of existence of the general solution and integrals* in [5, p. 306], the system of ordinary differential equations (7) has the solutions for $\xi^h(x) \in C^0$, and, the functions $\xi^i(x)/\xi^1(x)$, $i = 2, \ldots, n$, satisfy Lipschitz conditions. In this case there exist $(n-1)$ functional independent integrals $f^i(x) \in C^1$, $i = 2, \ldots, n$, which are solutions of equation (4) in a neighbourhood of the point p.

Moreover, from the differential equation (4) we can see that for $\xi^h(x) \in C^r$, $i = 2, \ldots, n$, there exist integrals $f^i(x) \in C^r$, $i = 2, \ldots, n$. A similar statement holds for equation (5), i.e. $f^1(x) \in C^r$.

From the above follow.

Proposition 1 *If $\xi^h(x) \in C^r$, $r \geq 1$, then there exist functionally independent solutions of* (4) *and* (5)*:*

$$f^h(x) \in C^r, \quad h = 1, 2, \ldots, n.$$

Theorem 4 *Let X be a vector field on M_n such that $X_p \neq 0$ at a point $p \in M$. If $\xi^h(x) \in C^r$, $r \geq 1$, then there is a coordinate system x' near p such that $X = \partial/\partial x'^1$ and the transformation $x' = f(x) \in C^r$.*

Remark 1 The proof for $X \in C^1$ can be given e.g. by means of local flows [16].

Remark 2 It is easy to show examples where $\xi^h(x) \in C^r$, $r \geq 1$, and solutions $f^i(x) \in C^{r+1}$ do not exist.

Remark 3 Finally, we show another approach to the transformation $x'^h = f^h(x)$ of Theorem 4.

Let $\xi'^h(x') \in C^r$ be a vector field in coordinates x' and $x'^h = x'^h(x)$ be the transformation of coordinates $x \mapsto x'$ for which $0 \mapsto 0$. Further we assume that $\xi^h(x) = \delta_1^h$. Then formula (3) has the following form

$$\partial_1 x'^h(x) = \xi'^h(x'(x)). \tag{9}$$

We can look at the partial differential equations (9) as ordinary differential equations in the value x^1 and real parameters $\tilde{x} = (x^2, \ldots, x^n)$:

$$dx'^h(x^1, \tilde{x})/dx^1 = \xi'^h(x'(x^1, \tilde{x})), \tag{10}$$

and we can use the integral form:

$$x'^h(x^1, \tilde{x}) = \varphi^h(\tilde{x}) + \int_0^{x^1} \xi'^h(x'(\tau^1, \tilde{x}))\, d\tau^1, \tag{11}$$

where $\varphi^h(\tilde{x})$ are functions. These functions are initial conditions for the differential equations (10). For these conditions we assume

$$x'^h(0, \tilde{x}) = \varphi^h(\tilde{x}), \quad h = 1, 2, \ldots, n. \tag{12}$$

Evidently, the points $(x^1, \tilde{x})$ belong to a certain neighbourhood of the origin 0.

As it is known [5,6], if φ^h and ξ'^h are continuous, then equations (11) (and also equation (10) with initial conditions (12)) has a solution $x'^h(x^1, \tilde{x})$. For this solution, evidently, exists the partial derivative $\partial_1 x'^h(x)$, unfortunately in general $\partial_i x'^h(x)$, $i = 2, \ldots, n$, can not exist, and in this case $x'^h(x) \notin C^1$.

From properties of integrals and convergence of series of functions with parameters, see [10, p. 300], after differentiation of the integral equations (11) we obtain that

$$\text{if} \quad \xi'^h(x'), \varphi^h(\tilde{x}) \in C^r \ (r \geq 0) \quad \text{then} \quad x'^h(x) \in C^r.$$

Often it can happen that $x'^h(x) \in C^{r+1}$.

We note that for the transformation of coordinates $x'(x)$ the initial functions $\varphi(\tilde{x})$ must satisfy the following conditions $\det \|\partial_i x'^h(0,\tilde{x})\| \neq 0$ where $(0,\tilde{x})$ is in neighbourhood of the origin 0. These conditions might be for example: $\varphi^1(\tilde{x}) = 0$ and $\varphi^i(\tilde{x}) = x^i$, $i > 1$.

This is the correct proof of Theorem 4.

Proposition 2 *The above general transformations $x' = f(x)$ depend on n functions with $(n-1)$ arguments.*

Proof. The general solution f of the homogeneous equation (4) is a functional composition of $(n-1)$ independent solutions of (6): $f = \Phi(f^2, f^3, \dots, f^n)$. The same holds for the solution of equation (5), because a general solution of the non-homogeneous equation (5) is a sum of one solution of (5) and the general solution f of the homogeneous equation (4).

From the above follows that the functions f^i of the transformation (8) can have a lower class of differentiability than C^{m+1}, it depends on the differentiability of the functions Φ.

In the law of the transformation the components of the transformed tensor depend of the components of the tensor T and also on $\partial_i x'^h$ (or $\partial'_i x^h$). From that follows that the introduced coordinate transformation $f\colon x \mapsto x'$ belongs to the class of differentiability C^{r+1} the components of the tensor fields $T(x) \in C^{r^*}$ are transformed by

$$T^{\cdots}_{\cdots}(x) \in C^{r^*} \longmapsto T'^{\cdots}_{\cdots}(x') \in C^{\min\{r^*, r\}}.$$

Because the transformation law of the affine connection (20) contains $\partial_{ij} x'^h$

$$\Gamma^h_{ij}(x) \in C^{r^*} \longmapsto \Gamma'^h_{ij}(x') \in C^{\min\{r^*, r-1\}}.$$

3 Pre-semigeodesic coordinates

Let $A_n = (M, \nabla)$ be an n-dimensional manifold M with affine connection ∇, and let $U \subset M$ be a coordinate neighborhood at the point $x_0 \in U$. (U, x) are coordinate maps on A_n.

It is well known that the curve $\ell\colon\ x^h = x^h(\tau)$ is a *geodesic*, if on it exists a parallel tangent vector. A geodesic ℓ is characterized by the following equation $\nabla_{\lambda(\tau)}\lambda(\tau) = 0$, where $\lambda(\tau) = dx^h(\tau)/d\tau$, ($\tau$ is a natural parameter on ℓ), which we can rewrite in local coordinates

$$\frac{d^2x^h(\tau)}{d\tau^2} + \Gamma^h_{ij}(x(\tau))\ \frac{dx^i(\tau)}{d\tau}\ \frac{dx^j(\tau)}{d\tau} = 0. \tag{13}$$

The coordinates in A_n are called *pre-semigeodesic coordinates* if one system of coordinate lines are geodesics and their natural parameter is just the first coordinate, see Definition 1.

Let the x^1-curves be geodesics $\ell = (\tau, x_0^2, x_0^3, \ldots, x_0^n)$ where τ is a natural parameter. Substituting this parametrization into the equations for geodesics (13) we obtain

$$\Gamma_{11}^h(x) = 0. \tag{14}$$

This condition is necessary and sufficient for a coordinate system to be pre-semigeodesic, see Theorem 2. *Proof* of Theorem 2. Let $\Gamma_{11}^h = 0$ hold for $h = 1, \ldots, n$. Then the local curves with parametrizations $\ell = (\tau, x_0^2, x_0^3, \ldots, x_0^n)$ satisfy

$$d\ell(\tau)/d\tau = (\partial_1)_{\ell(\tau)}, \qquad d^2\ell(\tau)/d\tau^2 = 0, \tag{15}$$

therefore they are solutions to the system (13).

Conversely, if the curves $\ell = (\tau, x_0^2, x_0^3, \ldots, x_0^n)$ are among the solutions to (13), then due to (15), we get $\Gamma_{11}^h = 0$.

Hence the pre-semigeodesic chart is fully characterized by the condition (14) that the curves $x^1 = \tau$, $x^i = \text{const}$, $i = 2, \ldots, n$, are geodesics of the given connection in the coordinate neighbourhood, see Theorem 1. The definition domain U of such a chart is "tubular", a tube along geodesics.

4 On the existence of pre-geodesic charts

We proved Theorem 3 that a pre-semigeodesic chart exists in the case if the components of the connection are twice differentiable.

Evidently, the existence of this chart is not excluded in the case when the components are only continuous.

Let (U, x) be a coordinate system at a point $p \in U \subset M$, and let $\Gamma_{ij}^h(x) \in C^r$, $r \geq 0$, be components of ∇ on (U, x).

In a neighbourhood of p we construct a set of geodesics, which go through the point $x_0 = (x_0^1, x_0^2, \ldots, x_0^n)$ of a hypersurface $\sigma \ni p$ in the direction $\lambda_0(x_0) \neq 0$, which is not tangent to σ.

Let σ and λ_0 be defined in the following way:

$$\sigma : x^1 = \varphi(x_0^2, \ldots, x_0^n),\ x^i = x_0^i,\ i > 1, \quad \text{and} \quad \lambda_0^h = \Lambda^h(x_0^2, \ldots, x_0^n). \tag{16}$$

Then the above considered geodesics are the solutions of the following ODE's

$$\begin{aligned} \frac{dx^h(\tau)}{d\tau} &= \lambda^h(\tau), \\ \frac{d\lambda^h(\tau)}{d\tau} &= -\Gamma_{\alpha\beta}^h(x(\tau))\lambda^\alpha(\tau)\lambda^\beta(\tau) \end{aligned} \tag{17}$$

for the initial conditions

$$\begin{aligned} x^h(0) &= (\varphi(x_0^2, \ldots, x_0^n), x_0^2, \ldots, x_0^n), \\ \lambda^h(0) &= \Lambda^h(x_0^2, \ldots, x_0^n) \end{aligned} \tag{18}$$

for any $(x_0^2, \ldots, x_0^n)$ in the neighbourhood of p.

Remark 4 From (17), (18) and from the theory of ODE's [6,7] follows:
1) If $\Gamma^h_{\alpha\beta}(x)$ are continuous, then by the Peano existence theorem locally exists a solution. 2) If $\Gamma^h_{\alpha\beta}(x)$ satisfy Lipschitz conditions, then by the Picard-Lindelöf theorem this solution is unique.

In the neighbourhood of p we have constructed a vector field $\lambda^h(x) \neq 0$ which is tangent to the considered geodesics.

In addition, by more detailed analysis it can be shown that $\lambda^h(x) \in C^r$ if $\Gamma^h_{ij}(x) \in C^r$ and moreover

$$\varphi(x^2, \dots, x^n) \in C^r \quad \text{and} \quad \Lambda^h(x^2, \dots, x^n) \in C^r.$$

Note that from the decreasing of the degree of differentiability of the functions φ and Λ^h follows the decreasing of the degree of differentiability of $\lambda^h(x)$.

As an example, we can take the initial conditions (18) in the form:

$$x^h(0) = (0, x_0^2, \dots, x_0^n) \quad \text{and} \quad \lambda^h(0) = \delta^h_1. \tag{19}$$

Theorem 4 ensures the existence of a coordinate system x' in which $\lambda'^h(x') = \delta^h_1$. So, this system x' is pre-semigeodesic, according to Theorems 1 and 2, there also exists a transformation $x' = x'(x) \in C^r$.

The components of a connection ∇ satisfy the well-known transformation law [4,13,14,17,20]:

$$\Gamma'^h_{ij}(x') = \left(\Gamma^\gamma_{\alpha\beta}(x(x')) \, \frac{\partial x^\alpha}{\partial x'^i} \, \frac{\partial x^\beta}{\partial x'^j} + \frac{\partial^2 x^\gamma}{\partial x'^i \partial x'^j} \right) \frac{\partial x'^h}{\partial x^\gamma}. \tag{20}$$

Evidently, we can prove:

$$\Gamma^h_{\alpha\beta}(x) \in C^r, C^\infty, C^\omega \quad \longmapsto \quad \Gamma'^h_{\alpha\beta}(x') \in C^{r-2}, C^\infty, C^\omega.$$

Thus Theorem 3 was proved.

Remark 5 Unfortunately, the existence of a solution $\lambda(x) \in C^0$ (if $\Gamma^h_{ij} \in C^0$) does not ensure the existence of a transformation $x' = x'(x) \in C^2$, which leads to the solution of our problem. In this case, the conditions for the transformations of connections are not fulfilled.

Remark 6 We show a short alternative approach of the methods for finding a transformation $x'^h = f^h(x)$, $0 \mapsto 0$, in Theorem 3.

Proof of Theorem 3. From (20) follows formula

$$\frac{\partial^2 x'^h}{\partial x^i \partial x^j} = \Gamma^\alpha_{ij}(x) \, \frac{\partial x'^h}{\partial x^\alpha} - \Gamma'^h_{\alpha\beta}(x'(x)) \, \frac{\partial x'^\alpha}{\partial x^i} \frac{\partial x'^\beta}{\partial x^j}.$$

We substitute from the last formula with $i = j = 1$, to the conditions $\Gamma^h_{11}(x) = 0$ and $x' = f(x)$ and we get

$$\frac{\partial^2 f^h}{\partial x^1 \partial x^1} = -\Gamma'^h_{\alpha\beta}(f(x)) \, \frac{\partial f^\alpha}{\partial x^1} \frac{\partial f^\beta}{\partial x^1}, \quad h = 1, \dots, n. \tag{21}$$

If $x^1 = t$ and if the other coordinates $\tilde{x} = (x^2, \dots, x^n)$ are supposed as parameters the system (21) is a system of ordinary differential equations with respect to the variable t.

Let the initial condition be

$$\begin{aligned} f^h(0,\tilde{x}) &= \varphi_0^h(\tilde{x}) \\ \frac{\partial f^h}{\partial x^1}\,(0,\tilde{x}) &= \varphi_1^h(\tilde{x}). \end{aligned} \tag{22}$$

To equations (21) and (22) Remark 4 applies.

In addition, for the transformation $x' = f(x)$ to be regular, it is necessary that the Jacobi matrix at a point $(0, \tilde{x})$ is regular. Then it is regular in some neighborhood of the origin 0. An example of suitable initial conditions ($\in C^\omega$) are

$$\varphi_0^1(\tilde{x}) = 0, \quad \varphi_0^h(\tilde{x}) = x^h, \ h > 1, \quad \varphi_1^h(\tilde{x}) = 1, \quad \varphi_1^h(\tilde{x}) = 0.$$

Unfortunately, from the existence of a solution does not necessary follows the existence of a transformation, which would lead to the solution of our problem.

The solution $f^h(x)$ may not be generally differentiable variables $x^2, \dots, x^n$. In order to realize the transformation of the components of the connection (20) it is necessary that the second derivative of $f^h(x)$ according to the variables $x^2, \dots, x^n$ exists.

It is known we can find the solution of (21) with initial conditions (22) by a method of successive iterations [6]:

$$\begin{aligned} f_{\sigma+1}^h(x^1,\tilde{x}) &= \varphi_0^h(\tilde{x}) + \int_0^{x_1} \lambda_\sigma^h(t,\tilde{x})\,dt\ , \\ \lambda_{\sigma+1}^h(x^1,\tilde{x}) &= \varphi_1^h(\tilde{x}) + \int_0^{x_1} \Gamma'^h_{\alpha\beta}(f_\sigma^i(t,\tilde{x}))\ \lambda_\sigma^\alpha(t,\tilde{x})\lambda_\sigma^\beta(t,\tilde{x})\,dt. \end{aligned} \tag{23}$$

In the neighbourhood of the point $(0, x^2, \dots, x^n)$ the iterations $f_{\sigma+1}^h(x^1, \tilde{x})$ and $\lambda_{\sigma+1}^h(x^1, \tilde{x})$ uniformly converge to the solutions $f^h(x)$ and $\lambda^h(x)$.

From the properties of the derivative of the integral of the parametric functions, see [10, p. 300], it follows that the first derivative of solution $f^h(x)$ exists, if $\Gamma'^h_{ij}(x'), \varphi_0^h(\tilde{x}), \varphi_1^h(\tilde{x}) \in C^1$. If we take $f_0^h = \lambda_0^h = 0$, then each successive iteration $f_\sigma^h, \lambda_\sigma^h$ will belong to the class C^1. Because iteration is uniformly convergent, and based on the above properties, the limits $f_\sigma^h \mapsto f^h$ and $\lambda_\sigma^h \mapsto \lambda^h$ also belong to class C^1.

Analogically, the solution $f^h(x) \in C^r$ exists, if $\Gamma'^h_{ij}(x'), \varphi_0^h(\tilde{x}), \varphi_1^h(\tilde{x}) \in C^r$.

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On the Theory of Geodesic Mappings of Einstein Spaces and their Generalizations

Josef Mikeš*, Irena Hinterleitner† and Vladimir A. Kiosak**

*Dept. of Algebra and Geometry, Fac. Sci., Palacky Univ., Tomkova 40, 779 00 Olomouc, Czech Rep.
†Dept. of Math., FSI VUT Brno, Strojnická 10, 616 69 Brno, Czech Rep.
**Dept. of Math., Ushinskogo 11/2, Odessa, Ukraine.

Abstract. In this paper we consider results of the theory of geodesic mappings of Einstein spaces and their generalizations.

In 1925 H. Brinkmann found the metric of equidistant spaces and obtained conditions, when these spaces are Einstein spaces, resp. spaces of constant curvature. We introduce the conditions on these spaces when they are semisymmetric, pseudosymmetric, Ricci semisymmetric, Ricci pseudosymmetric and spaces $V_n(B)$.

A diffeomorphism f between Riemannian spaces V_n and $\bar{V}_n$ is called a *geodesic mapping*, if any geodesic line in V_n is mapped into a geodesic line in $\bar{V}_n$. In 1954 N.S. Sinyukov proved that equidistant spaces admit geodesic mappings. Our constructions of a geodesic mapping of Einstein spaces with the Brinkmann metric proves that Petrov's conjecture is not true.

We formulate results by E. Beltrami, R. Couty, V.I. Golikov, S. Formella, V.A. Kiosak, T. Levi-Civita, J. Mikeš, A.Z. Petrov and A.V. Pogorelov about geodesic mappings of Einstein spaces and spaces of constant curvature.

Further we introduce results on geodesic mappings for Riemannian spaces, which are generalized Einstein spaces and spaces of constant curvature. For instance symmetric, recurrent, generalized recurrent, semisymmetric, pseudosymmetric, Ricci semisymmetric, Ricci pseudosymmetric spaces, spaces with harmonic curvature, etc. These results were obtained by many authors: R. Deszcz, V.A. Kiosak, J. Mikeš, N.S. Sinykov, E.N. Sinyukova, V.S. Sobchuk, etc.

Keywords: Einstein spaces, spaces of constant curvature, symmetric spaces, recurrent spaces, semisymmetric spaces, pseudosymmetric spaces, Ricci semisymmetric spaces, Ricci pseudosymmetric spaces, space with harmonic curvature, geodesic mappings, projective transformations.
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INTRODUCTION

Study of diffeomorphisms and automorphisms of geometrically generalized spaces constitute one of the current main directions in differential geometry. A large number of papers is devoted to conformal, geodesic, quasigeodesic, almost geodesic, holomorphically projective and other mappings. One line of thought is now the most important one, namely, the investigation of special affine-connected, Riemannian, Kählerian and Hermitian spaces (see [13, 31, 32, 38, 39, 41, 43, 47, 57]).

This paper is a survey of some recent results concerning **geodesic mappings of Einstein spaces** $\mathscr{E}_n$ and their generalizations.

If not oterwise stated, expressions in the present review are given locally in tensor form in the class of real sufficiently smooth functions. All the spaces are assumed to be connected. Let us present the basic notions of the theory of n-dimensional Riemannian spaces V_n, using the notations of [13, 31, 38, 39, 41, 43, 47].

The Riemannian space V_n, endowed with a local coordinate system $x = (x^1, x^2, \dots, x^n)$, is characterized by the regular symmetric metric tensor $g_{ij}(x)$. The signature of the metric form $ds^2 = g_{ij}(x)dx^i dx^j$ is assumed, in general, to be arbitrary. The space V_n belongs to the class C^r ($V_n \in C^r$) if $g_{ij}(x) \in C^r$.

In the Riemannian space V_n, endowed with the metric tensor $g_{ij}(x)$, are considered the Christoffel symbols of type I and II $\Gamma_{ijk} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$ and $\Gamma^h_{ij} = g^{h\alpha}\Gamma_{ij\alpha}$, respectively, and the Riemannian, Ricci and Weyl (of the projective curvature) tensors are defined as follows:

$$R^h_{ijk} = \partial_j \Gamma^h_{ki} + \Gamma^\alpha_{ki}\Gamma^h_{j\alpha} - \partial_k \Gamma^h_{ji} + \Gamma^\alpha_{ji}\Gamma^h_{k\alpha}, \qquad R_{ij} = R^\alpha_{ij\alpha}, \qquad W^h_{ijk} = R^h_{ijk} - \frac{1}{n-1}\,(\delta^h_k R_{ij} - \delta^h_j R_{ik}),$$

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where δ_i^h is the Kronecker symbol, $\partial_i = \partial/\partial x^i$ and g^{ij} are elements of the inverse matrix to g_{ij}. In V_n is considered the scalar curvature $R = R_{\alpha\beta} g^{\alpha\beta}$. Using g_{ij} and g^{ij}, we introduce in V_n the operations of lowering and raising indices, for example: $R_{hijk} = g_{h\alpha} R^{\alpha}_{ijk}$, $R^{h}{}_{ij}{}^{k}{}_{\cdot} = g^{k\alpha} R^{h}_{ij\alpha}$, $R^h_i = g^{h\alpha} R_{\alpha i}$.

EINSTEIN SPACES AND THEIR GENERALIZATIONS

As we know, Einstein spaces $\mathscr{E}_n$ are a special case of Riemannian spaces. These spaces were introduced by A. Einstein as model time-spaces [12, 13, 39, 41, 47, 57].

A Riemannian space V_n is called *Einstein space* $\mathscr{E}_n$, when its Ricci tensor is proportional to the metric tensor, i.e.

$$R_{ij} = \rho\, g_{ij},$$

where $\rho = \dfrac{R}{n}$. If $n > 2$, then R is constant.

Well-known *spaces of constant curvature* K (denoted by $\mathscr{S}_n$) are special Einstein spaces, where $K = \dfrac{-R}{n(n-1)}$. These spaces are characterized by the following conditions on the Riemannian tensor:

$$R_{hijk} = K\,(g_{hj} g_{ik} - g_{hk} g_{ij}).$$

It is known that all 3-dimensional Einstein spaces $\mathscr{E}_3$ are spaces of constant curvature. For dimensions $n > 3$ this is not true.

Spaces of constant curvature and Einstein spaces are generalized by the following special Riemannian spaces (see [4, 17, 18, 31, 41, 47, 50, 57])

symmetric spaces (S_n^1)	–	$R^h_{ijk,l} = 0$,
recurrent spaces (K_n^1)	–	$R^h_{ijk,l} = \varphi_l\, R^h_{ijk}$,
Ricci-symmetric spaces ($RicS_n^1$)	–	$R_{ij,l} = 0$,
Ricci-recurrent spaces ($RicK_n^1$)	–	$R_{ij,l} = \varphi_l\, R_{ij}$,
V_n with harmonic curvature (H_n)	–	$R^{\alpha}_{ijk,\alpha} = 0$ ($\Leftrightarrow$ $R_{ij,k} = R_{ik,j}$),
spaces L_n	–	$R_{ij,k} = a_k g_{ij} + b_i g_{jk} + b_j g_{ik}$.

Hereafter "," denotes the covariant derivative with respect to the connection of the space V_n and φ_l, a_k, b_i are nonvanishing covectors.

By using the tensor

$$Z^h_{ijk} = R^h_{ijk} - B(\delta^h_k g_{ij} - \delta^h_j g_{ik}),$$

where B is a function, let us define, for every tensor field T of the type $\binom{p}{q}$ in V_n, a tensor operation $\langle lm\rangle$ in the following way [24, 31]:

$$T^{h_1\ldots h_p}_{i_1\ldots i_q\langle lm\rangle} \equiv \sum_{r=1}^{q} T^{h_1\ldots h_p}_{i_1\ldots i_{r-1}\alpha\, i_{r+1}\ldots i_q} Z^{\alpha}_{i_r lm} - \sum_{r=1}^{p} T^{h_1\ldots h_{r-1}\alpha\, h_{r+1}\ldots h_p}_{i_1\ldots i_q} Z^{h_r}_{\alpha lm}.$$

Generalizations of the above mentioned spaces S_n^1 and $RicS_n^1$ are

semisymmetric space (Ps_n)	–	$R^h_{ijk\langle lm\rangle} = 0, \quad B = 0$,
pseudosymmetric space ($Ps_n(B)$)	–	$R^h_{ijk\langle lm\rangle} = 0$,
Ricci-semisymmetric space ($RicPs_n$)	–	$R_{ij\langle lm\rangle} = 0, \quad B = 0$,
Ricci-pseudosymmetric space ($RicPs_n(B)$)	–	$R_{ij\langle lm\rangle} = 0$.

Semisymmetric spaces were first considered in 1920 by P. A. Shirokov, E. Cartan, and A. Lichnerowicz when studying symmetric spaces. The name *semisymmetric spaces* was, however, introduced by N.S. Sinyukov (see [4, 45, 47]). He (see [31, 45, 47]) started to study semisymmetric spaces Ps_n and their geodesic mappings. Research on this subject was continued by J. Mikeš [20, 21, 22, 24, 30, 31, 47] and P. Venzi [54]. Many investigations have been devoted to the study of these spaces; comprehensive reviews of this problem are given by V. R. Kaigorodov [17, 18] and E. Boeckx, O. Kowalski, L. Vanhecke [4].

The research of pseudosymmetric and Ricci-pseudosymmetric spaces has been started by J. Mikeš (see [20, 21, 24, 30, 31, 36, 47]). These spaces were studied further by F. Defeverer, R. Deszcz, W. Grycak, M. Hotlos etc. [7, 8, 9, 10, 11].

GEODESIC MAPPINGS

A diffeomorphism $f\colon V_n \to \bar{V}_n$ is called a *geodesic mapping*, if any geodesic line in V_n maps into a geodesic line in $\bar{V}_n$. Beginning with E. Beltrami [2, 3] much effort was dedicated to these mappings, see [13, 19, 31, 41, 43, 47].

Consider a concrete mapping $f\colon V_n \to \bar{V}_n$, both spaces being considered with the general coordinate system x with respect to this mapping. This is a coordinate system where two corresponding points $M \in V_n$ and $f(M) \in \bar{V}_n$ have equal coordinates $x = (x^1, x^2, \dots, x^n)$; the corresponding geometric objects in $\bar{V}_n$ will be marked with a bar. For example, Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ are components of the affine connection on V_n and $\bar{V}_n$, respectively.

The Riemannian space V_n admits a geodesic mapping f onto the Riemannian space $\bar{V}_n$ if and only if in the common coordinate system x the following conditions hold

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \psi_i \delta^h_j + \psi_j \delta^h_i, \tag{1}$$

where $\psi_i(x)$ is a gradient, i.e. there is a function $\psi(x)$ for $\psi_i(x) = \partial \psi(x)/\partial x^i$.

If $\psi_i \not\equiv 0$, then a geodesic mapping is called *nontrivial*; otherwise it is said to be *trivial* or *affine*.

Given a geodesic mapping the following conditions hold:

$$\bar{R}^h_{ijk} = R^h_{ijk} + \delta^h_k \psi_{ij} - \delta^h_j \psi_{ik}; \quad \bar{R}_{ij} = R_{ij} + (n-1)\psi_{ij}; \quad \bar{W}^h_{ijk} = W^h_{ijk},$$

where $\psi_{ij} = \psi_{i,j} - \psi_i \psi_j$. The Weyl tensor of the projective curvature, W^h_{ijk}, is an invariant object of the geodesic mapping.

Condition (1) is equivalent to

$$\bar{g}_{ij,k} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi_j \bar{g}_{ik}, \tag{2}$$

where $\bar{g}_{ij}$ is the metric tensor of $\bar{V}_n$. Conditions (1) and (2) are called the *Levi-Civita equations*.

A Riemannian space V_n admits geodesic mappings onto $\bar{V}_n$ if and only if in V_n the linear differential equations in covariant derivatives (*Sinyukov's equations*)

$$a_{ij,k} = \lambda_i a_{jk} + \lambda_j a_{ik}, \tag{3}$$

have a solution with respect to the unknown regular symmetric tensor a_{ij} and the gradient vector λ_i; $\lambda_i \neq 0$ if and only if $\psi_i \neq 0$ [47]. The metric tensor $\bar{g}_{ij}$ of $\bar{V}_n$ and solutions of (3) are connected by the relations

$$a_{ij} = \mathrm{e}^{-2\psi} \bar{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}, \qquad \lambda_i = -\mathrm{e}^{-2\psi} \bar{g}^{\alpha\beta} \psi_\alpha g_{\beta i}, \qquad \|\bar{g}^{ij}\| = \|\bar{g}_{ij}\|^{-1}.$$

We shall denote a Riemannian space V_n by *space* $V_n(B)$, if it admits a nontrivial geodesic mapping with

$$\lambda_{i,j} = \mu\, g_{ij} + B\, a_{ij}, \tag{4}$$

where μ and B are some functions. Formulas (4) are equivalent to

$$\psi_{ij} = \bar{B}\, \bar{g}_{ij} - B\, g_{ij},$$

where $\bar{B}$ is a function [21, 24, 31].

From this it follows that a space $V_n(B)$ maps geodesically only on spaces $\bar{V}_n(\bar{B})$, moreover $B = \text{const} \Leftrightarrow \bar{B} = \text{const}$. If $B = \text{const}$, then $\mu_{,i} = 2B\lambda_i$, and if $B \equiv 0$, then $\mu \equiv \text{const}$. The spaces $V_n(B)$, $B = \text{const} \neq 0$, admit nontrivial projective transformations and the vector λ_i is not isotropic.

Under geodesic mappings from $V_n(B)$ onto $\bar{V}_n(\bar{B})$ the tensors Z^h_{ijk} and Z_{ij} are invariant:

$$\bar{Z}^h_{ijk} = Z^h_{ijk} \quad \text{and} \quad \bar{Z}_{ij} = Z_{ij},$$

where $Z_{ij} = Z^\alpha_{ij\alpha} \equiv R_{ij} - B(n-1)g_{ij}$.

The spaces $V_n(B)$ naturally generalize the space $V(K)$, introduced by A.S. Solodovnikov [16, 52].

EQUIDISTANT SPACES

A vector field ξ^h is called *concircular*, if $\xi^h_{,i} = \rho\delta^h_i$, where ρ is a function. If $\rho = \text{const}$, ξ^h is *convergent*. A space V_n with a concircular vector field is called *equidistant*, see [31, 46, 47, 56].

Equidistant spaces V_n, where the concircular vector fields are nonisotropic, can be endowed with a system of coordinates x, where the metric is of the form

$$ds^2 = \frac{1}{f(x^1)} dx^{1\,2} + f(x^1)\, d\tilde{s}^2, \tag{5}$$

where $f \in C^1$ $(f \neq 0)$ is a function, $d\tilde{s}^2 = \tilde{g}_{ab}(x^2, \dots, x^n)\, dx^a dx^b$ $(a, b = 2, \dots, n)$ is the metric form of a certain Riemannian space $\tilde{V}_{n-1}$ (see [5, 41, 47, 56]).

H.K. Brinkmann [5] showed that the space V_n with metric (5) is an Einstein space $\mathscr{E}_n$ (resp. $\mathscr{S}_n$) if and only if

$$f = Kx^{1\,2} + 2ax^1 + b, \tag{6}$$

where K, a, b are constants and $d\tilde{s}^2$ is a metric of an Einstein space $\tilde{\mathscr{E}}_{n-1}$ (resp. $\tilde{\mathscr{S}}_{n-1}$), moreover $K = \dfrac{-R}{n(n-1)}$, $\tilde{K} = \dfrac{-\tilde{R}}{(n-1)(n-2)} = bK^2 - a^2$, R and $\tilde{R}$ are the scalar curvatures of $\mathscr{E}_n$ and $\tilde{\mathscr{E}}_{n-1}$ (resp. $\mathscr{S}_n$ and $\tilde{\mathscr{S}}_{n-1}$).

An equidistant space V_n with metric (5) admits geodesic mappings onto the Riemannian space $\bar{V}_n$, whose metric form is

$$d\bar{s}^2 = \frac{p}{f\cdot(1+qf)^2} dx^{1\,2} + \frac{pf}{1+qf} d\tilde{s}^2, \tag{7}$$

where p, q are some constants such that $1 + qf \not\equiv 0$, $p \not\equiv 0$. If $qf' \not\equiv 0$, the mapping is nontrivial; otherwise it is trivial; here x are common coordinates for V_n and $\bar{V}_n$ [46].

If $f \neq \text{const}$ the space V_n is a space $V_n(\mathrm{B})$. A space V_n with (5) is $V_n(\mathrm{B})$, $B = \text{const}$, if and only if $f = Bx^{1\,2} + ax^1 + b$, where B, a, b are constants, $f' \neq 0$.

It can be shown that for all spaces $\mathscr{S}_n$ with constant curvature K there exists always the above mentioned coordinate system, in which the metric has the form (5). As we have said above, all $\mathscr{E}_3$ have constant curvature, moreover Einstein spaces $\mathscr{E}_4$ with metric (5) also have constant curvature [5, 41]. For Einstein spaces $\mathscr{E}_n$ $(n > 4)$ this is not the case in general. It is obvious that Einstein spaces $\mathscr{E}_n$ with a metric (5) admit nontrivial geodesic mappings.

In many papers these mentioned problems were studied in a semigeodesic coordinate system x, in which the equidistant spaces V_n have a metric tensor in the following form $ds^2 = \pm dx^{1\,2} + f(x^1)\, d\tilde{s}^2$, see [24, 31, 33, 35, 47].

GEODESIC MAPPINGS OF SPACES WITH CONSTANT CURVATURE

First let us consider geodesic mappings from spaces of constant curvature ($\mathscr{S}_n$), which are a special case of $\mathscr{E}_n$ and which, in 1865 [2, 3], were the initial objects with which the history of geodesic mappings began.

A theorem by E. Beltrami in modern formulation states that *a Riemannian space V_n, admitting a geodesic mapping onto a Euclidean space, is a space with constant curvature.* The proofs of this theorem (see [13, 19, 43, 47]) are given under the condition $V_n \in C^2$, i.e. $g_{ij}(x) \in C^2$.

There exists a more general theorem:

Theorem 1 (A.V. Pogorelov [42]) *Let in the Euclidean space a Riemannian metric be given by the linear element $ds^2 = g_{ij}(x)dx^i dx^j$, $g_{ij}(x) \in C^0$ in cartesian coordinates. Let the geodesic lines of the space with this metric be straight lines (segments of straight lines). Then this space has constant curvature.*

Locally it holds that between two spaces $\mathscr{S}_n$ and $\bar{\mathscr{S}}_n$ with constant curvature K and $\bar{K}$, respectively, there exists a nontrivial geodesic mapping, where for the tensor ψ_{ij} the formula $\psi_{ij} = Kg_{ij} - \bar{K}\bar{g}_{ij}$ holds [43]. Therefore an arbitrary space $\mathscr{S}_n$ with constant curvature K is a special case of a space $V_n(K)$.

It is proved, for the n-dimensional sphere S_n, that it admits global nontrivial projective transformations and nontrivial geodesic mappings [26]. By applying the global Γ-*transformation* (its local application is considered in ([47], p. 127]) to two spheres S_n and $\overline{S}_n$ that are in a nontrivial global geodesic correspondence, an infinite series of compact orientable properly Riemannian spaces with nonconstant curvature can be obtained, including some spaces L_n.

On the other hand, compact flat spaces do not admit global nontrivial geodesic mappings. Compact properly Riemannian spaces with constant negative curvature also do not admit global nontrivial geodesic mappings [48].

There are no global geodesic mappings between compact spaces with constant curvature and with different signatures of metrics. The above-mentioned property is proved when one of the spaces is properly Riemannian.

GEODESIC MAPPINGS OF EINSTEIN SPACES

The studies of geodesic mappings of Lorentzian four-dimensional Einstein spaces were initiated in 1961 by A.Z. Petrov [40], see [41]. A space V_n is called *Lorentzian*, if it has a metric with Minkowski signature. The following holds

Theorem 2 (V.I. Golikov and A.Z. Petrov, see [41]) *Lorentzian four-dimensional Einstein spaces with nonconstant curvature do not admit nontrivial geodesic mappings onto Lorentzian Riemannian spaces.*

These investigations are completed by the following:

Theorem 3 (J. Mikeš and V.A. Kiosak [33]) *Four-dimensional Einstein spaces with nonconstant curvature do not admit nontrivial geodesic mappings to Riemannian spaces.*

From this is follows that $\mathscr{E}_4$ with nonconstant curvature are characterized among Riemannian spaces by the position of their geodesic curves.

P. Venzi [55] proved that a properly Riemannian $\mathscr{E}_n$ admits geodesic mappings only onto an $\bar{\mathscr{E}}_n$. There exists a more general theorem generalizing the theorem by E. Beltrami:

Theorem 4 (J. Mikeš [23]) *If the Einstein space $\mathscr{E}_n$ admits a nontrivial geodesic mapping onto the Riemannian space $\bar{V}_n$, then $\bar{V}_n$ is an Einstein space.*

Einstein spaces $\mathscr{E}_n$ admitting nontrivial geodesic mappings are the spaces $V_n(\frac{R}{n(n-1)})$, and they always admit projective transformations (when $R \not\equiv 0$, they admit nontrivial projective transformations).

R. Couty [6] proved that under additional conditions compact $\mathscr{E}_n$ do not admit nontrivial projective transformations. Compact Ricci-flat spaces $V_n \in C^3$ do not admit global nontrivial geodesic mappings [25]. Geodesic mappings from a compact equiaffine Ricci-flat space onto an equiaffine Ricci-flat space are trivial. Hence compact equiaffine Ricci-flat spaces do not admit nontrivial projective transformations.

One cannot set global nontrivial geodesic mappings between compact Einstein spaces V_n and $\bar{V}_n$ with different signatures of metrics.

Geodesic mappings of Einstein spaces were investigated by S. Formella [14], and for Einstein-Finsler spaces by Z. Shen [44].

A.Z. PETROV'S CONJECTURE ON GEODESIC MAPPINGS OF EINSTEIN SPACES

A.Z. Petrov extended methods of studying geodesic mappings of four-dimensional Lorentzian-Einstein spaces to Einstein spaces of higher dimensions $n > 4$, and conjectured that *the Lorentzian-Einstein spaces $\mathscr{E}_n$ $(n > 4)$ which are distinct from the spaces of constant curvature, do not admit nontrivial geodesic mappings onto Lorentzian-Einstein spaces* ([41], pp. 355, 461).

Let us construct a counterexample to A.Z. Petrov's conjecture (see [33]).

Let $\mathscr{E}_n$ $(n > 4)$ be an equidistant Einstein space of nonconstant curvature with Brinkmann metric (5), satisfying condition (6). It is known that the space $\mathscr{E}_n$ with a coordinate system (5) admits a geodesic mapping onto the Einstein space $\bar{\mathscr{E}}_n$ with metric (7). If $qf' \neq 0$, the mapping is nontrivial. The coordinates x are common to this mapping. The signatures of the metrics of $\mathscr{E}_n$ and $\bar{\mathscr{E}}_n$ are different if $1 + qf < 0$, otherwise they coincide.

One can easily see that, under an appropriate choice of the constant q, it is possible to construct an example of a nontrivial geodesic mapping between Einstein spaces with Minkowski signature which have nonconstant curvatures and whose dimensions are greather than four. This provides a counterexample to the reduced Petrov conjecture.

GEODESIC MAPPINGS OF SEMISYMMETRIC SPACES AND THEIR GENERALIZATIONS

As we have said before spaces of constant curvature, Einstein spaces and spaces $V_n(B)$ form closed classes of geodesic mappings of Riemannian spaces.

A similar property was shown for pseudosymmetric ($Ps_n(B)$) and Ricci-pseudosymmetric ($RicPs_n(B)$) spaces:

Theorem 5 (J. Mikeš [21, 24]) *If a pseudosymmetric space $Ps_n(B)$ (resp. a Ricci pseudosymmetric space $RicPs_n(B)$) admits geodesic mappings onto $\bar{V}_n$, then the space $\bar{V}_n$ is a pseudosymmetric space $\bar{P}s_n(\bar{B})$ (resp. a Ricci pseudosymmetric space $Ric\bar{P}s_n(\bar{B})$), and $B, \bar{B}$ are constants.*

It can be shown that a Riemannian space V_n with metric (5) is a space $Ps_n(B)$ (resp. $RicPs_n(B)$), $B = \text{const}$, if $f = B{x^1}^2 + 2ax^1 + b$, B, a, b are constants, and $d\tilde{s}^2$ is a metric of space $\tilde{P}s_n(\tilde{B})$ (resp. $Ric\tilde{P}s_n(\tilde{B})$), and $\tilde{B} = bB^2 - a^2$.

The above mentioned questions concerning spaces were studied by F. Defeverer, R. Deszcz, W. Grycak, M. Hotlos [7, 8, 9, 10, 11].

Theorem 6 (J. Mikeš [20], see [21, 22, 47], P. Venzi [54]) *If a semisymmetric space Ps_n admits nontrivial geodesic mappings onto $\bar{V}_n$, then $\bar{V}_n$ is an equidistant pseudosymmetric space.*

Theorem 7 (J. Mikeš [22]) *If a Ricci semisymmetric space $RicPs_n \not\equiv \mathscr{E}_n$ admits nontrivial geodesic mappings onto $\bar{V}_n$, then this space is an equidistant space.*

Theorem 8 (J. Mikeš [25]) *Compact semisymmetric Riemannian spaces $Ps_n \in C^4$ with nonconstant curvature (resp. non-Einsteinian Ricci-semisymmetric Riemannian spaces $RicPs_n \in C^4$) do not admit either global nontrivial geodesic mappings or global non-affine projective transformations. onto $\overline{V}_n \in C^3$.*

In the works by E.N. Sinyukova [48], a series of results for global geodesic mappings of compact (Ricci-) semisymmetric Riemannian spaces with additional conditions is obtained.

The property of spaces $\mathscr{E}_4$, which is the subject of Theorem 3, is shared by many Riemannian spaces, which are generalization of $\mathscr{S}_n$ and $\mathscr{E}_n$. In 1954 N.S. Sinyukov [47] proved that the symmetric and recurrent Riemannian spaces V_n with nonconstant curvature do not admit nontrivial geodesic mappings.

V.R. Kaygorodov [17, 18] introduced into consideration the *generally recurrent spaces D_n^m*, defined by the conditions

$$R^h_{ijk,l_1l_2\cdots l_m} = \sum_{s=1}^{m} \overset{s}{\Omega}_{l_sl_{s+1}\cdots l_m}\, R^h_{ijk,l_1l_2\cdots l_{s-1}},$$

where $\overset{s}{\Omega}$ are some tensors. The spaces where $R^h_{ijk,l_1l_2\ldots l_m} = 0$ are called *m-symmetric spaces S_n^m*, and the spaces where $R^h_{ijk,l_1l_2\ldots l_m} = \Omega_{l_1l_2\ldots l_m}R^h_{ijk}$, $\Omega \not\equiv 0$, are called *m-recurrent spaces K_n^m*. Note that many spaces D_n^m are semisymmetric spaces Ps_n. In particular, S_n^1, S_n^2, $K_n^m \subset Ps_n$.

J. Mikeš [22, 31, 47] proved that the semisymmetric spaces considered above with nonconstant curvature do not admit nontrivial geodesic mappings: (a) K_n^m; (b) S_n^2; (c) D_n^2; (d) D_n^m, where $\Omega^2 \not\equiv 0$.

V.S. Sobchuk added to this list the semisymmetric spaces S_n^m. He also showed that the spaces of nonconstant curvature S_n^3, $n > 4$ (see [51]), S_n^4, $n > 4$ (see [37]), and S_n^m, $2n > m+3$ (see [29]), cannot be semisymmetric and do not admit nontrivial geodesic mappings.

This is true also for non-Einsteinian Ricci-symmetric ($R_{ij,k} = 0$, see [1, 22]), Ricci-2-symmetric ($R_{ij,kl} = 0$, see [22]), Ricci-3-symmetric ($R_{ij,klm} = 0$, $n > 4$, see [37]), Ricci-4-symmetric ($R_{ij,klmp} = 0$, $n > 4$, see [51]) and Ricci-m-symmetric ($R_{ij,l_1l_2\ldots l_m} = 0$, $2n > m+2$, see [29]) spaces.

GEODESIC MAPPINGS OF SPACES WITH HARMONIC CURVATURE

A Riemannian space V_n *with harmonic curvature* is defined as a space where $R^\alpha_{ijk,\alpha} = 0$ ($\Leftrightarrow R_{ij,k} = R_{ik,j}$). In particular, V_n with $R_{ij,k} = 0$ is Ricci symmetric $RicS_n^1$; in [22], it is proved that $RicS_n^1 \not\equiv \mathscr{E}_n$ do not admit nontrivial projective transformations, nor nontrivial geodesic mappings, see also [1].

Theorem 9 (V.S. Sobchuk [50]) *In spaces V_n with harmonic curvature admitting nontrivial geodesic mappings, there exist concircular vector fields and special coordinates (5), where $d\tilde{s}^2$ is a metric of some Einsteinian space with scalar* *curvature $\widetilde{R}$, and the function $f \not\equiv$ const satisfies the differential equation*

$$(n-1)(4ff''+(n-2)f'^2)+4e(\widetilde{R}-Rf)=0,$$

where R is the constant scalar curvature of V_n.

S. Tanno [53] studied projective transformations of complete Riemannian spaces V_n with harmonic curvature. His results are generalized by the following theorems:

Theorem 10 (J. Mikeš, Ž. Radulović [35]) *Non-Einsteinian spaces V_n with harmonic curvature do not admit a nontrivial geodesic mapping onto $\bar{V}_n$ with harmonic curvature.*

Theorem 11 (J. Mikeš, Ž. Radulović [35]) *Non-Einsteinian spaces V_n with harmonic curvature do not admit any non-affine projective transformations.*

GEODESIC MAPPINGS OF SPACES L_n

The Riemannian spaces V_n with nonconstant curvature R such that

$$R_{ij,k}=\sigma_k g_{ij}+\nu_i g_{jk}+\nu_j g_{ik}, \quad \text{where} \quad \sigma_k \equiv \frac{n}{(n-1)(n+2)}R_{,k}; \quad \nu_k \equiv \frac{n-2}{2n}\sigma_k,$$

are called the *spaces L_n* [47].

The tensor a_{ij}, which is a nontrivial solution of the basic geodesic mappings equations (3) in $\mathscr{S}_n$, is a metric tensor of the space L_n [47]. A similar circumstance is stated for nontrivial geodesic mappings of Einsteinian spaces [14, 15].

The general solution of (3) in the space L_n has the form

$$a_{ij}=c_1\,g_{ij}+c_2\left(R_{ij}-\frac{nR}{(n-1)(n+2)}g_{ij}\right),$$

where c_1, c_2 are constants. The same result has been partially proved earlier under the condition $\mathrm{Rang}\left\|R_{ij}-\frac{R}{n}g_{ij}\right\|>2$ in [49].

The local expression of the metric L_n is given by S. Formella [15].

The problems of global geodesic mappings of spaces L_n were considered in [27, 49]. It follows from Theorems 2 and 3 [49] that there is no compact properly Riemannian space L_n with the inequality

$$\left(R_{ij}-\frac{4R}{(n-1)(n+6)}g_{ij}\right)R^iR^j \geq 0$$

holding everywhere, where $R^i \equiv g^{i\alpha}R_{,\alpha}$.

In [27], the principal scheme of constructing a compact orientable space L_n admitting global nontrivial geodesic mappings is given.

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GEODESIC MAPPINGS ON COMPACT RIEMANNIAN MANIFOLDS WITH CONDITIONS ON SECTIONAL CURVATURE

Irena Hinterleitner

ABSTRACT. We found new criteria for sectional curvatures on compact Riemannian manifolds for which geodesic mappings are affine, and, moreover, homothetic.

1. Introduction

To the theory of geodesic mappings and their transformations have been devoted many papers, these results are formulated in a large number of research papers and monographs [**2**, **4**–**12**, **16**–**19**, **21**–**26**, **30**, **33**], etc.

In 1953, Takeno and Ikeda [**31**] considered geodesic mappings of spherically symmetric spaces V_4, in 1954 Sinyukov [**26**, p. 88] studied the case of symmetric and recurrent spaces and, in 1976 Mikeš ([**13**,**16**], [**21**, p. 206], [**26**, pp. 151–155]) proved that generalized recurrent (pseudo-) Riemannian spaces V_n with nonconstant curvature do not admit nontrivial geodesic mappings. In this topic Prvanović [**23**] and Sobchuk [**20**, **29**] also have been interested. These results were obtained "locally" and they are contained in [**14**, **16**, **21**, **26**].

Global results for geodesic mappings of compact Riemannian manifolds were obtained by Vrançeanu [**33**], Sinyukova [**27**, **28**], Mikeš [**15**, **16**], etc.

The above results are related to questions of projective rigidity of (pseudo-) Riemannian manifolds and also of manifolds with affine connections.

In [**10**] and [**11**] we proved that these mappings preserve the smoothness class of metrics of geodesically equivalent (pseudo-) Riemannian manifolds. In [**10**] it was sufficient to suppose the metrics to be of differentiability class C^2, and in [**11**] to be of class C^1.

We present new results on geodesic mappings of compact Riemannian manifolds with certain conditions on the sectional curvature of the Ricci directions.

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2. Geodesic mapping theory

Let $V_n = (M, g)$ and $\bar{V}_n = (\bar{M}, \bar{g})$ be n-dimensional (pseudo-) Riemannian manifolds with metrics g and $\bar{g}$, respectively.

DEFINITION 2.1. A diffeomorphism $f : V_n \to \bar{V}_n$ is called a *geodesic mapping* of V_n onto $\bar{V}_n$ if f maps any geodesic in V_n onto a geodesic in $\bar{V}_n$.

We restricted ourselves to the study of a coordinate neighborhood (U, x) of the points $x \in V_n$ and $f(x) \in \bar{V}_n$. The points x and $f(x)$ have the same coordinates $x = (x^1, \dots, x^n)$. We assume that V_n, $\bar{V}_n \in C^1$ $(g, \bar{g} \in C^1)$ if their components $g_{ij}(x), \bar{g}_{ij}(x) \in C^1$ on (U, x), respectively.

It is known [**12**], see [**6**, pp. 131–133], [**21**, p. 167], that V_n admits a geodesic mapping onto $\bar{V}_n$ if and only if the following Levi-Civita equations

$$\nabla_k \bar{g}_{ij} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi_j \bar{g}_{ik} \tag{2.1}$$

hold, where ∇ is the Levi-Civita connection on V_n and

$$\psi_i = \partial_i \Psi, \quad \Psi = \frac{1}{n+1} \ln \sqrt{|\det \bar{g} / \det g|}, \quad \partial_i = \partial / \partial x^i.$$

Sinyukov [**26**, p. 121], see [**21**, p. 167], proved that the Levi-Civita equations (2.1) are equivalent to

$$\nabla_k a_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik}, \tag{2.2}$$

where

$$\text{(a)}\ \ a_{ij} = e^{2\Psi}\, \bar{g}^{\alpha\beta} g_{i\alpha} g_{j\beta}; \quad \text{(b)}\ \ \lambda_i = -e^{2\Psi}\, \bar{g}^{\alpha\beta} \psi_\alpha g_{i\beta}, \tag{2.3}$$

and, moreover, $\lambda_i = \partial_i \Lambda$, $\Lambda = \frac{1}{2}\, a_{\alpha\beta} g^{\alpha\beta}$. Here $(\bar{g}^{ij}) = (\bar{g}_{ij})^{-1}$ and $(g^{ij}) = (g_{ij})^{-1}$.

On the other hand:

$$\bar{g}_{ij} = e^{2\Psi} \hat{g}_{ij}, \quad \Psi = \ln \sqrt{|\det \hat{g} / \det g|}, \quad (\hat{g}_{ij}) = (a_{\alpha\beta} g^{i\alpha} g^{j\beta})^{-1}.$$

Furthermore, we assume that $V_n = (M, g) \in C^2$ and $\bar{V}_n = (M, \bar{g}) \in C^2$. In this case, the integrability conditions of the equations (2.2), due to the Ricci identity

$$\nabla_l \nabla_k a_{ij} - \nabla_k \nabla_l a_{ij} = a_{i\alpha} R^\alpha_{jkl} + a_{j\alpha} R^\alpha_{ikl}, \tag{2.4}$$

have the following form

$$a_{i\alpha} R^\alpha_{jkl} + a_{j\alpha} R^\alpha_{ikl} = g_{ik} \nabla_l \lambda_j + g_{jk} \nabla_l \lambda_i - g_{il} \nabla_k \lambda_j - g_{jl} \nabla_k \lambda_i, \tag{2.5}$$

where R^h_{ijk} are components of the Riemannian tensor R on V_n, and after contraction with g^{ik} we get [**26**, p. 133]

$$n \nabla_l \lambda_j = \mu\, g_{jl} - a_{j\alpha} R^\alpha_l - a_{\alpha\beta} R_j{}^{\alpha\beta}{}_l, \tag{2.6}$$

where $\mu = \nabla_\alpha \lambda^\alpha$, $R^\alpha_i = g^{\alpha\beta} R_{\beta i}$ and $R_{ij} = R^\alpha_{i\alpha j}$ are components of the Ricci tensor Ric on V_n.

3. Integral formula

We introduce the vector field ξ on $V_n \in C^2$ in the following way

$$\xi^i = a^\alpha_\beta \nabla_\alpha a^{i\beta} - a^i_\beta \nabla_\alpha a^{\alpha\beta}, \tag{3.1}$$

where $a^l_i = g^{l\alpha} a_{\alpha i}$, $a^{ij} = a_{\alpha\beta} g^{i\alpha} g^{j\beta}$. Using formula (3.1), the Ricci identity (2.4) and Sinyukov's equations (2.2) we obtained that the divergence of the vector ξ has the following representation

$$\operatorname{div} \xi = \Phi(a) - (n-1)(n+2)\, \lambda_\alpha \lambda_\beta g^{\alpha\beta},$$

where $\Phi(a) = R_{ij} a^{ik} a^j_k - R_{ijkl} a^{ik} a^{jl}$.

Suppose that the Riemannian manifold (M, g) is compact and without boundary, then on the basis of the Gauß theorem $\int_M \operatorname{div} \xi \, d\nu = 0$ we obtain the integral formula

$$\int_M \Phi(a)\, d\nu = (n-1)(n+2) \int_M \lambda_\alpha \lambda_\beta g^{\alpha\beta} d\nu. \tag{3.2}$$

For applying the Gauss theorem it is necessary to require the orientability of M, if M is a non-orientable manifold, then we'll look at the oriented double cover.

Let $g(e_i, e_j) = \delta_{ij}$ and $a(e_i, e_j) = \alpha_i \delta_{ij}$ with the Kronecker symbol δ_{ij}, i.e., $\{e_1, \ldots, e_n\}$ is the orthonormal basis of eigenvectors to the eigenvalues $\alpha_1, \ldots, \alpha_n$ of the tensor $a = (a_{ij})$ of $T_x M$ at any point $x \in M$. As we can see from direct calculation, $\Phi(a)$ has the following form (see [**3**, p. 592]):

$$\Phi(a) = \sum_{i<j} K(e_i, e_j)(\alpha_i - \alpha_j)^2, \tag{3.3}$$

where $K(e_i, e_j)$ are sectional curvatures in the two-directions $e_i \wedge e_j$.

It is easy to see:

$$\begin{aligned}
\Phi(a) &= R_{ij} a^{ik} a^j_k - R_{ijkl} a^{ik} a^{jl} = \sum_{i,j} (\alpha_i)^2 R_{ijij} - \sum_{i,j} \alpha_i \alpha_j R_{ijij} \\
&= \sum_{i<j} ((\alpha_i)^2 + (\alpha_j)^2) \cdot R_{ijij} - 2 \sum_{i<j} \alpha_i \alpha_j R_{ijij} \\
&= \sum_{i<j} (\alpha_i - \alpha_j)^2 \cdot R_{ijij} = \sum_{i<j} (\alpha_i - \alpha_j)^2 \cdot K(e_i, e_j),
\end{aligned}$$

where

$$K(e_i, e_j) = \frac{R(e_i, e_j, e_i, e_j)}{g(e_i, e_i) \cdot g(e_j, e_j) - (g(e_i, e_j))^2} = R_{ijij}.$$

4. Principal orthonormal basis

Eisenhart [**6**, pp. 113–114] introduced a *principal direction* in a Riemannian manifold (M, g), as an eigenvector of the Ricci tensor. He showed that at any point $x \in M$ there exists the orthonormal basis $\{e_1, \ldots, e_n\}$ in which

$$g_{ij} = \delta_{ij} \quad \text{and} \quad R_{ij} = \rho_i \delta_{ij},$$

i.e., $e_1, \ldots, e_n$ are the vectors of the principal directions and $\rho_1, \ldots, \rho_n$ are their eigenvalues. This basis is called the *principal orthonormal basis*.

This means that the existence of this basis is a property only of the Riemannian manifold (M, g), independent of the solution a_{ij} of equation (2.2). Generally the set of principal orthonormal bases is a proper subset of the set of orthonormal bases. Because the vector field λ_i is gradient-like, formula (2.6) implies [**26**, p. 138]

$$a_{i\alpha} R^{\alpha}_{j} = a_{j\alpha} R^{\alpha}_{i}.$$

So the tensors a_{ij} and R_{ij} commute and have common eigenvectors. From this fact it follows that there exist a principal orthonormal basis in which $g_{ij} = \delta_{ij}$ and $a_{ij} = \alpha_i \delta_{ij}$ hold. This basis is called a *joint principal orthonormal basis*. Note that we do not restrict the signature of the Ricci tensor and the tensor a_{ij}. In the following we restrict ourselves to the study of formulas (3.2) and (3.3) on joint principal orthonormal bases.

5. Main Theorems

For the following we recall that a compact Riemannian manifold V_n admits a geodesic mapping onto a (pseudo-) Riemannian manifold $\bar{V}_n$.

If we assume that at each point $x \in M$ all sectional curvatures $K(e_i, e_j)$ are non-positive in the two-directions $e_i \wedge e_j$ of the joint principal orthonormal basis $\{e_1, \ldots, e_n\}$ of vectors of the main directions of the Ricci tensor, then from integral formula (3.2) it follows

$$\text{(5.1)} \qquad \text{(a)} \int_M \Phi(a)\, d\nu = 0 \quad \text{and} \quad \text{(b)} \int_M \lambda_\alpha \lambda_\beta g^{\alpha\beta}\, d\nu = 0.$$

From integral (5.1b) follows $\lambda_\alpha \lambda_\beta g^{\alpha\beta} = 0$ and this fact implies that λ_i is vanishing on M, i.e., $\lambda_1 = \cdots = \lambda_n = 0$. In this case, the geodesic mapping is *affine* (see [**21**, p. 150]). We proved the following theorem:

Theorem 5.1. *Assume a compact Riemannian manifold (M, g) without boundary of dimension $n \geqslant 2$. If at any point $x \in M$ the sectional curvature $K(e_i, e_j)$ is non-positive for any two-direction $e_i \wedge e_j$ from all the principal orthonormal basis $\{e_1, \ldots, e_n\}$ of vectors of the main direction of the Ricci tensor, then any geodesic mapping of (M, g) is affine.*

Moreover, we suppose at each point $x \in M$ the sectional curvature $K(e_i, e_j)$ is non-positive and that there is a certain point $x_0 \in M$ where the sectional curvature $K(e_i, e_j)$ in any two-direction $e_i \wedge e_j$ of the joint principal orthonormal basis $\{e_1, \ldots, e_n\}$ of vectors of the main directions of the Ricci tensor is negative. Then from integral (3.2) follows equation (5.1). On the basis of Theorem 5.1 it follows $\lambda_1 = \cdots = \lambda_n = 0$ and the geodesic mapping is affine.

Further, from integral (5.1a) follows $\Phi(a) = 0$ on M. Then from formula (3.3) at the point $x_0 \in M$ we obtain $\alpha_1 = \cdots = \alpha_n = \alpha$. Hence $a_{ij} = \alpha \delta_{ij}$, i.e., $a_{ij}(x_0) = \alpha g_{ij}(x_0)$.

In this case, the affine mapping is homothetic, i.e., $\bar{g} = \alpha' g$, where $\alpha' = \text{const}$. This fact follows from the uniqueness of solutions of the fundamental equations of affine mappings $V_n \to \bar{V}_n : \nabla_k \bar{g}_{ij} = 0$ with initial values $\bar{g}_{ij}(x_0) = \alpha' \, g_{ij}(x_0)$. This is equivalent to $a_{ij}(x_0) = \alpha g_{ij}(x_0)$, this fact follows from equation (2.3).

We proved the following theorem:

THEOREM 5.2. *Assume a compact Riemannian manifold (M, g) without boundary of dimension $n \geqslant 2$. If at any point $x \in M$ the sectional curvature $K(e_i, e_j)$ is non-positive and if there is a certain point $x_0 \in M$, where the sectional curvature $K(e_i, e_j)$ is negative in any two-direction $e_i \wedge e_j$ of all the principal orthonormal basis $\{e_1, \dots, e_n\}$ of vectors of the main directions of the Ricci tensor, then any geodesic mapping of (M, g) is homothetic.*

These Theorems generalize the results of Mikeš [**15**] (see [**16**]), which were obtained by means of modifications of integral inequalities obtained by Švec [**1**, p. 10].

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Brno University of Technology
Faculty of Civil Engineering
Deptartment of Mathematics
Brno, Czech Republic
hinterleitner.irena@seznam.cz

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GEODESIC MAPPINGS OF (PSEUDO-) RIEMANNIAN MANIFOLDS PRESERVE CLASS OF DIFFERENTIABILITY

IRENA HINTERLEITNER AND JOSEF MIKEŠ

Abstract. In this paper, we prove that geodesic mappings of (pseudo-) Riemannian manifolds preserve the class of differentiability ($C^r, r \geq 1$). Also, if the Einstein space V_n admits a non-trivial geodesic mapping onto a (pseudo-) Riemannian manifold $\bar{V}_n \in C^1$, then $\bar{V}_n$ is an Einstein space. If a four-dimensional Einstein space with non-constant curvature globally admits a geodesic mapping onto a (pseudo-) Riemannian manifold $\bar{V}_4 \in C^1$, then the mapping is affine and, moreover, if the scalar curvature is non-vanishing, then the mapping is homothetic, i. e. $\bar{g} = \text{const} \cdot g$.

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1. Introduction

The paper is devoted to the geodesic mapping theory of (pseudo-) Riemannian manifolds with respect to differentiability of their metrics. Most of the results in this area are formulated for "sufficiently" smooth, or analytic, geometric objects, as usual in differential geometry. It can be observed in most of the monographs and researches dedicated to the study of the theory of geodesic mappings and transformations, see [1, 3, 5–11, 13–19, 23–36].

Let $V_n = (M, g)$ and $\bar{V}_n = (\bar{M}, \bar{g})$ be (pseudo-) Riemannian manifolds, where M and $\bar{M}$ are n-dimensional manifolds with dimension $n \geq 2$, g and $\bar{g}$ are metrics. All the manifolds are assumed to be connected.

Definition 1. A diffeomorphism $f\colon V_n \to \bar{V}_n$ is called a *geodesic mapping* of V_n onto $\bar{V}_n$ if f maps any geodesic in V_n onto a geodesic in $\bar{V}_n$.

Hinterleitner and Mikeš [11] have proved the following theorem:

Theorem 1. *If the (pseudo-) Riemannian manifold V_n ($V_n \in C^r$, $r \geq 2$, $n \geq 2$) admits a geodesic mapping onto $\bar{V}_n \in C^2$, then $\bar{V}_n$ belongs to C^r.*

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Here and later, $V_n = (M, g) \in C^r$ means that $g \in C^r$, i. e., in a coordinate neighborhood (U, x) for the components of the metric g, $g_{ij}(x) \in C^r$ holds. If $V_n \in C^r$, then $M \in C^{r+1}$. This means that the atlas on the manifold M has the differentiability class C^{r+1}, i. e., for non-disjoint charts (U, x) and (U', x') on $U \cap U'$, it is true that the transformation $x' = x'(x) \in C^{r+1}$.

We suppose that the differentiability class r is equal to $0, 1, 2, \dots, \infty, \omega$, where $0, \infty$ and ω denote continuous, infinitely differentiable, and real analytic functions, respectively.

In the paper, we prove more general results. The following theorem holds:

Theorem 2. *If the (pseudo-) Riemannian manifold V_n ($V_n \in C^r$, $r \geq 1$, $n \geq 2$) admits a geodesic mapping onto $\bar{V}_n \in C^1$, then $\bar{V}_n$ belongs to C^r .*

Briefly, this means that *the geodesic mapping preserves the class of smoothness of the metric*.

Remark 1. It's easy to prove that the Theorems 1 and 2 are valid also for $r = \infty$ and for $r = \omega$. This follows from the theory of solvability of differential equations. Of course, we can apply this theorem only locally, because differentiability is a local property.

Remark 2. To require $V_n, \bar{V}_n \in C^1$ is a minimal requirement for geodesic mappings.

T. Levi-Civita [13] found metrics (*Levi-Civita metrics*) which admit geodesic mappings, see [1, 5], [25, p. 173], [27, p. 325]. From these metrics, we can easily see examples of non-trivial geodesic mappings $V_n \to \bar{V}_n$, where

- $V_n, \bar{V}_n \in C^r$ and $\notin C^{r+1}$ for $r \in \mathbb{N}$;
- $V_n, \bar{V}_n \in C^\infty$ and $\notin C^\omega$;
- $V_n, \bar{V}_n \in C^\omega$.

2. Geodesic mappings of Einstein manifolds

These results may be applied to geodesic mappings of Einstein manifolds V_n onto pseudo-Riemannian manifolds $\bar{V}_n \in C^1$.

Geodesic mappings of Einstein spaces have been studied by many authors starting by A. Z. Petrov (see [27]). Einstein spaces V_n are characterized by the condition $\mathrm{Ric} = \mathrm{const} \cdot g$.

An Einstein space V_3 is a space of constant curvature. It is known that Riemannian spaces of constant curvature form a closed class with respect to geodesic mappings (Beltrami theorem [5, 23, 25, 27, 29, 31]). In 1978 (see [15] and PhD. thesis [14], and see [16, 20, 22], [23, p. 125], [25, p. 188]), Mikeš proved that under the conditions $V_n, \bar{V}_n \in C^3$, the following theorem holds (locally):

Theorem 3. *If the Einstein space V_n admits a non-trivial geodesic mapping onto a (pseudo-) Riemannian manifold $\bar{V}_n$, then $\bar{V}_n$ is an Einstein space.*

Many properties of Einstein spaces appear when $V_n \in C^3$ and $n > 3$. Moreover, it is known (D. M. DeTurck and J. L. Kazdan [4], see [2, p. 145]), that Einstein space V_n belongs to C^ω, i. e., for all points of V_n a local coordinate system x exists, for which $g_{ij}(x) \in C^\omega$ (*analytic coordinate system*).

It implies global validity of Theorem 3 and, on the basis of Theorem 2, the following more general theorem holds:

Theorem 4. *If the Einstein space V_n admits a nontrivial geodesic mapping onto a (pseudo-) Riemannian manifold $\bar{V}_n \in C^1$, then $\bar{V}_n$ is an Einstein space.*

The present Theorem is true globally, because the function Ψ which determines the geodesic mapping is real analytic on an analytic coordinate system and so ψ $(= \nabla\Psi)$ is vanishing only on a point set of zero measure. This simplifies the proof given in [11].

Finally, based on the results (see [16,20–22], [23, p. 128], [25, p. 194]) for geodesic mappings of four-dimensional Einstein manifolds, the following theorem holds:

Theorem 5. *If a four-dimensional Einstein space V_4 with non-constant curvature globally admits a geodesic mapping onto a (pseudo-) Riemannian manifold $\bar{V}_4 \in C^1$, then the mapping is affine and, moreover, if the scalar curvature is non-vanishing, then the mapping is homothetic, i. e. $\bar{g} = \mathrm{const} \cdot g$.*

3. Geodesic mapping theory for $V_n \to \bar{V}_n$ of class C^1

Let us briefly recall some main facts of geodesic mapping theory of (pseudo-) Riemannian manifolds which were found by T. Levi-Civita [13], L. P. Eisenhart [5,6] and N. S. Sinyukov [31], see [1, 9–11, 14, 16, 18, 19, 23, 25–32, 34–36]. In these results, no details about the smoothness class of the metric were stressed. They were formulated "for sufficiently smooth" geometric objects.

Since a geodesic mapping $f\colon V_n \to \bar{V}_n$ is a diffeomorphism, we can suppose $\bar{M} = M$. A (pseudo-) Riemannian manifold $V_n = (M, g)$ admits a geodesic mapping onto $\bar{V}_n = (M, \bar{g})$ if and only if the *Levi-Civita equations*

$$\bar{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X \tag{3.1}$$

hold for any tangent fields X, Y and where ψ is a differential form on M. Here, ∇ and $\bar{\nabla}$ are Levi-Civita connections of g and $\bar{g}$, respectively. If $\psi \equiv 0$, then f is *affine* or *trivially geodesic*.

Let (U, x) be a chart from the atlas on M. Then, equation (3.1) on U has the following local form: $\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_i\delta^h_j + \psi_j\delta^h_i$, where Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ are the Christoffel symbols of V_n and $\bar{V}_n$, ψ_i are components of ψ and δ^h_i is the Kronecker delta. Equations (3.1) are equivalent to the following Levi-Civita equations

$$\nabla_k \bar{g}_{ij} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi \bar{g}_{ik} \tag{3.2}$$

where $\bar{g}_{ij}$ are components of $\bar{g}$.

It is known that

$$\psi_i = \partial_i \Psi, \quad \Psi = \frac{1}{2(n+1)} \ln \left| \frac{\det \bar{g}}{\det g} \right|, \quad \partial_i = \frac{\partial}{\partial x^i}.$$

N.S. Sinyukov proved that the Levi-Civita equations (3.1) and (3.2) are equivalent to ([31, p. 121], [16], [23, p. 108], [25, p. 167], [29, p. 63]):

$$\nabla_k a_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik}, \tag{3.3}$$

where

$$\text{(a)}\ a_{ij} = \mathrm{e}^{2\Psi} \bar{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}; \quad \text{(b)}\ \lambda_i = -\,\mathrm{e}^{2\Psi} \bar{g}^{\alpha\beta} g_{\beta i} \psi_\alpha. \tag{3.4}$$

From (3.3) follows $\lambda_i = \partial_i(\frac{1}{2} a_{\alpha\beta} g^{\alpha\beta})$, $(g^{ij}) = (g_{ij})^{-1}$ and $(\bar{g}^{ij}) = (\bar{g}_{ij})^{-1}$.

On the other hand [29, p. 63]:

$$\bar{g}_{ij} = \mathrm{e}^{2\Psi} \hat{g}_{ij}, \quad \Psi = \frac{1}{2} \ln \left| \frac{\det \hat{g}}{\det g} \right|, \quad (\hat{g}_{ij}) = (g^{i\alpha} g^{j\beta} a_{\alpha\beta})^{-1}. \tag{3.5}$$

We can rewrite equations (3.3) and (3.4) in the following equivalent form (see [18], [25, p. 150]):

$$\nabla_k a^{ij} = \lambda^i \delta^j_k + \lambda^j \delta^i_k, \tag{3.6}$$

where

$$\text{(a)}\ a^{ij} = \mathrm{e}^{2\Psi} \bar{g}^{ij} \quad \text{and} \quad \text{(b)}\ \lambda^i = -\psi_\alpha a^{\alpha i}. \tag{3.7}$$

Evidently, it follows

$$\lambda^i = \frac{1}{2} g^{ik}\, \partial_k (a^{\alpha\beta} g_{\alpha\beta}). \tag{3.8}$$

The above formulas (3.1), (3.2), (3.3), (3.6), are the criterion for geodesic mappings $V_n \to \bar{V}_n$ globally as well as locally. These formulas are true only under the condition $V_n, \bar{V}_n \in C^1$.

4. Geodesic mapping theory for $V_n \in C^2 \to \bar{V}_n \in C^1$

In this section, we prove the main Theorem 2 from above. It is easy to see that Theorem 2 follows from Theorem 1 and the following theorem.

Theorem 6. *If $V_n \in C^2$ admits a geodesic mapping onto $\bar{V}_n \in C^1$, then $\bar{V}_n \in C^2$.*

Proof. Below, we prove Theorem 6.

4.1. We will suppose that the (pseudo-) Riemannian manifold $V_n \in C^2$ admits the geodesic mapping onto the (pseudo-) Riemannian manifold $\bar{V}_n \in C^1$. Furthermore, we can assume that $\bar{M} = M$.

We study the coordinate neighborhood (U, x) of any point $p = (0, 0, \dots, 0)$ at M. Evidently, components $g_{ij}(x) \in C^2$ and $\bar{g}_{ij}(x) \in C^1$ on $U \subset M$. On (U, x), formulas (3.1)–(3.8) hold. From that fact, it follows that the functions $g^{ij}(x) \in C^2$, $\bar{g}^{ij}(x) \in C^1$, $\Psi(x) \in C^1$, $\psi_i(x) \in C^0$, $a^{ij}(x) \in C^1$, $\lambda^i(x) \in C^0$, and $\Gamma^h_{ij}(x) \in C^1$, where $\Gamma^h_{ij} = \frac{1}{2} g^{hk}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$ are Christoffel symbols.

4.2. It is easy to see that in a neighborhood of the point p in $V_n \in C^r$ there exist a semigeodesic coordinate system (U, x) for which the metric $g \in C^r$ has the following form (see [5], [25, p. 64])

$$\mathrm{d}s^2 = e(\mathrm{d}x^1)^2 + g_{ab}(x^1, \dots, x^n)\mathrm{d}x^a \mathrm{d}x^b, \quad e = \pm 1, \quad a, b > 1. \tag{4.1}$$

Evidently, for $a > 1$:

$$g_{11} = g^{11} = e = \pm 1, \quad g_{1a} = g^{1a} = 0 \text{ and } \Gamma^1_{11} = \Gamma^1_{1a} = \Gamma^a_{11} = 0. \tag{4.2}$$

We can construct such a coordinate system using a coordinate transformation of class C^{r+1} for a basis of non-isotropic hypersurfaces $\Sigma \in C^{r+1}$ in a neighborhood of $p \in \Sigma$. Moreover, we can assume at p that

$$g_{ij}(0) = e_i\, \delta_{ij}; \quad e_i = \pm 1. \tag{4.3}$$

4.3. We write equations (3.6) in the following form

$$\partial_k a^{ij} = \lambda^i \delta^j_k + \lambda^j \delta^i_k - a^{i\alpha}\Gamma^j_{\alpha k} - a^{j\alpha}\Gamma^i_{\alpha k}. \tag{4.4}$$

Because $a^{ij} \in C^1$ and $\Gamma^j_{\alpha k} \in C^1$ from equation (4.4), we have the existence of the derivative immediately

$$\partial_{kl}a^{ii},\ \partial_{kk}a^{ii},\ \partial_{ki}a^{ii}(\equiv \partial_{ik}a^{ii}),\ \partial_{kl}a^{ij},\ \partial_{kk}a^{ij},\ \partial_{ki}a^{ij}(\equiv \partial_{ik}a^{ij}),$$

for each set of different indices i, j, k, l. Derivatives do not depend on the order because they are continuous functions.

We compute formula (4.4) for $i = j = k$ and for $i \neq j = k$:

$$\partial_i a^{ii} = 2\lambda^i - 2a^{i\alpha}\Gamma^i_{\alpha i} \quad \text{and} \quad \partial_k a^{ik} = \lambda^i - a^{k\alpha}\Gamma^i_{\alpha k} - a^{i\alpha}\Gamma^k_{\alpha k}$$

where, for an index k, we do not carry out the Einstein summation and after eliminating λ^i, we obtain

$$\tfrac{1}{2}\, \partial_i a^{ii} - \partial_k a^{ik} = a^{k\alpha}\Gamma^i_{\alpha k} + a^{i\alpha}\Gamma^k_{\alpha k} - a^{i\alpha}\Gamma^i_{\alpha i} \tag{4.5}$$

Because there exists the partial derivative $\partial_{ik}a^{ii}$, formula (4.5) implies the existence of the partial derivatives $\partial_{kk}a^{ik}$.

4.4. In the semigeodesic coordinate system (4.1), we compute (4.4) for $i = j = k = 1$: $\lambda^1 = \frac{1}{2}\, \partial_1 a^{11}$, and from (3.8): $\lambda^1 = \frac{1}{2}\, \partial_1(a^{11} + e a^{\alpha\beta} g_{\alpha\beta})$, we obtain $\partial_1(a^{\alpha\beta} g_{\alpha\beta}) = 0$. Here and later $\alpha, \beta > 1$.

Further (4.4) for $i = j = 1$ and $k = 2$, we have the following expression $\partial_1 a^{12} + a^{1\gamma}\Gamma^2_{\gamma 1} + a^{2\gamma}\Gamma^1_{\gamma 1} = \lambda^2$. Using (3.8), we have

$$\partial_1 a^{12} = \tfrac{1}{2}\, g^{2\gamma} \cdot \partial_\gamma(a^{11} + a^{\alpha\beta} g_{\alpha\beta}) - a^{1\gamma}\Gamma^2_{\gamma 1}, \quad \gamma > 1,$$

and after integration, we obtain

$$a^{12} = \frac{1}{2}\left(\int_0^{x^1} g^{2\gamma}(\tau^1, x^2, \dots, x^n) d\tau^1\right) \cdot \partial_\gamma (a^{\alpha\beta} \cdot g_{\alpha\beta}) + \frac{1}{2}\int_0^{x^1} g^{2\gamma}(\tau^1, x^2, \dots, x^n) \cdot \partial_\gamma a^{11} d\tau^1 - \int_0^{x^1} a^{1\gamma} \Gamma^2_{\gamma 1} d\tau^1 + A(x^2, \dots, x^n). \quad (4.6)$$

As $a^{12}(0, x^2, \dots, x^n) \equiv A(x^2, \dots, x^n)$, the function $A \in C^1$.

After differentiating the formula (4.6) by x^2 and using the law of commutation of derivatives and integrals, see [12, p. 300], we can see that

$$\frac{\partial}{\partial x^2}\left\{\left(\int_0^{x^1} g^{2\gamma}(\tau^1, x^2, \dots, x^n) d\tau^1\right) \cdot \partial_\gamma (a^{\alpha\beta} \cdot g_{\alpha\beta})\right\} \quad (4.7)$$

exists. From (4.5) for $i = 2$ and $k = c \neq 2$, we obtain $\partial_c a^{c2} = \frac{1}{2}\partial_2 a^{22} + a^{c\delta}\Gamma^2_{\delta c} + a^{2\delta}\Gamma^c_{\delta c} - a^{2\delta}\Gamma^2_{\delta 2}$. Using this formula, we can rewrite the bracket (4.7) in the following form

$$\left\{\left(\int_0^{x^1} g^{2\gamma}(\tau^1, x^2, \dots, x^n) d\tau^1\right) \cdot g_{2\gamma} \cdot \partial_2 a^{22} + f\right\},$$

where f is the rest of this parenthesis, which is evidently differentiable by x^2.

Since the parenthesis and also the coefficients by $\partial_2 a^{22}$ are differentiable with respect to x^2, there exists $\partial_{22} a^{22}$ if

$$\left(\int_0^{x^1} g^{2\gamma}(\tau^1, x^2, \dots, x^n) d\tau^1\right) \cdot g_{2\gamma} \neq 0.$$

Using (3.3), this inequality is true for all x in a neighborhood of the point p excluding the point for which $x^1 = 0$.

For these reasons, in this domain, there exists the derivative $\partial_{22} a^{22}$ as well as all second derivatives a^{ij}. This follows from the derivative of the formula (4.5).

So, $a^{ij} \in C^2$ and $\lambda^i \in C^1$, from the formula (3.7b), it follows $\psi_i \in C^1$ and it means that $\Psi \in C^2$. From (3.7a) follows $\bar{g}^{ij} \in C^2$ and also $\bar{g}_{ij} \in C^2$. This is a proof of Theorem 6. $\square$

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Authors' addresses

Irena Hinterleitner
Brno University of Technology, Faculty of Civil Engineering, Dept. of Mathematics, Žižkova 17, 602 00, Brno, Czech Republic
E-mail address: hinterleitner.irena@seznam.cz

Josef Mikeš
Palacky University, Faculty of Science, Dept. of Algebra and Geometry, 17. listopadu 12, 77146, Olomouc, Czech Republic
E-mail address: josef.mikes@upol.cz

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Geodesic Mappings and Einstein Spaces

Irena Hinterleitner and Josef Mikeš

Abstract. In this paper we study fundamental properties of geodesic mappings with respect to the smoothness class of metrics. We show that geodesic mappings preserve the smoothness class of metrics. We study geodesic mappings of Einstein spaces.

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Keywords. Geodesic mapping, smoothness class, Einstein space.

1. Introduction

First we study the general dependence of geodesic mappings of (pseudo-) Riemannian manifolds in dependence on the smoothness class of the metric. We present well-known facts, which were proved by Beltrami, Levi-Civita, Weyl, Sinyukov, etc., see [1–5]. In these results no details about the smoothness class of the metric were discussed. They were formulated "for sufficiently smooth" geometric objects.

In the last section we present proofs of some facts about geodesic mappings of Einstein spaces.

2. Geodesic mappings theory for $V_n \to \bar{V}_n$ of class C^1

Assume the (pseudo-) Riemannian manifolds $V_n = (M, g)$ and $\bar{V}_n = (\bar{M}, \bar{g})$ with metrics g and $\bar{g}$, and Levi-Civita connections ∇ and $\bar{\nabla}$, respectively. Here V_n, $\bar{V}_n$ $\in C^1$, i.e., $g, \bar{g} \in C^1$ which means that their components $g_{ij}, \bar{g}_{ij} \in C^1$.

Definition 1. A diffeomorphism f: $V_n \to \bar{V}_n$ is called a *geodesic mapping* of V_n onto $\bar{V}_n$ if f maps any geodesic in V_n onto a geodesic in $\bar{V}_n$.

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Let there exist a geodesic mapping $f\colon V_n \to \bar V_n$. Since f is a diffeomorphism, we can assume the existence of local coordinate maps on M or $\bar M$, respectively, such that locally, $f\colon V_n \to \bar V_n$ maps points onto points with the same coordinates, and $\bar M = M$. A manifold V_n admits a geodesic mapping onto $\bar V_n$ if and only if the *Levi-Civita equations*

$$\bar\nabla_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X \tag{1}$$

hold for any tangent fields X, Y and where ψ is a differential form. If $\psi \equiv 0$ than f is *affine* or *trivially geodesic*.

In a local form: $\bar\Gamma^h_{ij} = \Gamma^h_{ij} + \psi_i\delta^h_j + \psi_j\delta^h_i$, where $\Gamma^h_{ij}(\bar\Gamma^h_{ij})$ are the Christoffel symbols of V_n and $\bar V_n$, ψ_i are components of ψ and δ^h_i is the Kronecker delta. Equations (1) are equivalent to the following equations

$$\bar g_{ij,k} = 2\psi_k\bar g_{ij} + \psi_i\bar g_{jk} + \psi\bar g_{ik} \tag{2}$$

where "," denotes the covariant derivative in V_n. It is known that

$$\psi_i = \partial_i\Psi, \quad \Psi = \frac{1}{2(n+1)}\ln\left|\frac{\det\bar g}{\det g}\right|, \quad \partial_i = \partial/\partial x^i.$$

Sinyukov [5] proved that the Levi-Civita equations are equivalent to

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik}, \tag{3}$$

where

$$a_{ij} = e^{2\Psi}\bar g^{\alpha\beta} g_{\alpha i} g_{\beta j}; \quad \lambda_i = -e^{2\Psi}\bar g^{\alpha\beta} g_{\beta i}\psi_\alpha.$$

From (3) follows $\lambda_i = \partial_i\lambda = \partial_i(\frac{1}{2}a_{\alpha\beta}g^{\alpha\beta})$. On the other hand [4, p. 63]:

$$\bar g_{ij} = e^{2\Psi}\tilde g_{ij}, \quad \Psi = \frac{1}{2}\ln\left|\frac{\det\tilde g}{\det g}\right|, \quad \|\tilde g_{ij}\| = \|g^{i\alpha}g^{j\beta}a_{\alpha\beta}\|^{-1}. \tag{4}$$

The above formulas are the criterion for geodesic mappings $V_n \to \bar V_n$ globally as well as locally.

3. Geodesic mappings theory for $V_n \to \bar V_n$ of class C^2

Let V_n and $\bar V_n \in C^2$, then for geodesic mappings $V_n \to \bar V_n$ the Riemann and the Ricci tensors transform in the following way

$$\text{(a)}\quad \bar R^h_{ijk} = R^h_{ijk} + \delta^h_k\psi_{ij} - \delta^h_j\psi_{ik}; \quad \text{(b)}\quad \bar R_{ij} = R_{ij} + (n-1)\psi_{ij}, \tag{5}$$

where $\psi_{ij} = \psi_{i,j} - \psi_i\psi_j$, and the Weyl tensor of projective curvature, which is defined in the following form $W^h_{ijk} = R^h_{ijk} - \frac{1}{n-1}\left(\delta^h_k R_{ij} - \delta^h_j R_{ik}\right)$, is invariant.

The integrability conditions of the Sinyukov equations (3) have the following form

$$a_{i\alpha}R^\alpha_{jkl} + a_{j\alpha}R^\alpha_{ikl} = g_{ik}\lambda_{j,l} + g_{jk}\lambda_{i,l} - g_{il}\lambda_{j,k} - g_{jl}\lambda_{i,k}. \tag{6}$$

After contraction with g^{jk} we get [5]

$$n\lambda_{i,l} = \mu g_{il} + a_{i\alpha}R^\alpha_l - a_{\alpha\beta}R^\alpha{}_{il}{}^\beta \tag{7}$$

where $R^\alpha{}_{ij}{}^\beta = g^{\beta k}R^\alpha{}_{ij\beta}$; $R^\alpha_j = g^{\alpha\beta}R_{\beta j}$ and $\mu = \lambda_{i,j}g^{ij}$.

4. Geodesic mapping between $V_n \in C^r$ $(r > 2)$ and $\bar{V}_n \in C^2$

Theorem 2. *If $V_n \in C^r$ $(r > 2)$ admits geodesic mappings onto $\bar{V}_n \in C^2$, then $\bar{V}_n \in C^r$.*

The proof of this Theorem follows from the following lemmas.

Lemma 3. *Let $\lambda^h \in C^1$ be a vector field and ϱ a function. If $\partial_i \lambda^h - \varrho\, \delta_i^h \in C^1$ then $\lambda^h \in C^2$ and $\varrho \in C^1$.*

Proof. The condition $\partial_i \lambda^h - \varrho\, \delta_i^h \in C^1$ can be written in the following form

$$\partial_i \lambda^h - \varrho \delta_i^h = f_i^h(x), \tag{8}$$

where $f_i^h(x)$ are functions of class C^1. Evidently, $\varrho \in C^0$. For fixed but arbitrary indices $h \neq i$ we integrate (8) with respect to dx^i:

$$\lambda^h = \Lambda^h + \int_{x_o^i}^{x^i} f_i^h(x^1, \ldots, x^{i-1}, t, x^{i+1}, \ldots, x^n)\, dt,$$

where Λ^h is a function, which does not depend on x^i.

Because of the existence of the partial derivatives of the functions λ^h and the above integrals (see [5, p. 300]), also the derivatives $\partial_h \Lambda^h$ exist; in this proof we don't use Einstein's summation convention. Then we can write (8) for $h = i$:

$$\varrho = -f_h^h + \partial_h \Lambda^h + \int_{x_o^i}^{x^i} \partial_h f_i^h(x^1, \ldots, x^{i-1}, t, x^{i+1}, \ldots, x^n)\, dt. \tag{9}$$

Because the derivative with respect to x^i of the right-hand side of (9) exists, the derivative of the function ϱ exists, too. Obviously $\partial_i \varrho = \partial_h f_i^h - \partial_i f_h^h$, therefore $\varrho \in C^1$ and from (8) follows $\lambda^h \in C^2$. □

In a similar way we can prove the following: *if $\lambda^h \in C^r$ $(r \geq 1)$ and $\partial_i \lambda^h - \varrho \delta_i^h \in C^r$ then $\lambda^h \in C^{r+1}$ and $\varrho \in C^r$.*

Lemma 4. *If $V_n \in C^3$ admits a geodesic mapping onto $\bar{V}_n \in C^2$, then $\bar{V}_n \in C^3$.*

Proof. In this case Sinyukov's equations (3) and (7) hold. According to the assumptions $g_{ij} \in C^3$ and $\bar{g}_{ij} \in C^2$. By a simple check-up we find $\Psi \in C^2$, $\psi_i \in C^1$, $a_{ij} \in C^2$, $\lambda_i \in C^1$ and $R^h_{ijk}, R^h{}_{ij}{}^k, R_{ij}, R_i^h \in C^1$.

From the above-mentioned conditions we easily convince ourselves that we can write equation (7) in the form (8), where $\lambda^h = g^{h\alpha}\lambda_\alpha \in C^1$, $\varrho = \mu/n$ and $f_i^h = (-\lambda^\alpha \Gamma^h_{\alpha i} - g^{h\gamma} a_{\alpha\gamma} R_i^\alpha + g^{h\gamma} a_{\alpha\beta} R^\alpha{}_{i\gamma}{}^\beta)/n \in C^1$.

From Lemma 3 follows that $\lambda^h \in C^2$, $\varrho \in C^1$, and evidently $\lambda_i \in C^2$. Differentiating (3) twice we convince ourselves that $a_{ij} \in C^3$. From this and formula (4) follows that also $\Psi \in C^3$ and $\bar{g}_{ij} \in C^3$. □

Further we notice that for geodesic mappings between V_n and $\bar{V}_n$ of class C^3 holds the third set of Sinyukov equations:

$$(n-1)\mu_{,k} = 2(n+1)\lambda_\alpha R_k^\alpha + a_{\alpha\beta}(2R^\alpha{}_{k,}{}^\beta - R^{\alpha\beta}{}_{,k}). \tag{10}$$

If $V_n \in C^r$ and $\bar{V}_n \in C^2$, then by Lemma 4, $\bar{V}_n \in C^3$ and (10) hold. Because Sinyukov's system (3), (7) and (10) is closed, we can differentiate equations (3) $(r-1)$ times. So we convince ourselves that $a_{ij} \in C^r$, and also $\bar{g}_{ij} \in C^r$ ($\equiv \bar{V}_n \in C^r$).

Remark 5. Because for holomorphically projective mappings of Kähler (and also hyperbolic and parabolic Kähler) spaces hold equations analogical to (3) and (7), see [7, 9, 12], from Lemma 3 follows an analog to Theorem 2 for these mappings.

5. On geodesic mappings of Einstein spaces

Geodesic mappings of Einstein spaces were studied by many authors starting with A.Z. Petrov (see [10]). Einstein spaces V_n are characterized by the condition $Ric = \mathrm{const} \cdot g$, so $V_n \in C^2$ would be sufficient. But many properties of Einstein spaces appear when $V_n \in C^3$ and $n > 3$. An Einstein space V_3 is a space of constant curvature.

We continue with geodesic mappings of Einstein spaces $V_n \in C^3$. On basis of Theorem 2 it is natural to suppose that $\bar{V}_n \in C^3$. In 1978 (see PhD thesis [3] and [4]) Mikeš proved that under these conditions the following theorem holds:

Theorem 6. *If the Einstein space V_n admits a nontrivial geodesic mapping onto a (pseudo-) Riemannian space $\bar{V}_n$, then $\bar{V}_n$ is an Einstein space.*

Proof. Let the Einstein space $V_n \in C^3$ (for which $R_{ij} = -K\,(n-1)\,g_{ij}$) admit a nontrivial geodesic mapping onto $\bar{V}_n \in C^2$. Then the Sinyukov equations (3) hold; their integrability conditions have the form (6). Taking (3) into account, we differentiate (6) with respect to x^m, contract the result with g^{lm}, and then we alternate with respect to i, k. By (8), we get $\lambda_\alpha R^\alpha_{ijk} = g_{ij}\xi_k - g_{ik}\xi_j$, where ξ_i is some vector. Contracting the latter with g^{ij} and using (8) we see that $\xi_i = K\lambda_i$, that is, the formula reads $\lambda_\alpha R^\alpha_{ijk} = K(g_{ij}\lambda_k - g_{ik}\lambda_j)$.

We contract (6) with λ^l. Considering the last formula, we get

$$g_{ki}\Lambda_{j\alpha}\lambda^\alpha + g_{kj}\Lambda_{i\alpha}\lambda^\alpha - \lambda_i\Lambda_{jk} - \lambda_j\Lambda_{ik} = 0, \tag{11}$$

where $\Lambda_{ij} = \lambda_{i,j} - K a_{ij}$. It is easy to show that $\lambda^\alpha\Lambda_{\alpha i} = \mu\lambda_i$, where μ is a function. Since $\lambda_i \neq 0$, we find from (11) that

$$\lambda_{i,j} = \mu\, g_{ij} + K\, a_{ij}. \tag{12}$$

Differentiating (4b) and considering (2), (3), (4), it is easy to get the following equation:

$$\psi_{ij} \equiv \psi_{i,j} - \psi_i\psi_j = \bar{K}\, g_{ij} - K\bar{g}_{ij}, \tag{13}$$

where $\bar{K}$ is a function. Then from (5b), by virtue of the last relation, and considering $R_{ij} = -K\,(n-1)\,g_{ij}$, we get that $\bar{R}_{ij} = (n-1)\bar{K}\,\bar{g}_{ij}$. Hence $\bar{V}_n$ is an Einstein space. The theorem is proved. □

Theorem 6 was proved "locally" but it is easy to show that when the domain of validity of equations (13) borders with a domain where $\psi_i \equiv 0$, then in this domain $\psi_i \equiv 0$. Assume a point x_0 on the borders between these domains, then $\psi_i(x_0) = 0$ and $\psi_{ij} = 0$. Indeed **a)** If $K \neq 0$ or $\bar{K} \neq 0$ then $\bar{g}_{ij}(x_0) = \bar{K}/K\ g_{ij}(x_0)$. From these properties follows that the system of equations (2) and (13) has a unique solution $\bar{g}_{ij} = \bar{K}/K\ g_{ij}$ and $\psi_i = 0$. **b)** If $K = \bar{K} = 0$ then equations (13): $\psi_{i,j} = \psi_i\psi_j$ have a unique solution for $\psi_i(x_0) = 0$: $\psi_i = 0$.

This Theorem was used for geodesic mappings of 4-dimensional Einstein spaces (Mikeš, Kiosak [8]) and to find metrics of Einstein spaces that admit geodesic mappings (Formella, Mikeš [2]), etc.

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Irena Hinterleitner
Dept. of Math., Brno University of Technology,
Žižkova 17, CZ 602 00 Brno, Czech Republic
e-mail: `hinterleitner.i@fce.vutbr.cz`

Josef Mikeš
Dept. of Algebra and Geometry, Palacky University,
17. listopadu 12, CZ 77146 Olomouc, Czech Republic
e-mail: `josef.mikes@upol.cz`

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GEODESIC MAPPING ONTO KÄHLERIAN SPACES OF THE FIRST KIND

Milan Zlatanović, Niš, Irena Hinterleitner, Brno,
Marija Najdanović, Niš

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Abstract. In the present paper a generalized Kählerian space $\mathbb{G}\underset{1}{\mathbb{K}}_N$ of the first kind is considered as a generalized Riemannian space $\mathbb{G}\mathbb{R}_N$ with almost complex structure F_i^h that is covariantly constant with respect to the first kind of covariant derivative.

Using a non-symmetric metric tensor we find necessary and sufficient conditions for geodesic mappings $f\colon \mathbb{G}\mathbb{R}_N \to \mathbb{G}\underset{1}{\overline{\mathbb{K}}}_N$ with respect to the four kinds of covariant derivatives. These conditions have the form of a closed system of partial differential equations in covariant derivatives with respect to unknown components of the metric tensor and the complex structure of the Kählerian space $\mathbb{G}\underset{1}{\mathbb{K}}_N$.

Keywords: geodesic mapping; equitorsion geodesic mapping; generalized Kählerian space

MSC 2010: 53B05, 53B35

1. Introduction

Geodesic mappings of Kählerian manifolds have been studied by many authors. We continue the general idea by introducing the notion of generalized Kählerian spaces of the first kind $\mathbb{G}\underset{1}{\mathbb{K}}_N$, which generalize Kählerian spaces in the spirit of Einstein's Unified Field Theory and Moffat's non-symmetrical gravitational theory. This paper is devoted to the study of geodesic mappings of *generalized Riemannian spaces* to *generalized Kählerian spaces of the first kind* $\mathbb{G}\underset{1}{\mathbb{K}}_N$,

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The main results of the paper: New explicit formulas of geodesic mappings onto $\underset{1}{\mathbb{GK}}_N$ are given in Subsection 3.1, new explicit formulas of equitorsion geodesic mappings onto $\underset{1}{\mathbb{GK}}_N$ in Subsection 3.2.

In a similar way we can consider generalized Kählerian spaces of the second, the third and the fourth kind.

2. Generalized Kählerian spaces of the first kind

2.1. Generalized Riemannian spaces. A *generalized Riemannian space* $\mathbb{GR}_N$ in the sense of Eisenhart's definition [5] is a differentiable N-dimensional manifold, equipped with a non-symmetric metric tensor g_{ij} (i.e. $g_{ij} \neq g_{ji}$). The symmetric and the antisymmetric parts of g_{ij} are

$$g_{\underline{ij}} = \frac{1}{2}(g_{ij} + g_{ji}) = \frac{1}{2}g_{(ij)}, \quad g_{\underset{\vee}{ij}} = \frac{1}{2}(g_{ij} - g_{ji}) = \frac{1}{2}g_{[ij]}.$$

The lowering and the rising of indices are defined by the tensors $g_{\underline{ij}}$ and $g^{\underline{ij}}$, respectively, where $g^{\underline{ij}}$ is defined by the equation

$$g_{\underline{ij}}g^{\underline{jk}} = \delta_i^k \tag{2.1}$$

(δ_i^k is the Kronecker symbol). From (2.1) we have that the matrix $(g^{\underline{ij}})$ is inverse to $(g_{\underline{ij}})$, wherefrom it is necessary that $g = \det(g_{\underline{ij}}) \neq 0$. Connection coefficients of this space are generalized Christoffel symbols of the second kind, where

$$\Gamma^i_{jk} = g^{\underline{ip}}\Gamma_{p.jk}, \quad \Gamma_{i.jk} = \frac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j}), \quad g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k}.$$

Generally $\Gamma^i_{jk} \neq \Gamma^i_{kj}$. Therefore, one can define the symmetric and the antisymmetric part of Γ^i_{jk}, respectively, by

$$\Gamma^i_{\underline{jk}} = \frac{1}{2}(\Gamma^i_{jk} + \Gamma^i_{kj}) = \frac{1}{2}\Gamma^i_{(jk)}, \quad \Gamma^i_{\underset{\vee}{jk}} = \frac{1}{2}(\Gamma^i_{jk} - \Gamma^i_{kj}) = \frac{1}{2}\Gamma^i_{[jk]}.$$

The quantity $\Gamma^i_{\underset{\vee}{jk}}$ is the *torsion tensor* of the spaces $\mathbb{GR}_N$.

The use of a non-symmetric metric tensor and a non-symmetric connection became especially topical after the appearance of the papers of A. Einstein [2]–[4] related to the attempt to formulate a Unified Field Theory (UFT). We remark that in UFT the symmetric part $g_{\underline{ij}}$ of g_{ij} is related to gravitation, and the antisymmetric one $g_{\underset{\vee}{ij}}$ to electromagnetism. More recently the ides of a non-symmetric metric tensor appears in Moffat's non-symmetric gravitational theory [17]. In Moffat's theory the antisymmetric part represents a Proca field (massive Maxwell field) which is part of the gravitational interaction, contributing to the rotation of galaxies.

Based on the non-symmetry of the connection in a generalized Riemannian space one can define four kinds of covariant derivatives. For example, for a tensor a^i_j in $\mathbb{GR}_N$ we have

$$a^i_{j\underset{1}{|}m} = a^i_{j,m} + \Gamma^i_{pm}a^p_j - \Gamma^p_{jm}a^i_p, \quad a^i_{j\underset{2}{|}m} = a^i_{j,m} + \Gamma^i_{mp}a^p_j - \Gamma^p_{mj}a^i_p,$$
$$a^i_{j\underset{3}{|}m} = a^i_{j,m} + \Gamma^i_{pm}a^p_j - \Gamma^p_{mj}a^i_p, \quad a^i_{j\underset{4}{|}m} = a^i_{j,m} + \Gamma^i_{mp}a^p_j - \Gamma^p_{jm}a^i_p.$$

By applying four kinds of covariant derivatives of tensors, it is possible to construct several Ricci type identities. In these identities 12 curvature tensors appear as well as 15 quantities, which are not tensors, named "curvature pseudotensors" by S. M. Minčić [12], [13]. In the case of the space $\mathbb{GR}_N$ we have five independent curvature tensors.

2.2. Generalized Kählerian space of the first kind. Kählerian spaces and their mappings were investigated by many authors, for example T. Otsuki and Y. Tasiro [18], [25], K. Yano [26], J. Mikeš, V. V. Domashev [1], [6], [7], [8], [9], [10], [11], [22], M. Prvanović [19], N. Pušić [21], S. S. Pujar [20], M. S. Stanković at al. [16], [24], and many others.

An N-dimensional Riemannian space with metric tensor g_{ij} is a *Kählerian space* $\mathbb{K}_N$ if there exists an almost complex structure F^i_j such that

$$F^h_p F^p_i = -\delta^h_i,$$
$$g_{pq}F^p_i F^q_j = g_{ij}, \quad g^{ij} = g^{pq}F^i_p F^j_q,$$
$$F^h_{i;j} = 0,$$

where ";" denotes the covariant derivative with respect to the metric tensor g_{ij}.

Definition 2.1. A generalized N-dimensional Riemannian space with non-symmetric metric tensor g_{ij} is a *generalized Kählerian space of the first kind* $\underset{1}{\mathbb{GK}}_N$ if there exists an almost complex structure F^i_j such that

$$F^h_p F^p_i = -\delta^h_i, \tag{2.2}$$
$$g_{\underline{pq}}F^p_i F^q_j = g_{\underline{ij}}, \quad g^{\underline{ij}} = g^{\underline{pq}}F^i_p F^j_q, \tag{2.3}$$
$$F^h_{i\underset{1}{|}j} = 0, \quad F^h_{i;j} = 0,$$

where "$\underset{1}{|}$" denotes the covariant derivative of the first kind with respect to the connection Γ^i_{jk} ($\Gamma^i_{jk} \neq \Gamma^i_{kj}$) and ";" denotes the covariant derivative with respect to the symmetric part of the metric tensor $\Gamma^i_{\underline{jk}}$.

From (2.3), using (2.2), we get $F_{ij} = -F_{ji}$ and $F^{ij} = -F^{ji}$, where we denote $F_{ji} = F^p_j g_{\underline{pi}}$, $F^{ji} = F^j_p g^{\underline{pi}}$.

The following theorem holds.

Theorem 2.1 ([23]). *For the almost complex structure F^i_j of $\mathbb{G}\underset{1}{\mathbb{K}}_N$ the relations*

$$F^h_{i\underset{2}{|}j} = 0, \quad F^h_{i\underset{3}{|}j} = 2F^h_p \Gamma^p_{\underset{\vee}{ij}}, \quad F^h_{i\underset{4}{|}j} = 2F^p_i \Gamma^h_{\underset{\vee}{jp}}$$

are valid, where $\Gamma^h_{\underset{\vee}{ij}}$ is the torsion tensor.

3. Geodesic mapping

3.1. Geodesic mapping between generalized Kählerian spaces of the first kind. In this part we consider geodesic mappings $f\colon \mathbb{GR}_N \to \mathbb{G}\underset{1}{\overline{\mathbb{K}}}_N$.

Definition 3.1. A diffeomorphism $f\colon \mathbb{GR}_N \to \mathbb{G}\underset{1}{\overline{\mathbb{K}}}_N$ is *geodesic*, if geodesics of the space $\mathbb{GR}_N$ are mapped to geodesics of the space $\mathbb{G}\underset{1}{\overline{\mathbb{K}}}_N$.

At the corresponding points M and $\overline{M}$ we can put

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + P^i_{jk} \quad (i,j,k = 1,\dots,N), \tag{3.1}$$

where P^i_{jk} is *the deformation tensor* of the connection Γ of $\mathbb{GR}_N$ corresponding to the mapping $f\colon \mathbb{GR}_N \to \mathbb{G}\underset{1}{\overline{\mathbb{K}}}_N$.

Theorem 3.1 ([14]). *A necessary and sufficient condition for the mapping $f\colon \mathbb{GR}_N \to \mathbb{G}\underset{1}{\overline{\mathbb{K}}}_N$ to be geodesic is that the deformation tensor P^i_{jk} from* (3.1) *has the form*

$$P^i_{jk} = \delta^i_j \psi_k + \delta^i_k \psi_j + \xi^i_{jk}, \tag{3.2}$$

where

$$\psi_i = \frac{1}{N+1}(\bar{\Gamma}^\alpha_{i\alpha} - \Gamma^\alpha_{i\alpha}), \quad \xi^i_{jk} = P^i_{\underset{\vee}{jk}} = \frac{1}{2}(P^i_{jk} - P^i_{kj}).$$

We remark that in $\mathbb{G}\underset{1}{\mathbb{K}}_N$ the following equations are valid:

$$\Gamma^\alpha_{\underset{\vee}{i\alpha}} = 0, \quad \xi^\alpha_{i\alpha} = 0, \quad F^\alpha_\alpha = 0.$$

In [11] Mikeš et al. proved necessary and sufficient conditions for geodesic mappings of a Riemannian space onto a Kählerian space.

Theorem 3.2. *The Riemannian space $\mathbb{R}_N$ admits a nontrivial geodesic mapping onto the Kählerian space $\overline{\mathbb{K}}_N$ with metric $\overline{g}_{ij}$ and complex structure $\overline{F}_i^h$ satisfying*

$$\overline{g}_{ij} = \overline{g}_{ji}, \quad \det(\overline{g}_{ij}) \neq 0, \quad \overline{F}_i^p \overline{g}_{pj} + \overline{F}_j^p \overline{g}_{pi} = 0, \quad \overline{F}_p^h \overline{F}_i^p = -\delta_i^h,$$

if and only if, in the common coordinate system x with respect to the mapping, the conditions

a) $\overline{g}_{ij;k} = 2\psi_k \overline{g}_{ij} + \psi_i \overline{g}_{jk} + \psi_j \overline{g}_{ik}$;
b) $\overline{F}^h_{i;k} = \overline{F}^h_k \psi_i - \delta^h_k \overline{F}^\alpha_i \psi_\alpha$

hold, where $\psi_i \neq 0$.

Our idea is to find the corresponding equations with respect to the four kinds of covariant derivative.

In all the following theorems concerning mappings from a generalized Riemannian space onto a generalized Kählerian space, $\overline{g}_{ij}$ and $\overline{F}_i^j$ denote the metric and the almost complex structure of $\mathbb{G}\underset{1}{\overline{\mathbb{K}}}_N$, respectively, satisfying

$$\overline{g}_{ij} \neq \overline{g}_{ji}, \quad \det(\overline{g}_{ij}) \neq 0, \quad \overline{F}_i^p \overline{g}_{\underline{pj}} + \overline{F}_j^p \overline{g}_{\underline{pi}} = 0, \quad \overline{F}_p^h \overline{F}_i^p = -\delta_i^h. \tag{3.3}$$

Theorem 3.3. *The generalized Riemannian space $\mathbb{GR}_N$ admits a nontrivial geodesic mapping onto the generalized Kählerian space $\mathbb{G}\underset{1}{\overline{\mathbb{K}}}_N$ if and only if, in the common coordinate system x with respect to the mapping, the conditions*

$$\begin{aligned} &\text{a)} \quad \overline{g}_{ij\underset{1}{|}k} = \overline{g}_{ij\underset{\vee 1}{\overline{|}}k} + 2\psi_k \overline{g}_{ij} + \psi_i \overline{g}_{jk} + \psi_j \overline{g}_{ik} + \xi^\alpha_{ik} \overline{g}_{\alpha j} + \xi^\alpha_{jk} \overline{g}_{i\alpha}; \\ &\text{b)} \quad \overline{F}^h_{i\underset{1}{|}k} = \overline{F}^h_k \psi_i - \delta^h_k \overline{F}^\alpha_i \psi_\alpha - \xi^h_{\alpha k} \overline{F}^\alpha_i + \xi^\alpha_{ik} \overline{F}^h_\alpha, \end{aligned} \tag{3.4}$$

hold with respect to the first kind of covariant derivatives, where $\psi_i \neq 0$.

P r o o f. Equation (3.4) a) guarantees the existence of a geodesic mapping from the generalized Riemannian space $\mathbb{GR}_N$ onto the generalized Riemannian space $\mathbb{G}\overline{\mathbb{R}}_N$ with metric tensor $\overline{g}_{ij}$ with respect to the first kind of covariant derivatives (see [15]).

Formula (3.4) b) implies that the structure $\overline{F}_i^h$ in $\mathbb{G}\overline{\mathbb{R}}_N$ is covariantly constant with respect to the first kind of covariant derivative. The algebraic conditions (3.3) guarantee that $\overline{g}_{ij}$ and $\overline{F}_i^h$ are the metric tensor and the structure of $\mathbb{G}\underset{1}{\overline{\mathbb{K}}}_N$, respectively.

The deformation tensor is determined by equation (3.2), i.e.,

$$\overline{\Gamma}^h_{ij} - \Gamma^h_{ij} = \psi_i \delta^h_j + \psi_j \delta^h_i + \xi^h_{ij}. \tag{3.5}$$

For the structure $\overline{F}$, we have the following equations:

$$\overline{F}^h_{i\underset{1}{|}k} = \overline{F}^h_{i,k} + \Gamma^h_{pk}\overline{F}^p_i - \Gamma^p_{ik}\overline{F}^h_p, \quad \overline{F}^h_{i\underset{2}{|}k} = \overline{F}^h_{i,k} + \Gamma^h_{kp}\overline{F}^p_i - \Gamma^p_{ki}\overline{F}^h_p. \tag{3.6}$$

Replacing Γ^h_{ij} from (3.5) in (3.6), we get

$$\begin{aligned}
\overline{F}^h_{i\underset{1}{|}k} &= \overline{F}^h_{i,k} + (\overline{\Gamma}^h_{pk} - \psi_p\delta^h_k - \psi_k\delta^h_p - \xi^h_{pk})\overline{F}^p_i - (\overline{\Gamma}^p_{ik} - \psi_i\delta^p_k - \psi_k\delta^p_i - \xi^p_{ik})\overline{F}^h_p \\
&= \overline{F}^h_{i,k} + \overline{\Gamma}^h_{pk}\overline{F}^p_i - \psi_p\delta^h_k\overline{F}^p_i - \psi_k\delta^h_p\overline{F}^p_i - \xi^h_{pk}\overline{F}^p_i - \overline{\Gamma}^p_{ik}\overline{F}^h_p \\
&\quad + \psi_i\delta^p_k\overline{F}^h_p + \psi_k\delta^p_i\overline{F}^h_p + \xi^p_{ik}\overline{F}^h_p \\
&= \overline{F}^h_{i\underset{1}{\overline{|}}k} - \psi_p\delta^h_k\overline{F}^p_i - \psi_k\delta^h_p\overline{F}^p_i - \xi^h_{pk}\overline{F}^p_i + \psi_i\delta^p_k\overline{F}^h_p + \psi_k\delta^p_i\overline{F}^h_p + \xi^p_{ik}\overline{F}^h_p \\
&= \overline{F}^h_{i\underset{1}{\overline{|}}k} - \psi_p\delta^h_k\overline{F}^p_i - \psi_k\overline{F}^h_i - \xi^h_{pk}\overline{F}^p_i + \psi_i\overline{F}^h_k + \psi_k\overline{F}^h_i + \xi^p_{ik}\overline{F}^h_p \\
&= \underbrace{\overline{F}^h_{i\underset{1}{\overline{|}}k}}_{0} - \psi_p\delta^h_k\overline{F}^p_i + \psi_i\overline{F}^h_k - \xi^h_{pk}\overline{F}^p_i + \xi^p_{ik}\overline{F}^h_p,
\end{aligned}$$

where "$|$", and "$\overline{|}$" are covariant derivatives in $\mathbb{GR}_N$ and $\mathbb{G}\underset{1}{\overline{\mathbb{K}}}_N$, respectively. □

Theorem 3.4. *The generalized Riemannian space $\mathbb{GR}_N$ admits a nontrivial geodesic mapping onto the generalized Kählerian space $\mathbb{G}\underset{1}{\overline{\mathbb{K}}}_N$ if and only if, in the common coordinate system x with respect to the mapping, the conditions*

a) $\overline{g}_{ij\underset{2}{|}k} = \overline{g}_{\underset{\vee}{ij}\underset{2}{\overline{|}}k} + 2\psi_k\overline{g}_{ij} + \psi_i\overline{g}_{jk} + \psi_j\overline{g}_{ik} + \xi^\alpha_{ki}\overline{g}_{\alpha j} + \xi^\alpha_{kj}\overline{g}_{i\alpha}$;

b) $\overline{F}^h_{i\underset{2}{|}k} = \overline{F}^h_k\psi_i - \delta^h_k\overline{F}^\alpha_i\psi_\alpha - \xi^h_{k\alpha}\overline{F}^\alpha_i + \xi^\alpha_{ki}\overline{F}^h_\alpha$,

hold with respect to the second kind of covariant derivatives, where $\psi_i \neq 0$.

P r o o f. For the second kind of covariant derivatives in $\mathbb{GR}_N$, we have

$$\begin{aligned}
\overline{F}^h_{i\underset{2}{|}k} &= \overline{F}^h_{i,k} + (\overline{\Gamma}^h_{kp} - \psi_k\delta^h_p - \psi_p\delta^h_k - \xi^h_{kp})\overline{F}^p_i - (\overline{\Gamma}^p_{ki} - \psi_k\delta^p_i - \psi_i\delta^p_k - \xi^p_{ki})\overline{F}^h_p \\
&= \overline{F}^h_{i,k} + \overline{\Gamma}^h_{kp}\overline{F}^p_i - \psi_k\delta^h_p\overline{F}^p_i - \psi_p\delta^h_k\overline{F}^p_i - \xi^h_{kp}\overline{F}^p_i - \overline{\Gamma}^p_{ki}\overline{F}^h_p \\
&\quad + \psi_k\delta^p_i\overline{F}^h_p + \psi_i\delta^p_k\overline{F}^h_p + \xi^p_{ki}\overline{F}^h_p \\
&= \overline{F}^h_{i\underset{2}{\overline{|}}k} - \psi_k\delta^h_p\overline{F}^p_i - \psi_p\delta^h_k\overline{F}^p_i - \xi^h_{kp}\overline{F}^p_i + \psi_k\delta^p_i\overline{F}^h_p + \psi_i\delta^p_k\overline{F}^h_p + \xi^p_{ki}\overline{F}^h_p \\
&= \overline{F}^h_{i\underset{2}{\overline{|}}k} - \psi_k\overline{F}^h_i - \psi_p\delta^h_k\overline{F}^p_i - \xi^h_{kp}\overline{F}^p_i + \psi_k\overline{F}^h_i + \psi_i\overline{F}^h_k + \xi^p_{ki}\overline{F}^h_p \\
&= \psi_i\overline{F}^h_k - \psi_p\delta^h_k\overline{F}^p_i - \xi^h_{kp}\overline{F}^p_i + \xi^p_{ki}\overline{F}^h_p.
\end{aligned}$$

□

In a similar way, we can prove the corresponding theorems for the third and the fourth kind of covariant derivative:

Theorem 3.5. *The generalized Riemannian space $\mathbb{GR}_N$ admits a nontrivial geodesic mapping onto the generalized Kählerian space $\mathbb{G}\underset{1}{\overline{\mathbb{K}}}_N$ if and only if, in the common coordinate system x with respect to the mapping, the conditions*

a) $\overline{g}_{ij\underset{3}{|}k} = \overline{g}_{\underset{\vee}{ij}\underset{3}{\overline{|}}k} + 2\psi_k\overline{g}_{ij} + \psi_i\overline{g}_{jk} + \psi_j\overline{g}_{ik} + \xi^{\alpha}_{ik}\overline{g}_{\alpha j} + \xi^{\alpha}_{kj}\overline{g}_{i\alpha}$;

b) $\overline{F}^h_{i\underset{3}{|}k} = \overline{F}^h_{i\underset{3}{\overline{|}}k} - \psi_p\delta^h_k\overline{F}^p_i + \psi_i\overline{F}^h_k - \xi^h_{pk}\overline{F}^p_i + \xi^p_{ki}\overline{F}^h_p$,

hold with respect to the third kind of covariant derivatives, where $\psi_i \neq 0$.

Theorem 3.6. *The generalized Riemannian space $\mathbb{GR}_N$ admits a nontrivial geodesic mapping onto the generalized Kählerian space $\mathbb{G}\underset{1}{\overline{\mathbb{K}}}_N$ if and only if, in the common coordinate system x with respect to the mapping, the conditions*

a) $\overline{g}_{ij\underset{4}{|}k} = \overline{g}_{\underset{\vee}{ij}\underset{4}{\overline{|}}k} + 2\psi_k\overline{g}_{ij} + \psi_i\overline{g}_{jk} + \psi_j\overline{g}_{ik} + \xi^{\alpha}_{ki}\overline{g}_{\alpha j} + \xi^{\alpha}_{jk}\overline{g}_{i\alpha}$;

b) $\overline{F}^h_{i\underset{4}{|}k} = \overline{F}^h_{i\underset{4}{\overline{|}}k} - \psi_p\delta^h_k\overline{F}^p_i + \psi_i\overline{F}^h_k - \xi^h_{kp}\overline{F}^p_i + \xi^p_{ik}\overline{F}^h_p$,

hold with respect to the fourth kind of covariant derivatives, where $\psi_i \neq 0$.

3.2. Equitorsion geodesic mapping. Equitorsion mappings play an important role in the theories of geodesic, conformal and holomorphically projective transformations between two spaces of non-symmetric affine connection.

Definition 3.2 ([14]). A mapping $f\colon \mathbb{GR}_N \to \mathbb{G}\underset{1}{\overline{\mathbb{K}}}_N$ is an *equitorsion geodesic mapping* if the torsion tensors of the spaces $\mathbb{GR}_N$ and $\mathbb{G}\underset{1}{\overline{\mathbb{K}}}_N$ are equal. Then from (3.1), (3.2) and (3.5):

$$\overline{\Gamma}^h_{\underset{\vee}{ij}} - \Gamma^h_{\underset{\vee}{ij}} = \xi^h_{ij} = 0,$$

where $\underset{\vee}{ij}$ denotes an antisymmetrization with respect to i, j.

In the case of these mappings, the previous Theorems 3.3–3.6 become:

Theorem 3.7. *The generalized Riemannian space $\mathbb{GR}_N$ admits a nontrivial equitorsion geodesic mapping onto the generalized Kählerian space $\mathbb{G}\underset{1}{\overline{\mathbb{K}}}_N$ if and only if, in the common coordinate system x with respect to the mapping, the conditions*

a) $\overline{g}_{\underline{ij}\underset{1}{|}k} = 2\psi_k\overline{g}_{\underline{ij}} + \psi_i\overline{g}_{\underline{jk}} + \psi_j\overline{g}_{\underline{ik}}$;

b) $\overline{F}^h_{i\underset{1}{|}k} = \overline{F}^h_k\psi_i - \delta^h_k\overline{F}^p_i\psi_p$,

hold with respect to the first kind of covariant derivatives, where $\psi_i \neq 0$.

Theorem 3.8. *The generalized Riemannian space $\mathbb{GR}_N$ admits a nontrivial equitorsion geodesic mapping onto the generalized Kählerian space $\mathbb{G}\underset{1}{\overline{\mathbb{K}}}_N$ if and only if, in the common coordinate system x with respect to the mapping, the conditions*

a) $\overline{g}_{\underline{ij}\underset{2}{|}k} = 2\psi_k\overline{g}_{\underline{ij}} + \psi_i\overline{g}_{\underline{jk}} + \psi_j\overline{g}_{\underline{ik}}$;

b) $\overline{F}^h_{i\underset{2}{|}k} = \overline{F}^h_k\psi_i - \delta^h_k\overline{F}^p_i\psi_p$,

hold with respect to the second kind of covariant derivatives, where $\psi_i \neq 0$.

Theorem 3.9. *The generalized Riemannian space $\mathbb{GR}_N$ admits a nontrivial equitorsion geodesic mapping onto the generalized Kählerian space $\mathbb{G}\underset{1}{\overline{\mathbb{K}}}_N$ if and only if, in the common coordinate system x with respect to the mapping, the conditions*

a) $\overline{g}_{\underline{ij}\underset{3}{|}k} = 2\psi_k\overline{g}_{\underline{ij}} + \psi_i\overline{g}_{\underline{jk}} + \psi_j\overline{g}_{\underline{ik}}$;

b) $\overline{F}^h_{i\underset{3}{|}k} = \overline{F}^h_{\bar{i}\underset{3}{|}k} - \psi_p\delta^h_k\overline{F}^p_i + \psi_i\overline{F}^h_k$,

hold with respect to the third kind of covariant derivatives, where $\psi_i \neq 0$.

Theorem 3.10. *The generalized Riemannian space $\mathbb{GR}_N$ admits a nontrivial equitorsion geodesic mapping onto the generalized Kählerian space $\mathbb{G}\underset{1}{\overline{\mathbb{K}}}_N$ if and only if, in the common coordinate system x with respect to the mapping, the conditions*

a) $\overline{g}_{\underline{ij}\underset{4}{|}k} = 2\psi_k\overline{g}_{\underline{ij}} + \psi_i\overline{g}_{jk} + \psi_j\overline{g}_{\underline{ik}}$;

b) $\overline{F}^h_{i\underset{4}{|}k} = \overline{F}^h_{\bar{i}\underset{4}{|}k} - \psi_p\delta^h_k\overline{F}^p_i + \psi_i\overline{F}^h_k$,

hold with respect to the fourth kind of covariant derivatives, where $\psi_i \neq 0$.

4. Conclusion

We have shown that the notions of geodesic and equitorsion geodesic mappings from Riemannian to Kählerian spaces can be generalized to the case of a non-symmetric metric, and we have given necessary and sufficient conditions for nontrivial such mappings.

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Authors' addresses: Milan Zlatanović, Department of Mathematics, Faculty of Science and Mathematics, University of Niš, Višegradska 33, 18000 Niš, Serbia, e-mail: zlatmilan@pmf.ni.ac.rs; Irena Hinterleitner, University of Technology, Faculty of Civil Engineering, Žižkova 17, 602 00 Brno, Czech Republic, e-mail: irenka1@post.cz; Marija Najdanović, Department of Mathematics, Faculty of Science and Mathematics, University of Niš, Višegradska 33, 180 00 Niš, Serbia, e-mail: marijamath@yahoo.com.

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Geodesic Mappings and Differentiability of Metrics, Affine and Projective Connections

Irena Hinterleitner[a], Josef Mikeš[b]

[a] *Brno University of Technology, Faculty of Civil Engineering, Dept. of Math.*
[b] *Palacky University Olomouc, Dept. Algebra and Geometry*

Abstract. In this paper we study fundamental equations of geodesic mappings of manifolds with affine and projective connection onto (pseudo-) Riemannian manifolds with respect to the smoothness class of these geometric objects. We prove that the natural smoothness class of these problems is preserved.

1. Introduction and Basis Definitions

To theory of geodetic mappings and transformations were devoted many papers, these results are formulated in large number of researchs and monographs [1], [2], [3], [4], [5], [7], [8], [9], [10], [11], [12], [13], [14], [16], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [30], [31], [32], [33], [34], [35], [36], [37], etc.

First we studied the general dependence of geodesic mappings of manifolds with affine and projective connection onto (pseudo-) Riemannian manifolds in dependence on the smoothness class of these geometric objects. We presented well known facts, which were proved by H. Weyl [37], T. Thomas [35], J. Mikeš and V. Berezovski [21], see [5], [20], [25], [26], [30], [32], [36].

In these results no details about the smoothness class of the metric, resp. connection, were stressed. They were formulated as "for sufficiently smooth" geometric objects.

In the paper [14, 15] we proved that these mappings preserve the smoothness class of metrics of geodetically equivalent (pseudo-) Riemannian manifolds. We prove that this property generalizes in a natural way for a more general case.

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Email addresses: hinterleitner.irena@seznam.cz (Irena Hinterleitner), josef.mikes@upol.cz (Josef Mikeš)

2. Geodesic Mapping Theory for Manifolds with Affine and Projective Connections

Let $A_n = (M, \nabla)$ and $\bar{A}_n = (\bar{M}, \bar{\nabla})$ be manifolds with affine connections ∇ and $\bar{\nabla}$, respectively, without torsion.

Definition 2.1. *A diffeomorphism $f\colon A_n \to \bar{A}_n$ is called a geodesic mapping of A_n onto $\bar{A}_n$ if f maps any geodesic in A_n onto a geodesic in $\bar{A}_n$.*

A manifold A_n admits a geodesic mapping onto $\bar{A}_n$ if and only if the *Levi-Civita equations* (H. Weyl [37], see [5, p. 56], [25, p. 130], [26, p. 166], [32, p. 72]):

$$\bar{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X \tag{1}$$

hold for any tangent fields X, Y and where ψ is a differential form on M $(= \bar{M})$. If $\psi \equiv 0$ then f is *affine* or *trivially geodesic*.

Eliminating ψ from the formula (1) T. Thomas [35], see [5, p. 98], [25, p. 132], obtained that equation (1) is equivalent to

$$\bar{\Pi}(X, Y) = \Pi(X, Y) \text{ for all tangent vectors } X, Y, \tag{2}$$

where

$$\Pi(X, Y) = \nabla(X, Y) - \frac{1}{n+1}\,(trace(V \to \nabla_V X) \cdot Y + trace(V \to \nabla_V Y) \cdot X)$$

is the *Thomas' projective parameter* or *Thomas' object of projective connection*.

A geometric object Π that transforms according to a similar transformation law as Thomas' projective parameters is called a *projective connection*, and manifolds on which an object of projective connection is defined is called a *manifold with projective connection*, denoted by P_n. Such manifolds represent an obvious generalization of affine connection manifolds.

A projective connection on P_n will be denoted by $\blacktriangledown$. Obviously, $\blacktriangledown$ is a mapping $TP_n \times TP_n \to TP_n$, i.e. $(X, Y) \mapsto \blacktriangledown_X Y$. Thus, we denote a manifold M with projective connection $\blacktriangledown$ by $P_n = (M, \blacktriangledown)$. See [5, p. 99], [6].

We restricted ourselves to the study of a coordinate neighborhood (U, x) of the points $p \in A_n$ (P_n) and $f(p) \in \bar{A}_n$ $(\bar{P}_n)$. The points p and $f(p)$ have the same coordinates $x = (x^1, \ldots, x^n)$.

We assume that A_n, $\bar{A}_n$, P_n, $\bar{P}_n \in C^r$ $(\nabla, \bar{\nabla}, \blacktriangledown, \bar{\blacktriangledown} \in C^r)$ if their components $\Gamma^h_{ij}(x), \bar{\Gamma}^h_{ij}(x), \Pi^h_{ij}(x), \bar{\Pi}^h_{ij}(x) \in C^r$ on (U, x), respectively. Here C^r is the smoothness class. On the other hand, the manifold M which these structures exist, must have a class smoothness C^{r+2}. This means that the atlas on M is of class C^{r+2}, i.e. for the non disjunct charts (U, x) and (U', x') on $(U \cap U')$ it is true that the transformation $x' = x'(x) \in C^{r+2}$.

Formulae (1) and (2) in the common system (U, x) have the local form:

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \psi_i(x)\delta^h_j + \psi_j(x)\delta^h_i \quad \text{and} \quad \bar{\Pi}^h_{ij}(x) = \Pi^h_{ij}(x),$$

respectively, where ψ_i are components of ψ and δ^h_i is the Kronecker delta.

It is seen that in a manifold $A_n = (M, \nabla)$ with affine connections ∇ there exists a projective connection $\blacktriangledown$ (i.e. Thomas projective parameter) with the same smoothness. The opposite statement is not valid, for example if $\nabla \in C^r$ ($\Rightarrow \blacktriangledown \in C^r$ and also $\bar{\blacktriangledown} \in C^r$) and $\psi(x) \in C^0$, then $\bar{\nabla} \in C^0$.

In the paper [12] we presented a construction that the existing ∇ on M guarantees on $P_n = (M, \blacktriangledown)$. Moreover, the following theorem holds:

Theorem 2.2. *An arbitrary manifold $P_n = (M, \blacktriangledown) \in C^r$ admits a global geodesic mapping onto a manifold $\bar{A}_n = (M, \bar{\nabla}) \in C^r$ and, moreover, for which a formula* trace$(V \to \bar{\nabla}_V)X = \nabla_X G$ *holds for arbitrary X and a function G on M, i.e. $\bar{A}_n$ is an equiaffine manifold and $\bar{\nabla}$ is an equiaffine connection. Moreover, if $r \geq 1$ the Ricci tensor on $\bar{A}_n$ is symmetric.*

Proof. It is known that on the whole manifold $M \in C^{r+2}$ exists globally a sufficiently smooth metric $\hat{g} \in C^{r+1}$. For our purpose it is sufficient if $\hat{g} \in C^{r+1}$, i.e. the components $\hat{g}_{ij}$ of $\hat{g}$ in a coordinate domain of M are functions of type C^{r+1}. We denote by $\hat{\nabla}$ the Levi-Civita connection of $\hat{g}_{ij}$, and, evidently, $\hat{\nabla} \in C^r$.

We define $\tau(X) = \frac{1}{n+1}\, trace(V \mapsto \hat{\nabla}_V X)$ and we construct $\bar{\nabla}$ in the following way

$$\bar{\nabla}_X Y = \blacktriangledown_X Y + \tau(X)\cdot Y + \tau(Y)\cdot X. \tag{3}$$

It is easily seen that $\bar{\nabla}$ constructed in this way is an affine connection on M. The components of the object $\bar{\nabla}$ in the coordinate system (U, x) can be written in the form: $\bar{\Gamma}^h_{ij}(x) = \Pi^h_{ij}(x) + \tau_i(x)\cdot\delta^h_j + \tau_j(x)\cdot\delta^h_i$ where Π^h_{ij} and $\bar{\Gamma}^h_{ij}$ are components of the projective connection $\blacktriangledown$ and the affine connection $\bar{\nabla}$, respectively, and $\tau_i = \frac{1}{n+1}\,\partial G/\partial x^i$, $G = \ln\sqrt{|\det\|\hat{g}_{ij}\||}$. It is obvious that P_n is geodesically mapped onto $\bar{A}_n = (M, \bar{\nabla})$, and, evidently because $\bar{\Gamma}^h_{ij} \in C^r$, $\bar{A}_n \in C^r$.

Insofar as $\Pi^\alpha_{\alpha i}(x) = 0$, then $\bar{\Gamma}^\alpha_{\alpha i}(x) = \partial G/\partial x^i$, i.e. trace$(V \to \bar{\nabla}_V)X = \nabla_X G$. Hence follows that $\bar{A}_n$ has an equiaffine connection [26, p. 151]. Moreover, if $\nabla \in C^1$ then the Ricci tensor *Ric* is symmetric ([25, p. 35], [26, p. 151]). □

3. Geodesic Mappings from Equiaffine Manifolds onto (pseudo-) Riemannian Manifolds

Let manifold $A_n = (M, \nabla) \in C^0$ admit a geodesic mapping onto a (pseudo-) Riemannian manifold $\bar{V}_n = (M, \bar{g}) \in C^1$, i.e. components $\bar{g}_{ij}(x) \in C^1(U)$. It is known [21], see [25, p. 145], that equations (1) are equivalent to the following Levi-Civita equations

$$\nabla_k \bar{g}_{ij} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi \bar{g}_{ik}. \tag{4}$$

If A_n is an equiaffine manifold then ψ have the following form

$$\psi_i = \partial_i \Psi, \quad \Psi = \frac{1}{n+1}\ln\sqrt{|\det\bar{g}|} - \rho, \quad \partial_i\rho = \frac{1}{n+1}\,\Gamma^\alpha_{\alpha i}, \quad \partial_i = \partial/\partial x^i,$$

and Mikeš and Berezovski [32], see [25, p. 150], proved that the Levi-Civita equations (1) and (4) are equivalent to

$$\nabla_k a^{ij} = \lambda^i \delta^j_k + \lambda^j \delta^i_k, \tag{5}$$

where

$$\text{(a)}\ \ a^{ij} = \mathrm{e}^{2\Psi}\,\bar{g}^{ij}; \quad \text{(b)}\ \ \lambda^i = -\,\mathrm{e}^{2\Psi}\,\bar{g}^{i\alpha}\psi_\alpha. \tag{6}$$

Here $\|\bar{g}^{ij}\| = \|\bar{g}_{ij}\|^{-1}$. On the other hand:

$$\bar{g}_{ij} = \mathrm{e}^{2\Psi}\hat{g}_{ij}, \quad \Psi = \ln\sqrt{|\det\hat{g}|} - \rho, \quad \|\hat{g}_{ij}\| = \|a^{ij}\|^{-1}. \tag{7}$$

Using the equation $\Pi^h_{ij}(x) = \Gamma^h_{ij}(x)$ (see (37.4) in [5, p. 105]), where Π is a projective connection and Γ is *normal* affine connection (it is also equi-affine), we after substitution $\Gamma^h_{ij}(x) \mapsto \Pi^h_{ij}(x)$ into (5) have equation (2.3) in [4], immediately.

Furthermore, we assume that $A_n = (M, \nabla) \in C^1$ and $\bar{V}_n = (M, \bar{g}) \in C^2$. In this case, the integrability conditions of the equations (5) from the Ricci identity $\nabla_l\nabla_k a^{ij} - \nabla_k\nabla_l a^{ij} = -a^{i\alpha}R^j_{\alpha kl} - a^{j\alpha}R^i_{\alpha kl}$ have the following form

$$-a^{i\alpha}R^j_{\alpha kl} - a^{j\alpha}R^i_{\alpha kl} = \delta^i_k\nabla_l\lambda^j + \delta^j_k\nabla_l\lambda^i - \delta^i_l\nabla_k\lambda^j - \delta^j_l\nabla_k\lambda^i, \tag{8}$$

where R^h_{ijk} are components of the curvature (Riemannian) tensor R on A_n, and after contraction of the indices i and k we get [21]

$$n\nabla_l\lambda^j = \mu\,\delta^j_l - a^{j\alpha}R_{\alpha l} - a^{\alpha\beta}R^j_{\alpha\beta l}\,, \tag{9}$$

where $\mu = \nabla_\alpha\lambda^\alpha$ and $R_{ij} = R^\alpha_{i\alpha j}$ are components of the Ricci tensor *Ric* on A_n.

4. Main Theorems

Let $V_n = (M, g) \in C^r$ be the (pseudo-) Riemannian manifold. If $r \geq 1$ then its natural affine connection $\nabla \in C^{r-1}$ (i.e. the Levi-Civita connection) and projective connection $\blacktriangledown \in C^{r-1}$; hence $A_n = (M, \nabla)$ and $P_n = (M, \blacktriangledown)$ be manifolds with affine and projective connection, respectively. The following theorems are true.

Theorem 4.1. *If $P_n \in C^{r-1}$ ($r > 2$) admits geodesic mappings onto a (pseudo-) Riemannian manifold $\bar{V}_n \in C^2$, then $\bar{V}_n \in C^r$.*

Theorem 4.2. *If $A_n \in C^{r-1}$ ($r > 2$) admits geodesic mappings onto a (pseudo-) Riemannian manifold $\bar{V}_n \in C^2$, then $\bar{V}_n \in C^r$.*

Based on the previous comments (at the end of the second section), it will be sufficient to prove the validity of the second Theorem. Moreover, the manifold A_n can be an equiaffine manifold.

The proof of the Theorem 4.2 follows from the following lemmas.

Lemma 4.3 ([13]). *Let $\lambda^h \in C^1$ be a vector field and ϱ a function. If $\partial_i\lambda^h - \varrho\,\delta^h_i \in C^1$ then $\lambda^h \in C^2$ and $\varrho \in C^1$.*

Proof. The condition $\partial_i\lambda^h - \varrho\,\delta^h_i \in C^1$ can be written in the following form

$$\partial_i\lambda^h - \varrho\delta^h_i = f^h_i(x), \tag{10}$$

where $f^h_i(x)$ are functions of class C^1. Evidently, $\varrho \in C^0$. For fixed but arbitrary indices $h \neq i$ we integrate (10) with respect to dx^i:

$$\lambda^h = \Lambda^h + \int_{x^i_o}^{x^i} f^h_i(x^1, \dots, x^{i-1}, t, x^{i+1}, \dots, x^n)\, dt,$$

where Λ^h is a function, which does not depend on x^i.

Because of the existence of the partial derivatives of the functions λ^h and the above integrals (see [17, p. 300]), also the derivatives $\partial_h\Lambda^h$ exist; in this proof we don't use Einstein's summation convention. Then we can write (10) for $h = i$:

$$\varrho = -f^h_h + \partial_h\Lambda^h + \int_{x^i_o}^{x^i} \partial_h f^h_i(x^1, \dots, x^{i-1}, t, x^{i+1}, \dots, x^n)\, dt. \tag{11}$$

Because the derivative with respect to x^i of the right-hand side of (11) exists, the derivative of the function ϱ exists, too. Obviously $\partial_i\varrho = \partial_h f^h_i - \partial_i f^h_h$, therefore $\varrho \in C^1$ and from (10) follows $\lambda^h \in C^2$. □

In a similar way we can prove the following: *if $\lambda^h \in C^r$ ($r \geq 1$) and $\partial_i\lambda^h - \varrho\delta^h_i \in C^r$ then $\lambda^h \in C^{r+1}$ and $\varrho \in C^r$.*

Lemma 4.4. *If $A_n \in C^2$ admits a geodesic mapping onto $\bar{V}_n \in C^2$, then $\bar{V}_n \in C^3$.*

Proof. In this case Mikeš's and Berezovsky's equations (5) and (9) hold. According to the assumptions, $\Gamma^h_{ij} \in C^2$ and $\bar{g}_{ij} \in C^2$. By a simple check-up we find $\Psi \in C^2$, $\psi_i \in C^1$, $a_{ij} \in C^2$, $\lambda^i \in C^1$ and $R^h_{ijk}, R_{ij} \in C^1$.

From the above-mentioned conditions we easily convince ourselves that we can write equation (9) in the form (10), where

$$\varrho = \mu/n \quad \text{and} \quad f^h_i = (-\lambda^\alpha\Gamma^h_{\alpha i} + a^{j\alpha}R_{\alpha l} - a^{\alpha\beta}R^j_{\alpha\beta l})/n \in C^1.$$

From Lemma 4.3 follows that $\lambda^h \in C^2$, $\varrho \in C^1$, and evidently $\lambda^i \in C^2$. Differentiating (5) twice we convince ourselves that $a^{ij} \in C^3$. From this and formula (7) follows that also $\Psi \in C^3$ and $\bar{g}_{ij} \in C^3$. □

Further we notice that for geodesic mappings from $A_n \in C^2$ onto $\bar{V}_n \in C^3$ holds the third set of Mikeš's and Berezovsky's equations [21]:

$$(n-1)\nabla_k\mu = -2(n+1)\lambda^\alpha R_{\alpha k} + a^{\alpha\beta}(R_{\alpha\beta,k} - 2R_{\alpha k,\beta}). \tag{12}$$

If $A_n \in C^{r-1}$ and $\bar{V}_n \in C^2$, then by Lemma 4.4, $\bar{V}_n \in C^3$ and (12) hold. Because Mikeš's and Berezovsky's system (5), (9) and (12) is closed, we can differentiate equations (5) r times. So we convince ourselves that $a^{ij} \in C^r$, and also $\bar{g}_{ij} \in C^r$ ($\equiv \bar{V}_n \in C^r$). □

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SPECIAL MAPPINGS OF EQUIDISTANT SPACES

HINTERLEITNER Irena, (CZ)

Abstract. In this paper we study special mappings of equidistant spaces in a canonical coordinate system. Concretely for conformal, affine, geodesic, harmonic, conformally-projective harmonic and equivolume mappings.

Key words and phrases. Equidistant space, conformal mapping, affine mapping, geodesic mapping, harmonic mapping, conformally-projective harmonic mapping, equivolume mapping.

Mathematics Subject Classification. Primary 53B20, 53B30.

1 Introduction

The theory of conformal, affine, geodesic, harmonic and other mappings is an interesting part of differential geometry of Riemannian and pseudo-Riemannian spaces, see [1]-[14].

In this paper we study special mappings of equidistant spaces in a canonical coordinate system, concretely for conformal, affine, geodesic, harmonic, conformally-projective harmonic and equivolume mappings.

For the calculations in this paper we will make use of tensorial analysis in local form, all used functions are continuous and sufficiently differentiable. The dimension n of the studied spaces is larger than two, unless stated otherwise. All spaces are linearly connected.

In this paper we use notions from the theory of Riemannian spaces as in the monographies and reviews [1]-[10].

2 Equidistant spaces

Assume a *Riemannian space* V_n, determined by the symmetric and regular metric tensor $g_{ij}(x)$ and endowed with a local coordinate system $x = (x^1, x^2, \dots, x^n)$.

In the following, under the notion "Riemannian" we understand "true" Riemannian metrics with positive signature as well as pseudo-Riemannian ones with negative signature, like in [8, 9, 10], for example.

Christoffel symbols of types I and II are introduced on V_n by the following formulas $\Gamma_{ijk} \equiv \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$ and $\Gamma^h_{ij} \equiv g^{h\alpha}\Gamma_{ij\alpha}$, where $\partial_i \equiv \partial/\partial x^i$, g^{ij} is the inverse matrix to g_{ij}. Christoffel symbols of type II are the *natural connection* (the *Levi-Civita connection*) of Riemannian spaces.

A vector field ξ^h is called *concircular*, if

$$\xi^h_{,i} = \varrho\, \delta^h_i,$$

where ϱ is a function, δ^h_i is the Kronecker delta, "," denotes the *covariant derivative* with respect to the connection of the space V_n. If $\varrho = \text{const}$, ξ^h is *convergent*. A Riemannian space V_n with concircular vector field is called *equidistant*, see [5, 10, 13].

In equidistant spaces V_n, where the concircular vector fields are nonisotropic (i.e. $g_{ij}\xi^i\xi^j \neq 0$), we can introduce a system of so-called canonical coordinates x, where the metric is of the form

$$\mathrm{d}s^2 = a(x^1)\,(\mathrm{d}x^1)^2 + b(x^1)\,\mathrm{d}\tilde{s}^2, \tag{1}$$

and $a, b \in C^1$ are non-zero functions, $\mathrm{d}\tilde{s}^2 = \tilde{g}_{ab}(x^2, \dots, x^n)\,\mathrm{d}x^a\mathrm{d}x^b$ is the metric form of certain Riemannian spaces $\tilde{V}_{n-1}$. Here and after the indices $a, b, c, \dots$ assume values from 2 to n.

Under the regular transformation $\bar{x}^1 = \bar{x}^1(x^1)$; $\bar{x}^a = \bar{x}^a(x^2, \dots, x^n)$, the principal form of the metric does not change (1). Firstly W.H. Brinkmann [2] found a metric (1) in the form (1) with $a = 1/f(x^1)$ and $b = f(x^1)$, and often the metric of the equidistant spaces is written in the form with $a = \pm 1$ and $b = f(x^1)$, see [1, 4, 5, 7, 8, 10, 13].

It is known, that curves $\ell = \{(t, \overset{\circ}{x}{}^2, \dots, \overset{\circ}{x}{}^n),\ t \in \mathbb{R}\}$ are geodesics and they form a *geodesic congruence*. The hypersurface $S = \{x^1 = \text{const}\}$ is orthogonal to this congruence and there is a conformal to $\tilde{V}_{n-1}$.

We will compute the non-zero components of the Christoffel symbols for the metric form (1):

$$\begin{aligned}
&\Gamma_{111} = \frac{1}{2}\,a'; \quad \Gamma_{1ab} = \Gamma_{a1b} = \frac{1}{2}\,b'\,\tilde{g}_{ab}; \quad \Gamma_{ab1} = -\frac{1}{2}\,b'\,\tilde{g}_{ab}; \quad \Gamma_{abc} = b\,\tilde{\Gamma}_{abc};\\
&\Gamma^1_{11} = \frac{1}{2}\,\frac{a'}{a}; \quad \Gamma^b_{1a} = \Gamma^b_{a1} = \frac{1}{2}\,\frac{b'}{b}\,\delta^b_a; \quad \Gamma^1_{ab} = -\frac{1}{2}\,\frac{b'}{a}\,\tilde{g}_{ab}; \quad \Gamma^c_{ab} = \tilde{\Gamma}^c_{ab},
\end{aligned} \tag{2}$$

where $\tilde{\Gamma}_{abc}$ and $\tilde{\Gamma}^c_{ab}$ are Christoffel symbols of type I and II, respectively, of $\tilde{V}_{n-1}$.

3 Special mappings of Riemannian spaces

Consider then a map $f\colon V_n \to \bar{V}_n$ in a *common coordinate system* x, i.e. the point $M \in V_n$ and its image $f(M) \in \bar{V}_n$ have the same coordinates $x = (x^1, x^2, \dots, x^n)$; the corresponding geometric objects in V_n will be marked with a bar. For example, $\bar{\Gamma}^h_{ij}$ is the Christoffel symbol of $\bar{V}_n$.

Definition 3.1 ([2, 3, 8, 10, 13]) The mapping $f\colon V_n \to \bar{V}_n$ is *conformal* if and only if, in the common coordinate system x with respect to the mapping, the condition $\bar{g}_{ij}(x) = \mathrm{e}^{2\sigma(x)} g_{ij}(x)$ holds, where $\sigma(x)$ is a function on V_n.

Definition 3.2 ([3, 5, 8, 10]) The diffeomorphism $f\colon V_n \to \bar{V}_n$ is called a *geodesic mapping* if f maps any geodesic line of V_n into a geodesic line of $\bar{V}_n$.

The mapping from V_n onto $\bar{V}_n$ is *geodesic* if and only if, in the common coordinate system x with respect to the mapping, the conditions

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \delta^h_i \psi_j + \delta^h_j \psi_i \tag{3}$$

hold, where $\psi_i(x)$ is a gradient-like covector, δ^h_i is the Kronecker delta. If $\psi_i \not\equiv 0$, then a geodesic mapping is called *nontrivial*; otherwise it is said to be *trivial* or *affine*.

Harmonic mappings as introduced by many authors, for example [11, 12], preserve the solutions of the Laplace equation. The diffeomorphism from V_n onto $\bar{V}_n$ is harmonic if and only if, in the common coordinate system x with respect to the mapping, the following conditions hold [11]

$$(\bar{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x))\, g^{ij} = 0. \tag{4}$$

Definition 3.3 ([4]) The composition of conformal and geodesic (projective) mappings in the case when it is harmonic is called *conformally-projective harmonic*.

A diffeomorphism from V_n onto $\bar{V}_n$ is a *conformally-projective harmonic mapping* if and only if in the common coordinate system x the following condition holds

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \varphi_i \delta^h_j + \varphi_j \delta^h_i - \frac{2}{n}\, \varphi^h g_{ij}, \tag{5}$$

where φ_i is a gradient-like covector and $\varphi^h = g^{h\alpha}\varphi_\alpha$.

Finally we consider *equivolume mappings*, which were defined and studied by T.V. Zudina and S.E. Stepanov [14]. This mapping $f\colon V_n \to \bar{V}_n$ is characterized by the following condition

$$\bar{\Gamma}^\alpha_{i\alpha}(x) = \Gamma^\alpha_{i\alpha}(x). \tag{6}$$

4 Special mapping for equidistant spaces

Consider a *special mapping* f between equidistant spaces V_n and $\bar{V}_n$, where the equidistant space V_n has a metric of the form (1) and the equidistant space $\bar{V}_n$ has an analogous metric

$$\mathrm{d}\bar{s}^2 = A(x^1)\,(\mathrm{d}x^1)^2 + B(x^1)\,\mathrm{d}\hat{s}^2, \tag{7}$$

where $A, B \in C^1$ are non-zero functions, $\mathrm{d}\hat{s}^2 = \hat{g}_{ab}(x^2, \dots, x^n)\,\mathrm{d}x^a \mathrm{d}x^b$ $(a, b = 2, \dots, n)$ is the metric form of certain Riemannian spaces $\hat{V}_{n-1}$.

The equdistant space V_n is defined by non-zero differentiable functions $a(x^1), b(x^1)$ and the metric $\mathrm{d}\tilde{s}^2$ of $\tilde{V}_{n-1}$. Analogically, the image of the metric under the special f in $\bar{V}_n$ is defined by non-zero differentiable functions $A(x^1)$, $B(x^1)$, and the metric form $\mathrm{d}\hat{s}^2$ of $\hat{V}_{n-1}$.

Under this map the geodesic curves $\ell = \{(t, \overset{\circ}{x}{}^2, \dots, \overset{\circ}{x}{}^n),\ t \in \mathbb{R}$ of V_n map into the geodesics of $\bar{V}_n$ and the orthogonal surfaces on this geodesic congruence of V_n also map into the orthogonal surfaces on the corresponding geodesic congruence of $\bar{V}_n$.

The deformation tensor $P^h_{ij}(x) = \bar{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x)$ of the mapping $f\colon V_n \to \bar{V}_n$ has in this case the following form

$$\begin{gathered} P^1_{11} = \frac{1}{2}\left(\frac{A'}{A} - \frac{a'}{a}\right); \quad P^1_{1a} = P^1_{a1} = P^c_{11} = 0; \quad P^c_{1a} = P^c_{a1} = \frac{1}{2}\left(\frac{B'}{B} - \frac{b'}{b}\right)\delta^c_a; \\ P^1_{ab} = -\frac{1}{2}\left(\frac{B'}{A}\hat{g}_{ab} - \frac{b'}{a}\tilde{g}_{ab}\right); \quad P^c_{ab} = \hat{\Gamma}^c_{ab} - \tilde{\Gamma}^c_{ab}, \end{gathered} \tag{8}$$

where $\tilde{\Gamma}^c_{ab}$ and $\hat{\Gamma}^c_{ab}$ are the Christoffel symbols of $\tilde{V}_{n-1}$ and $\hat{V}_{n-1}$.

4.1 Conformal mappings. By simple analysis we obtain the following lemma.

Lemma 4.1 *The special mapping f between equidistant spaces V_n and $\bar{V}_n$ is conformal if and only if there exists a function $\varrho(x^1) \neq 0$ and the metric of $\bar{V}_n$ has the following form*

$$\mathrm{d}\bar{s}^2 = \varrho(x^1)\,\mathrm{d}s^2 = \varrho(x^1)\left(a(x^1)\,(\mathrm{d}x^1)^2 + b(x^1)\,\mathrm{d}\tilde{s}^2\right).$$

4.2 Affine mappings. An affine mapping $f\colon\ V_n \to \bar{V}_n$ is characterized by the condition $P^h_{ij} = 0$. From (8) follows $A = \alpha \cdot a$ and $B = \beta \cdot b$, where $\alpha, \beta = \text{const} \neq 0$. After a detailed analysis we obtain the following lemma.

Lemma 4.2 *The special mapping f between equidistant spaces V_n and $\bar{V}_n$ is affine if and only if*
1) *the metric of $\bar{V}_n$ has the following form $\mathrm{d}\bar{s}^2 = \text{const} \cdot \mathrm{d}s^2$, i.e. f is homothetic, and*
2) *if $b(x^1) = \text{const}$, the metric of $\bar{V}_n$ has the following form $\mathrm{d}\bar{s}^2 = \alpha\, a(x^1)\,(\mathrm{d}x^1)^2 + \mathrm{d}\hat{s}^2$, and the space $\hat{V}_{n-1}$ with the metric $\mathrm{d}\hat{s}^2$ is affine to $\tilde{V}_{n-1}$.*

4.3 Geodesic mappings. Rewriting the necessary and sufficient condition (3) of the geodesic mapping $V_n \to \bar{V}_n$in terms of the deformation tensor in the form $P^h_{ij} = \psi_i\delta^h_j + \psi_j\delta^h_i$ we obtain

$$\begin{array}{rclclcl}
P^1_{11} &=& \frac{1}{2}(\frac{A'}{A} - \frac{a'}{a}) &=& \psi_1\delta^1_1 + \psi_1\delta^1_1 = 2\psi_1; &\Rightarrow& \frac{A'}{A} - \frac{a'}{a} = 4\psi_1\\
P^c_{1a} &=& \frac{1}{2}(\frac{B'}{B} - \frac{b'}{b})\delta^c_a &=& \psi_a\delta^c_1 + \psi_1\delta^c_a = \psi_1\delta^c_a; &\Rightarrow& \frac{B'}{B} - \frac{b'}{b} = 2\psi_1\\
P^1_{ab} &=& -\frac{1}{2}(\frac{B'}{A}\hat{g}_{ab} - \frac{b'}{a}\tilde{g}_{ab}) &=& \psi_a\delta^1_b + \psi_b\delta^1_a = 0; &\Rightarrow& \frac{B'}{A} - \frac{b'}{a} = 0\\
P^c_{ab} &=& \hat{\Gamma}^c_{ab} - \tilde{\Gamma}^c_{ab} &=& \psi_a\delta^c_b + \psi_b\delta^c_a = 0; &\Rightarrow& \psi_a = 0.
\end{array}$$

By analysis of these equations we obtain the following theorem.

Theorem 4.3 *The special mapping f between equidistant spaces V_n and $\bar{V}_n$ is non-trivially geodesic if and only if $\hat{V}_{n-1}$ is homothetic to $\tilde{V}_{n-1}$, and the metric of $\bar{V}_n$ has the following form*

$$\mathrm{d}\bar{s}^2 = \frac{p\,a(x^1)}{(1+q\,b(x^1))^2}\,(\mathrm{d}x^1)^2 + \frac{p\,b(x^1)}{1+q\,b(x^1)}\,\mathrm{d}\tilde{s}^2,$$

where p, q are some constants such that $p \neq 0$, $1 + q\,b(x^1) \neq 0$, and $q\,b'(x^1) \not\equiv 0$. From this follows $\psi = -\frac{1}{2}\ln|1 + q\,b(x^1)|$.

4.4 Harmonic mappings. For harmonic mappings we can rewrite condidtion (4) in the form $P^h_{\alpha\beta}g^{\alpha\beta} = 0$, leading to the following differential equations:

$$(a)\quad \frac{1}{a}\left(\frac{A'}{A} - \frac{a'}{a}\right) - \frac{1}{b}\left(\frac{B'}{A}\,\hat{g}_{ab}\tilde{g}^{ab} - (n-1)\,\frac{b'}{a}\right) = 0, \qquad (b)\quad \left(\hat{\Gamma}^c_{ab} - \tilde{\Gamma}^c_{ab}\right)\tilde{g}^{ab} = 0 \qquad (9)$$

Analyzing the last equations we have the following theorem.

Theorem 4.4 *The special mapping f between equidistant spaces V_n and $\bar{V}_n$ is harmonic if and only if the mapping of the subspace $\tilde{V}_{n-1}$ to $\hat{V}_{n-1}$is harmonic and the metric of $\bar{V}_n$ has the following form $\mathrm{d}\bar{s}^2 = c \cdot a(x^1)\,b^{1-n}(x^1)\,\mathrm{d}(x^1)^2 + B\,\mathrm{d}\hat{s}^2$, where $c, B = \text{const} \neq 0$; moreover, if $\hat{g}_{ab}\tilde{g}^{ab} = \text{const}$, then for arbitrary functions $B(x^1) \in C^1$ there is a function $A(x^1)$ satisfying an ordinary differential equation (9a).*

Remark. If $\hat{V}_{n-1} = \tilde{V}_{n-1}$ (i.e. $\hat{g}_{ab} = \tilde{g}_{ab}$) then (9b) is satisfed automatically and $\hat{g}_{ab}\tilde{g}^{ab} = n-1$. In this case the family of harmonic mappings $f\colon V_n \to \bar{V}_n$ depends on the one function $A(x^1) \neq 0$, because for a concrete function, we can find $B(x^1)$ by simple integration of equation (9a).

4.5 Conformally-projective harmonic mappings. From the necessary and sufficient condition (5) of conformally-projective harmonic mappings $V_n \to \bar{V}_n$, rewritten in the form $P^h_{ij} = \varphi_i \delta^h_j + \varphi_j \delta^h_i - \frac{2}{n}\,\varphi^h g_{ij}$, and with the help of (8) we obtain that $\varphi = \varphi(x^1)$ and

$$\frac{A'}{A} - \frac{a'}{a} = \frac{4}{n}\,(n-1)\varphi'; \quad \frac{B'}{B} - \frac{b'}{b} = 2\varphi'; \quad \frac{B'}{A}\hat{g}_{ab} - \frac{b'}{a}\tilde{g}_{ab} = \frac{4b}{na}\,\varphi'\,\tilde{g}_{ab}; \quad \hat{\Gamma}^c_{ab} = \tilde{\Gamma}^c_{ab}\,. \tag{10}$$

For $B = \mathrm{const}$ we obtain that $\varphi = \mathrm{const}$, and $f\colon V_n \to \bar{V}_n$ is affine, see 4.2.

In the case $B \neq \mathrm{const}$ from (10) it follows that $\hat{g}_{ab} = \mathrm{const}\,\tilde{g}_{ab}$, i.e. $\hat{V}_{n-1}$ and $\tilde{V}_{n-1}$ are homothetic. After analyzing of equations (10) we have the following theorem

Theorem 4.5 *The special non-affine mapping f between equidistant spaces V_n and $\bar{V}_n$ is conformal-projective harmonic if and only if $\tilde{V}_{n-1}$ admits a homothetic mapping on $\hat{V}_{n-1}$, and the metric of $\bar{V}_n$ has the following form*

$$\mathrm{d}\bar{s}^2 = \alpha \cdot a(x^1)\,\mathrm{e}^{4\,\frac{n-1}{n}\varphi(x^1)}\,(\mathrm{d}x^1)^2 + \beta \cdot b(x^1)\;\mathrm{e}^{2\varphi(x^1)}\mathrm{d}\tilde{s}^2, \tag{11}$$

where α, β are non-zero constants, and the function $\varphi(x^1)$ satisfes the following ordinary differential equation

$$\beta\, n\,(b' + 2b\varphi') \cdot \mathrm{e}^{2\,\frac{2-n}{n}\,\varphi} - \alpha n\, b' - 4\alpha\, b\,\varphi' = 0.$$

4.6 Equivolume mappings. From the rewritten form of the necessary and sufficient condition (6) of equivolume mappings $V_n \to \bar{V}_n$ $P^{\alpha}_{i\,\alpha} = 0$, and with help of (8) we obtain the following formulas

$$\frac{A'}{A} - \frac{a'}{a} + (n-1)\left(\frac{B'}{B} - \frac{b'}{b}\right) = 0; \quad \hat{\Gamma}^c_{ac} = \tilde{\Gamma}^c_{ac}\,. \tag{12}$$

After analyzing the equations (12) we have following theorem

Theorem 4.6 *The special non-affine mapping f between equidistant spaces V_n and $\bar{V}_n$ is equivolume if and only if $\tilde{V}_{n-1}$ admits an equivolume mapping on $\hat{V}_{n-1}$, and the metric of $\bar{V}_n$ has the following form*

$$\mathrm{d}\bar{s}^2 = \alpha \cdot a\left(\frac{b}{B}\right)^{n-1}(\mathrm{d}x^1)^2 + B\,\mathrm{d}\hat{s}^2,$$

where α is a non-zero constant, and $B(x^1)$ is a non-zero differentiable function.

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Current address

Mgr. Irena Hinterleitner
Inst. of Math., FSI VUT Brno, Technická 2, 616 69 Brno, Czech Republic,
e-mail: Hinterleitner.Irena@seznam.cz.

On Global Geodesic Mappings of Ellipsoids

Irena Hinterleitner

Department of Mathematics, Brno University of Technology, Faculty of Civil Engineering, Žižkova 17, 602 00 Brno, Czech Rep.

Abstract. In this paper we study geodesic deformations of ellipsoids of revolution. We present a one-parameter family of geodesic mappings that deform ellipsoids to surfaces of revolution, which are generally of a different type.

Keywords: geodesic mapping, geodesic deformation, surface of revolution, ellipsoid
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INTRODUCTION

In the present work we study geodesic mappings and deformations of ellipsoids of revolution. The possibility of geodesic mappings of second order surfaces and surfaces of revolution was shown by U. Dini [5], see [6]. In the paper [7] a globally geodesic deformation of a sphere was constructed, and in [13] the existence of geodesic mappings of an ellipsoid was proved. Geodesic deformations of rotational surfaces were studied in [8]. Here we construct explicitely rotational surfaces, which arise from geodesic deformation of a rotational ellipsoid and show that these surfaces cannot be ellipsoids.

GEODESIC MAPPINGS OF SURFACES OF REVOLUTION

A deformation of a surface is called *geodesic* if it preserves geodesics. These notations were introduced in [2, 6, 9, 10, 11, 12].

Assume a rotational surface $\mathscr{S}_2$ in the Euclidean 3-space E_3 given by the equations $x = r(w)\cos t$,
$y = r(w)\sin t,\ \ z = z(w),\ \ w \in [w_1, w_2],\ \ t \in [0, 2\pi)$. Its metric has the form

$$\mathrm{d}s^2 = a(w)\,\mathrm{d}w^2 + b(w)\,\mathrm{d}t^2, \tag{1}$$

where $a(w)$ and $b(w)$ are the differentiable functions $a(w) = r'^2(w) + z'^2(w)$ and $b(w) = r^2(w)$.

As it is known [3, 4], the surface $\mathscr{S}_2$ with the metric (1) maps geodesically onto surfaces $\bar{\mathscr{S}}_2$ with the metric

$$\mathrm{d}\bar{s}^2 = \frac{p\,a(w)}{(1+qb(w))^2}\,\mathrm{d}w^2 + \frac{p\,b(w)}{1+qb(w)}\,\mathrm{d}t^2, \tag{2}$$

where p and q are real parameters, t and w are common coordinates.

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Now we suppose that a certain one-parameter family of rotational surfaces $\bar{\mathscr{S}}_2$ with $x = \bar{r}(w)\cos t$, $y = \bar{r}(w)\sin t$, $z = \bar{z}(w)$ is obtained from the original surface $\mathscr{S}_2$ by the following particular transformations

$$\bar{r}(w) = \frac{r(w)}{\sqrt{1+ar^2(w)}}, \qquad \bar{z}(w) = \int_{w_1}^{w} \sqrt{\frac{1+ar^2(\tau)-r'^2(\tau)}{(1+ar^2(\tau))^3}}\,\mathrm{d}\tau \tag{3}$$

with parameter a. It was proved that (3) describes a one-parameter family of geodesic deformations.

The coordinate w is the same as before, therefore it is *not* the length parameter of the curve $(\bar{r},\bar{z})$. The functions $\bar{r}$ and $\bar{z}$ introduced above must satisfy the conditions of smoothness of the surfaces at the poles $w = w_1$ and $w = w_2$, where $\bar{r} = 0$, namely $\frac{\mathrm{d}\bar{r}}{\mathrm{d}w} = \pm 1$ and $\frac{\mathrm{d}\bar{z}}{\mathrm{d}w} = 0$. They hold, provided they are satisfied for r and z.

APPLICATION TO ROTATIONAL ELLIPSOIDS

In the foregoing section we have seen a class of nontrivial geodesic mappings between smooth surfaces of revolution, which are homeomorphic to a sphere. Now we take as a concrete example a rotational ellipsoid, embedded into the 3-dimensional Euclidian space, and investigate its deformation by the considered geodesic mappings. This is done in a local coordinate patch, covering one half of the surface. Rather than in terms of the arc length w we formulate it in terms of the angular variable φ,

$$r(\varphi) = k\sin\varphi, \qquad z(\varphi) = 1-\cos\varphi. \tag{4}$$

The squared element of the arc length is $\mathrm{d}w^2 = \mathrm{d}r^2 + \mathrm{d}z^2 = (k^2\cos^2\varphi + \sin^2\varphi)\,\mathrm{d}\varphi^2$. We choose $w_1 = w(\varphi = 0) = 0$, so that the origin of φ and the arc length coincide, then $w_2 = w(\varphi = \pi)$ is half of the circumference of the ellipse. The condition $r(w_1) = r(w_2) = 0$ is fulfilled and $\frac{\mathrm{d}r}{\mathrm{d}w} = \frac{\mathrm{d}r}{\mathrm{d}\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}w} = \frac{k\cos\varphi}{\sqrt{k^2\cos^2\varphi+\sin^2\varphi}}$, so also $\frac{\mathrm{d}r}{\mathrm{d}w}(w_1) = 1$ and $\frac{\mathrm{d}r}{\mathrm{d}w}(w_2) = -1$ are satisfied.

The transformation (3) in terms of φ is

$$\bar{r}(\varphi) = \frac{k\sin\varphi}{\sqrt{1+ak^2\sin^2\varphi}} \tag{5}$$

and

$$\bar{z}(\varphi) = \int_0^{\varphi} \sqrt{\frac{1+ar^2(\varphi')-r'^2(\varphi')}{(1+ar^2(\varphi'))^3}}\,\frac{\mathrm{d}w}{\mathrm{d}\varphi'}\,\mathrm{d}\varphi'. \tag{6}$$

Note that here and in the following r' means always the derivative with respect to w, even when written as function of φ', so $r'(\varphi')$ is $\frac{\mathrm{d}r}{\mathrm{d}\varphi'}\frac{\mathrm{d}\varphi'}{\mathrm{d}w}$.

Explicitly we find

$$\bar{r}'(\varphi) = \frac{k\cos\varphi}{(k^2\cos^2\varphi+\sin^2\varphi)^{\frac{1}{2}}(1+ak^2\sin^2\varphi)^{\frac{3}{2}}}, \tag{7}$$

the maximal value of $\bar{r}$, $\bar{r}_{\max} = \frac{k}{\sqrt{1+ak^2}}$, occurs at $\varphi = \frac{\pi}{2}$, like for the original ellipsoid.

Instead of solving the integral of (7) explicitly, we consider the derivative $\frac{\mathrm{d}\bar{z}}{\mathrm{d}\bar{r}}$, which gives a differential equation for the curve, and eliminate the parameter φ. This is done in several steps: First we express $\frac{\mathrm{d}\bar{z}}{\mathrm{d}\bar{r}}$ in the form $\frac{\mathrm{d}\bar{z}}{\mathrm{d}\varphi}/\frac{\mathrm{d}\bar{r}}{\mathrm{d}\varphi}$. From (5) and (6) we get

$$\frac{\mathrm{d}\bar{r}}{\mathrm{d}\varphi} = \frac{k\cos\varphi}{(1+ak^2\sin^2\varphi)^{\frac{3}{2}}} \quad \text{and} \quad \frac{\mathrm{d}\bar{z}}{\mathrm{d}\varphi} = \sqrt{\frac{1+ar^2(\varphi)-r'^2(\varphi)}{(1+ar^2(\varphi))^3}}\,\frac{\mathrm{d}w}{\mathrm{d}\varphi}.$$

Then from the definitions (4) and the explicit equation of the ellipse $(1-z)^2+\frac{r^2}{k^2}=1$ we express $\sin\varphi$ and $\cos\varphi$ in terms of r and find

$$\frac{\mathrm{d}\bar{z}}{\mathrm{d}\bar{r}} = \frac{r\sqrt{1+ak^4+ar^2-a^2k^2r^2}}{k\sqrt{k^2-r^2}}$$

in terms of r.

Now we insert the inverse of (5), $r = \frac{\bar{r}}{\sqrt{1-a\bar{r}^2}}$ to express this derivative in terms of $\bar{r}$,

$$\frac{\mathrm{d}\bar{z}}{\mathrm{d}\bar{r}} = \frac{\bar{r}\sqrt{\frac{1}{k^2}+ak^2-a(1+ak^2)\bar{r}^2}}{\sqrt{1-a\bar{r}^2}\sqrt{k^2-(1+ak^2)\bar{r}^2}}.$$

At last, for a direct comparison with the corresponding differential equation for an ellipse, $\frac{\mathrm{d}z}{\mathrm{d}r} = \frac{r}{k\sqrt{k^2-r^2}}$, we carry out a scale transformation $\hat{r} = \bar{r}\sqrt{1+ak^2}$, $\hat{z} = \bar{z}\sqrt{1+ak^2}$, so that the maximal value of $\hat{r}$ is equal to k, like the maximal value of r in the case of the ellipse and the radial extensions of both surfaces are the same. In terms of these variables, finally,

$$\frac{\mathrm{d}\hat{z}}{\mathrm{d}\hat{r}} = \frac{\hat{r}}{k\sqrt{k^2-\hat{r}^2}} \cdot \sqrt{\frac{1+ak^2(k^2-\hat{r}^2)}{1+a(k^2-\hat{r}^2)}}. \tag{8}$$

From this we can see that the transformed curve is of a different type than an ellipse. At the maximal values of the radial variables, i. e. at the “equator”, both the derivatives $\frac{\mathrm{d}z}{\mathrm{d}r}$ for the ellipse and $\frac{\mathrm{d}\hat{z}}{\mathrm{d}\hat{r}}$ for the deformed curve go to infinity, corresponding to the fact that r and t provide only a local chart for one half of the surface.

An interesting feature of these transformations is that they leave circles ($k=1$) invariant (up to a scale factor $\sqrt{1+a}$). In the limit of a large transformation parameter a the modification factor in (8) goes to k and the transformed curve approaches a circle.

The metric of the resulting surface of revolution is,

$$\mathrm{d}s^2 = \left(1+\frac{\mathrm{d}\hat{z}^2}{\mathrm{d}\hat{r}^2}\right)\mathrm{d}\hat{r}^2 + \hat{r}^2\mathrm{d}t^2 = \frac{k^2+ak^4+\left(\frac{1}{k^2}-ak^2-1\right)\hat{r}^2}{(k^2-\hat{r}^2)(1+ak^2-a\hat{r}^2)}\,\mathrm{d}\hat{r}^2 + \hat{r}^2\,\mathrm{d}t^2.$$

This form of the metric in terms of $\hat{r}$ is local and applies only to the lower or the upper half of the surface. It can be generalized without problems to higher dimensions, when the circles with constant $\hat{z}$ are replaced by higher-dimensional spheres. Then dt has only to be replaced by the solid angle element dΩ of the corresponding dimension.

This metric can be pulled back to the original ellipsoid by simply expressing $\hat{r}$ in terms of r,

$$\mathrm{d}s^2 = (1+ak^2)\left[\frac{k^2+\left(\frac{1}{k^2}-1\right)r^2}{(k^2-r^2)(1+ar^2)^2}\,\mathrm{d}r^2 + \frac{r^2}{1+ar^2}\,\mathrm{d}t^2\right], \tag{9}$$

whereas the metric on the original ellipsoid is

$$\mathrm{d}s^2 = \frac{k^2+(\frac{1}{k^2}-1)r^2}{k^2-r^2}\,\mathrm{d}r^2 + r^2\,\mathrm{d}t^2. \tag{10}$$

For an explicit expression of the deformed surfaces, we calculate the equations of the "meridians" in the form $\hat{z}(r)$, where $\hat{z}$ and $\hat{r}$ are cartesian coordinates of a cross-section through the rotation axis. For this purpose we integrate (8), from now on we drop the hats on r and z. We begin with the substitution $\sin^2\varphi = \frac{k^2-r^2}{k^2-\frac{1}{a}-r^2}$. Then

$$z(r) = -\frac{1}{k\sqrt{a}}\int_{\varphi(0)}^{\varphi(r)} \frac{\sqrt{1-(1-k^2)\sin^2\varphi'}}{\cos^2\varphi'}\,\mathrm{d}\varphi', \tag{11}$$

where $\varphi(0) = \arcsin\sqrt{\frac{ak^2}{1+ak^2}}$ and $\varphi(r) = \arcsin\sqrt{\frac{a(k^2-r^2)}{1+ak^2-ar^2}}$. Integrating (11) by parts gives

$$\int_{\varphi(0)}^{\varphi(r)} \sqrt{1-(1-k^2)\sin^2\varphi'}\,\frac{\mathrm{d}\varphi'}{\cos^2\varphi'} = \sqrt{1-(1-k^2)\sin^2\varphi}\,\tan\varphi\Big|_{\varphi(0)}^{\varphi(r)}$$
$$-\frac{1}{(1-k^2)}\int_{\varphi(0)}^{\varphi(r)} \sqrt{1-(1-k^2)\sin^2\varphi'}\,\mathrm{d}\varphi' + \frac{1}{(1-k^2)}\int_{\varphi(0)}^{\varphi(r)} \frac{\mathrm{d}\varphi'}{\sqrt{1-(1-k^2)\sin^2\varphi'}}$$

where the last two integrals are the standard elliptic integrals of the second and first kind [1] with arguments Φ and κ

$$E(\Phi,\kappa) = \int_0^\Phi \sqrt{1-\kappa^2\sin^2\phi}\,\mathrm{d}\phi \quad\text{and}\quad F(\Phi,\kappa) = \int_0^\Phi \frac{\mathrm{d}\phi}{\sqrt{1-\kappa^2\sin^2\phi}}.$$

Inserting back r gives finally

$$z(r) = -\frac{\sqrt{k^2-r^2}}{k}\sqrt{\frac{1+ak^4-ak^2r^2}{1+ak^2-ar^2}} + \sqrt{\frac{1+ak^4}{1+ak^2}} +$$
$$\frac{1}{\sqrt{a}\,k(1-k^2)}\left[E\left(\arcsin\sqrt{\frac{k^2-r^2}{k^2+\frac{1}{a}-r^2}},\sqrt{1-k^2}\right) - E\left(\arcsin\frac{k}{\sqrt{k^2+\frac{1}{a}}},\sqrt{1-k^2}\right)\right.$$

$$-F\left(\arcsin\sqrt{\frac{k^2-r^2}{k^2+\frac{1}{a}-r^2}},\sqrt{1-k^2}\right)\;+\;F\left(\arcsin\frac{k}{\sqrt{k^2+\frac{1}{a}}},\sqrt{1-k^2}\right)\Bigg].$$

We have considered two aspects of geodesic mappings of ellipsoids. The last equation and (8) describe the geodesic deformations in E_3. An interesting property is that on a sphere as a special case of an ellipsoid these transformations act as identity, whereas they act highly non trivially on general ellipsoids. In the limit of large transformation parameters the transformed surfaces approach a sphere as limiting surface.

The second aspect, represented by (9) and (10), concerns geodesic transformations of the metric on a manifold homeomorphic to the sphere, in accordance with [13], where it is shown by application of a classical theorem by Dini [5] that there is (up to homothety) a one-parameter family of geodesically equivalent metrics on $\mathscr{S}_2$. Our result can be summarized in form of a theorem.

Theorem 1 *Rotational ellipsoids admit global nontrivial geodesic deformations under which they remain rotational surfaces. The resulting surfaces are not ellipsoids.*

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ON THE MOBILITY DEGREE OF (PSEUDO-) RIEMANNIAN SPACES WITH RESPECT TO CONCIRCULAR MAPPINGS

OLENA CHEPURNA AND IRENA HINTERLEITNER

Abstract. In this paper we study the mobility degree of (pseudo-) Riemannian spaces with respect to concircular mappings. We assume that the smoothness class of differentiability is C^2.

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Keywords: geodesic mappings, (pseudo-)Riemannian manifold, smoothness class

1. INTRODUCTION

Under a geodesic circle we understand a curve for which the first curvature is constant and the second curvature is zero. K. Yano [14] introduced a conformal mapping of (pseudo-) Riemannian spaces which preserves geodesic circles and is called *concircular*.

These mappings are studied in many papers. In the present paper, we show results connected with basic notations under the conditions of minimal differentiability of metrics and geometric objects which define concircular mappings and also concircular vector fields.

2. FUNDAMENTAL EQUATIONS OF CONCIRCULAR MAPPINGS

Let $V_n = (M, g)$ and $\bar{V}_n = (\bar{M}, \bar{g})$ be n-dimensional (pseudo-) Riemannian manifolds with the metric tensors g and $\bar{g}$, respectively, $n > 2$.

Definition 1. A *conformal mapping* is a diffeomorphism of V_n onto $\bar{V}_n$ such that for all points $x \in M$ $(\equiv \bar{M})$ the following relation is satisfied

$$\bar{g}(x) = e^{2\sigma(x)} g(x), \tag{2.1}$$

where σ is a function on M.

If σ is constant, then the mapping is *homothetic*, and, moreower, if $\sigma = 0$, then the mapping is *isometric*. See [1, 7, 9, 10, 12].

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As we have checked (see [14], [9, p. 117]), if a pseudo-Riemannian space admits concircular mappings, then the function of conformality $\vartheta \overset{\text{def}}{=} \mathrm{e}^{-\sigma}$ satisfies

$$\nabla\nabla\vartheta = \rho \cdot g, \tag{2.2}$$

where ρ is a function and ∇ is the Levi-Civita connection with respect to the metric g. In a local coordinate neighbourhood (U, x), $U \subset M$, it has the form $\nabla_i \vartheta_j = \rho\, g_{ij}$, where g_{ij} are components of g and $\vartheta_j = \nabla_j \vartheta$. A vector field ϑ_i is called *equidistant* (Sinyukov [12, p. 92], see [9, p. 82]).

The integrability conditions of the last set of equations read

$$\vartheta_\alpha R^\alpha_{ijk} = g_{ij} \nabla_k \rho - g_{ik} \nabla_j \rho, \tag{2.3}$$

where R^h_{ijk} are components of the Riemann tensor of V_n. Using contraction, we get:

$$\nabla_i \rho = -\frac{1}{n-1}\, \vartheta_\alpha R^\alpha_i, \tag{2.4}$$

where $R^h_i = g^{h\alpha} R_{\alpha i}$ and $R_{ij} = R^\alpha_{i\alpha j}$ are components of the Ricci tensor on V_n.

Remark 1. In many papers, the Ricci tensor was defined with the opposite sign, for example, [2–8, 12].

Contracting the integrability condition (2.3) with $g^{i\beta}\vartheta_\beta$, we obtain easily $\nabla_k \rho = B\, \vartheta_k$, where B is a function. Because ϑ_k is gradient-like: $\vartheta_k = \nabla_k \vartheta$, then it implies that $\rho = \rho(\vartheta)$ and $B = B(\vartheta)$.

After this, the condition (2.3) acquires the following form

$$\vartheta_\alpha R^\alpha_{ijk} = B\, (g_{ij} \vartheta_k - g_{ik} \vartheta_j). \tag{2.5}$$

As was shown earlier [13] (see [3, 4, 6, 9]), these equations are satisfied if

$$V_n, \bar{V}_n \in C^2 \text{ (i. e. } g_{ij}(x), \bar{g}_{ij}(x) \in C^2),\ \vartheta(x) \in C^3,\ \vartheta_i(x) \in C^2 \text{ and } \varrho(x) \in C^1.$$

3. Fundamental equations of concircular mappings for minimal differentiable conditions

We can write formula (2.2) in the following form

$$\nabla_j \vartheta^i \equiv \frac{\partial \vartheta^i}{\partial x^j} + \Gamma^i_{\alpha j} \vartheta^\alpha = \varrho \cdot \delta^i_j, \tag{3.1}$$

where $\vartheta^i = g^{i\alpha}\vartheta_\alpha$, δ^i_j is the Kronecker symbol and Γ^h_{ij} are the Christoffel symbols. It is easily seen that formulas (3.1) and also (2.2) are true when

$$V_n, \bar{V}_n \in C^1 \text{ (i. e. } g_{ij}(x), \bar{g}_{ij}(x) \in C^1),\ \vartheta(x) \in C^2,\ \vartheta_i(x) \in C^1 \text{ and } \varrho(x) \in C^0.$$

The following lemma holds.

Lemma 1 (Hinterleitner and Mikeš [2]). *Let $\lambda^h \in C^1$ be a vector field and ρ a function. If*

$$\frac{\partial \lambda^h}{\partial x^i} - \rho\, \delta^h_i \in C^1,$$

then $\lambda^h \in C^2$ and $\rho \in C^1$.

If $\Gamma^h_{ij} \in C^1$ holds, which is equivalent to $V_n \in C^2$ (i. e., $g_{ij} \in C^2$), then from formula (3.1) follows $\dfrac{\partial \vartheta^i}{\partial x^j} - \varrho \cdot \delta^i_j \in C^1$, and from Lemma 1 we get:

$$\vartheta^i(x) \in C^2 (\equiv \vartheta_i(x) \in C^2 \equiv \vartheta(x) \in C^3) \text{ and } \varrho(x) \in C^1.$$

From this viewpoint, we specify and generalize the results involving concircular vector fields below. Evidently, in this case, the above formulas from (2.3) to (2.5) are satisfied.

The system of equations

$$\begin{aligned} \nabla_i \vartheta_j &= \rho \cdot g_{ij}, \\ \nabla_i \rho &= -\frac{1}{n-1}\, \vartheta_\alpha R^\alpha_i \end{aligned} \tag{3.2}$$

is closed. It is a system of linear differential equations with respect to the co-vector ϑ_i and function ϱ, of Cauchy type, in first order covariant derivatives with coefficients uniquely determined by the metric g of the (pseudo-) Riemannian space V_n. For any family of initial values $\vartheta_i(x_0) = \vartheta^\circ_i$ and $\rho(x_0) = \rho^\circ$ of the functions under consideration in the given point x_0, it admits at most one solution. Consequently, the number of free parameters in the general solution of the system is at most $n+1$. See [6, 13].

Definition 2. The upper bound for the number of substantial parameters in the general solution of the system of equations (2.2) is called *the mobility degree under concircular mappings* of the (pseudo-) Riemannian manifold V_n.

Since the system is linear, it admits at most $n+1$ linearly independent solutions corresponding to constant coefficients. It is obvious that the mobility degree under concircular mappings of the space coincides with the cardinality of the system of independent (substantial) concircular vector fields of the space.

It is known that only spaces with constant curvature admit the maximal number of $n+1$ linearly independent concircular vector fields. Hence, under concircular mappings, only the spaces of constant curvature have the maximal mobility degree. This holds locally.

It follows from the analysis of the system of equations (3.2) that if $V_n \in C^r$, $r \geq 2$, then $\vartheta_i \in C^r$ and $\rho \in C^{r-1}$. It follows that the function ϑ belongs to C^{r+1}. From this and the formula (2.1), we obtain the following theorem.

Theorem 1. *If the (pseudo-) Riemannian manifold V_n ($V_n \in C^r$, $r \geq 2$, $n > 2$) admits a concircular mapping onto $\bar{V}_n \in C^2$, then $\bar{V}_n$ belongs to C^r. Moreover, the function ϑ of conformality V_n and $\bar{V}_n$: $\bar{g} = \vartheta^{-2} \cdot g$ belongs to C^{r+1}.*

We suppose that the differentiability class r is equal to $2, 3, \ldots, \infty, \omega$, where ∞ and ω denote infinitely differentiable and real analytic functions, respectively.

We can construct examples of such concircular mappings $V_n \to \bar{V}_n$ in the form of equidistant metrics, see [9, p. 79]:

$$\bar{g} = \frac{1}{\vartheta^2} \cdot g, \quad g = \pm(dx^1)^2 + \text{const} \cdot \sqrt{|\vartheta'|} \cdot d\tilde{s}^2,$$

where $d\tilde{s}^2(x^2, \ldots, x^n)$ is a C^r metric of an $(n-1)$-dimensional (pseudo-) Riemannian space $\tilde{V}_{n-1}$ and $\vartheta(x^1)$ is a C^{r+1} function and $\vartheta > 0$, $\vartheta' \neq 0$.

4. A (pseudo-) Riemannian space which admits at least two linearly independent concircular vector fields

Below we prove the following properties of concircular fields.

Lemma 2. *The non-vanishing concircular vector field $\vartheta_i(x)$ can be equal to zero only on point sets of zero measure.*

Proof. Let us suppose that Lemma 2 is not true. Thus there exists a point $x_0 \in M$ in the neighborhood $U_{x_0} \subset M$ of which the concircular vector field $\vartheta_i(x)$ is vanishing. From (3.2) follows that $\rho(x) = 0$ on U_{x_0}. From that follows the initial conditions at the point x_0: $\vartheta_i(x_0) = 0$ and $\rho(x_0) = 0$. The system of linear equations (3.2) with these initial conditions has only the trivial solution $\vartheta_i(x) = 0$ and $\rho(x) = 0$ on all of M. □

By mathematical induction we have the following lemma.

Lemma 3. *The set of r ($r < n$) linear independent concircular vector fields*

$$\{\overset{1}{\vartheta}_i, \overset{2}{\vartheta}_i, \ldots, \overset{r}{\vartheta}_i\} \tag{4.1}$$

on V_n can be linearly dependent only on point sets of zero measure.

Proof. Successively we are able to substitute $r = 1, 2, \ldots, n-1$. Let (4.1) be linearly independent (excluding at point sets of zero measure) concircular vector fields on V_n which satisfy the equations

$$\overset{s}{\vartheta}_{i,j} = \overset{s}{\rho} g_{ij},$$

where $\overset{s}{\rho}$ are functions on V_n.

Let these vectors be linearly independent at the point $x_0 \in M$, then these are linearly independent at a point x in a certain neighborhood U_{x_0}. Finally, let ϑ_i be a concircular vector field on M and

$$\vartheta_i(x) = \textstyle\sum_{s=1}^{r} \overset{s}{\alpha}(x) \cdot \overset{s}{\vartheta}_i(x) \text{ for } x \in U_{x_0} \tag{4.2}$$

where $\overset{s}{\alpha}(x)$ are functions on U_{x_o}. Because $\overset{s}{\vartheta}_i(x) \in C^1$, the functions $\overset{s}{\alpha}(x)$ are differentiable. Covariantly differentiating (4.2) with respect to x^j we find

$$(\rho - \sum_{s=1}^{r} \overset{s}{\alpha} \cdot \overset{s}{\rho})\, g_{ij} = \sum_{s=1}^{r} \nabla_j \overset{s}{\alpha} \cdot \overset{s}{\vartheta}_i.$$

This implies that $\rho = \sum_{s=1}^{r} \overset{s}{\alpha} \cdot \overset{s}{\rho}$ and $\nabla_j \overset{s}{\alpha} = 0$ (i. e., $\overset{s}{\alpha}$ = const) on U_{x_o}.

For the initial conditions

$$\vartheta_i(x_o) = \sum_{s=1}^{r} \overset{s}{\alpha} \cdot \overset{s}{\vartheta}_i(x_o),$$

$$\rho(x_o) = \sum_{s=1}^{r} \overset{r}{\alpha} \cdot \overset{r}{\rho}(x_o),$$

the equations (3.2) have only one solution: $\vartheta_i(x) = \sum_{s=1}^{r} \overset{s}{\alpha} \cdot \overset{s}{\vartheta}_i(x)$ on V_n. □

We are going to prove the following

Theorem 2. *If a (pseudo-) Riemannian space $V_n \in C^2$ $(n > 2)$ admits at least two linearly independent concircular vector fields $\vartheta_i(x) \in C^1$ with constant coefficients, then B is a constant, uniquely determined by the metric of the space V_n.*

Remark 2. In [6] and [4, p. 88] a similar theorem was published, but the proof was done only for $V_n \in C^3$, $\vartheta_i(x) \in C^3$ and $\varrho(x) \in C^2$, and, moreover, it has local validity. This also concerns the following Theorems 3, 4 and 5. On the basis of Lemmas 2 and 3 these Theorems are valid globally.

Proof. Assume in V_n exist at least two linearly independent concircular vector fields with constant coefficients ϑ_i and $\tilde{\vartheta}_i$, with correspondent functions B and $\tilde{B}$, respectively. Then the following is satisfied (see (3.1)):

$$\vartheta_\alpha R^\alpha_{ijk} = B(g_{ij}\vartheta_k - g_{ik}\vartheta_j), \tag{4.3}$$

$$\tilde{\vartheta}_\alpha R^\alpha_{ijk} = \tilde{B}(g_{ij}\tilde{\vartheta}_k - g_{ik}\tilde{\vartheta}_j). \tag{4.4}$$

Multiplying (4.3) by $\tilde{\vartheta}_\alpha g^{\alpha k}$ and contracting over k we get by (4.4)

$$(B - \tilde{B})(g_{ij}\vartheta_\alpha\tilde{\vartheta}^\alpha - \tilde{\vartheta}_i\vartheta_j) = 0.$$

Suppose $B \neq \tilde{B}$. Then $g_{ij}\vartheta_\alpha\tilde{\vartheta}^\alpha - \tilde{\vartheta}_i\vartheta_j = 0$. From the last formula we get $\vartheta_\alpha\tilde{\vartheta}^\alpha = 0$ and $\tilde{\vartheta}_i\vartheta_j = 0$, a contradiction, since the vector fields are non-zero.

Hence $B = \tilde{B}$ holds. That is, the function B is uniquely defined by the metric of the space V_n itself. Because ϑ_k and $\tilde{\vartheta}_k$ are gradient-like covector fields ($\vartheta_k = \nabla_k\vartheta$ and $\tilde{\vartheta}_k = \nabla_k\tilde{\vartheta}$) from the equality $B = \tilde{B}$ the fact $B(\vartheta) = \tilde{B}(\tilde{\vartheta})$ follows. Note that ϑ and $\tilde{\vartheta}$ are indenpendent variables, then from this fact follows: B is constant. □

Note that the above theorem is analogous to some results proven earlier under the additional assumptions $V_n, \bar{V}_n \in C^3$, [5,6,13].

Theorem 3. *There are no (pseudo-) Riemannian spaces $V_n \in C^2$, except spaces of constant curvature, which admit more than $(n-2)$ linearly independent concircular vector fields $\vartheta_i(x) \in C^1$ corresponding to constant coefficients.*

Remark 3. In [4, p. 86], [3,5], a similar theorem was published but the proof was done only for $V_n \in C^3$, $\vartheta_i(x) \in C^3$ and $\varrho_i(x) \in C^2$.

Proof. Let us suppose the opposite. Let V_n be a space which is not of constant curvature and yet admits more than $(n-2)$ linearly independent concirrcular vector fields with constant coefficients. The conditions (2.5) read

$$\vartheta_\alpha Z^\alpha_{ijk} = 0, \tag{4.5}$$

where

$$Z^h_{ijk} \overset{\text{def}}{=} R^h_{ijk} - B(\delta^h_k g_{ij} - \delta^h_j g_{ik}).$$

We can write the tensor Z^h_{ijk} as

$$Z^h_{ijk} = \sum_{s=1}^{m} \underset{s}{b}^h \overset{s}{\Omega}_{ijk},$$

where $b_s{}^h$ are some linearly independent vectors, and $\overset{s}{\Omega}_{ijk}$ are linearly independent tensors. Since V_n is not of constant curvature, $m \geq 2$ holds.

From the conditions (4.5), we obtain

$$\vartheta_\alpha \underset{1}{b}^\alpha = 0, \quad \vartheta_\alpha \underset{2}{b}^\alpha = 0, \quad \dots \quad , \vartheta_\alpha \underset{m}{b}^\alpha = 0. \tag{4.6}$$

Since $m \geq 2$, among the equations of the system (4.6) there are at least two substantial equations. From the previous facts it follows that there exist less or equal to $n-2$ linearly independent vector fields ϑ_i, a contradiction. This proves Theorem 3. □

From Theorem 3 and results in [6], the following two theorems are obtained:

Theorem 4. *Let $V_n \in C^2$, $(n > 2)$, be (pseudo-) Riemannian spaces in which there are $(n-2)$ linearly independent concircular vector fields $\vartheta_i(x) \in C^1$. Then the Riemannian tensor has the following expression*

$$R_{hijk} = B\,(g_{hk} g_{ij} - g_{hj} g_{ik}) + e(a_h b_i - a_i b_h)(a_j b_k - a_k b_j),$$

where a_i and b_i are non-colinear and pairwise orthogonal covectors, $e = \pm 1$, and $B = \text{const.}$

Theorem 5. *The (pseudo-) Riemannian space* $V_n \in C^3$ $(n > 3)$ *admits* $(n-2)$ *linearly independent concircular vector fields* $\vartheta_i(x) \in C^1$ *if and only if in* V_n *the relations [11]*

$$\begin{aligned} R_{hijk} &= B(g_{hk}g_{ij} - g_{hj}g_{ik}) + e(a_h b_i - a_i b_h)(a_j b_k - a_k b_j), \\ a_{i,j} &= \overset{1}{\xi}_j a_i + \overset{2}{\xi}_j b_i + c_i a_j; \\ b_{i,j} &= \overset{3}{\xi}_j a_i + \overset{4}{\xi}_j b_i + c_i b_j; \\ c_{i,j} &= \overset{5}{\xi}_j a_i + \overset{6}{\xi}_j b_i + c_i c_j - B g_{ij} \end{aligned}$$

are satisfied, where a_i *and* b_i *are non-colinear and pairwise orthogonal covectors;* c_i, $\overset{s}{\xi}_j$ $(s = 1, \dots, 6)$ *are some covectors;* $e = \pm 1$, *and* $B = \text{const}$.

Remark. This theorem was proved locally for $V_n \in C^3$, $\vartheta_i \in C^3$, $\varrho \in C^2$, in [6]. The detailed local proof is contained in the dissertation [3, p. 94-95], [4, p. 88-92].

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Authors' addresses

Olena Chepurna
Odessa State Economical University, Department of Mathematics, Odessa, Ukraine
E-mail address: culeshova@ukr.net

Irena Hinterleitner
Brno University of Technology, Faculty of Civil Engineering, Department of Mathematics, Žižkova 17, 602 00, Brno, Czech Republic
E-mail address: hinterleitner.irena@seznam.cz

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On Equitorsion Concircular Tensors of Generalized Riemannian Spaces

Milan Zlatanović[a], Irena Hinterleitner[b], Marija Najdanović[c]

[a]*Faculty of Sciences and Mathematics, University of Niš, Serbia*
[b]*Department of Mathematics, Faculty of Civil Engineering, Brno University of Technology, Czech Republic*
[c]*Faculty of Sciences and Mathematics, University of Niš, Serbia and College of Vocational Studies, Kruševac, Serbia*

Abstract. In this paper we consider concircular vector fields of manifolds with non-symmetric metric tensor. The subject of our paper is an equitorsion concircular mapping. A mapping $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ is an equitorsion if the torsion tensors of the spaces $\mathbb{GR}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$ are equal.

For an equitorsion concircular mapping of two generalized Riemannian spaces $\mathbb{GR}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$, we obtain some invariant curvature tensors of this mapping $\underset{\theta}{Z}$, $\theta = 1, 2, \ldots, 5$, given by equations (3.14, 3.21, 3.28, 3.31, 3.38). These quantities are generalizations of the concircular tensor Z given by equation (2.5).

1. Introduction

The use of non-symmetric basic tensors and non-symmetric connection became especially actual after appearance of the works of A. Einstein [2]-[4] related to the Unified Field Theory (UFT). Remark that in the UFT the symmetric part $g_{\underline{ij}}$ of the basic tensor g_{ij} is related to gravitation, and antisymmetric one $g_{\underset{\vee}{ij}}$ to electromagnetism.

A generalized Riemannian space $\mathbb{GR}_N$ in the sense of Eisenhart's definition [5] is a differentiable N-dimensional manifold, equipped with non-symmetric basic tensor g_{ij}.

Let us consider two N-dimensional generalized Riemannian spaces $\mathbb{GR}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$ with basic tensors g_{ij} and $\overline{g}_{ij}$, respectively. Generalized Christoffel symbols of the first kind of the spaces $\mathbb{GR}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$ are given by

$$\Gamma_{i.jk} = \frac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j}) \quad \text{and} \quad \overline{\Gamma}_{i.jk} = \frac{1}{2}(\overline{g}_{ji,k} - \overline{g}_{jk,i} + \overline{g}_{ik,j}), \tag{1.1}$$

where, for example, $g_{ij,k} = \partial g_{ij}/\partial x^k$. Connection coefficients of these spaces are generalized Christoffel symbols of the second kind $\Gamma^i_{jk} = g^{\underline{ip}}\Gamma_{p.jk}$ and $\overline{\Gamma}^i_{jk} = \overline{g}^{\underline{ip}}\overline{\Gamma}_{p.jk}$ respectively, where $(g^{\underline{ij}}) = (g_{\underline{ij}})^{-1}$ and $\underline{ij}$ denotes symmetrization with division of the indices i and j. Generally the generalized Christoffel symbols

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Email addresses: zlatmilan@yahoo.com (Milan Zlatanović), Hinterleitner.Irena@seznam.cz (Irena Hinterleitner), marijamath@yahoo.com (Marija Najdanović)

are not symmetric, i.e. $\Gamma^i_{jk} \neq \Gamma^i_{kj}$. We suppose that $g = \det(g_{ij}) \neq 0$, $\overline{g} = \det(\overline{g}_{ij}) \neq 0$, $\underline{g} = \det(g_{\underline{ij}}) \neq 0$, $\underline{\overline{g}} = \det(\overline{g}_{\underline{ij}}) \neq 0$.

A diffeomorpism $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ is a *conformal mapping* if for the basic tensors g_{ij} and $\overline{g}_{ij}$ of these spaces the condition

$$\overline{g}_{ij} = e^{2\psi} g_{ij} \tag{1.2}$$

is satisfied, where ψ is an arbitrary function of x, and the spaces are considered in the common system of local coordinates x^i.

In this case for the Christoffel symbols of the first kind of the spaces $\mathbb{GR}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$ the relation

$$\overline{\Gamma}_{i.jk} = e^{2\psi}(\Gamma_{i.jk} + g_{ji}\psi_{,k} - g_{jk}\psi_{,i} + g_{ik}\psi_{,j}) \tag{1.3}$$

is satisfied and for the Christoffel symbols of the second kind we have

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + g^{\underline{ip}}(g_{jp}\psi_{,k} - g_{jk}\psi_{,p} + g_{pk}\psi_{,j}), \tag{1.4}$$

where $\psi_{,k} = \partial\psi/\partial x^k$. Let us denote $\psi_k = \psi_{,k}$ and $\psi^i = g^{\underline{ip}}\psi_p$. Now, from (1.4) we have

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + g^{\underline{ip}}(g_{\underline{jp}}\,\psi_k - g_{\underline{jk}}\,\psi_p + g_{\underline{pk}}\,\psi_j) + g^{\underline{ip}}(g_{\underset{\vee}{jp}}\psi_k - g_{\underset{\vee}{jk}}\psi_p + g_{\underset{\vee}{pk}}\psi_j),$$

i.e.

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j\,\psi_k + \delta^i_k\,\psi_j - \psi^i g_{\underline{jk}} + \xi^i_{jk}, \tag{1.5}$$

where

$$\xi^i_{jk} = g^{\underline{ip}}(g_{\underset{\vee}{jp}}\,\psi_k - g_{\underset{\vee}{jk}}\,\psi_p + g_{\underset{\vee}{pk}}\,\psi_j) = -\xi^i_{kj}, \qquad \psi_i = \frac{1}{N}(\overline{\Gamma}^p_{\underline{jp}} - \Gamma^p_{\underline{jp}}). \tag{1.6}$$

and $\underset{\vee}{ij}$ denotes an antisymmetrisation with division. In the corresponding points $M(x)$ and $\overline{M}(x)$ of a conformal mapping we can put

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + P^i_{jk} \quad (i, j, k = 1, ..., N), \tag{1.7}$$

where P^i_{jk} is *the deformation tensor* of the connection Γ of $\mathbb{GR}_N$ according to the conformal mapping $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$.

Notice that in $\mathbb{GR}_N$ we have

$$\Gamma^p_{\underset{\vee}{ip}} = 0, \tag{1.8}$$

(eq. (2.10) in [14]).

Based on the non-symmetry of the connection in a generalized Riemannian space one can define four kinds of covariant derivatives. For example, for a tensor a^i_j in $\mathbb{GR}_N$ we have

$$a^i_{j\underset{1}{|}m} = a^i_{j,m} + \Gamma^i_{pm}a^p_j - \Gamma^p_{jm}a^i_p, \qquad a^i_{j\underset{2}{|}m} = a^i_{j,m} + \Gamma^i_{mp}a^p_j - \Gamma^p_{mj}a^i_p,$$
$$a^i_{j\underset{3}{|}m} = a^i_{j,m} + \Gamma^i_{pm}a^p_j - \Gamma^p_{mj}a^i_p, \qquad a^i_{j\underset{4}{|}m} = a^i_{j,m} + \Gamma^i_{mp}a^p_j - \Gamma^p_{jm}a^i_p.$$

Here we denoted by $\underset{\theta}{|}$ a covariant derivative of the kind θ ($\theta \in \{1, 2, 3, 4\}$) in $\mathbb{GR}_N$.

In the case of the space $\mathbb{GR}_N$ we have five independent curvature tensors [24]:

$$\begin{aligned}
\underset{1}{K}{}^i_{jmn} &= \Gamma^i_{jm,n} - \Gamma^i_{jn,m} + \Gamma^p_{jm}\Gamma^i_{pn} - \Gamma^p_{jn}\Gamma^i_{pm},\\
\underset{2}{K}{}^i_{jmn} &= \frac{1}{2}(\Gamma^i_{jm,n} - \Gamma^i_{jn,m} + \Gamma^i_{mj,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{np} + \Gamma^p_{mj}\Gamma^i_{pn} - \Gamma^p_{jn}\Gamma^i_{mp} - \Gamma^p_{nj}\Gamma^i_{pm}),\\
\underset{3}{K}{}^i_{jmn} &= \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{pm} + \Gamma^p_{nm}(\Gamma^i_{pj} - \Gamma^i_{jp}),\\
\underset{4}{K}{}^i_{jmn} &= \frac{1}{2}(\Gamma^i_{jm,n} - \Gamma^i_{jn,m} + \Gamma^i_{mj,n} - \Gamma^i_{nj,m} + \Gamma^p_{mj}\Gamma^i_{pn} + \Gamma^p_{jm}\Gamma^i_{np} - \Gamma^p_{jn}\Gamma^i_{pm} - \Gamma^p_{nj}\Gamma^i_{mp}),\\
\underset{5}{K}{}^i_{jmn} &= \frac{1}{2}(\Gamma^i_{jm,n} + \Gamma^i_{mj,n} - \Gamma^i_{jn,m} - \Gamma^i_{nj,m} + 2\Gamma^p_{\underline{jm}}\Gamma^i_{\underline{pn}} - 2\Gamma^p_{\underline{jn}}\Gamma^i_{\underline{mp}} + \Gamma^p_{\underset{\vee}{nm}}\Gamma^i_{\underset{\vee}{pm}}).
\end{aligned}$$

We use the conformal mapping $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ to obtain the tensors $\underset{\theta}{\overline{K}}{}^i_{jmn}$ $(\theta = 1, ..., 5)$, where for example

$$\underset{1}{\overline{K}}{}^i_{jmn} = \overline{\Gamma}^i_{jm,n} - \overline{\Gamma}^i_{jn,m} + \overline{\Gamma}^p_{jm}\overline{\Gamma}^i_{pn} - \overline{\Gamma}^p_{jn}\overline{\Gamma}^i_{pm}. \tag{1.9}$$

2. Concircular vector field

In 1940. K. Yano [23] considered the conformal mapping $\overline{g}_{ij} = \psi^2 g_{ij}$ of two Riemannian spaces. In this case, he proved that geodesics are invariant under this mapping if and only if

$$\psi_{;ij} - \psi_i\psi_j = \omega g_{ij}, \tag{2.1}$$

where $(;)$ is a covariant derivative, g_{ij} a symmetric metric tensor, ω an invariant and ψ_i is a gradient vector.

When N. S. Sinyukov studied geodesic mappings of symmetric spaces [18], he wrote this condition in terms of $\xi = e^{-\psi}$. It is easy to see that the formula (2.1) transformes to

$$\xi_{i;j} = \rho g_{ij}, \tag{2.2}$$

where $\rho = -\omega e^{-\psi}$, $\xi_{;i} = \xi_i$. The vector field ξ_i, was called *concircular* vector field by K. Yano [23] . In the case when $\rho = const.$, ξ is called *convergent*, and in the case $\rho = B\xi + C$, $(B, C = const.)$, ξ is called *special concircular*. A space with concircular vector field was called *equidistant space* by N.S. Sinyukov.

Definition 2.1. [1] *A generalized Riemannian space $\mathbb{GR}_N$ with a non-symmetric metric tensor g_{ij} is called an* **equidistant space***, if its adjoint Riemannian space $\mathbb{R}_N$ is an equidistant space, i.e. if there exists a non-vanishing one-form φ in $\mathbb{GR}_N$, $\varphi_i \neq 0$ satisfying*

$$\varphi_{i;j} = \rho g_{\underline{ij}}, \tag{2.3}$$

where $(;)$ denotes the covariant derivative with respect to the symmetric part of the connection of the space $\mathbb{GR}_N$. For $\rho \neq 0$ equidistant spaces belong to the **primary type***, and for $\rho \equiv 0$ to the* **particular**.

The following definition is a consequence of the previous definition

Definition 2.2. A Concircular mapping *$f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ is a conformal mapping if the following equation is valid*

$$\psi_{ij} = \psi_{;ij} - \psi_i\psi_j = \omega g_{\underline{ij}}, \tag{2.4}$$

where $\psi_i = \frac{1}{N}(\overline{\Gamma}^p_{\underline{jp}} - \Gamma^p_{\underline{jp}})$, ω is an invariant, and $(;)$ is the covariant derivative with respect to the connection $\Gamma^i_{\underline{jk}}$.

In the case of a concircular mapping $f : \mathbb{R}_N \to \overline{\mathbb{R}}_N$ of two Riemannian spaces $\mathbb{R}_N$ and $\overline{\mathbb{R}}_N$, we have an invariant geometric object

$$\boxed{Z^i{}_{jmn} = R^i{}_{jmn} - \frac{R}{N(N-1)}(\delta^i_n\, g_{jm} - \delta^i_m\, g_{jn}),} \tag{2.5}$$

where $R^i{}_{jmn}$ is the Riemann-Christoffel curvature tensor of the space $\mathbb{R}_N$, R_{jm} the Ricci tensor and R the scalar curvature. The object $Z^i{}_{jmn}$ is called the *concircular curvature tensor*.

3. Equitorsion concircular curvature tensors

For a concircular mapping $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$, it is not possible to find a generalization of the concircular curvature tensor. For that reason, we define a special concircular mapping.

Definition 3.1. *A concircular mapping $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ is* **equitorsion** *if the torsion tensors of the spaces $\mathbb{GR}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$ are equal at corresponding points.*

According to (1.7), this means that

$$\overline{\Gamma}^i_{\underset{\vee}{jk}} - \Gamma^i_{\underset{\vee}{jk}} = \xi^i_{jk} = 0. \tag{3.1}$$

3.1. *Equitorsion concircular curvature tensor of the first kind*

Using (1.7), we get a relation between the first kind curvature tensors of the spaces $\mathbb{GR}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$:

$$\underset{1}{\overline{K}}{}^i_{jmn} = \underset{1}{K}{}^i_{jmn} + P^i_{\underline{jm};n} - P^i_{\underline{jn};m} + P^p_{\underline{jm}} P^i_{\underline{pn}} - P^p_{\underline{jn}} P^i_{\underline{pm}} + P^i_{\underline{pn}} \Gamma^p_{\underset{\vee}{jm}} - P^p_{\underline{jn}} \Gamma^i_{\underset{\vee}{pm}} - P^i_{\underline{pm}} \Gamma^p_{\underset{\vee}{jn}} + P^p_{\underline{jm}} \Gamma^i_{\underset{\vee}{pn}}\ . \tag{3.2}$$

Substituting the deformation tensor P with respect to (1.5, 1.7), and using (2.4), we obtain

$$\begin{aligned}\underset{1}{\overline{K}}{}^i_{jmn} = \underset{1}{K}{}^i_{jmn} &+ 2\delta^i_m\, \omega g_{\underline{jn}} - 2\delta^i_n\, \omega g_{\underline{jm}} + (\delta^i_m\, g_{\underline{jn}} - \delta^i_n\, g_{\underline{jm}})\triangle\psi \\ &+ \psi_p \delta^i_n\, \Gamma^p_{\underset{\vee}{jm}} - 2\psi_j \Gamma^i_{\underset{\vee}{nm}} - \psi_p \delta^i_m\, \Gamma^p_{\underset{\vee}{jn}} - 2\psi^i g_{\underline{pn}}\, \Gamma^p_{\underset{\vee}{jm}} + \psi^p g_{\underline{jn}}\, \Gamma^i_{\underset{\vee}{pm}} - \psi^p g_{\underline{jm}}\, \Gamma^i_{\underset{\vee}{pn}},\end{aligned} \tag{3.3}$$

where we denoted

$$\psi^i_j = g^{\underline{ip}} \psi_{pj}, \quad \triangle\psi = g^{\underline{pq}} \psi_p \psi_q = \psi_p \psi^p. \tag{3.4}$$

Contracting with respect to the indices i and n in (3.3) we get

$$\underset{1}{\overline{K}}_{jm} = \underset{1}{K}_{jm} - 2(N-1)\omega g_{\underline{jm}} - (N-1)\triangle\psi g_{\underline{jm}} + (N-2)\psi_p \Gamma^p_{\underset{\vee}{jm}} + 2\psi^p \Gamma_{\underset{\vee}{m.jp}}, \tag{3.5}$$

In case of concircular mappings, it is easy to prove the following formula

$$\overline{g}^{\underline{ij}} = e^{-2\psi}\, g^{\underline{ij}}\ . \tag{3.6}$$

In (3.5) multiplying by $g^{\underline{jm}}$ and contracting with respect to the indices j and then m we get

$$e^{2\psi} \underset{1}{\overline{K}} = \underset{1}{K} + 2N(1-N)\omega + N(1-N)\triangle\psi, \tag{3.7}$$

where $\underset{1}{\overline{K}} = \overline{g}^{\underline{pq}} \underset{1}{\overline{K}}_{pq}$, and $\underset{1}{K} = g^{\underline{pq}} \underset{1}{K}_{pq}$ are scalar curvatures of the first kind of the spaces $\mathbb{G}\overline{\mathbb{R}}_N$ and $\mathbb{GR}_N$ respectively. From (3.7), we have

$$\omega = \frac{1}{2N(1-N)}(e^{2\psi} \underset{1}{\overline{K}} - \underset{1}{K}) - \frac{1}{2}\triangle\psi. \tag{3.8}$$

It is easy to see that for concircular mappings the following formula is valid

$$g^{\underline{pi}} g_{\underline{jn}} = \overline{g}^{\underline{pi}} \overline{g}_{\underline{jn}}. \tag{3.9}$$

From (1.2) follows

$$\psi_i = \frac{1}{2N}\left(\frac{\partial}{\partial x^i}\ln \overline{g} - \frac{\partial}{\partial x^i}\ln g\right), \tag{3.10}$$

where $g = \det(g_{ij})$, $\overline{g} = \det(\overline{g}_{ij})$. From (3.1) and (3.10) we obtain

$$\Gamma_{j.\underset{\vee}{nm}} \psi^i = \frac{1}{2N}\overline{\Gamma}_{j.\underset{\vee}{nm}} \overline{g}^{\underline{ip}} \frac{\partial}{\partial x^p}\ln \overline{g} - \frac{1}{2N}\Gamma_{j.\underset{\vee}{nm}} g^{\underline{ip}} \frac{\partial}{\partial x^p}\ln g \tag{3.11}$$

and

$$\Gamma^i_{\underset{\vee}{qn}} g_{\underline{mj}}\, \psi^q = \frac{1}{2N}\overline{\Gamma}^i_{\underset{\vee}{qn}} \overline{g}_{\underline{mj}} \overline{g}^{\underline{pq}} \frac{\partial}{\partial x^p}\ln \overline{g} - \frac{1}{2N}\Gamma^i_{\underset{\vee}{qn}} g_{\underline{mj}} g^{\underline{pq}} \frac{\partial}{\partial x^p}\ln g. \tag{3.12}$$

Taking into account (3.10, 3.11, 3.12), we can write the relation (3.3) in the form

$$\underset{1}{\overline{Z}}{}^i_{jmn} = \underset{1}{Z}{}^i{}_{jmn}, \tag{3.13}$$

where

$$\begin{aligned} \underset{1}{Z}{}^i_{imn} &= \underset{1}{K}{}^i_{jmn} - \frac{1}{N(N-1)} K(\delta^i_n\, g_{jm} - \delta^i_m\, g_{jn}) \\ &+ \frac{1}{2N}\Big(-\delta^i_n\, \Gamma^p_{\underset{\vee}{jm}} + 2\delta^p_j\, \Gamma^i_{\underset{\vee}{nm}} + \delta^i_m\, \Gamma^p_{\underset{\vee}{jn}} + 2 g^{\underline{ip}} g_{\underline{qn}}\, \Gamma^q_{\underset{\vee}{jm}} - g^{\underline{pq}} g_{\underline{jn}}\, \Gamma^i_{\underset{\vee}{qm}} + g^{\underline{pq}} g_{\underline{jm}}\, \Gamma^i_{\underset{\vee}{qn}}\Big)\frac{\partial}{\partial x^p}\ln g. \end{aligned} \tag{3.14}$$

and analogously for the geometrical object $\underset{1}{\overline{Z}}{}^i_{jmn} \in \mathbb{G}\overline{\mathbb{R}}_N$. The tensor $\underset{1}{Z}{}^i{}_{jmn}$ is an invariant of equitorsion concircular mappings, and one can call it **the equitorsion concircular curvature tensor of the first kind.** So, the following theorem is proved:

Theorem 3.1. *Let the generalized Riemannian spaces $\mathbb{GR}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$ be defined by virtue of their non-symmetric basic tensors g_{ij} and $\overline{g}_{ij}$ respectively. The equitorsion concircular curvature tensor of the first kind* $\underset{1}{Z}{}^i{}_{jmn}$ (3.14) *is an invariant of the equitorsion concircular mapping $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$.*

3.2. Equitorsion concircular curvature tensor of the second kind

For the second kind curvature tensors of the spaces $\mathbb{GR}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$ we get the relation

$$\underset{2}{\overline{K}}{}^i_{jmn} = \underset{2}{K}{}^i_{jmn} + P^i_{\underline{jm};n} - P^i_{\underline{jn};m} + P^p_{\underline{jm}} P^i_{\underline{pn}} - P^p_{\underline{jn}} P^i_{\underline{pm}} \tag{3.15}$$

i.e., using (1.5, 1.7, 2.4) one obtains

$$\underset{2}{\overline{K}}{}^i_{jmn} = \underset{2}{K}{}^i_{jmn} + 2\delta^i_m\, \omega g_{\underline{jn}} - 2\delta^i_n\, \omega g_{\underline{jm}} + (\delta^i_m\, g_{\underline{jn}} - \delta^i_n\, g_{\underline{jm}})\triangle\psi. \tag{3.16}$$

Contracting with respect to the indices i and n in (3.16) we get

$$\underset{2}{\overline{K}}_{jm} = \underset{2}{K}_{jm} - 2(N-1)\omega g_{\underline{jm}} - (N-1)\triangle\psi g_{\underline{jm}}. \tag{3.17}$$

In the previous equation multiplying by $g^{\underline{jm}}$ and contracting with respect to j and then to m, we get

$$e^{2\psi}\underset{2}{\overline{K}} = \underset{2}{K} + 2N(1-N)\omega + N(1-N)\Delta\psi, \tag{3.18}$$

where $\underset{2}{\overline{K}} = \overline{g}^{\underline{pq}}\underset{2}{\overline{K}}_{pq}$, and $\underset{2}{K} = g^{\underline{pq}}\underset{2}{K}_{pq}$ are scalar curvatures of the second kind of the spaces $\mathbb{G}\overline{\mathbb{R}}_N$ and $\mathbb{G}\mathbb{R}_N$ respectively. From (3.18), we have

$$\omega = \frac{1}{2N(1-N)}(e^{2\psi}\underset{2}{\overline{K}} - \underset{2}{K}) - \frac{1}{2}\Delta\psi. \tag{3.19}$$

And finally, taking into account (3.10, 3.11, 3.12), we can write the relation (3.16) in the form

$$\underset{2}{\overline{Z}}{}^i_{jmn} = \underset{2}{Z}{}^i{}_{jmn}, \tag{3.20}$$

where

$$\underset{2}{Z}{}^i{}_{jmn} = \underset{2}{K}{}^i_{jmn} - \frac{1}{N(N-1)}\underset{2}{K}(\delta^i_n\, g_{jm} - \delta^i_m\, g_{jn}) \tag{3.21}$$

and analogously for $\underset{2}{\overline{Z}}{}^i_{jmn} \in \mathbb{G}\overline{\mathbb{R}}_N$. The tensor $\underset{2}{Z}{}^i{}_{jmn}$ is an invariant of equitorsion concircular mappings, and one can call it **the equitorsion concircular curvature tensor of the second kind.** So, we have:

Theorem 3.2. *Starting from the curvature tensor $\underset{2}{K}{}^i_{jmn}$, one obtains an invariant tensor $\underset{2}{Z}{}^i{}_{jmn}$ with respect to the equitorsion concircular mapping $f : \mathbb{G}\mathbb{R}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ in the form* (3.21).

3.3. *Equitorsion concircular curvature tensor of the third kind*

In the case of the third kind curvature tensors of the spaces $\mathbb{G}\mathbb{R}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$ we get the relation

$$\begin{aligned}\underset{3}{\overline{K}}{}^i_{jmn} = \underset{3}{K}{}^i_{jmn} &+ P^i_{\underline{jm};n} - P^i_{\underline{jn};m} + P^p_{\underline{jm}}P^i_{\underline{pn}} - P^p_{\underline{jn}}P^i_{\underline{pm}} \\ &+ P^i_{\underline{pn}}\Gamma^p_{\underset{\vee}{jm}} - P^p_{\underline{jn}}\Gamma^i_{\underset{\vee}{pm}} + P^i_{\underline{pm}}\Gamma^p_{\underset{\vee}{jn}} - P^p_{\underline{jm}}\Gamma^i_{\underset{\vee}{pn}} - 2P^p_{\underline{nm}}\Gamma^i_{\underset{\vee}{jp}},\end{aligned} \tag{3.22}$$

i.e., using (1.5, 1.7, 2.4) one obtains

$$\begin{aligned}\underset{3}{\overline{K}}{}^i_{jmn} = \underset{3}{K}{}^i_{jmn} &+ 2\delta^i_m\,\omega g_{\underline{jn}} - 2\delta^i_n\,\omega g_{\underline{jm}} + (\delta^i_m\, g_{\underline{jn}} - \delta^i_n\, g_{\underline{jm}})\Delta\psi \\ &- 2\psi_n\Gamma^i_{\underset{\vee}{jm}} + \psi_p\delta^i_n\,\Gamma^p_{\underset{\vee}{jm}} - 2\psi_m\Gamma^i_{\underset{\vee}{jn}} + \psi_p\delta^i_m\,\Gamma^p_{\underset{\vee}{jn}} + \psi^p g_{\underline{jn}}\,\Gamma^i_{\underset{\vee}{pm}} + 2\psi^p g_{\underline{mn}}\,\Gamma^i_{\underset{\vee}{jp}} + \psi^p g_{\underline{jm}}\,\Gamma^i_{\underset{\vee}{pn}}.\end{aligned} \tag{3.23}$$

Contracting (3.23) with respect to the indices i and n, the previous equation becomes

$$\underset{3}{\overline{K}}_{jm} = \underset{3}{K}_{jm} - 2(N-1)\omega g_{\underline{jm}} - (N-1)\Delta\psi g_{\underline{jm}} + (N-2)\psi_p\Gamma^p_{\underset{\vee}{jm}} + 2\psi^p\Gamma_{\underset{\vee}{m.jp}}, \tag{3.24}$$

Multiplying (3.24) by $\overline{g}^{\underline{jm}} = e^{-2\psi}g_{\underline{jm}}$ and contracting we get

$$e^{2\psi}\underset{3}{\overline{K}} = \underset{3}{K} + 2N(1-N)\omega + N(1-N)\Delta\psi, \tag{3.25}$$

where $\underset{3}{\overline{K}} = \overline{g}^{\underline{pq}}\underset{3}{\overline{K}}_{pq}$, and $\underset{3}{K} = g^{\underline{pq}}\underset{3}{K}_{pq}$ are scalar curvatures of the third kind of the spaces $\mathbb{G}\overline{\mathbb{R}}_N$ and $\mathbb{G}\mathbb{R}_N$ respectively. From (3.25), we have

$$\omega = \frac{1}{2N(1-N)}(e^{2\psi}\underset{3}{\overline{R}} - \underset{3}{R}) - \frac{1}{2}\Delta\psi, \tag{3.26}$$

Finally,

$$\underset{3}{\overline{Z}}{}^i_{jmn} = \underset{3}{Z}{}^i{}_{jmn} \tag{3.27}$$

where

$$\begin{aligned}\underset{3}{Z}{}^{i}_{jmn} &= \underset{3}{R}{}^{i}{}_{jmn} - \frac{1}{N(N-1)}\underset{3}{K}(\delta^{i}_{n}\, g_{jm} - \delta^{i}_{m}\, g_{jn}) \\ &+\frac{1}{2N}\Big(2\delta^{p}_{n}\,\underset{\vee}{\Gamma}{}^{i}_{jm}-\delta^{i}_{n}\,\underset{\vee}{\Gamma}{}^{p}_{jm}+2\delta^{p}_{m}\,\underset{\vee}{\Gamma}{}^{i}_{jn}-\delta^{i}_{m}\,\underset{\vee}{\Gamma}{}^{p}_{jn}-g^{\underline{pq}}g_{\underline{jn}}\,\underset{\vee}{\Gamma}{}^{i}_{qm}-2g^{\underline{pq}}g_{\underline{mn}}\,\underset{\vee}{\Gamma}{}^{i}_{jq}-g^{\underline{pq}}g_{\underline{jm}}\,\underset{\vee}{\Gamma}{}^{i}_{qn}\Big)\frac{\partial}{\partial x^{p}}\ln g.\end{aligned} \tag{3.28}$$

And analogously for $\underset{3}{\overline{Z}}{}^{i}_{jmn}$ of the space $\mathbb{G}\overline{\mathbb{R}}_N$. The tensor $\underset{3}{Z}{}^{i}{}_{jmn}$ is an invariant of equitorsion concircular mappings, and one can call it **the equitorsion concircular curvature tensor of the third kind.** Now we have proved

Theorem 3.3. *From the curvature tensor $\underset{3}{K}{}^{i}_{jmn}$, we obtain an invariant tensor $\underset{3}{Z}{}^{i}{}_{jmn}$ according to the equitorsion concircular mapping $f : \mathbb{G}\mathbb{R}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ in the form* (3.28).

3.4. Equitorsion concircular curvature tensor of the fourth kind

For curvature tensors of the fourth kind we get

$$\underset{4}{\overline{K}}{}^{i}_{jmn} = \underset{4}{K}{}^{i}_{jmn} + P^{i}_{\underline{jm};n} - P^{i}_{\underline{jn};m} + P^{p}_{\underline{jm}}P^{i}_{\underline{pn}} - P^{p}_{\underline{jn}}P^{i}_{\underline{pm}} \tag{3.29}$$

i.e.

$$\underset{4}{\overline{K}}{}^{i}_{jmn} = \underset{4}{K}{}^{i}_{jmn} + 2\delta^{i}_{m}\,\omega g_{\underline{jn}} - 2\delta^{i}_{n}\,\omega g_{\underline{jm}} + (\delta^{i}_{m}\, g_{\underline{jn}} - \delta^{i}_{n}\, g_{\underline{jm}})\triangle\psi. \tag{3.30}$$

Using the same procedure like in the previous cases, in this case an invariant object of the equitorsion concircular mapping is in the form

$$\underset{4}{Z}{}^{i}{}_{jmn} = \underset{4}{K}{}^{i}_{jmn} - \frac{1}{N(N-1)}\underset{4}{K}(\delta^{i}_{n}\, g_{jm} - \delta^{i}_{m}\, g_{jn}) \tag{3.31}$$

where $\underset{4}{K}_{jm}$ is the Ricci curvature tensor of the fourth kind and $\underset{4}{K}$ a scalar curvature of the fourth kind. The object $\underset{4}{Z}{}^{i}{}_{jmn}$ is a tensor and we call it **equitorsion concircular curvature tensor of the fourth kind** of the equitorsion mapping. So, the next theorem is valid:

Theorem 3.4. *From the curvature tensor $\underset{4}{K}{}^{i}_{jmn}$, one obtains an invariant tensor $\underset{4}{Z}{}^{i}{}_{jmn}$* (3.31) *of the equitorsion mapping of generalized Riemannian spaces.*

3.5. Equitorsion concircular curvature tensor of the fifth kind

For the curvature tensors of the fifth kind of the spaces $\mathbb{G}\mathbb{R}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$ we have

$$\underset{5}{\overline{K}}{}^{i}_{jmn} = \underset{5}{K}{}^{i}_{jmn} + P^{i}_{\underline{jm};n} - P^{i}_{\underline{jn};m} + P^{p}_{\underline{jm}}P^{i}_{\underline{pn}} - P^{p}_{\underline{jn}}P^{i}_{\underline{pm}} \tag{3.32}$$

i.e.

$$\underset{5}{\overline{K}}{}^{i}_{jmn} = \underset{5}{K}{}^{i}_{jmn} + 2\delta^{i}_{m}\,\omega g_{\underline{jn}} - 2\delta^{i}_{n}\,\omega g_{\underline{jm}} + (\delta^{i}_{m}\, g_{\underline{jn}} - \delta^{i}_{n}\, g_{\underline{jm}})\triangle\psi. \tag{3.33}$$

Contracting with respect to the indices i, n and denoting

$$\underset{5}{K}{}^{p}_{jmp} = \underset{5}{K}_{jm}, \quad \underset{5}{\overline{K}}{}^{p}_{jmp} = \underset{5}{\overline{K}}_{jm}, \tag{3.34}$$

we obtain

$$\underset{5}{\overline{K}}_{jm} = \underset{5}{K}_{jm} - 2(N-1)\omega g_{\underline{jm}} - (N-1)\triangle\psi g_{\underline{jm}}\,. \tag{3.35}$$

wherefrom, multiplying by $\overline{g}^{\underline{jm}} = e^{-2\psi} g_{\underline{jm}}$ and contracting with respect to the indices j and m one obtains

$$\omega = \frac{1}{2N(1-N)}(e^{2\psi}\underset{5}{\overline{K}} - \underset{5}{K}) - \frac{1}{2}\Delta\psi. \tag{3.36}$$

After eliminating ω from (3.33) we can write

$$\underset{5}{\overline{Z}}{}^i_{jmn} = \underset{5}{Z}{}^i{}_{jmn}, \tag{3.37}$$

where

$$\underset{5}{Z}{}^i{}_{jmn} = \underset{5}{K}{}^i_{jmn} - \frac{1}{N(N-1)}\underset{5}{K}(\delta^i_n g_{jm} - \delta^i_m g_{jn}). \tag{3.38}$$

The object $\underset{5}{Z}{}^i{}_{jmn}$ is an invariant of the concircular equitorsion mapping. We call it **equitorsion concircular curvature tensor of the fifth kind**. So, the following theorem is proved:

Theorem 3.5. *Starting from the curvature tensor $\underset{5}{K}{}^i_{jmn}$, we obtain an invariant tensor $\underset{5}{Z}{}^i{}_{jmn}$* (3.38) *of the equitorsion concircular mapping $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$.*

4. Concluding remarks

For $g_{ij}(x) = g_{ji}(x)$ the space $\mathbb{GR}_N$ reduces to the Riemannian space $\mathbb{R}_N$. The curvature tensors $\underset{\theta}{K}$, $\theta = 1, \ldots, 5$ in a generalized Riemannian space reduce to the single curvature tensor R in Riemannian space (in the symmetric case).

In the case of equitorsion concircular mapping of the Riemannian spaces (in the symmetric case) $\underset{\theta}{Z}$, $(\theta = 1, \cdots, 5)$, given by the formulas (3.14, 3.21, 3.28, 3.31, 3.38) reduce to the concircular curvature tensor [18, 23]

$$Z^i{}_{jmn} = R^i{}_{jmn} - \frac{R}{N(N-1)}(\delta^i_n g_{jm} - \delta^i_m g_{jn}). \tag{4.1}$$

All these new quantities can be quite interesting for further investigation.

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Conformally-projective harmonic diffeomorphisms of equidistant manifolds

Irena Hinterleitner

Inst. of Math., FSI VUT Brno, Technická 2, 616 69 Brno, Czech Republic, e-mail: Hinterleitner.Irena@seznam.cz

Abstract

In this paper we study compositions of conformal and geodesic diffeomorphisms, which are at the same time harmonic mappings (conformally-projective harmonic diffeomorphisms). Conformally-projective harmonic diffeomorphisms of equidistant manifolds are shown. As an explicit example we will show the Friedmann cosmological models.

Keywords: conformal mappings, projective mappings, harmonic mappings, conformally-projective harmonic diffeomorphisms, equidistant manifolds, Friedmann models.

2000 Mathematics Subject Classification: 53B20, 53C43.

1. Introduction

The theory of conformal, geodesic and harmonic mappings can be viewed as an interesting part of differential geometry of Riemannian and pseudo-Riemannian spaces, see [1–10]. Harmonic mappings are extremal with respect to the natural energy functionals of sigma models, see J.C. Wood [9].

S.E. Stepanov and I.G. Shandra [8] studied harmonic diffeomorphisms. In this paper compositions of conformal and geodesic mappings, which are harmonic, are studied. We shall call such a composition conformally-projective harmonic. We study particularly conformally-projective harmonic diffeomorphisms of equidistant manifolds.

2. Special diffeomorphisms of Riemannian spaces

Consider an n-dimensional *Riemannian manifold* V_n (*Riemannian space*) endowed with the metric g, which in any coordinate neighborhood $U \subset V_n$ with local coordinates $x \equiv (x^1, x^2, \dots, x^n)$ is determined by the components $g_{ij}(x)$, $i, j = 1, 2, \dots, n$, which form a symmetric and non-singular matrix. We use notions from the theory of Riemannian spaces as in the articles [1, 2, 3, 5, 6, 8].

The signature of the metric is assumed, in general, to be arbitrary, i.e. under the notion of a Riemannian space V_n we understand "classical" Riemannian spaces, as well as pseudo-Riemannian spaces, like in [5, 6], for example.

Christoffel symbols of types I and II are introduced by the formulas: $\Gamma_{ijk} \equiv \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$ and $\Gamma^h_{ij} \equiv g^{h\alpha}\Gamma_{ij\alpha}$, where $\partial_i \equiv \partial/\partial x^i$, g^{ij} is the inverse matrix to g_{ij}. Christoffel symbols of type II are the *natural connection* (the *Levi-Civita connection*) of Riemannian spaces, with respect to which the metric tensor is covariantly constant, i.e. $g_{ij,k} = 0$. Hereafter "," denotes the *covariant derivative* with respect to the connection of the space V_n.

We study special diffeomorphisms f between Riemannian spaces V_n and $\bar{V}_n$, and we restrict ourselves to coordinate neighbourhoods $U \subset V_n$ and $\bar{U} = f(U) \subset \bar{V}_n$. In U and $\bar{U}$ we introduce a *common coordinate system* x with respect to the diffeomorphism f, so that the point $M \in U$ and its image $f(M) \in \bar{U}$ have the same coordinates $x = (x^1, x^2, \dots, x^n)$, i.e. f is represented by the identity map from U to $\bar{U}$; see, for example, [2, 3, 4, 5, 6, 8] According geometric objects in $\bar{V}_n$ will be denoted by a bar. For example, $\bar{\Gamma}^h_{ij}$ are Christoffel symbols in $\bar{V}_n$.

Definition 1 *(see [1, 2, 5, 6, 8, 10]). The mapping $f\colon V_n \longrightarrow \bar{V}_n$ is conformal if and only if, in the common coordinate system x with respect to the mapping, the condition*

$$\bar{g}_{ij}(x) = \mathrm{e}^{2\sigma(x)} g_{ij}(x) \tag{1}$$

holds, where $\sigma(x)$ is a function on V_n.

Under conformal mappings the following conditions hold [1, 2, 5, 6, 10]:

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \delta^h_i \sigma_j + \delta^h_j \sigma_i - \sigma^h g_{ij}, \tag{2}$$

where $\sigma_i = \partial_i \sigma(x)$, $\sigma^h = \sigma_\alpha g^{\alpha h}$ and δ^h_i is the Kronecker delta.

Definition 2 *(see [2, 3, 4, 5, 6]). The diffeomorphism $f\colon V_n \longrightarrow \bar{V}_n$ is called a geodesic (or projective) mapping if f maps any geodesic line of V_n into a geodesic line of $\bar{V}_n$.*

A diffeomorphism from V_n onto $\bar{V}_n$ is *geodesic* if and only if the conditions

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \delta^h_i \psi_j + \delta^h_j \psi_i \tag{3}$$

hold, where $\psi_i(x)$ is a gradient vector, i.e. $\psi_i = \partial_i \psi(x)$ for some function ψ. If $\psi_i \not\equiv 0$, a geodesic mapping is called *nontrivial*; otherwise it is said to be *trivial* or *affine*. See [2, 3, 4, 5, 6].

Definition 3 *(see [9]). A harmonic diffeomorphism is a diffeomorphism that preserves Laplace's equation.*

A diffeomorphism from V_n onto $\bar{V}_n$ is harmonic if and only if the following conditions hold ([8])

$$(\bar{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x))\, g^{ij} = 0. \tag{4}$$

Definition 4 *We shall call a composition of a conformal and a geodesic (projective) diffeomorphism between Riemannian spaces, which is harmonic,conformally-projective harmonic.*

It follows from [8] that a diffeomorphism from an n-dimensional Riemannian space V_n onto a Riemannian space $\bar{V}_n$ is conformally-projective harmonic if and only if the following conditions hold

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \varphi_i\delta^h_j + \varphi_j\delta^h_i - \frac{2}{n}\,\varphi^h g_{ij}, \tag{5}$$

where $\varphi_i = \partial_i\varphi(x)$ is a gradient-like vector, $\varphi^h = g^{h\alpha}\varphi_\alpha$. Conditions (5) are derived by a combination of (2) and (3) under the assumption of (4).

3. Equidistant manifolds

Definition 5 *A vector field ξ^h is called concircular [10], if $\xi^h_{,i} = \varrho\,\delta^h_i$, where ϱ is an invariant. A Riemannian space V_n with concircular vector field is called equidistant [6], see [3, 4].*

A Riemannian space V_n is equidistant with a non-isotropic concircular vector field ξ^h (non-isotropic means $\xi^\alpha\xi^\beta g_{\alpha\beta} \neq 0$) if and only if in V_n exists a system of coordinates x, where the metric has the form (see [1, 4, 5, 6, 10])

$$ds^2 = \frac{1}{f(x^1)}{dx^1}^2 + f(x^1)\, d\tilde{s}^2, \tag{6}$$

where $f \in C^1$ $(f \neq 0)$ is a function, $d\tilde{s}^2 = \tilde{g}_{ab}(x^2, \dots, x^n)\, dx^a dx^b$ $(a, b = 2, \dots, n)$ is the metric form of the Riemannian subspace $\tilde{V}_{n-1}$, given by $x^1 =$ const.

An equidistant manifold V_n with metric (6) admits geodesic diffeomorphisms given by the identity map onto the Riemannian space $\bar{V}_n$, whose metric form is

$$d\bar{s}^2 = \frac{p}{f\cdot(1+qf)^2}{dx^1}^2 + \frac{p\,f}{1+qf}d\tilde{s}^2, \tag{7}$$

where p, q are some constants such that $1+qf \neq 0$, $p \neq 0$. If $qf' \not\equiv 0$, the mapping is nontrivial; otherwise it is trivial, and x are common coordinates for V_n and $\bar{V}_n$, see [4]. The function $\psi(x)$, which defines a geodesic mapping (see (3)), has the following form: $\psi(x) = -\frac{1}{2}\ln|1+q\,f|$.

H.W. Brinkmann [1] (see [4, 5]) showed that the space V_n with metric (6) is an Einstein space $\mathcal{E}_n$ (resp. a space $\mathcal{S}_n$ with constant curvature K) if and only if holds:

Condition 1. $f = K\,{x^1}^2 + 2a\,x^1 + b$, where K, a and b are constants and $d\tilde{s}^2$ is a metric of an Einstein space $\tilde{\mathcal{E}}_{n-1}$ (resp. a space $\tilde{\mathcal{S}}_{n-1}$ with constant curvature $\tilde{K}$), moreover $\tilde{K} = \dfrac{\tilde{R}}{(n-1)(n-2)} = b\,K^2 - a^2$, where $K = \dfrac{R}{n(n-1)}$.
Here R and $\tilde{R}$ are the scalar curvatures of $\mathcal{E}_n$ and $\tilde{\mathcal{E}}_{n-1}$ (resp. $\mathcal{S}_n$ and $\tilde{\mathcal{S}}_{n-1}$).

4. Conformally-projective harmonic diffeomorphisms of equidistant manifolds

The following theorem holds:

Theorem 6 *An equidistant manifold V_n with the metric*

$$ds^2 = (1 + q\,f(x^1))^{\frac{2}{n-2}} \left(\frac{1}{f(x^1)} d{x^1}^2 + f(x^1)\, d\tilde{s}^2 \right), \tag{8}$$

where $f \in C^1$ $(f \neq 0)$ is a function and $d\tilde{s}^2 = \tilde{g}_{ab}(x^2, \ldots, x^n)\, dx^a dx^b$ $(a, b = 2, \ldots, n)$ is the metric of some $(n-1)$-dimensional Riemannian space $\tilde{V}_{n-1}$, is mapped by the identity map conformally-projectively harmonically on to the Riemannian space $\bar{V}_n$ with the metric (7).

Proof. Let (8) and (7) be the metric forms of the Riemannian spaces V_n and $\bar{V}_n$. We calculate the Christoffel symbols Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ of these spaces. Formula (5) holds for $\varphi = -\dfrac{n}{2(n-2)} \ln|1 + q\,f|$.

Analysing formulas (1)-(8) we can convince ourselves that the following holds:

Proposition 7 *The equidistant manifold V_n with metric (8) is conformally mapped onto a Riemannian space with metric (6), which is geodesically mapped onto a Riemannian space $\bar{V}_n$ with metric (7).*

Proposition 8 *By comparison of the metrics (8) and (7) we find that, dependent on the choice of the parameter q, the signatures of the two metrics can be the same or not.*

Proposition 9 *There are spaces with a metric of the form (8), satisfying Condition 1, admitting conformally-projectively harmonic mappings onto an Einstein space, resp. a space of constant curvature.*

By a detailed analysis we can convince ourselves of the existence of compact Riemannian spaces, for which global non affine conformally-projective harmonic mappings exist.

5. Equidistant manifolds on geodesic coordinate system and Friedmann metrics

Upon a suitable transformation of the coordinate x^1 we can rewrite the metrics (6), (7) and (8) in the form:

$$ds^2 = e dx^{1^2} + f(x^1)\, d\tilde{s}^2, \tag{9}$$

where $e = \pm 1$, $f \in C^1$ $(f \neq 0)$ is a function, $d\tilde{s}^2 = \tilde{g}_{ab}(x^2, \ldots, x^n) dx^a dx^b$ $(a, b = 2, \ldots, n)$ is the metric of a certain Riemannian subspace $\tilde{V}_{n-1}$ (see [3, 6]). Generally this function f is not the function, which figures in (6), (7) and (8). It is known that this coordinate system x is *geodesic* (see [2, 5, 6]).

The *Friedmann metric* is a metric (9) with $\tilde{V}_{n-1}$ being a space with constant curvature, modeling a spatially homogenous and isotropic universe. The function f describes the evolution in the time coordinate x^1 [7].

An equidistant space V_n with metric (9) referred to coordinates x admits geodesic mappings onto a Riemannian space $\bar{V}_n$, whose metric form is [3]

$$d\bar{s}^2 = \frac{ep}{(1+qf)^2}\, dx^{1^2} + \frac{p\, f}{1+qf}\, d\tilde{s}^2, \tag{10}$$

where p, q are some constants such that $1 + qf \neq 0$, $p \neq 0$. If $qf' \not\equiv 0$, the mapping is nontrivial; otherwise it is affine. The function $\psi(x)$, which defines a geodesic mapping, has also the form $\psi(x) = -\frac{1}{2} \ln |1 + q\, f|$.

Theorem 10 *An equidistant manifold V_n with the metric*

$$ds^2 = (1 + q\, f(x^1))^{\frac{2}{n-2}} \left(e\, dx^{1^2} + f(x^1)\, d\tilde{s}^2\right), \tag{11}$$

where $f \in C^1$ $(f \neq 0)$ is a function, $d\tilde{s}^2 = \tilde{g}_{ab}(x^2, \ldots, x^n)\, dx^a dx^b$ $(a, b = 2, \ldots, n)$ is the metric of some $(n-1)$-dimensional Riemannian space $\tilde{V}_{n-1}$, is mapped by the identity map conformally-projectively harmonically on the Riemannian space $\bar{V}_n$ with the metric (10).

The proof of Theorem 10 is analogous to that of Theorem 6 for that same function φ. The manifold V_n with metric (11) is conformally mapped onto a Riemannian space with metric (9), which is geodesically mapped onto a Riemannian space $\bar{V}_n$ with metric (10).

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On F-planar mappings of spaces with affine connections

Irena Hinterleitner
Department of Mathematic, FSI, Technical University of Brno
hinterleitner.irena@seznam.cz

Josef Mikeš[i]
Department of Algebra and Geometry, Fac. Sci., Palacky University of Olomouc
mikes@inf.upol.cz

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Abstract. In this paper we study F-planar mappings of n-dimensional or infinitely dimensional spaces with a torsion-free affine connection. These mappings are certain generalizations of geodesic and holomorphically projective mappings.
Here we make fundamental equations on F-planar mappings for dimensions $n > 2$ more precise.

Keywords: F-planar mapping, space with affine connections

MSC 2000 classification: primary 53C20, secondary 53C20

Introduction

In many papers geodesic mappings and their generalizations, like quasi-geodesic, holomorphically-projective, F-planar, 4-planar, mappings, were considered. One of the basic tasks was and is the derivation of the fundamental equations of these mappings. They were shown in the most various ways, see [1]-[7].

Unless otherwise specified, all spaces, connections and mappings under consideration are differentiable of a sufficiently high class. The dimension n of the spaces being considered is higher than *two*, as a rule. This fact is not specially stipulated. All spaces are assumed to be connected.

Here we show a method that simplifies and generalizes many of the results. Our results are valid also for infinite dimensional spaces with Banach bases $(n = \infty)$.

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1 F-planar curves

We consider an n-dimensional ($n > 2$) or infinite dimensional ($n = \infty$) space A_n with a torsion-free affine connection ∇, and an affinor structure F, i.e. a tensor field of type $\binom{1}{1}$.

If $n = \infty$ we assume that A_n is locally homeomorphic to a Banach space E_∞. In connection with local studies we assume the existence of a coordinate neighbourhood U in the Euclidean space E_n, resp. $U \subset E_\infty$.

1 Definition (J. Mikeš, N.S. Sinyukov [4])**.** A curve ℓ, which is given by the equations

$$\ell = \ell(t), \quad \lambda(t) = d\ell(t)/dt \ (\neq 0), \quad t \in I \tag{1}$$

where t is a parameter, is called *F-planar*, if its tangent vector $\lambda(t_0)$, for any initial value t_0 of the parameter t, remains, under parallel translation along the curve ℓ, in the distribution generated by the vector functions λ and $F\lambda$ along ℓ.

In particular, if $F = \varrho\, I$ we obtain the definition of a geodesic parametrized by an arbitrary parameter, see [4]. Here ϱ is a function and I is the identity operator.

In accordance with this definition, ℓ is F-planar if and only if the following condition holds [4]:

$$\nabla_{\lambda(t)}\, \lambda(t) = \varrho_1(t)\, \lambda(t) + \varrho_2(t)\, F\lambda(t), \tag{2}$$

where ϱ_1 and ϱ_2 are some functions of the parameter t.

2 F-planar mappings between two spaces with affine connection

We suppose two spaces A_n and $\bar{A}_n$ with torsion-free affine connections ∇ and $\bar{\nabla}$, respectively. Affine structures F and $\bar{F}$ are defined on A_n, resp. $\bar{A}_n$.

2 Definition (J. Mikeš, N.S. Sinyukov [4])**.** A diffeomorphism $f\colon A_n \to \bar{A}_n$ between two manifolds with affine connections is called *F-planar* if any F-planar curve in A_n is mapped onto an $\bar{F}$-planar curve in $\bar{A}_n$.

Important convention. Due to the diffeomorphism f we always suppose that ∇, $\bar{\nabla}$, and the affinors F, $\bar{F}$ are defined on A_n. Moreover, we always identify a given curve $\ell\colon I \to A_n$ and its tangent vector function $\lambda(t)$ with their images $\bar{\ell} = f \circ \ell$ and $\bar{\lambda} = f_*(\lambda(t))$ in $\bar{A}_n$.

Two principially different cases are possible for the investigation:

$$\text{a)} \quad \bar{F} = a\, F + b\, I; \tag{3}$$

$$\text{b)} \quad \bar{F} \neq a\,F + b\,I, \tag{4}$$

a, b are some functions.

Naturally, case a) characterizes F-planar mappings *which preserve F-structures.* In case b) the structures of F and $\bar{F}$ are essentially distinct. The following holds.

3 Theorem. *An F-planar mapping f from A_n onto $\bar{A}_n$ preserve F-structures and is characterized by the following condition*

$$P(X,Y) = \psi(X)\,Y + \psi(Y)\,X + \varphi(X)\,FY + \varphi(Y)\,FX \tag{5}$$

for any vector fields X, Y, where $P \overset{def}{=} \bar{\nabla} - \nabla$ is the deformation tensor field of f, ψ, φ are some linear forms.

Let us recall that on each tangent space $T_x A_n$, $P(X,Y)$ is a symmetric bilinear mapping $T_x A_n \times T_x A_n \to T_x A_n$ and a tensor field of type $\binom{1}{2}$.

Theorem 3 was proved by J. Mikeš and N. S. Sinyukov [4] for finite dimension $n > 3$. Here we can show a more rational proof of this Theorem for $n > 3$ and also a proof for $n = 3$. We show a counter example for $n = 2$.

3 F-planar mappings which preserve F-structures

First we prove the following proposition

4 Theorem. *An F-planar mapping f from A_n onto $\bar{A}_n$ which preserves F-structures is characterized by condition* (5)*.*

In the sequel we shall need the following lemma:

5 Lemma. *Let V be an n-dimensional vector space, $Q\colon V \times V \to V$ be a symmetric bilinear mapping and $F\colon V \to V$ a linear mapping. If, for each vector $\lambda \in V$*

$$Q(\lambda,\lambda) = \varrho_1(\lambda)\,\lambda + \varrho_2(\lambda)\,F(\lambda) \tag{6}$$

holds, where $\varrho_1(\lambda)$, $\varrho_2(\lambda)$ are functions on V, then there are linear forms ψ and φ such that the condition

$$Q(X,Y) = \psi(X)\,Y + \psi(Y)\,X + \varphi(X)\,F(Y) + \varphi(Y)\,F(X) \tag{7}$$

holds for any $X, Y \in V$.

Proof. Formula (6) has the following coordinate expression

$$Q^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = \varrho_1(\lambda)\,\lambda^h + \varrho_2(\lambda)\,F^h_\alpha\lambda^\alpha, \tag{8}$$

where $\lambda^i, F^h_i, Q^h_{ij}$ are the components of λ, F, Q.

By multiplying (8) with $\lambda^i F^j_\alpha \lambda^\alpha$ and antisymmetrizing the indices h, i and j we obtain

$$\left\{ Q^{[h}_{\alpha\beta}\delta^i_\gamma F^{j]}_\delta \right\} \lambda^\alpha\lambda^\beta\lambda^\gamma\lambda^\delta = 0, \tag{9}$$

where square brackets denote the alternation of indices. The term in curly brackets does not depend on λ and (9) holds for any vector $\lambda \in V$, therefore

$$Q^{[h}_{(\alpha\beta}\delta^i_\gamma F^{j]}_{\delta)} = 0 \tag{10}$$

holds, where the round brackets denote symmetrization of indices.

It is natural to assume that $F^h_i \neq a\,\delta^h_i$ with $a = \text{const}$. By virtue of this there exist some vectors ξ^h such that $\xi^\alpha F^h_\alpha \neq b\,\xi^h$, $b = \text{const}$. Introducing $P^h_i \overset{\text{def}}{=} P^h_{i\alpha}\xi^\alpha$, $P^h \overset{\text{def}}{=} P^h_\alpha\xi^\alpha$ and $F^h \overset{\text{def}}{=} F^h_\alpha\xi^\alpha$, we contract (10) with $\xi^\alpha\xi^\beta\xi^\gamma\xi^\delta$. Since $F^h \neq b\,\xi^h$, we obtain $P^h = 2a\,\xi^h + 2b\,F^h$, where a, b are certain constants. Contracting (10) with $\xi^\beta\xi^\gamma\xi^\delta$, and taking into account the precending, we have $P^h_i = a\,\delta^h_i + b\,F^h_i + a_i\,\xi^h + b_i\,F^h$, where a_i, b_i are some components of linear forms. Analogously, contracting (10) with $\xi^\gamma\xi^\delta$, we have

$$Q^h_{ij} = \psi_i\delta^h_j + \psi_j\delta^h_i + \varphi_i F^h_j + \varphi_j F^h_i + \xi^h a_{ij} + F^h b_{ij}, \tag{11}$$

where ψ_i, φ_i are components of a 1-form ψ, φ defined on V, and a_{ij}, b_{ij} are components of a symmetric 2-form defined on V.

In case that $a_{ij} = b_{ij} = 0$, evidently from (11) we obtain formula (7).

Now we will suppose that either $a_{ij} \neq 0$, or $b_{ij} \neq 0$. Since ξ^h and F^h are noncollinear, it is evident that

$$\xi^h a_{ij} + F^h b_{ij} \neq 0. \tag{12}$$

Formula (10) by virtue of (11) has the form

$$\Omega^{[hi}_{(\alpha\beta\gamma}F^{j]}_{\delta)} = 0, \tag{13}$$

where $\Omega^{hi}_{\alpha\beta\gamma} \overset{\text{def}}{=} (\xi^h a_{\alpha\beta} + F^h b_{\alpha\beta})\delta^i_\gamma - (\xi^i a_{\alpha\beta} + F^i b_{\alpha\beta})\delta^h_\gamma$. It is possible to show that there exists some vector ε^h for which $\Omega^{hi}_{\alpha\beta\gamma}\varepsilon^\alpha\varepsilon^\beta\varepsilon^\gamma \neq 0$, otherwise (12) would be violated.

Contracting (13) with $\varepsilon^\alpha\varepsilon^\beta\varepsilon^\gamma\varepsilon^\delta$, we have $F^h_\alpha\varepsilon^\alpha = a\,\xi^h + b\,F^h + c\,\varepsilon^h$, with a, b, c being constants. Analogously, contracting (13) with $\varepsilon^\beta\varepsilon^\gamma\varepsilon^\delta$, we obtain that F^h_i is represented in the following manner:

$$F^h_i = a\,\delta^h_i + a_i\,\xi^h + b_i\,F^h + c_i\,\varepsilon^h, \tag{14}$$

where a_i, b_i, c_i are components of 1-forms.

Formula (13) by virtue of (14) has the form

$$\omega^{[hi}_{(\alpha\beta\gamma}\delta^{j]}_{\delta)} = 0, \tag{15}$$

where

$$\omega^{hi}_{\alpha\beta\gamma} \stackrel{\text{def}}{=} \xi^{[h}F^{i]}(a_{(\alpha\beta}b_{\gamma)} - b_{(\alpha\beta}a_{\gamma)}) + \xi^{[h}\varepsilon^{i]}a_{(\alpha\beta}c_{\gamma)} + F^{[h}\varepsilon^{i]}b_{(\alpha\beta}c_{\gamma)}.$$

a) If $n > 3$ then $\omega^{hi}_{\alpha\beta\gamma} = 0$ follows from (13), and because ξ^h, F^h and ε^h are linear independent, we obtain $a_{(\alpha\beta}c_{\gamma)} = 0$ and $b_{(\alpha\beta}c_{\gamma)} = 0$. Therefore $c_i = 0$ and

$$F^h_i = a\,\delta^h_i + a_i\,\xi^h + b_i\,F^h. \tag{16}$$

b) If $n = 3$ the matrix F^h_i has always the previous form (16) while ξ^h, F^h and ε^h are not linear dependent.

Then formula (13) becomes (15), whereas $\omega^{hi}_{\alpha\beta\gamma} \stackrel{\text{def}}{=} \xi^{[h}F^{i]}(a_{(\alpha\beta}b_{\gamma)} - b_{(\alpha\beta}a_{\gamma)})$. For $n > 2$ it follows $\omega^{hi}_{\alpha\beta\gamma} = 0$ and consequently

$$a_{(\alpha\beta}b_{\gamma)} = b_{(\alpha\beta}a_{\gamma)}. \tag{17}$$

If a_α and b_α are linear indepedent, then from (17) we obtain

$$a_{ij} = a_{(i}\omega_{j)} \quad \text{and} \quad b_{ij} = b_{(i}\omega_{j)},$$

where ω_i are components of a 1-form. Afterwards it is possible to show that on the basis of (16) formula (11) assumes the following form

$$Q^h_{ij} = (\psi_i - a\omega_i)\delta^h_j + (\psi_j - a\omega_j)\delta^h_i + (\varphi_i + a\omega_i)F^h_j + (\varphi_j + a\omega_j)F^h_i,$$

i.e. formula (7) also holds.

Now there remains the case that a_α and b_α are linear depedent. For example, $b_\alpha = \alpha\,a_\alpha$, $\alpha \neq 0$. Then from (17) follows $b_{\alpha\beta} = \alpha\,a_{\alpha\beta}$. We denote $\Lambda^h = \xi^h + \alpha\,F^h$, $\omega_i = \psi_i + \alpha\,\varphi_i$, $\omega_{ij} = a_{ij} + a_{(i}\varphi_{j)}$, from (11) and (16) we obtain that Q^h_{ij} and F^h_i are represented by

$$Q^h_{ij} = \psi_i\delta^h_j + \psi_j\delta^h_i + \Lambda^h\omega_{ij} \quad \text{and} \quad F^h_i = a\delta^h_i + \Lambda^h a_i. \tag{18}$$

Then formula (8) appears in the following way

$$\Lambda^h\,(\omega_{\alpha\beta}\lambda^\alpha\lambda^\beta - \varrho_2(\lambda)\,a_\alpha\lambda^\alpha) = \lambda^h\,(\varrho_1(\lambda) + a\,\varrho_2(\lambda) - 2\psi_\alpha\lambda^\alpha).$$

From this it follows that

$$\omega_{\alpha\beta}\lambda^\alpha\lambda^\beta = \varrho_2(\lambda)\,a_\alpha\lambda^\alpha, \qquad \forall\lambda^h \neq \alpha\,\Lambda^h.$$

By simple analysis we obtain that $\omega_{ij} = a_{(i}\sigma_{j)}$, where σ_i are components of a 1-form.

Then due to (18) we have $Q^h_{ij} = (\psi_i - a\sigma_i)\delta^h_j + (\psi_j - a\sigma_j)\delta^h_i + \sigma_i F^h_j + \sigma_j F^h_i$. Evidently Lemma 5 is proved. $\boxed{QED}$

Proof of Theorem 4. It is obvious that geodesics are a special case of F-planar curves. Let a geodesic in A_n, which satisfies the equations (1) and $\nabla_\lambda \lambda = 0$, be mapped onto an F-planar curve in $\bar{A}_n$, which satisfies equations (1) and

$$\bar{\nabla}_\lambda \lambda = \bar{\varrho}_1(t)\,\lambda + \bar{\varrho}_2(t) F\lambda.$$

Here $\bar{\varrho}_1, \bar{\varrho}_2$ are functions of the parameter t.

Because the deformation tensor satisfies $P(\lambda,\lambda) = \bar{\nabla}_\lambda\lambda - \nabla_\lambda\lambda$, we have

$$P(\lambda(t),\lambda(t)) = \bar{\varrho}_1(t)\,\lambda + \bar{\varrho}_2(t) F\lambda.$$

It follows from the previous formula that in each point $x \in A_n$

$$P(\lambda,\lambda) = \varrho_1(\lambda)\,\lambda + \varrho_2(\lambda) F\lambda.$$

for each tangent vector $\lambda \in T_x$; $\varrho_1(\lambda), \varrho_2(\lambda)$ are functions dependent on λ.

Based on Lemma 5 it follows that there exist linear forms ψ and φ, for which formula (5) holds. $\boxed{QED}$

4 F-planar mappings which do not preserve F-structures

We now assume that the structures F and $\bar{F}$ are essentially distinct, i.e.

$$\bar{F}_i^h \neq a\delta_i^h + b\,F_i^h.$$

a) It is obvious, that geodesics are a special case of F-planar curves. Let a geodesic in A_n, which satisfies the equations (1) and $\nabla_\lambda\lambda = 0$, be mapped onto an $\bar{F}$-planar curve in $\bar{A}_n$, which satisfies the equations (1) and

$$\bar{\nabla}_\lambda \lambda = \bar{\varrho}_1(t)\,\lambda + \bar{\varrho}_2(t) \bar{F}\lambda.$$

Here $\bar{\varrho}_1, \bar{\varrho}_2$ are functions of the parameter t.

For the deformation tensor we have $P(\lambda(t),\lambda(t)) = \bar{\varrho}_1(t)\,\lambda + \bar{\varrho}_2(t)\bar{F}\lambda$. It follows from the previous formula that in each point $x \in A_n$

$$P(\lambda,\lambda) = \varrho_1(\lambda)\,\lambda + \varrho_2(\lambda) \bar{F}\lambda.$$

for each tangent vector $\lambda \in T_x$; $\varrho_1(\lambda), \varrho_2(\lambda)$ are functions dependent on λ.

Based on Lemma 5 it follows, that there exist linear forms ψ and φ, for which formula

$$P(X,Y) = \psi(X)\,Y + \psi(Y)\,X + \varphi(X)\,\bar{F}Y + \varphi(Y)\,\bar{F}X \tag{19}$$

holds.

b) Let a special F-planar curve in A_n, which satisfies the equations (1) and $\nabla_\lambda \lambda = F\lambda$, be mapped onto an $\bar{F}$-planar curve in $\bar{A}_n$, which satisfies the equations (1) and

$$\bar{\nabla}_\lambda \lambda = \bar{\varrho}_1(t)\,\lambda + \bar{\varrho}_2(t)\bar{F}\lambda.$$

Here $\bar{\varrho}_1, \bar{\varrho}_2$ are functions of the parameter t.

For the deformation tensor we have $P(\lambda(t), \lambda(t)) = F\lambda + \bar{\varrho}_1(t)\,\lambda + \bar{\varrho}_2(t)\bar{F}\lambda$. It follows from the previous formula that in each point $x \in A_n$

$$P(\lambda, \lambda) = F\lambda + \varrho_1(\lambda)\,\lambda + \varrho_2(\lambda)\bar{F}\lambda.$$

for each tangent vector $\lambda \in T_x$; $\varrho_1(\lambda), \varrho_2(\lambda)$ are functions dependent on λ.

Applying (19) we obtain

$$F\lambda = \tilde{\varrho}_1(\lambda)\,\lambda + \tilde{\varrho}_2(\lambda)\bar{F}\lambda.$$

Analyzing this expression like in Lemma 5 we convince ourselves that formula (3) holds. In this way we prove

6 Theorem. *Any F-planar mapping of a space with affine connection A_n onto $\bar{A}_n$ preserves F-structures.*

5 F-planar mappings for dimension n = 2

It is easy to see that for $n = 2$ Theorems 3 and 4 do not hold. If they would hold, the functions ϱ_1 and ϱ_2, appearing in (6), would be linear in λ.

In the case

$$F_i^h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

for example, these functions have the forms

$$\varrho_1(\lambda) = \frac{\lambda^1 P^1_{\alpha\beta}\lambda^\alpha\lambda^\beta + \lambda^2 P^2_{\alpha\beta}\lambda^\alpha\lambda^\beta}{(\lambda^1)^2 + (\lambda^2)^2} \quad \text{and} \quad \varrho_2(\lambda) = \frac{\lambda^1 P^2_{\alpha\beta}\lambda^\alpha\lambda^\beta - \lambda^2 P^1_{\alpha\beta}\lambda^\alpha\lambda^\beta}{(\lambda^1)^2 + (\lambda^2)^2},$$

which are not linear in general.

On the other hand an arbitrary diffeomorphism from A_2 onto $\bar{A}_2$ is an F-planar mapping with (6) being valid for the above functions ϱ_1 and ϱ_2.

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Infinitesimal F-Planar Transformations

I. Hinterleitner[1*], J. Mikeš[2], and J. Stránská[2]**

[1] *Institute of Mathematics, FSI VUT Brno, Technická 2, 616 69 Brno, Czech Republic*

[2] *Palacky University, Tomkova 40, 779 00 Olomouc, Czech Republic*

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Abstract—The theory of F-planar maps of Riemannian spaces and affinely connected spaces developed by J. Mikeš and N. S. Sinyukov [1–6] naturally extends the theory of geodesic and holomorphic projective maps. In the present paper we find basic equations of infinitesimal F-planar maps and study these equations. The F-planar maps are maps between spaces endowed with affinor structures. The geometry of Riemannian spaces and affinely connected spaces endowed by affinor structures was investigated by A. P. Shirokov (see, e.g., [7–14]) who also studied maps between spaces of this type ([13, 14]).

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1. DEFINITION OF INFINITESIMAL F-PLANAR TRANSFORMATIONS

Let us consider an n-dimensional torsion-free affinely connected space A_n, where, along with the object of linear connection Γ, an affinor F (a tensor field of type $\binom{1}{1}$) is given. Denote by $x = (x^1, x^2, \ldots, x^n)$ a coordinate system on A_n. In what follows we suppose that $n > 2$.

A curve ℓ in A_n given by equations $x^h = x^h(t)$ is said to be *F-planar* if under the parallel translation along ℓ the tangent vector $\lambda^h \equiv dx^h(t)/dt$ remains in the 2-dimensional plane spanned by the vectors λ^h and $F^h_\alpha \lambda^\alpha$.

A curve ℓ is F-planar if and only if [1–6]

$$\frac{d\lambda^h}{dt} + \Gamma^h_{\alpha\beta}(x(t))\lambda^\alpha\lambda^\beta = \varrho_1(t)\lambda^h + \varrho_2(t)F^h_\alpha\lambda^\alpha, \tag{1}$$

where $\Gamma^h_{ij}(x)$ are the components of Γ, and $\varrho_1(t)$, $\varrho_2(t)$ are functions of the parameter t. In case $\varrho_2(t) \equiv 0$, the curve ℓ is geodesic. A. Z. Petrov's quasi-geodesic curves [15], the analytic curves of Kähler, hyperbolic Kähler, and parabolic Kähler spaces provide examples of F-planar curves [1, 5].

An infinitesimal transformation of an affinely connected space A_n is given with respect to the coordinates as follows:

$$\overline{x}^h = x^h + \varepsilon\xi^h(x), \tag{2}$$

where x^h are the coordinates of a point in A_n and $\overline{x}^h$ are the coordinates of its image, ε is an infinitesimal parameter which does not depend on x^h, and ξ^h is the displacement vector.

An infinitesimal transformation (2) of the space A_n will be said to be *F-planar* if it maps F-planar curves of A_n onto curves which are F-planar in their *principal parts*.

*E-mail: Hinterleitner.Irena@seznam.cz.
**E-mail: Mikes@risc.upol.cz.

If an object $\mathcal{A}$ depends not only on $x \in A_n$ but also on the infinitesimal parameter ε, i.e., $\mathcal{A} = \mathcal{A}(x, \varepsilon)$, then the *principal part* of $\mathcal{A}$ is $\underset{0}{\mathcal{A}}(x) + \underset{1}{\mathcal{A}}(x)\varepsilon$ in the expansion in series with respect to ε:

$$\mathcal{A}(x, \varepsilon) = \underset{0}{\mathcal{A}}(x) + \underset{1}{\mathcal{A}}\varepsilon + \underset{2}{\mathcal{A}}(x)\varepsilon^2 + \cdots$$

In our case curves obtained by the transformation from F-planar curves satisfy the equation of F-planar curves under condition that the terms containing higher powers of ε (i.e., $\varepsilon^2, \varepsilon^3, \dots$) are dropped.

2. BASIC EQUATIONS OF INFINITESIMAL F-PLANAR TRANSFORMATIONS

Theorem 1. *A differential operator $X = \xi^\alpha(x)\partial_a$ ($\partial_a = \partial/\partial x^a$) determines an infinitesimal F-planar transformation of an affinely connected space A_n if and only if*

$$\text{a) } L_\xi \Gamma^h_{ij} = \psi_i \delta^h_j + \psi_j \delta^h_i + \varphi_i F^h_j + \varphi_j F^h_i; \quad \text{b) } L_\xi F^h_i = a\delta^h_i + bF^h_i, \tag{3}$$

where ψ_i and φ_i are covectors, a and b are functions, δ^h_i is the Kronecker delta, and L_ξ is the Lie derivative with respect to ξ.

First, let us formulate two lemmas which will be used in the proof of this theorem. Let A^h_{ij} $(= A^h_{ji})$, A^h_i, F^h_i, ψ_i, φ_i, α, and β $(h, i, j = 1, 2, \dots, n)$ be constants. Let $\lambda^1, \lambda^2, \dots, \lambda^n$ be coordinates of a vector λ, and $a(\lambda)$ and $b(\lambda)$ be functions depending on λ.

Lemma 1 ([6]). *Equations $A^h_{ij}\lambda^i\lambda^j = a(\lambda)\lambda^h + b(\lambda)F^h_i\lambda^i$ hold identically with respect to an arbitrary vector λ if and only if $A^h_{ij} = \psi_i\delta^h_j + \psi_j\delta^h_i + \varphi_iF^h_j + \varphi_jF^h_i$.*

Lemma 2 (ibid.). *Equations $A^h_i\lambda^i = a(\lambda)\lambda^h + b(\lambda)F^h_i\lambda^i$ hold identically with respect to an arbitrary vector λ if and only if $A^h_i = \alpha\delta^h_i + \beta F^h_i$.*

Proof. Let us consider an infinitesimal F-planar transformation of an affinely connected space A_n determined by Eqs. (2). Suppose that $F^h_i \neq \varrho\delta^h_i$. Let ℓ be an F-planar curve of the space A_n given by equations $x^h = x^h(t)$ and (1). The curve $\overline{\ell}$ corresponding to ℓ under transformation (2) has equations

$$\overline{x}^h(t) = x^h(t) + \varepsilon\xi^h(x(t)). \tag{4}$$

The infinitesimal transformation (2) is F-planar if $\overline{\ell}$ is F-planar in the principal part. Hence, $\overline{x}^h(t)$ given by (4) satisfy in the principal part Eqs. (1), which in this case take the form

$$\frac{d\overline{\lambda}^h(t)}{dt} + \Gamma^h_{\alpha\beta}(\overline{x}(t))\overline{\lambda}^\alpha(t)\overline{\lambda}^\beta(t) = \overline{\varrho}_1(t)\overline{\lambda}^h(t) + \overline{\varrho}_2(t)F^h_\alpha(\overline{x}(t))\overline{\lambda}^\alpha(t). \tag{5}$$

Let us find the objects involved in (5). From Eqs. (4) we find the tangent vector $\overline{\lambda}^h(t)$ of the curve $\overline{\ell}$:

$$\overline{\lambda}^h(t) \equiv \frac{\mathrm{d}\overline{x}^h(t)}{\mathrm{d}t} = \frac{\mathrm{d}x^h(t)}{\mathrm{d}t} + \varepsilon\frac{\partial\xi^h(x(t))}{\partial x^\alpha}\frac{\mathrm{d}x^\alpha(t)}{\mathrm{d}t} = \lambda^h(t) + \varepsilon\lambda^\alpha(t)\partial_\alpha\xi^h(x(t)).$$

For the object of affine connection Γ and the structure F, at the point $\overline{x}$ we have

$$\Gamma^h_{ij}(\overline{x}) = \Gamma^h_{ij}(x) + \varepsilon\frac{\partial\Gamma^h_{ij}(x)}{\partial x^\gamma}\xi^\gamma(x) + \boxed{\varepsilon^2} \text{ and } F^h_i(\overline{x}) = F^h_i(x) + \varepsilon\frac{\partial F^h_i(x)}{\partial x^\gamma}\xi^\gamma(x) + \boxed{\varepsilon^2}.$$

Hereafter $\boxed{\varepsilon^2}$ stands for the terms containing higher powers of the parameter ε.

Let us expand $\overline{\varrho}_1(t)$ and $\overline{\varrho}_2(t)$ in power series with respect to ε:

$$\overline{\varrho}_1(t) = \overline{\varrho}_{1,0}(t) + \overline{\varrho}_{1,1}(t)\varepsilon + \boxed{\varepsilon^2} \text{ and } \overline{\varrho}_2(t) = \overline{\varrho}_{2,0}(t) + \overline{\varrho}_{2,1}(t)\varepsilon + \boxed{\varepsilon^2}.$$

We substitute these expressions into (5) and obtain

$$\begin{aligned}
\frac{\mathrm{d}\lambda^h}{\mathrm{d}t} &+ \varepsilon\left(\partial_{\alpha\beta}\xi^h\lambda^\alpha\lambda^\beta + \frac{\mathrm{d}\lambda^\alpha}{\mathrm{d}t}\partial_\alpha\xi^h\right)\\
&+ (\Gamma^h_{\alpha\beta} + \varepsilon\xi^\gamma\partial_\gamma\Gamma^h_{\alpha\beta} + \boxed{\varepsilon^2})(\lambda^\alpha + \varepsilon\lambda^\gamma\partial_\gamma\xi^\alpha)(\lambda^\beta + \varepsilon\lambda^\gamma\partial_\gamma\xi^\beta)\\
&= (\overline{\varrho}_{1,0} + \varepsilon\overline{\varrho}_{1,1} + \boxed{\varepsilon^2})(\lambda^h + \varepsilon\lambda^\gamma\partial_\gamma\xi^h)\\
&+ (\overline{\varrho}_{2,0} + \varepsilon\overline{\varrho}_{2,1} + \boxed{\varepsilon^2})(F^h_\alpha + \varepsilon\xi^\gamma\partial_\gamma F^h_\alpha + \boxed{\varepsilon^2})(\lambda^\alpha + \varepsilon\lambda^\gamma\partial_\gamma\xi^\alpha).
\end{aligned}$$

Since the curve ℓ is F-planar, we can use (1) to eliminate $\frac{\mathrm{d}\lambda^h}{\mathrm{d}t}$ from the previous relation:

$$\begin{aligned}
&- \Gamma^h_{\alpha\beta}\lambda^\alpha\lambda^\beta + \varrho_1\lambda^h + \varrho_2\lambda^\alpha F^h_\alpha + \varepsilon(\partial_{\alpha\beta}\xi^h\lambda^\alpha\lambda^\beta - \Gamma^\gamma_{\alpha\beta}\partial_\gamma\xi^h\lambda^\alpha\lambda^\beta + \varrho_1\lambda^\alpha\partial_\alpha\xi^h + \varrho_2\lambda^\alpha F^\beta_\alpha\partial_\beta\xi^h)\\
&+ (\Gamma^h_{\alpha\beta} + \varepsilon\xi^\gamma\partial_\gamma\Gamma^h_{\alpha\beta} + \boxed{\varepsilon^2})(\lambda^\alpha + \varepsilon\lambda^\gamma\partial_\gamma\xi^\alpha)(\lambda^\beta + \varepsilon\lambda^\gamma\partial_\gamma\xi^\beta)\\
&= (\overline{\varrho}_{1,0} + \varepsilon\overline{\varrho}_{1,1} + \boxed{\varepsilon^2})(\lambda^h + \varepsilon\lambda^\gamma\partial_\gamma\xi^h)\\
&+ (\overline{\varrho}_{2,0} + \varepsilon\overline{\varrho}_{2,1} + \boxed{\varepsilon^2})(F^h_\alpha + \varepsilon\xi^\gamma\partial_\gamma F^h_\alpha + \boxed{\varepsilon^2})(\lambda^\alpha + \varepsilon\lambda^\gamma\partial_\gamma\xi^\alpha). \qquad (6)
\end{aligned}$$

It is evident that (6) holds true at each point $x \in A_n$. Therefore we can assume that ϱ_1, ϱ_2, $\overline{\varrho}_{1,0}$, $\overline{\varrho}_{1,1}$, $\overline{\varrho}_{2,0}$, and $\overline{\varrho}_{2,1}$ are functions of the point $x = (x^1, x^2, \dots, x^n)$ as well as of the tangent vector $\lambda = (\lambda^1, \lambda^2, \dots, \lambda^n)$.

The constant term in (6), i.e., the term which does not depend on ε, vanishes. Hence, after calculation, we get

$$(\overline{\varrho}_{1,0} - \varrho_1)\lambda^h + (\overline{\varrho}_{2,0} - \varrho_2)\lambda^\alpha F^h_\alpha = 0.$$

This relation holds true for all vectors λ^h at a given point x. Hence we have $\overline{\varrho}_{1,0} = \varrho_1$ and $\overline{\varrho}_{2,0} = \varrho_2$.

The linear (with respect to ε) term in (6) can be rewritten as follows:

$$\begin{aligned}
&(\partial_{\alpha\beta}\xi^h - \Gamma^\gamma_{\alpha\beta}\partial_\gamma\xi^h + \xi^\gamma\partial_\gamma\Gamma^h_{\alpha\beta} + \Gamma^h_{\gamma\alpha}\partial_\beta\xi^\gamma + \Gamma^h_{\gamma\beta}\partial_\alpha\xi^\gamma)\lambda^\alpha\lambda^\beta\\
&\qquad + \varrho_2(F^\beta_\alpha\partial_\beta\xi^h - F^h_\gamma\partial_\alpha\xi^\gamma - \xi^\gamma\partial_\gamma F^h_\alpha)\lambda^\alpha - \overline{\varrho}_{1,1}\lambda^h - \overline{\varrho}_{2,1}F^h_\alpha\lambda^\alpha.
\end{aligned}$$

This term also vanishes, so, using the definition of Lie derivative, we get

$$L_\xi\Gamma^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = \varrho_2 L_\xi F^h_\alpha\lambda^\alpha + \overline{\varrho}_{1,1}\lambda^h + \overline{\varrho}_{2,1}F^h_\alpha\lambda^\alpha. \qquad (7)$$

Here L_ξ stands for the Lie derivative with respect to ξ.

The transformation under consideration maps F-planar curves to F-planar curves up to the second order. Certainly, this is true also for geodesics, which are characterized by Eqs. (1) with $\varrho_2(t) \equiv 0$. In this case (7) turns into

$$L_\xi\Gamma^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = \overline{\varrho}_{1,1}\lambda^h + \overline{\varrho}_{2,1}F^h_\alpha\lambda^\alpha.$$

These equations hold true at any point and for any vector λ^h. By virtue of Lemma 1, from these equations we get (3a)).

By (3a)), under condition $\varrho_2 \equiv -1$ (this is possible because each F-planar curve is mapped onto an F-planar curve) relations (7) can be rewritten as follows:

$$L_\xi F^h_\alpha\lambda^\alpha = (\overline{\varrho}_{1,1} - 2\psi_\alpha\lambda^\alpha)\lambda^h + (\overline{\varrho}_{2,1} - 2\varphi_\beta\lambda^\beta)F^h_\alpha\lambda^\alpha.$$

From these relations (which hold true for any λ^h) by Lemma 2 (3b)) follows.

Thus we have proved the necessity. The sufficiency can be proved in a direct way. □

Note that, in case $F^h_i = \varrho\delta^h_i$, or $\varphi_i = 0$, each infinitesimal F-planar transformation is an infinitesimal geodesic transformation. These transformations were studied by L. P. Eisenhart [16], see also [17–19]

3. F-PLANAR TRANSFORMATIONS

We will show that the infinitesimal F-planar transformations are closely related to the F-planar transformations [2, 5].

Recall (ibid.) that a transformation $\overline{x}^h = \overline{x}^h(t)$ of an affinely connected space A_n which maps F-planar curves to F-planar curves is called *F-planar*.

In [2] and [5] it is proved that an *infinitesimal operator $X = \xi^\alpha(x)\partial_\alpha$ determines a one-parameter Lie group of F-planar transformations of an affinely connected space A_n if and only if* (3) *holds true*.

In [2] this statement is proved under condition that $n > 3$ and Rank $\|F - \alpha I\| > 1$. However, by a more detailed considerations, as, for example, in [6], one can prove that this statement holds also for $n > 2$. Thus, we have the following

Theorem 2. *In an affinely connected space A_n a one-parameter Lie group of F-planar transformations exists if and only if in A_n an infinitesimal F-planar transformation exists, and these transformations have the same differential operator.*

As it is known, the Lie derivatives $L_\xi\Gamma^h_{ij}$ and $L_\xi F^h_i$ can be written as follows:

$$L_\xi\Gamma^h_{ij} = \xi^h_{,ij} - \xi^\alpha R^h_{ij\alpha} \quad \text{and} \quad L_\xi F^h_i = \xi^\alpha F^h_{i,\alpha} + \xi^h_{,\alpha}F^\alpha_i - \xi^\alpha_{,i}F^h_\alpha.$$

Here the comma stands for the covariant derivative in the space A_n, and R^h_{ijk} stands for the curvature tensor of A_n.

Hence Eqs. (3) can be written as follows:

$$\xi^h_{,ij} = \xi^\alpha R^h_{ij\alpha} + \psi_i\delta^h_j + \psi_j\delta^h_i + \varphi_i F^h_j + \varphi_j F^h_i \quad \text{and} \quad \xi^\alpha F^h_{i,\alpha} + \xi^h_{,\alpha}F^\alpha_i - \xi^\alpha_{,i}F^h_\alpha = a\delta^h_i + bF^h_i. \tag{8}$$

In the space A_n, under conditions $n > 3$ and Rank $\|F - \alpha I\| > 1$, the basic Eqs. (3), which determine F-planar transformations and infinitesimal F-planar transformations, can be represented as a closed system of linear differential equations (written in terms of covariant derivatives) of Cauchy type in $n^2 + 3n$ unknown functions:

$$\xi^h(x), \quad \xi^h_i(x), \quad \psi_i(x), \quad \varphi_i(x). \tag{9}$$

This system has at most one solution (9) for the initial conditions at a point $x_o \in A_n$:

$$\xi^h(x_o) = \overset{\circ}{\xi}{}^h, \quad \xi^h_i(x_o) = \overset{\circ}{\xi}{}^h_i, \quad \psi_i(x_o) = \overset{\circ}{\psi}_i, \quad \varphi_i(x_o) = \overset{\circ}{\varphi}_i.$$

Hence, in A_n the dimension r of the group of F-planar transformations is lesser than or equal to $N = n^2 + 3n$, and the dimension of the space of infinitesimal F-planar transformations is lesser than or equal to N.

The above mentioned system can be written as follows:

$$\begin{aligned}
&\text{a) } \xi^h_{,i} = \xi^h_i;\\
&\text{b) } \xi^h_{i,j} = \xi^\alpha R^h_{ij\alpha} + \psi_i\delta^h_j + \psi_j\delta^h_i + \varphi_i F^h_j + \varphi_j F^h_i;\\
&\text{c) } \psi_{i,j} = {}^1Q_{ij\alpha}\xi^\alpha + {}^2Q^\beta_{ij\alpha}\xi^\alpha_\beta + {}^3Q^\beta_{ij}\varphi_\beta;\\
&\text{d) } \varphi_{i,j} = {}^4Q_{ij\alpha}\xi^\alpha + {}^5Q^\beta_{ij\alpha}\xi^\alpha_\beta + {}^6Q^\beta_{ij}\varphi_\beta,
\end{aligned} \tag{10}$$

where ${}^\sigma Q$ ($\sigma = \overline{1,6}$) are tensor objects composed from the geometric objects of the space A_n, i.e., from the affine connection Γ^h_{ij} and the affinor F^h_i.

We set $\xi^h_i = \xi^h_{,i}$, then obtain (10a)). Eqs. (10b)) are in fact (3a)) written in the form (8).

The integrability conditions for (10b)) are written as follows:

$$\begin{aligned}
&\delta^h_i(\psi_{j,k} - \psi_{k,j}) + \delta^h_j\psi_{i,k} - \delta^h_k\psi_{i,j} + F^h_i(\varphi_{j,k} - \varphi_{k,j}) + F^h_j\varphi_{i,k} - F^h_k\varphi_{i,j}\\
&\quad = \xi^\alpha R^h_{ijk,\alpha} + \xi^\alpha_i R^h_{\alpha jk} + \xi^\alpha_j R^h_{i\alpha k} + \xi^\alpha_k R^h_{ij\alpha} - \xi^h_\alpha R^\alpha_{ijk} - \varphi_i(F^h_{j,k} - F^h_{k,j}) + F^h_{i,j}\varphi_k - F^h_{i,k}\varphi_j.
\end{aligned} \tag{11}$$

One can verify that Eqs. (11) have unique solution (as linear algebraic equations) with respect to unknown $\psi_{i,j}$ and $\varphi_{j,k}$. From this one can get Eqs. (10c)) and (10d)), where the left-hand side is uniquely determined.

Eqs. (3b)) are linear algebraic equations with respect to ξ^h and ξ^h_i with coefficients defined in A_n (one can show that a and b are certain linear functions in ξ^h and ξ^h_i). The integrability conditions for Eqs. (10b)) (these are (11)) are linear algebraic equations in ξ^h, ξ^h_i, ψ_i, and φ_i with coefficients defined in A_n.

Now assume that the affine connection object of A_n and the structure F are analytic. Let us denote by (A_0) the integrability conditions for Eqs. (10) combined with Eqs. (3b)). Then the system (A_1) of first prolongations of the equations (A_0), the system (A_2) of second prolongations, and so on, consist of linear algebraic equations with respect to the unknown tensors ξ^h, ξ^h_i, ψ_i, and φ_i with coefficients defined in A_n.

From the analytic theory of differential equations it follows

Theorem 3. *An affinely connected space A_n $(n > 3)$ endowed with an affinor structure F such that* $\mathrm{Rank}\,\|F - \alpha I\| > 1$, *admits an F-planar transformation and an infinitesimal F-planar transformation if and only if the system of linear equations* $(A_0), (A_1), (A_2), \ldots, (A_{N-1})$ *has a non-trivial solution* (9).

The maximal number $r \leq N \equiv n(n+3)$ of essential parameters on which the general solution of equation system (10) depends, is the dimension of the group of F-planar transformations of A_n.

Using Eqs. (3b)) and their differential prolongations one can prove that the maximum $r = N$ cannot be achieved and, moreover, in fact $r \leq N - 2(n-2) \equiv n(n+1) + 4$.

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ON HOLOMORPHICALLY PROJECTIVE MAPPINGS OF e-KÄHLER MANIFOLDS

Irena Hinterleitner

Abstract. In this paper we study fundamental equations of holomorphically projective mappings of e-Kähler spaces (i.e. classical, pseudo- and hyperbolic Kähler spaces) with respect to the smoothness class of metrics. We show that holomorphically projective mappings preserve the smoothness class of metrics.

1. Introduction

First we study the general dependence of holomorphically projective mappings of classical, pseudo- and hyperbolic Kähler manifolds (shortly *e-Kähler*) in dependence on the smoothness class of the metric. We present well known facts, which were proved by Domashev, Kurbatova, Mikeš, Prvanović, Otsuki, Tashiro etc., see [2, 3, 6, 7, 8, 9, 10, 11, 12, 15, 16, 17, 18, 19]. In these results no details about the smoothness class of the metric were stressed. They were formulated "for sufficiently smooth" geometric objects.

2. Kähler manifolds

In the following definition we introduce generalizations of Kähler manifolds.

Definition 1. An n-dimensional (pseudo-)Riemannian manifold (M, g) is called an *e-Kähler manifold* K_n, if beside the metric tensor g, a tensor field F ($\neq \mathrm{Id}$) of type $(1,1)$ is given on the manifold M_n, called a *structure* F, such that the following conditions hold:

$$F^2 = e\,\mathrm{Id}\,; \quad g(X, FX) = 0\,; \quad \nabla F = 0\,, \tag{1}$$

where $e = \pm 1$, X is an arbitrary vector of TM_n, and ∇ denotes the covariant derivative in K_n.

If $e = -1$, K_n is a (*pseudo-*)*Kähler space* (also *elliptic Kähler space*) and F is a *complex structure*. As *A-spaces*, these spaces were first considered by P. A. Shirokov, see [14]. Independently they were studied by E. Kähler [5].

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If $e = +1$, K_n is a *hyperbolic Kähler space* (also *para Kähler space*, see [1]) and F is a *product structure*. The spaces K_n^+ were considered by P. K. Rashevskij [13].

The e-Kähler spaces introduced here are called shortly "Kähler" in the literature [10, 16]. By our definition we want to give a unified notation for all clases.

3. Holomorphically projective mapping theory for $K_n \to \bar{K}_n$ of class C^1

Assume the e-Kähler manifolds $K_n = (M, g, F)$ and $\bar{K}_n = (\bar{M}, \bar{g}, \bar{F})$ with metrics g and $\bar{g}$, structures F and $\bar{F}$, Levi-Civita connections ∇ and $\bar{\nabla}$, respectively. Here K_n, $\bar{K}_n \in C^1$, i.e. g, $\bar{g} \in C^1$ which means that their components g_{ij}, $\bar{g}_{ij} \in C^1$.

Likewise, as in [11] we introduce the following notations.

Definition 2. A curve ℓ in K_n which is given by the equation $\ell = \ell(t)$, $\lambda = d\ell/dt$, $(\neq 0)$, $t \in I$, where t is a parameter is called *analytically planar*, if under the parallel translation along the curve, the tangent vector λ belongs to the two-dimensional distribution $D = \mathrm{Span}\,\{\lambda,\, F\lambda\}$ generated by λ and its conjugate $F\lambda$, that is, it satisfies

$$\nabla_t \lambda = a(t)\lambda + b(t)F\lambda\,,$$

where $a(t)$ and $b(t)$ are some functions of the parameter t.

Particularly, in the case $b(t) = 0$, an analytically planar curve is a geodesic.

Definition 3. A diffeomorphism $f\colon K_n \to \bar{K}_n$ is called a *holomorphically projective mapping* of K_n onto $\bar{K}_n$ if f maps any analytically planar curve in K_n onto an analytically planar curve in $\bar{K}_n$.

Assume a holomorphically projective mapping $f\colon K_n \to \bar{K}_n$. Since f is a diffeomorphism, we can suppose local coordinate charts on M or $\bar{M}$, respectively, such that locally, $f\colon K_n \to \bar{K}_n$ maps points onto points with the same coordinates, and $\bar{M} = M$.

A manifold K_n admits a holomorphically projective mapping onto $\bar{K}_n$ if and only if the following equations [10, 16]:

$$\bar{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X + e\psi(FX)FY + e\psi(FY)FX \tag{2}$$

hold for any tangent fields X, Y and where ψ is a differential form. If $\psi \equiv 0$ than f is *affine* or *trivially holomorphically projective.* Beside these facts it was proved [10, 16] that $\bar{F} = \pm F$; for this reason we can suppose that $\bar{F} = F$. In local form:

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_i\delta^h_j + \psi_j\delta^h_i + e\psi_{\bar{i}}\delta^h_{\bar{j}} + e\psi_{\bar{j}}\delta^h_{\bar{i}}\,,$$

where Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ are the Christoffel symbols of K_n and $\bar{K}_n$, ψ_i, F^h_i are components of ψ, F and δ^h_i is the Kronecker delta, $\psi_{\bar{i}} = \psi_\alpha F^\alpha_i$, $\delta^h_{\bar{i}} = F^h_i$.

Here and in the following we will use the conjugation operation of indices in the way

$$A_{\dots\,\bar{i}\,\dots} = A_{\dots\,k\,\dots}F^k_i\,.$$

Equations (2) are equivalent to the following equations

$$\begin{aligned}\nabla_Z \bar g(X,Y) &= 2\psi(Z)\bar g(X,Y) + \psi(X)\bar g(Y,Z) + \psi(Y)\bar g(X,Z)\\ &\qquad - e\psi(FX)\bar g(FY,Z) - e\psi(FY)\bar g(FX,Z)\,.\end{aligned} \tag{3}$$

In local form:

$$\bar g_{ij,k} = 2\psi_k \bar g_{ij} + \psi_i \bar g_{jk} + \psi \bar g_{ik} - e\psi_{\bar i}\bar g_{\bar j k} - e\psi_{\bar j}\bar g_{\bar i k}\,,$$

where “ , ” denotes the covariant derivative on K_n. It is known that

$$\psi_i = \partial_i \Psi, \quad \Psi = \frac{1}{2(n+2)}\ln\left|\frac{\det \bar g}{\det g}\right|, \quad \partial_i = \partial/\partial x^i\,.$$

Domashev, Kurbatova and Mikeš [3, 6, 16] proved that equations (2) and (3) are equivalent to

$$\begin{aligned}\nabla_Z a(X,Y) &= \lambda(X)g(Y,Z) + \lambda(Y)g(X,Z)\\ &\qquad - e\lambda(FX)g(FY,Z) - e\lambda(FY)g(FX,Z)\,.\end{aligned} \tag{4}$$

In local form:

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} - e\lambda_{\bar i} g_{\bar j k} - e\lambda_{\bar j} g_{\bar i k}\,,$$

where

$$\text{(a)}\ \ a_{ij} = \mathrm{e}^{2\Psi}\bar g^{\alpha\beta} g_{\alpha i} g_{\beta j}; \qquad \text{(b)}\ \ \lambda_i = -\,\mathrm{e}^{2\Psi}\bar g^{\alpha\beta} g_{\beta i}\psi_\alpha\,. \tag{5}$$

From (4) follows $\lambda_i = \partial_i\lambda = \partial_i(\frac14\, a_{\alpha\beta}g^{\alpha\beta})$. On the other hand [10]:

$$\bar g_{ij} = \mathrm{e}^{2\Psi}\tilde g_{ij}, \quad \Psi = \frac12 \ln\left|\frac{\det\tilde g}{\det g}\right|, \qquad \|\tilde g_{ij}\| = \|g^{i\alpha}g^{j\beta}a_{\alpha\beta}\|^{-1}\,. \tag{6}$$

The above formulas are the criterion for holomorphically projective mappings $K_n \to \bar K_n$, globally as well as locally.

4. Holomorphically projective mapping theory for $K_n \to \bar K_n$ of class C^2

Let K_n and $\bar K_n \in C^2$ be e-Kähler manifolds, then for holomorphically projective mappings $K_n \to \bar K_n$ the Riemann and the Ricci tensors transform in this way

$$\begin{aligned}&\text{(a)}\quad \bar R^h_{ijk} = R^h_{ijk} + \delta^h_k\psi_{ij} - \delta^h_j\psi_{ik} - e\delta^h_{\bar k}\psi_{i\bar j} + e\delta^h_{\bar j}\psi_{i\bar k} + 2e\delta^h_{\bar i}\psi_{j\bar k}\,;\\ &\text{(b)}\quad \bar R_{ij} = R_{ij} - (n+2)\psi_{ij}\,,\end{aligned} \tag{7}$$

where $\psi_{ij} = \psi_{i,j} - \psi_i\psi_j + \psi_{\bar i}\psi_{\bar j}$ $(\psi_{ij} = \psi_{ji} = -e\psi_{\bar i\bar j})$.

The tensor of holomorphically projective curvature, which is defined in the following form

$$P^h_{ijk} = R^h_{ijk} + \frac{1}{n+2}\big(\delta^h_k R_{ij} - \delta^h_j R_{ik} - e\delta^h_{\bar k}R_{i\bar j} + e\delta^h_{\bar j}R_{i\bar k} + 2e\delta^h_{\bar i}R_{j\bar k}\big)\,, \tag{8}$$

is invariant with respect to holomorphically projective mappings, i.e. $\bar P^h_{ijk} = P^h_{ijk}$.

The integrability conditions of equations (4) have the following form

$$a_{i\alpha}R^{\alpha}_{jkl} + a_{j\alpha}R^{\alpha}_{ikl} = g_{ik}\lambda_{j,l} + g_{jk}\lambda_{i,l} - g_{il}\lambda_{j,k} - g_{jl}\lambda_{i,k} \tag{9}$$
$$- eg_{\bar{i}k}\lambda_{\bar{j},l} - eg_{\bar{j}k}\lambda_{\bar{i},l} + eg_{\bar{i}l}\lambda_{\bar{j},k} + eg_{\bar{j}l}\lambda_{\bar{i},k}\,.$$

We make the remark that the formulas introduced above, (7), (8) and (9), are not valid when $K_n \notin C^2$ or $\bar{K}_n \notin C^2$.

After contraction with g^{jk} we get:

$$a_{i\alpha}R^{\alpha}_{k} + a_{\alpha\beta}R^{\alpha\ \beta}_{ik} = e\lambda_{\bar{i},\bar{k}} - (n-1)\lambda_{i,k},$$

where $R^{\alpha}{}_{il}{}^{\beta} = g^{\beta k}R^{\alpha}{}_{ilk}$; $R^{\alpha}_{l} = g^{\alpha j}R_{jl}$ and $\mu = \lambda_{\alpha,\beta}g^{\alpha\beta}$.

We contract this formula with $F^{i}_{i'}F^{k}_{k'}$ and from the properties of the Riemann and the Ricci tensors of K_n we obtain

$$\lambda_{\bar{i},\bar{k}} = -e\lambda_{i,k}\,, \tag{10}$$

and ([3, 9, 10, 15])

$$n\lambda_{i,k} = \mu g_{ik} + a_{i\alpha}R^{\alpha}_{k} + a_{\alpha\beta}R^{\alpha}{}_{ik}{}^{\beta}\,. \tag{11}$$

Because λ_i is a gradient-like covector, from equation (11) follows $a_{i\alpha}R^{\alpha}_{j} = a_{j\alpha}R^{\alpha}_{i}$.

From (10) follows that the vector field $\lambda_{\bar{i}}$ ($\equiv \lambda_\alpha F^{\alpha}_{i}$) is a Killing vector field, i.e. $\lambda_{\bar{i},j} + \lambda_{\bar{j},i} = 0$.

5. Holomorphically projective mappings between $K_n \in C^r$ $(r > 2)$ and $\bar{K}_n \in C^2$

We proof the following theorem

Theorem 1. *If $K_n \in C^r$ $(r > 2)$ admits holomorphically projective mappings onto $\bar{K}_n \in C^2$, then $\bar{K}_n \in C^r$.*

The proof of this theorem follows from the following lemmas.

Lemma 1 (see [4]). *Let $\lambda^h \in C^1$ be a vector field and ϱ a function. If*

$$\partial_i\lambda^h - \varrho\,\delta^h_i \in C^1 \tag{12}$$

then $\lambda^h \in C^2$ and $\varrho \in C^1$.

In a similar way we can prove the following: *if $\lambda^h \in C^r$ $(r \geq 1)$ and $\partial_i\lambda^h - \varrho\delta^h_i \in C^r$ then $\lambda^h \in C^{r+1}$ and $\varrho \in C^r$.*

Lemma 2. *If $K_n \in C^3$ admits a holomorphically projective mapping onto $\bar{K}_n \in C^2$, then $\bar{K}_n \in C^3$.*

Proof. In this case equations (4) and (11) hold. According to the assumptions $g_{ij} \in C^3$ and $\bar{g}_{ij} \in C^2$. By a simple check-up we find $\Psi \in C^2$, $\psi_i \in C^1$, $a_{ij} \in C^2$, $\lambda_i \in C^1$ and $R^{h}_{ijk}, R^{h}{}_{ij}{}^{k}, R_{ij}, R^{h}_{i} \in C^1$.

From the above-mentioned conditions we easily convince ourselves that we can write equation (11) in the form (12), where

$\lambda^h = g^{h\alpha}\lambda_\alpha \in C^1$, $\varrho = \mu/n$ and $f^h_i = \dfrac{1}{n}\,(-\lambda^\alpha\Gamma^h_{\alpha i} - g^{h\gamma}a_{\alpha\gamma}R^{\alpha}_{i} + g^{h\gamma}a_{\alpha\beta}R^{\alpha}{}_{i\gamma}{}^{\beta}) \in C^1$.

From Lemma 1 follows that $\lambda^h \in C^2$, $\varrho \in C^1$, and evidently $\lambda_i \in C^2$. Differentiating (4) twice we convince ourselves that $a_{ij} \in C^3$. From this and formula (6) follows that also $\Psi \in C^3$ and $\bar{g}_{ij} \in C^3$. □

Further we notice that for holomorphically projective mappings between e-Kähler manifolds K_n and $\bar{K}_n$ of class C^3 holds the following third set of equations [6, 8, 9, 15, 10, 16]:

$$\mu_{,k} = 2\lambda_\alpha R^\alpha_k\,. \tag{13}$$

If $K_n \in C^r$ and $\bar{K}_n \in C^2$, then by Lemma 2, $\bar{K}_n \in C^3$ and (13) holds. Because the system (4), (11) and (13) is closed, we can differentiate equations (4) $(r-1)$ times. So we convince ourselves that $a_{ij} \in C^r$, and also $\bar{g}_{ij} \in C^r$ ($\equiv \bar{K}_n \in C^r$).

Remark. Moreover, in this case from equation (13) follows that the function $\mu \in C^{r-1}$.

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Brno University of Technology, Faculty of Civil Engineering,
Department of Mathematics, Žižkova 17, 602 00 Brno,
Czech Republic
E-mail: `hinterleitner.irena@seznam.cz`

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4-planar Mappings of Quaternionic Kähler Manifolds

Irena Hinterleitner

Abstract. In this paper we study fundamental equations of 4-planar mappings of almost quaternionic manifolds with respect to the smoothness class of metrics. We show that 4-planar mappings preserve the smoothness class of metrics.

Mathematics Subject Classification (2010). Primary 53B20; Secondary 53B21; 53B30; 53B35; 53C26.

Keywords. 4-planar mappings, almost quaternionic manifolds.

1. Introduction

4-planar and 4-quasiplanar mappings of almost quaternionic spaces have been studied in [1, 2] and [3]. These mappings generalize the geodesic, quasigeodesic and holomorphically projective mappings of Riemannian and Kählerian spaces, see [4–23]. Almost quaternionic structures were studied by many authors for example [24–26]. Generalisations of the above-introduced mappings were studied in [27–32].

First we study the general dependence of 4-planar mappings of almost quaternionic manifolds in dependence on the smoothness class of the metric. We present well-known facts, which were proved by Kurbatova, see [1], without stress on details about the smoothness class of the metric. They were formulated "for sufficiently smooth" geometric objects. In the present article we want to make this issue more precise.

2. Almost quaternionic and quaternionic Kähler manifolds

Under an *almost quaternionic* space we understand a differentiable manifold M_n with almost complex structures $\overset{1}{F}$ and $\overset{2}{F}$ defined on it, satisfying

$$\overset{1}{F}{}^h_\alpha \overset{1}{F}{}^\alpha_i = -\delta^h_i; \quad \overset{2}{F}{}^h_\alpha \overset{2}{F}{}^\alpha_i = -\delta^h_i; \quad \overset{1}{F}{}^h_\alpha \overset{2}{F}{}^\alpha_i + \overset{2}{F}{}^h_\alpha \overset{1}{F}{}^\alpha_i = 0,$$

where δ^h_i is the Kronecker symbol, see, e.g., [4, 24].

The tensor $\overset{3}{F}{}_i^h \equiv \overset{1}{F}{}_i^\alpha \overset{2}{F}{}_\alpha^h$ is further an almost complex structure. The relations among the tensors $\overset{1}{F}$, $\overset{2}{F}$ and $\overset{3}{F}$ are the following

$$\overset{1}{F}{}_i^h = \overset{2}{F}{}_i^\alpha \overset{3}{F}{}_\alpha^h = -\overset{3}{F}{}_i^\alpha \overset{2}{F}{}_\alpha^h;\ \overset{2}{F}{}_i^h = \overset{3}{F}{}_i^\alpha \overset{1}{F}{}_\alpha^h = -\overset{1}{F}{}_i^\alpha \overset{3}{F}{}_\alpha^h;\ \overset{3}{F}{}_i^h = \overset{1}{F}{}_i^\alpha \overset{2}{F}{}_\alpha^h = -\overset{2}{F}{}_i^\alpha \overset{1}{F}{}_\alpha^h.$$

Each pair chosen from the three structures $\overset{1}{F}$, $\overset{2}{F}$ and $\overset{3}{F}$ determines an *almost quaternionic structure*. The tensors ${}^*\overset{1}{F}$, ${}^*\overset{2}{F}$, ${}^*\overset{3}{F}$ and $\overset{1}{F}$, $\overset{2}{F}$, $\overset{3}{F}$ define the same almost quaternionic structure if ${}^*\overset{\sigma}{F} = \sum_{\rho=1}^{3} \alpha_\rho \overset{\rho}{F}$ where α_ρ are some functions.

An almost quaternionic manifold A_n is called a *quaternionic Kähler manifold*, if there exists a metric g such that $(g, \overset{s}{F})$, $s = 1, 2, 3$ are Kähler spaces, so that

$$g(X, \overset{s}{F}X) = 0, \quad \text{and} \quad \nabla \overset{s}{F} = 0,$$

for any $X \in TA_n$ and $s = 1, 2, 3$. Here and in the following ∇ is an affine connection with components Γ on A_n.

Let $A_n(\Gamma, \overset{1}{F}, \overset{2}{F}, \overset{3}{F})$ be a space with affine connection Γ without torsion with almost quaternionic structures $(\overset{1}{F}, \overset{2}{F}, \overset{3}{F})$.

Definition 1. A curve ℓ in A_n which is given by the equation $\ell = \ell(t)$, $\lambda = d\ell/dt$, $(\neq 0)$, $t \in I$, where t is a parameter, is called 4-*planar*, if under the parallel translation along the curve, the tangent vector λ belongs to the four-dimensional distribution $D = \mathrm{Span}\{\lambda, \overset{1}{F}\lambda, \overset{2}{F}\lambda, \overset{3}{F}\lambda\}$, that is, it satisfies

$$\nabla_t \lambda = a(t)\lambda + b(t)\overset{1}{F}\lambda + c(t)\overset{2}{F}\lambda + d(t)\overset{3}{F}\lambda,$$

where $a(t)$, $b(t)$, $c(t)$ and $d(t)$ are some functions of the parameter t.

Particularly, in the case $b(t) = c(t) = d(t) = 0$, a 4-planar curve is a geodesic. Evidently, a 4-planar curve with respect to the structure $(\overset{1}{F}, \overset{2}{F}, \overset{3}{F})$ is 4-planar with respect to the structure $({}^*\overset{1}{F}, {}^*\overset{2}{F}, {}^*\overset{3}{F})$, too.

3. 4-planar mappings

Consider two almost quaternionic manifolds with affine connections without torsion A_n and $\bar{A}_n$ with connection components Γ and $\bar{\Gamma}$, respectively. Let an almost quaternionic structure $(\overset{1}{F}, \overset{2}{F}, \overset{3}{F})$ be defined on A_n.

Definition 2. A diffeomorphism f: $A_n \to \bar{A}_n$ is called a 4-*planar mapping*, if it maps any 4-planar curve in A_n onto a 4-planar curve in $\bar{A}_n$.

Assume a 4-planar mapping f: $A_n \to \bar{A}_n$. Since f is a diffeomorphism, we can introduce local coordinate charts on M or $\bar{M}$, respectively, such that locally f: $A_n \to \bar{A}_n$ maps points onto points with the same coordinates, and $\bar{M} = M$.

A manifold A_n admits a 4-planar mapping onto $\bar{A}_n$ if and only if the following equations [3]:

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{s=0}^{3} \left\{ \underset{s}{\psi}(X) \overset{s}{F} Y + \underset{s}{\psi}(Y) \overset{s}{F} X \right\} \tag{1}$$

hold for any tangent fields X, Y and where $\underset{s}{\psi}$ are differential forms; $\overset{0}{F} = \mathrm{Id}$. If $\underset{s}{\psi} \equiv 0$, $(s = 0, 1, 2, 3)$ then f is *affine*.

Beside these facts it was proved [3] that the quaternionic structure of A_n and $\bar{A}_n$ is preserved; for this reason we can assume that $\overset{s}{\bar{F}} = \overset{s}{F}$. This was a priori assumed in the definition and results by Kurbatova [1].

Equation (1) in the common coordinate system x with respect to the mapping, has the following form

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \sum_{s=0}^{3} \underset{s}{\psi}_{(i} \overset{s}{F}{}^h_{j)}$$

where Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ are components of ∇ and $\bar{\nabla}$, $\underset{s}{\psi}_i(x)$ are components of $\underset{s}{\psi}$, $(i\, j)$ denotes a symmetrization without division by 2.

Finally we will assume that the space $A_n(\Gamma, \overset{1}{F}, \overset{2}{F}, \overset{3}{F})$ is mapped onto the (pseudo-) Riemannian space $\bar{V}_n(\bar{g})$. *A mapping $f\colon A_n \to \bar{V}_n$ is 4-planar if and only if the metric tensor $\bar{g}_{ij}(x)$ satisfies the following equations:*

$$\bar{g}_{ij,k} = \sum_{s=0}^{3} \left(\underset{s}{\psi}_k \, \bar{g}_{\alpha(i} \overset{s}{F}{}^\alpha_{j)} + \underset{s}{\psi}_{(i} \, \bar{g}_{j)\alpha} \overset{s}{F}{}^\alpha_k \right) \tag{2}$$

where the comma is the covariant derivative in A_n (see [3]),

4. 4-planar mapping theory for $K_n \to \bar{K}_n$ of class C^1

Let us consider the quaternionic Kähler manifolds $K_n = (M, g, F)$ and $\bar{K}_n = (\bar{M}, \bar{g}, \bar{F})$ with metrics g and $\bar{g}$, structures $F = (\overset{1}{F}, \overset{2}{F}, \overset{3}{F})$ and $\bar{F} = (\overset{1}{\bar{F}}, \overset{2}{\bar{F}}, \overset{3}{\bar{F}})$, Levi–Civita connections ∇ and $\bar{\nabla}$, respectively. Here K_n, $\bar{K}_n \in C^1$, i.e., $g, \bar{g} \in C^1$ which means that their components g_{ij}, $\bar{g}_{ij} \in C^1$.

We further assume that K_n admits a 4-planar mapping onto $\bar{K}_n$. Then we can consider $\bar{M} = M$ and $\overset{s}{\bar{F}} = \overset{s}{F}$ for $s = 1, 2, 3$.

In the present case we can simplify formula (2) as follows:

$$\bar{g}_{ij,k} = 2\psi_k \, \bar{g}_{ij} + \sum_{s=1}^{3} \left(\underset{s}{\psi}_i \, \bar{g}_{j\alpha} \overset{s}{F}{}^\alpha_k + \underset{s}{\psi}_j \, \bar{g}_{i\alpha} \overset{s}{F}{}^\alpha_k \right). \tag{3}$$

Here and in the following $\psi_k \equiv \underset{0}{\psi}{}_k$. When $n > 4$, it was proved in [1] that $\underset{s}{\psi}{}_i = -\frac{n}{n-4}\psi_\alpha \overset{s}{F}{}_i^\alpha$, $\quad s = 1, 2, 3$. Moreover, ψ is gradient-like, that is

$$\psi_i = \partial\Psi/\partial x^i \quad \text{and} \quad \Psi = \frac{n^2-4}{2(n-4)} \ \ln\left|\frac{\det\bar g}{\det g}\right|.$$

Kurbatova [9] proved that equations (3) are equivalent to

$$a_{ij,k} = \lambda_\alpha \check{Q}^{\alpha\beta}_{(ij)}\, g_{\beta k}, \tag{4}$$

where

$$\check{Q}^{\alpha\beta}_{ij} = \delta^\alpha_i \delta^\beta_j + \frac{n}{n-4} \ \sum_{s=1}^{3} \overset{s}{F}{}^\alpha_i \overset{s}{F}{}^\beta_j,$$

and

$$\text{(a)} \ \ a_{ij} = \mathrm{e}^{2\Psi}\bar g^{\alpha\beta} g_{\alpha i} g_{\beta j}; \quad \text{(b)} \ \ \lambda_i = -\mathrm{e}^{2\Psi}\bar g^{\alpha\beta} g_{\beta i}\psi_\alpha.$$

In addition, the formula

$$a_{\alpha\beta} \ \overset{s}{F}{}^\alpha_i \overset{s}{F}{}^\beta_j = a_{ij}$$

holds. From (4) follows $\lambda_i = \partial_i\lambda = \partial_i(\mathrm{const}\cdot a_{\alpha\beta}g^{\alpha\beta})$. On the other hand

$$\bar g_{ij} = \mathrm{e}^{2\Psi}\tilde g_{ij}, \quad \Psi = \frac{1}{2}\ln\left|\frac{\det\tilde g}{\det g}\right|, \quad \|\tilde g_{ij}\| = \|g^{i\alpha}g^{j\beta}a_{\alpha\beta}\|^{-1}. \tag{5}$$

The above formulas are the criterion for 4-planar mappings $K_n \to \bar K_n$, globally as well as locally.

5. 4-planar mapping theory for $K_n \to \bar K_n$ of class C^2

Let K_n and $\bar K_n \in C^2$ be quaternionic Kähler manifolds, then the integrability conditions of equations (4) have the following form

$$a_{ij,kl} - a_{ij,lk} \equiv a_{i\alpha}R^\alpha_{jkl} + a_{j\alpha}R^\alpha_{ikl} = \lambda_{\alpha l}\check{Q}^{\alpha\beta}_{(ij)}\, g_{\beta k} - \lambda_{\alpha k}\check{Q}^{\alpha\beta}_{(ij)}\, g_{\beta l}.$$

Here R^h_{ijk} are components of the Riemann tensor.

After contraction with g^{jk} we get [1]:

$$n\lambda_{i,k} = \mu g_{ik} + a_{\alpha\beta}B^{\alpha\beta}_{ik}, \tag{6}$$

where

$$\mu = \lambda_{\alpha\beta}g^{\alpha\beta}, \quad B^{\alpha\beta}_{il} = \hat Q^{\beta\gamma}_{(i\delta)} R^{\alpha\ \ \delta}_{\cdot\,\gamma\,\cdot\ \,l} \quad R^{\alpha\ \ \delta}_{\cdot\,\gamma\,\cdot\ \,l} = g^{\beta\delta}R^\alpha_{i\delta l},$$

$$\hat Q^{\beta\gamma}_{i\delta} = \frac{n(n-4)}{16(n-1)} \left(\frac{4-3n}{n} \ \delta^\beta_i \delta^\gamma_\delta + \sum_{s=1}^{3} \overset{s}{F}{}^\beta_i \overset{s}{F}{}^\gamma_\delta\right).$$

6. 4-planar mappings between $K_n \in C^r$ $(r > 2)$ and $\bar{K}_n \in C^2$

We demonstrate the following theorem

Theorem 1. *If $K_n \in C^r$ $(r > 2)$ admits 4-planar mappings onto $\bar{K}_n \in C^2$, then $\bar{K}_n \in C^r$.*

The proof of this theorem follows from the following lemmas.

Lemma 1 (see [7]). *Let $\lambda^h \in C^1$ be a vector field and ρ a function. If*

$$\partial_i \lambda^h - \rho\, \delta_i^h = f_i^h \in C^1 \tag{7}$$

then $\lambda^h \in C^2$ and $\rho \in C^1$.

Lemma 2. *If $K_n \in C^3$ admits a 4-planar mapping onto $\bar{K}_n \in C^2$, then $\bar{K}_n \in C^3$.*

Proof. In this case equations (4) and (6) hold. According to the foregoing assumptions, $g_{ij} \in C^3$ and $\bar{g}_{ij} \in C^2$. By a simple check-up we find $\Psi \in C^2$, $\psi_i \in C^1$, $a_{ij} \in C^2$, $\lambda_i \in C^1$ and $R^h_{ijk} \in C^1$.

From the above-mentioned conditions we easily convince ourselves that we can write equation (6) in the form (7), where

$$\lambda^h = g^{h\alpha} \lambda_\alpha \in C^1, \ \ \rho = \mu/n \text{ and } f_i^h = \frac{1}{n}\, g^{hl} a_{\alpha\beta} B^{\alpha\beta}_{il} \in C^1.$$

From Lemma 1 it follows that $\lambda^h \in C^2$, $\rho \in C^1$, and evidently $\lambda_i \in C^2$. Differentiating (4) twice we show that $a_{ij} \in C^3$. From this and formula (5) follows that also $\Psi \in C^3$ and $\bar{g}_{ij} \in C^3$. □

Further we notice that for 4-planar mappings between quaternionic Kähler manifolds K_n and $\bar{K}_n$ of class C^3 holds the following third set of equations (after simple modifications of [1]):

$$(n-1)\mu_{,k} = \lambda_\alpha C^\alpha_{\gamma[\delta k]} g^{\gamma\delta} + a_{\alpha\beta} B^{\alpha\beta}_{\gamma[\delta,k]} g^{\gamma\delta}, \tag{8}$$

where $C^\alpha_{ilk} = \check{Q}^{\alpha\delta}_{\gamma\beta} B^{\gamma\beta}_{i[l}\, g_{k]\delta}$.

If $K_n \in C^r$ and $\bar{K}_n \in C^2$, then by Lemma 2, $\bar{K}_n \in C^3$ and (8) holds. Because the system (4), (6) and (8) is closed, we can differentiate equations (4) $(r-1)$ times. So we convince ourselves that $a_{ij} \in C^r$, and also $\bar{g}_{ij} \in C^r$ $(\equiv \bar{K}_n \in C^r)$.

Remark 1. Moreover, in this case from equation (8) follows that the function $\mu \in C^{r-1}$.

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Irena Hinterleitner
Department of Mathematics
Faculty of Civil Engineering
Brno University of Technology
Žižkova 17
CZ-602 00 Brno, Czech Rep.
e-mail: hinterleitner.irena@seznam.cz

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Holomorphically projective mappings of (pseudo-) Kähler manifolds preserve the class of differentiability

Irena Hinterleitner[a]

[a] *Department of Mathematics, Faculty of Civil Engineering, Brno University of Technology, Czech Republic*

Abstract. In this paper we study fundamental equations of holomorphically projective mappings of (pseudo-) Kähler manifolds with respect to the smoothness class of metrics C^r, $r \geq 1$. We show that holomorphically projective mappings preserve the smoothness class of metrics.

subclass: 53B20; 53B21; 53B30; 53B35; 53C26

1. Introduction

First we study the general dependence of holomorphically projective mappings of classical and pseudo-Kähler manifolds (shortly *Kähler*) in dependence on the smoothness class of the metric. We present well known facts, which were proved by Otsuki, Tashiro [31], Tashiro, Ishihara [44], Domashev, Mikeš [8], Mikeš [19, 20], A.V. Aminova, D. Kalinin [2–5], etc., see [6, 9, 25, 27, 28, 35, 36, 45]. To the theory of holomorphically projective mappings and their generalization are devoted many publications, eg. [1, 7, 10, 11, 15–18, 21–23, 26, 29, 30, 32, 33, 38–41]. In these results no details about the smoothness class of the metric were stressed. They were formulated "for sufficiently smooth" geometric objects.

The following results are connected to the paper [12] where it was proved that holomorphically projective mappings preserve the smoothness class C^r of the metrics in the case $r \geq 2$. In the following paper we generalize this result to the case $r \geq 1$.

2. Main results

Let $K_n = (M, g, F)$ and $\bar{K}_n = (\bar{M}, \bar{g}, \bar{F})$ be (pseudo-) Kähler manifolds, where M and $\bar{M}$ are n-dimensional manifolds with dimension $n \geq 4$, g and $\bar{g}$ are metrics, F and $\bar{F}$ are structures. All the manifolds are assumed to be connected.

Definition 2.1. A diffeomorphism $f\colon K_n \to \bar{K}_n$ is called a *holomorphically projective mapping* of K_n onto $\bar{K}_n$ if f maps any holomorphically planar curve in K_n onto a holomorphically planar curve in $\bar{K}_n$.

We obtain the following theorem.

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Email address: hinterleitner.irena@seznam.cz (Irena Hinterleitner)

Theorem 2.2. *If the (pseudo-) Kähler manifold K_n ($K_n \in C^r$, $r \geq 1$) admits a holomorphically projective mapping onto $\bar{K}_n \in C^1$, then $\bar{K}_n$ belongs to C^r.*

Briefly, this means that:

holomorphically projective mappings preserve the class of smoothness of the metric.

The analogous property for geodesic mappings of (pseudo-) Riemannian manifolds is proved in [13].

Here and later $K_n = (M, g, F) \in C^r$ denotes that $g \in C^r$, i.e. in a coordinate neighborhood (U, x) for the components of the metric g holds $g_{ij}(x) \in C^r$. If $K_n \in C^r$ then $M \in C^{r+1}$. This means that the atlas on the manifold M has the differentiability class C^{r+1}, i.e. for non disjoint charts (U, x) and (U', x') on $U \cap U'$ it is true that the transformation $x' = x'(x) \in C^{r+1}$.

The differentiability class r is equal to $0, 1, 2, \ldots, \infty, \omega$, where $0, \infty$ and ω denotes continuous, infinitely differentiable, and real analytic functions respectively.

Remark 2.3. It's easy to prove that the Theorem 2.2 is valid also for $r = \infty$ and for $r = \omega$. This follows from the theory of solvability of differential equations. Of course we can apply this theorem only locally, because differentiability is a local property.

Remark 2.4. A minimal requirement for holomorphically projective mappings is K_n, $\bar{K}_n \in C^1$.

Mikeš, see [19, 21, 22, 24, 25], [28, p. 82] found equidistant Kähler metrics g in canonical coordinates x:

$$g_{ab} = g_{a+m\,b+m} = \partial_{ab} f + \partial_{a+m\,b+m} f \quad \text{and} \quad g_{a\,b+m} = \partial_{a\,b+m} f - \partial_{a+m\,b} f,$$

where $a = 1, 2, \ldots, m$, $m = n/2$, $f = \exp(2x^1) \cdot G(x^2, x^3, \ldots, x^m, x^{2+m}, x^{3+m}, \ldots, x^{2m})$, $G \in C^3$, which admit holomorphically projective mappings. Evidently, if $G \in C^{r+2}$ ($r \in \mathbb{N}$), $G \in C^\infty$ and C^ω, then $K_n \in C^r$, $K_n \in C^\infty$ and $K_n \in C^\omega$, respectively. From these metrics we can easily see examples of non trivial holomorphically projective mappings $K_n \to \bar{K}_n$, where

$$K_n, \bar{K}_n \in C^r \text{ and } \notin C^{r+1} \text{ for } r \in \mathbb{N}; \qquad K_n, \bar{K}_n \in C^\infty \text{ and } \notin C^\omega; \qquad K_n, \bar{K}_n \in C^\omega.$$

3. (Pseudo-) Kähler manifolds

In the following definition we introduce generalizations of Kähler manifolds.

Definition 3.1. An n-dimensional (pseudo-) Riemannian manifold ($n \geq 4$) is called a *(pseudo-) Kähler manifold* K_n=(M, g, F), if beside the metric tensor g, a tensor field F of type $(1, 1)$ is given on the manifold M, called a *structure F*, such that the following conditions hold:

$$F^2 = -Id; \quad g(X, FX) = 0; \quad \nabla F = 0, \tag{1}$$

where X is an arbitrary vector of TM, and ∇ denotes the covariant derivative in K_n.

These spaces were first considered as *A-spaces* by P.A. Shirokov, see [34]. Independently such spaces with positive definite metric were studied by E. Kähler [14]. The tensor field F is called a *complex structure* [45].

The following lemma specifies the properties of the differentiability of geometrical objects on (pseudo-) Kähler manifolds.

Lemma 3.2. *If $K_n = (M, g, F) \in C^r$, i.e. $g \in C^r$, then $F \in C^r$, for $r \in \mathbb{N}$ and $r = \infty, \omega$.*

Proof. Let $K_n \in C^r$, i.e. the components of metric $g_{ij}(x) \in C^r$ in a coordinate chart x. It is a priori valid that $F_i^h \in C^1$. The formula $\nabla F = 0$ can be written $\partial_k F_i^h = F_a^h \Gamma_{ik}^a - F_i^a \Gamma_{ak}^h$, where $\Gamma_{ijk} = {}^1\!/_2\, (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$, $\partial_k = \partial/\partial x^k$, and $\Gamma_{ij}^h = g^{hk}\Gamma_{ijk}$ are Christoffel symbols of the first and second kind, respectively. It holds that Γ_{ijk} and $\Gamma_{ij}^h \in C^{r-1}$. From this equation follows immediately $F_i^h(x) \in C^r$, i.e. $F \in C^r$. □

Moreover, due to the differentiability of $g \in C^r$ according to (1), each point has a coordinate neighborhood $(U, x) \in C^{r+1}$ in which the structure F has the following canonical form:

$$F_b^{a+m} = -F_{b+m}^a = \delta_b^a, \quad F_b^a = F_{b+m}^{a+m} = 0, \quad a, b = 1, \cdots, m; \ m = \frac{n}{2}. \tag{2}$$

We get, as an immediate consequence, that the dimension is even, $n = 2m$. Such a coordinate system will be called *canonical*.

Due to the conditions (1) and (2), the components of the metric tensor and Christoffel symbols of the second kind in a canonical coordinate system satisfy

$$g_{a+m,b+m} = g_{ab}, \quad g_{ab+m} = -g_{a+mb}, \quad \text{and} \quad \Gamma_{bc}^a = \Gamma_{b+mc+m}^{a+m} = -\Gamma_{b+mc+m}^a, \qquad \Gamma_{b+mc+m}^{a+m} = \Gamma_{b+mc}^a = -\Gamma_{bc}^{a+m}. \tag{3}$$

Obviously, the coordinate transformation $x'^h = x'^h(x)$ preserves a canonical coordinate system if and only if the Jacobi matrix $J = (\partial x'^h / \partial x^i)$ satisfies

$$\frac{\partial x'^{a+m}}{\partial x^{b+m}} = \frac{\partial x'^a}{\partial x^b} \quad \text{and} \quad \frac{\partial x'^{a+m}}{\partial x^b} = -\frac{\partial x'^a}{\partial x^{b+m}}. \tag{4}$$

Let us set $z^a = x^a + ix^{a+m}$, $z'^a = x'^a + ix'^{a+m}$ (where i is the imaginary unit). Then (4) can be interpreted as Cauchy-Riemann conditions for the complex functions $z'^a = z'^a(z^1, \cdots, z^m)$, and we will call this transformation *analytic*.

4. Holomorphically projective mappings $K_n \to \bar{K}_n$ of class C^1

Assume the (pseudo-) Kähler manifolds $K_n = (M, g, F)$ and $\bar{K}_n = (\bar{M}, \bar{g}, \bar{F})$ with metrics g and $\bar{g}$, structures F and $\bar{F}$, Levi-Civita connections ∇ and $\bar{\nabla}$, respectively. Here K_n, $\bar{K}_n \in C^1$, i.e. $g, \bar{g} \in C^1$ which means that their components g_{ij}, $\bar{g}_{ij} \in C^1$.

Likewise, as in [31], see [6], [35, p. 205], [36], [25], [28, p. 240], we introduce the following notations.

Definition 4.1. A curve ℓ in K_n which is given by the equation $\ell = \ell(t)$, $\lambda = d\ell/dt$ $(\neq 0), t \in I$, where t is a parameter is called *holomorphically planar*, if under the parallel translation along the curve, the tangent vector λ belongs to the two-dimensional distribution $D = Span\,\{\lambda, F\lambda\}$ generated by λ and its conjugate $F\lambda$, that is, it satisfies

$$\nabla_t \lambda = a(t)\lambda + b(t)F\lambda,$$

where $a(t)$ and $b(t)$ are some functions of the parameter t.

Particularly, in the case $b(t) = 0$, a holomorphically planar curve is a geodesic.

We recall the Definition 2.1: A diffeomorphism f: $K_n \to \bar{K}_n$ is called a *holomorphically projective mapping* of K_n onto $\bar{K}_n$ if f maps any holomorphically planar curve in K_n onto a holomorphically planar curve in $\bar{K}_n$.

Assume a holomorphically projective mapping f: $K_n \to \bar{K}_n$. Since f is a diffeomorphism, we can suppose local coordinate charts on M or $\bar{M}$, respectively, such that locally f: $K_n \to \bar{K}_n$ maps points onto points with the same coordinates, and $\bar{M} = M$.

A manifold K_n admits a holomorphically projective mapping onto $\bar{K}_n$ if and only if the following equations [28, 36]:

$$\bar{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X - \psi(FX)FY - \psi(FY)FX \tag{5}$$

hold for any tangent fields X, Y and where ψ is a differential form. In local form:

$$\bar{\Gamma}_{ij}^h = \Gamma_{ij}^h + \psi_i \delta_j^h + \psi_j \delta_i^h - \psi_{\bar{i}} \delta_{\bar{j}}^h - \psi_{\bar{j}} \delta_{\bar{i}}^h,$$

where Γ_{ij}^h and $\bar{\Gamma}_{ij}^h$ are the Christoffel symbols of K_n and $\bar{K}_n$, ψ_i, F_i^h are components of ψ, F and δ_i^h is the Kronecker delta, $\psi_{\bar{i}} = \psi_a F_i^a$, $\delta_{\bar{i}}^h = F_i^h$. Here and in the following we will use the conjugation operation of indices in the way

$$A_{\dots\bar{i}\dots}^{\dots} = A_{\dots k \dots}^{\dots} F_i^k, \qquad A_{\dots}^{\dots\bar{i}\dots} = A_{\dots}^{\dots k \dots} F_k^i.$$

If $\psi \equiv 0$, then f is *affine* or *trivially holomorphically projective*. Beside these facts it was proved [28, 36] that $\bar{F} = \pm F$; for this reason we can suppose that $\bar{F} = F$.

It is known that
$$\psi_i = \nabla_i \Psi, \quad \Psi = \frac{1}{2(n+2)} \ln \left| \frac{\det \bar{g}}{\det g} \right|.$$

Equations (5) are equivalent to the following equations

$$\nabla_Z \bar{g}(X,Y) = 2\psi(Z)\bar{g}(X,Y) + \psi(X)\bar{g}(Y,Z) + \psi(Y)\bar{g}(X,Z) + \psi(FX)\bar{g}(FY,Z) + \psi(FY)\bar{g}(FX,Z). \tag{6}$$

In local form:
$$\nabla_k \bar{g}_{ij} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi \bar{g}_{ik} + \psi_{\bar{i}} \bar{g}_{\bar{j}k} + \psi_{\bar{j}} \bar{g}_{\bar{i}k},$$

where $\bar{g}_{ij}$ are components of the metric $\bar{g}$ on $\bar{K}_n$.

The above formulas are well known for $\bar{F} = F$, see [31], [6], [35, p. 206], [36], [25], [28, p. 240-242].

Domashev and Mikeš ([8], see [35, p. 212], [36], [25], [28, p. 246]) proved that equations (5) and (6) are equivalent to

$$\nabla_Z a(X,Y) = \lambda(X)g(Y,Z) + \lambda(Y)g(X,Z) + \lambda(FX)g(FY,Z) + \lambda(FY)g(FX,Z); \tag{7}$$

in local form:
$$\nabla_k a_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik} + \lambda_{\bar{i}} g_{\bar{j}k} + \lambda_{\bar{j}} g_{\bar{i}k},$$

where

$$\text{(a)}\ \ a_{ij} = \mathrm{e}^{2\Psi} \bar{g}^{ab} g_{ai} g_{bj}; \quad \text{(b)}\ \ \lambda_i = -\,\mathrm{e}^{2\Psi} \bar{g}^{ab} g_{bi} \psi_a. \tag{8}$$

From (7) follows $\lambda_i = \nabla_i \Lambda$ and $\Lambda = \frac{1}{4}\, a_{bc} g^{bc}$. On the other hand [28]:

$$\bar{g}_{ij} = \mathrm{e}^{2\Psi} \tilde{g}_{ij}, \quad \Psi = \frac{1}{2} \ln \left| \frac{\det \tilde{g}}{\det g} \right|, \quad \|\tilde{g}_{ij}\| = \|g^{ib} g^{jc} a_{bc}\|^{-1}. \tag{9}$$

The above formulas are the criterion for holomorphically projective mappings $K_n \to \bar{K}_n$, globally as well as locally.

5. Holomorphically projective mapping for $K_n \in C^2 \to \bar{K}_n \in C^1$

I. Hinterleitner [12] proved the theorem:

Theorem 5.1. *If a (pseudo-) Kähler manifold $K_n \in C^r$, $r \geq 2$, admits a holomorphically projective mapping onto $\bar{K}_n \in C^2$, then $\bar{K}_n \in C^r$.*

It is easy to see that Theorem 2.2 follows from Theorem 5.1 and the following theorem.

Theorem 5.2. *If $K_n \in C^2$ admits a holomorphically projective mapping onto $\bar{K}_n \in C^1$, then $\bar{K}_n \in C^2$.*

Proof. We will suppose that the (pseudo-) Kähler manifold $K_n = (M, g, F) \in C^2$ admits a holomorphically projective mapping f onto the (pseudo-) Kähler manifold $\bar{K}_n = (\bar{M}, \bar{g}, \bar{F}) \in C^1$. Furthermore, we can assume that $\bar{M} = M$ and $\bar{F} = F$. The corresponding points $x \in M$ and $\bar{x} = f(x) \in \bar{M}$ have common coordinates $(x^1, x^2, \ldots, x^n)$, shortly x, in the coordinate chart (U, x), $U \subset M$, .

We study the coordinate neighborhood (U, x) of any point p at M. Moreover, we suppose that the coordinate system x is canonical (2). On (U, x) formulae (5)–(9) hold, and formula (7) may be written in the following form

$$\partial_k a^{ij} = \lambda^i \delta^j_k + \lambda^j \delta^i_k + \bar{\lambda}^i F^j_k + \bar{\lambda}^j F^i_k - f^{ij}_k, \tag{10}$$

where $a^{ij} = a_{bc}g^{bi}g^{cj}$, $\lambda^i = \lambda_a g^{ia}$, $\bar{\lambda}^i = \lambda^a F^i_a$, and $f^{ij}_k = a^{ib}\Gamma^j_{bk} + a^{jb}\Gamma^i_{bk}$.

The components $g_{ij}(x) \in C^2$ and $\bar{g}_{ij}(x) \in C^1$ on $U \subset M$ and from that facts follows that the functions $g^{ij}(x) \in C^2$, $\bar{g}^{ij}(x) \in C^1$, $\Psi(x) \in C^1$, $\psi_i(x) \in C^0$, $a^{ij}(x) \in C^1$, $\lambda^i(x) \in C^0$, and $\Gamma^h_{ij}(x) \in C^1$. It is easy to see, that $f^{ij}_k \in C^1$.

In the canonical coordinate system x we can calculate the following derivative for fixed different indices $a, b = 1, \dots, m$, $m = n/2$:

$$\begin{aligned} \partial_b a^{ab} &= \lambda^a - f^{ab}_b, & \partial_{b+m} a^{ab} &= -\lambda^{a+m} - f^{ab}_{b+m}, \\ \partial_b a^{ab+m} &= \lambda^{a+m} - f^{ab+m}_b, & \partial_{b+m} a^{ab+m} &= -\lambda^a - f^{ab+m}_{b+m}. \end{aligned} \tag{11}$$

Eliminating λ^a and λ^{a+m} we obtain the equations

$$\begin{aligned} \partial_b a^{ab} - \partial_{b+m} a^{ab+m} &= -f^{ab}_b + f^{ab+m}_b \\ \partial_{b+m} a^{ab} + \partial_b a^{ab+m} &= -f^{ab}_{b+m} - f^{ab+m}_b. \end{aligned} \tag{12}$$

We denote $w = a^{ab} + i \cdot a^{ab+m}$, $z = x^b + i \cdot x^{b+m}$, where i is the imaginary unit. Then (12) can be rewritten

$$\partial_z w = F \equiv (-f^{ab}_b + f^{ab+m}_b) + i \cdot (-f^{ab}_{b+m} - f^{ab+m}_b),$$

and because $F \in C^1$, then exists $\partial^2_{z\bar{z}} w$.

So there are the second partial derivatives of the functions a^{ab} and a^{ab+m} of the variables x^b and x^{b+m}; and, clearly, also of x^a and x^{a+m}. After this from formula (11) follows that $\lambda^h \in C^1$; and equations (10) implies that $a^{ij}, a_{ij} \in C^2$. Finally, formula (9) shows that $\bar{g}_{ij} \in C^2$. □

6. Holomorphically projective mapping $K_n \to \bar{K}_n$ of class C^2

Let K_n and $\bar{K}_n \in C^2$ be (pseudo-) Kähler manifolds, then for holomorphically projective mappings $K_n \to \bar{K}_n$ the Riemann and the Ricci tensors transform in the following way

$$\begin{aligned} &\text{(a)} \quad \bar{R}^h_{ijk} = R^h_{ijk} + \delta^h_k \psi_{ij} - \delta^h_j \psi_{ik} + \delta^h_{\bar{k}} \psi_{i\bar{j}} - \delta^h_{\bar{j}} \psi_{i\bar{k}} - 2\delta^h_{\bar{i}} \psi_{j\bar{k}}; \\ &\text{(b)} \quad \bar{R}_{ij} = R_{ij} - (n+2)\psi_{ij}, \end{aligned} \tag{13}$$

where $\psi_{ij} = \psi_{i,j} - \psi_i \psi_j + \psi_{\bar{i}} \psi_{\bar{j}}$ $(\psi_{ij} = \psi_{ji} = \psi_{\bar{i}\bar{j}})$. Here the Ricci tensor is defined by $R_{ik} = R^a_{iak}$. In many papers it is defined with the opposite sign [19, 25, 35, 46], etc.

The tensor of the holomorphically projective curvature, which is defined in the following form

$$P^h_{ijk} = R^h_{ijk} + \frac{1}{n+2}\left(\delta^h_k R_{ij} - \delta^h_j R_{ik} + \delta^h_{\bar{k}} R_{i\bar{j}} - \delta^h_{\bar{j}} R_{i\bar{k}} - 2\delta^h_{\bar{i}} R_{j\bar{k}}\right), \tag{14}$$

is invariant with respect to holomorphically projective mappings, i.e. $\bar{P}^h_{ijk} = P^h_{ijk}$.

The above mentioned formulae can be found in the papers [6, 28, 35].

The integrability conditions of equations (7) have the following form

$$a_{ia} R^a_{jkl} + a_{ja} R^a_{ikl} \;=\; g_{ik}\nabla_l \lambda_j + g_{jk}\nabla_l \lambda_i - g_{il}\nabla_k \lambda_j - g_{jl}\nabla_k \lambda_i + g_{\bar{i}k}\nabla_l \lambda_{\bar{j}} + g_{\bar{j}k}\nabla_l \lambda_{\bar{i}} - g_{\bar{i}l}\nabla_k \lambda_{\bar{j}} - g_{\bar{j}l}\nabla_k \lambda_{\bar{i}}. \tag{15}$$

After contraction with g^{jl} we get:

$$a_{ib} R^b_k + a_{bc} R^b{}_{ik}{}^c = -\nabla_{\bar{k}} \lambda_{\bar{i}} - (n-1)\nabla_k \lambda_i,$$

where $R^b{}_{il}{}^c = g^{ck} R^b{}_{ilk}$; $R^b_l = g^{bj} R_{jl}$ and $\mu = \nabla_c \lambda_b g^{bc}$.

We contract this formula with $F^i_{i'}F^k_{k'}$ and from the properties of the Riemann and the Ricci tensors of K_n we obtain

$$\nabla_{\bar{k}}\lambda_{\bar{i}} = \nabla_k\lambda_i, \tag{16}$$

and ([8, 25, 28, 35])

$$n\nabla_k\lambda_i = \mu g_{ik} - a_{ib}R^b_k - a_{bc}R^b{}_{ik}{}^c\ . \tag{17}$$

Because λ_i is a gradient-like covector, from equation (17) follows $a_{ib}R^b_j = a_{jb}R^b_i$.

From (16) follows that the vector field $\lambda_{\bar{i}}$ $(\equiv \lambda_a F^a_i)$ is a Killing vector field, i.e. $\nabla_j\lambda_{\bar{i}} + \nabla_i\lambda_{\bar{j}} = 0$. But the other side of the equations (16) can be written in the form $\nabla_a\lambda^h F^a_i = \nabla_i\lambda^a F^h_a$. In the canonical coordinate system x they are given by

$$\partial_b\lambda^a - \partial_{b+m}\lambda^{a+m} = 0 \text{ and } \partial_{b+m}\lambda^a + \partial_b\lambda^{a+m} = 0, \quad a, b = 1, \ldots, m,\ m = n/2.$$

These are Cauchy-Riemann equations, which implies that the functions $\lambda^h(x)$ are real analytic. After this differentiation of the Killing equations we obtain $\nabla_j(\nabla_i\bar{\lambda}^h) = \bar{\lambda}^a R^h_{ija}$, and by contraction with F^i_h, we finally obtain

$$\nabla_j\mu = -2\lambda^a R_{ai}.$$

These equations were found earlier under the assumption $K_n \in C^3$ and $\bar{K}_n \in C^3$, [20], see [35, p. 212], [28, pp. 247–248].

From that we proof the following theorem

Theorem 6.1. *A Kähler manifold $K_n \in C^2$ admits holomorphically projective mappings onto $\bar{K}_n \in C^1$ if and only if the system of differential equations*

$$\begin{aligned} \nabla_k a_{ij} &= \lambda_i g_{jk} + \lambda_j g_{ik} + \lambda_{\bar{i}} g_{\bar{j}k} + \lambda_{\bar{j}} g_{\bar{i}k}, \\ n\nabla_k\lambda_i &= \mu g_{ik} - a_{ib}R^b_k - a_{bc}R^b{}_{ik}{}^c, \\ \nabla_j\mu &= -2\lambda^b R_{bj}, \end{aligned} \tag{18}$$

has a solution a_{ij}, λ_i and μ satisfying the following conditions

$$a_{ij} = a_{ij} = a_{\bar{i}\bar{j}}, \quad \det(a_{ij}) \neq 0. \tag{19}$$

Remark 6.2. *Moreover if $K_n \in C^r$, it follows that $\bar{K}_n \in C^r$, the function $\lambda_i \in C^r$ and $\mu \in C^{r-1}$.*

Remark 6.3. *If $K_n \in C^\infty$, then $\bar{K}_n \in C^\infty$, and if $K_n \in C^\omega$, then $\bar{K}_n \in C^\omega$.*

Theorem 6.1 was proved in the case $K_n, \bar{K}_n \in C^3$, see [20].

The family of differential equations (18) is linear with coefficients of intrinsic character in K_n and independent of the choice of coordinates. If the metric tensor g and the structure tensor F of the Kähler manifold K_n are real then for the initial data

$$a_{ij}(x_0) = \overset{o}{a}_{ij}, \qquad \lambda_i(x_0) = \overset{o}{\lambda}_i, \qquad \mu(x_0) = \overset{o}{\mu},$$

the system (18) has at most one solution. Accounting that the initial data must satisfy (19), it follows that the general solution of (18) depends on r_{hpm} significant parameters, where $r_{hpm} \leq (n/2 + 1)^2$.

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