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# Connections on Higher Order Principal Prolongations

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# Connections on higher order principal prolongations

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# CONTENTS

# 1 INTRODUCTION

We introduce a part of the theory of gauge-natural operators and mention some possible applications of modern differential geometry in mathematical physics. First we recall some basic facts from differential geometry such as parallel transport, geodesics and linear connections. We also show a direct application of differential geometry in classical mechanics.

Generalizing linear connections on manifolds we get connections on fibered manifolds and adding a general structure group $G$ we introduce the principal bundles and principal connections. In what follows we use the theory of jets and the corresponding notation used in [12].

For the study of principal connections on higher order principal prolongations we use some techniques of finding general connections on higher order jet prolongations by means of a linear connection on the base manifold. These in fact are natural operators transforming a connection on the fibered manifold $Y \to M$ into a connection on the $r$-th order jet prolongation by means of a linear connection on the base manifold, see [7]. We are going to show the constructions on the first order jet prolongation in detail and then we generalize for higher orders. We also distinguish between the holonomic, semiholonomic and nonholonomic higher order jet prolongations, respectively.

Finally, given a principal bundle $P \to M$, it is well known that the $r$-th principal prolongation $W^r P$ of $P$ has many applications in differential geometry. For example, if $E$ is an arbitrary fiber bundle associated to $P$, then the $r$-th order jet prolongation $J^r E$ of $E$ is associated to $W^r P$. The gauge-natural bundle functor $W^r$ plays a fundamental role also in the theory of gauge-natural bundles. By [12], every gauge-natural bundle is a fibre bundle associated to the bundle $W^r P$ of certain order $r$. We show several constructions of principal connections on $W^r P$ by means of a linear connection on the base manifold $M$. Similarly to general connections, these are the gauge-natural operators transforming a connection on the principal bundle into a connection on its principal prolongation by means of a linear connection on a base manifold. We again distinguish between the holonomic, semiholonomic and nonholonomic higher order principal prolongations, respectively. We provide the exact coordinate formula for connections on the second order principal prolongation and show several constructions of connections on higher order principal prolongations.

The theory of gauge-natural operators, gauge-natural bundles and principal connections can be applied in physics, for example in quantum mechanics for studying the spin structures, see [14].

The aim of the thesis is to introduce the constructions of connections on higher order principal prolongations. An important part is devoted to prolongation of general connections, which is then used as a tool to achieve the main goal.

## 2 BASIC DEFINITIONS

### 2.1 CLASSICAL DIFFERENTIAL GEOMETRY

Let $(M, g)$ be an $n$-dimensional Riemannian manifold and denote by $(g_{ij}(x))$, $i, j = 1, \ldots, n$ the coordinate matrix of the metric $g$. We can define

$$\Gamma_{ij}^{k} = \frac{1}{2} \sum_{l=1}^{n} \widetilde{g}^{kl} \left( \frac{\partial g_{il}}{\partial u^{j}} + \frac{\partial g_{lj}}{\partial u^{i}} - \frac{\partial g_{ij}}{\partial u^{l}} \right),$$

where $(\widetilde{g}^{ij}(x))$ stands for the inverse matrix of $(g_{ij}(x))$ and $(u^i),\ i = 1, \ldots, n$ are local coordinates of $M$. $\Gamma_{ij}^{k}$ are called the Christoffel symbols of the Riemannian metric $g$. We recall that a vector field is a section of the tangent bundle $M \to TM$.

**Definition 2.1.** We say that a vector field $v(t) = (v^i(t))$ is parallel transported along the path $p : I \to M,\ p(t) = p^i(t)$, if

$$\frac{\mathrm{d}v^i}{\mathrm{d}t} + \sum_{j,k=1}^{n} \Gamma_{jk}^{i}(p(t)) v^j \frac{\mathrm{d}p^k}{\mathrm{d}t} = 0. \tag{2.1}$$

Symbol $I$ means a real interval.

This definition is independent of the choice of coordinates on $M$. As the vector $v(t_0)$ determines an initial condition in $p(t_0)$, the system of differential equations (2.1) determines the parallel transport of $v(t_0)$ uniquely. But if we transport the vector along different paths, we obtain different results, see Figure 2.1, where we move a vector from the point $S$ to the point $B$.

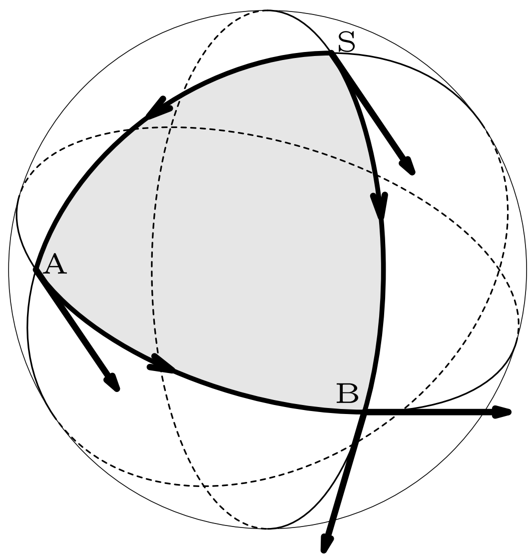



Figure 2.1: Parallel transport

Further, we can define a covariant derivative along the given path.

**Definition 2.2.** Given a system $v(t)$ of tangent vectors along the path $p(t)$ on the Riemannian manifold $(M, g)$, we define a covariant derivative $\frac{\nabla v(t)}{\mathrm{d}t} = \left(\frac{\nabla v^i(t)}{\mathrm{d}t}\right)$ by the coordinate expression

$$\frac{\nabla v^i}{\mathrm{d}t} = \frac{\mathrm{d}v^i}{\mathrm{d}t} + \sum_{j,k=1}^{n} \Gamma^i_{jk}(p(t))v^j \frac{\mathrm{d}p^k}{\mathrm{d}t}.$$

The operation of covariant derivative can be easily extended to the derivative along vector fields. Let us start with two vector fields $X, Y$ and their coordinate expression $X^i(x), Y^i(x)$ on $(M, g)$. At the point $x \in M$ consider a path $p(t)$ such that $\frac{\mathrm{d}p(0)}{\mathrm{d}t} = X(x)$. Further, consider a system of tangent vectors $Y(p(t))$ along the path $p(t)$. We directly obtain the coordinate expression of the vector $\frac{\nabla Y(p(0))}{\mathrm{d}t}$ in the form

$$\sum_{j,k=1}^{n} \left(\frac{\partial Y^i(x)}{\partial x^j} + \Gamma^i_{kj}(x)Y^k(x)\right) X^j(x). \tag{2.2}$$

**Definition 2.3.** The vector field (2.2) is called the covariant derivative of the vector field $Y$ with respect to the vector field $X$ and we denote it by $\nabla_X Y$.

Finally, generalizing the parallel transport for an arbitrary manifold $M$, we can define the linear connection. We introduce the axiomatic approach given by J. L. Koszul.

**Definition 2.4.** Let $\chi(M)$ be the set of all vector fields on the manifold $M$, that is the set of all smooth sections of the tangent bundle $TM$. Consider a mapping

$$\nabla : \chi(M) \times \chi(M) \to \chi(M), \quad (X, Y) \mapsto \overline{\nabla}_X Y$$

satisfying

(i) $\overline{\nabla}_X(Y_1 + Y_2) = \overline{\nabla}_X Y_1 + \overline{\nabla}_X Y_2$

(ii) $\overline{\nabla}_X(fY) = (Xf)Y + f\overline{\nabla}_X Y$

(iii) $\overline{\nabla}_{X_1+X_2} Y = \overline{\nabla}_{X_1} Y + \overline{\nabla}_{X_2} Y$

(iv) $\overline{\nabla}_{fX} Y = f\overline{\nabla}_X Y$

for arbitrary vector fields $X, Y, Y_1, Y_2$ and a real function $f : M \to \mathbb{R}$. Such a mapping $\nabla$ is called a linear connection on a manifold $M$.

Given the local coordinates $x^i$ on $M$ and the corresponding base tangent vectors $\frac{\partial}{\partial x^i}$, we set

$$\overline{\nabla}_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = \sum_{k=1}^{n} \Gamma^k_{ij}(x) \frac{\partial}{\partial x^k},$$

where $\Gamma^k_{ij}$ denote the Christoffel symbols. Using the Koszul axioms we deduce that the vector field $\overline{\nabla}_X Y$ has the coordinate expression (2.2), so that it is again called a covariant derivative of the vector field $Y$ with respect to the vector field $X$ and denoted by $\nabla_X Y$. This suggests the concept of a constant vector field along the integral curves of a vector field $Y$ and, consequently, along any smooth curve on $M$. Thus if $M$ is connected (and hence path connected), one has the notion of parallel transport of tangent vectors between any two points of $M$ along a particular curve connecting these points. The idea of parallel transport represents the original concept leading to the theory of connections.

If we consider a linear connection on a manifold $M$ as a map $\nabla : \mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathfrak{X}(M)$, it is easy to define the torsion of such a connection.

**Definition 2.5.** The map $T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \quad X,Y \in \mathfrak{X}(M)$$

is called the *torsion of a linear connection* $\nabla$. If $T = 0$ we say that $\nabla$ is torsion-free linear connection.

The torsion is a tensor field of type $(1,2)$ and this allows us to define the following concept.

**Definition 2.6.** Linear connection $\overline{\nabla} = \nabla - T$ is called the *(classical) conjugate connection* to $\nabla$.

If the coordinate expression of a connection $\nabla$ is

$$\mathrm{d}\dot{x}^i = \Gamma^i_{jk}\dot{x}^j\mathrm{d}\dot{x}^k,$$

then the equations of the conjugate connection to $\nabla$ are

$$\mathrm{d}\dot{x}^i = \Gamma^i_{kj}\dot{x}^j\mathrm{d}\dot{x}^k,$$

where the symbol $\dot{x}^i$ denotes the induced coordinates on the tangent bundle $TM$.

Now we can generalize the former definition of the parallel transport.

**Definition 2.7.** We say that the vector field $Y = Y^i(t)$ is parallel transported along the path $p(t)$, if the equation

$$\frac{\nabla Y^i}{\mathrm{d}t} + \sum_{j,k=1}^{n} \Gamma^i_{kj}(p(t))Y^k\frac{\mathrm{d}p^j}{\mathrm{d}t} = 0$$

is satisfied.

The path $p : I \to M$ is called the geodetic path of a linear connection $\nabla$, if the system $\gamma(t) = \frac{\mathrm{d}p(t)}{\mathrm{d}t}$ of the vectors tangent to $p(t)$ can be included into a vector field $Y = Y^i(t)$ on $M$ in such a way that $Y$ is parallel transported along $p(t)$. The geodetic path $p(t) = (x^i(t))$ satisfies the system of differential equations

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} + \sum_{j,k=1}^{n} \Gamma^i_{jk}(x)\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t} = 0, \quad i = 1,\dots,n.$$

The path $p(t)$ is called a geodetic curve, shortly a geodesic, if there exists a parametrization $\gamma(t)$ of $p(t)$ such that $\gamma(t)$ is a geodetic path.

**Theorem 2.1.** *Let $(M, g)$ be a Riemannian space and consider the linear connection corresponding to the metric $g$. Then there exists a neighborhood $U$ of any $x \in M$ such that the geodesic given by two points $p, q \in U$ is unique and it determines the shortest path in $U$ connecting the points $p, q$.*

The following example shows the application of geodesics in classical mechanics. But first, we recall some basic facts from classical differential geometry.

Let $S$ be a surface with parametrization $f(u, v)$ and let $T_xS$ be a space of all tangent vectors to $S$ at the point $x \in S$. Further, denote by

$$g_{11} = \frac{\partial f}{\partial u} \cdot \frac{\partial f}{\partial u}, \quad g_{22} = \frac{\partial f}{\partial v} \cdot \frac{\partial f}{\partial v}, \quad g_{12} = g_{21} = \frac{\partial f}{\partial u} \cdot \frac{\partial f}{\partial v}$$

the scalar products of the tangent base vectors. Then the matrix $(g_{ij})$ determines a quadratic form $\varphi_1$ called the first fundamental form and the numbers $g_{ij}$ are called the coefficients of the first fundamental form. Consider a unitary normal vector $\mathbf{n}$ at the point $x \in S$. If we denote

$$h_{11} = \mathbf{n} \cdot \frac{\partial^2 f}{\partial u^2}, \quad h_{22} = \mathbf{n} \cdot \frac{\partial^2 f}{\partial v^2}, \quad h_{12} = h_{21} = \mathbf{n} \cdot \frac{\partial^2 f}{\partial u \partial v}$$

and consider a tangent vector $a = (a^1, a^2) \in T_xS$, then the expression $\varphi_2(a) := \sum_{i,j=1}^{2} a^i a^j h_{ij}$ determines a quadratic form on $T_xS$. Then $\varphi_2$ is called the second fundamental form of the surface $S$. Numbers $h_{ij}$ are called the coefficients of the second fundamental form.

Now let us denote $f_1 := \frac{\partial f}{\partial u}$, $f_2 := \frac{\partial f}{\partial v}$ and $f_{ij}$ the second order derivatives. Consider a unitary vector $\mathbf{n}$ normal to the surface $S$ and let us recall that $\Gamma^k_{ij}$ denote the classical Christoffel symbols.

**Theorem 2.2.** *At each point of a surface $S$ the following equations hold:*

$$f_{ij} = \sum_{k=1}^{2} \Gamma^k_{ij} f_k + h_{ij} \cdot \mathbf{n}. \tag{2.3}$$

*These equations are called the Gauss equations.*

**Example 2.1.** *We examine the motion of a mass point of weight $m$ bonded to a surface $S = f(u^1, u^2) \subset E_3$ in the force field $F = F(x_1, x_2, x_3)$. Suppose the point moves along the path $p(t)$ with parametrization $(u^1(t), u^2(t))$. The corresponding expression of such path on the surface $S$ is $x = x(u^1(t), u^2(t))$. The bondage of the point moving on the surface $S$ is realized by the force $G$, which is at each point perpendicular to $S$. This situation practically corresponds to the frictionless motion of a ball in a bowl caused by the gravity force $F$. Let us remark that the surface really has to be bowl-shaped so that the ball stays in it. Otherwise the ball could leave the surface and we would loose the bondage condition. The motion equation is then of the form*

*$m\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = F + G$. Let us denote the velocity by $v(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}$. By second differentiation together with the Gauss equations we obtain*

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\nabla v}{\mathrm{d}t} + h(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}, \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}) \cdot \mathbf{n},$$

*where $h$ is the second fundamental form of $S$, $\mathbf{n}$ is its unitary normal vector and $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$ denotes the tangent vector to the surface $S$ at the point $p(t)$. Let us decompose the force $F = f^1\mathbf{x}_1 + f^2\mathbf{x}_2 + f \cdot \mathbf{n}$ into its tangential and vertical part and write $G = \gamma \cdot \mathbf{n}$, where $\gamma$ is a real function. Using the Einstein summation convention, we obtain*

$$m\frac{\nabla v}{\mathrm{d}t} - f^i\mathbf{x}_i + \left(m \cdot h(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}, \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}) - f - \gamma\right)\mathbf{n} = 0.$$

*As the velocity in the vertical direction is zero, we have*

$$m\frac{\nabla v}{\mathrm{d}t} - f^1\mathbf{x}_1 - f^2\mathbf{x}_2 = 0$$

*and thus a system of differential equations*

$$m\frac{\mathrm{d}^2u^i}{\mathrm{d}t^2} + m\Gamma^i_{jk}\frac{\mathrm{d}u^j}{\mathrm{d}t}\frac{\mathrm{d}u^k}{\mathrm{d}t} - f^i = 0.$$

*Furthermore, consider the initial conditions $u^i(t_0) = u^i_0$ and $\frac{\mathrm{d}u^i(t_0)}{\mathrm{d}t} = v^i_0$. We find that the motion is uniquely determined if we know the initial location of the velocity vector tangential to the surface $S$. If $F$ is zero vector field, the motion is realized along geodesics and the vector of initial velocity is parallel transported along them.*

## 2.2 CONNECTIONS

One of the most important notions in differential geometry and mathematical physics is that of a connection. In Definition 2.4 we recalled the notion of a linear connection.

Generalizing the idea of linear connections to fibered manifolds, we come to the following definition of a general connection.

**Definition 2.8.** *A general connection* on the fiber bundle $(E, p, M, S)$ is a vector-valued 1-form $\Gamma \in \Omega^1(E; VE)$ with values in the vertical bundle $VE$ such that $\Gamma \circ \Gamma = \Gamma$ and $\mathrm{Im}\Gamma = VE$.

**Remark (1).** Geometrically, a general connection $\Gamma$ on the fiber bundle $p : E \to M$ is defined simply as a projection $TE \to VE$, where $TE$ denotes the tangent bundle of $E$, see [12] for details.

**Remark (2).** In the following, we omit the word general and we specify the special characteristics of the connections if needed, e.g. principal connections.

**Definition 2.9.** Let $\Gamma \in \Omega^1(E; VE)$ be a connection on a fiber bundle $(E, p, M, S)$. Then the *curvature* $C \in \Omega^2(E, VE)$ of $\Gamma$ is given by

$$C(X, Y) = \frac{1}{2}[\Gamma, \Gamma](X, Y)$$

for any vector fields $X, Y$ on $E$, where $[\ ,\ ]$ means the Frölicher-Nijenhuis bracket.
**Remark (1).** Note that for vector fields $X, Y \in \mathfrak{X}(M)$ and their horizontal lifts $\Gamma X, \Gamma Y \in \mathfrak{X}(E)$ with respect to the connection $\Gamma$ on $E$ we have

$$C(\Gamma X, \Gamma Y) = [\Gamma X, \Gamma Y] - \Gamma([X, Y]).$$

Thus $C$ is an obstruction against integrability of the horizontal subbundle.
**Remark (2).** Equivalently to Definition 2.9, we can define the curvature of a connection $\Gamma$ on $E \to M$ as a map

$$C(\Gamma) : E \times_M \wedge^2 TM \to VE$$

given by

$$C(\Gamma)(y, X, Y) = (\Gamma([X, Y]) - [\Gamma X, \Gamma Y])\,(y) \quad \text{for} \quad y \in E,\ X, Y \in \mathfrak{X}(M),$$

where $\Gamma X$ means the $\Gamma$-lift of the vector field $X$. Thus in the following, the curvature of a connection $\Gamma$ will be denoted by $C(\Gamma)$.

Equivalently to the Definition 2.8, any connection on the fiber bundle $(Y, p, M, S)$ is determined by the horizontal projection $\chi = \mathrm{id}_{TY} - \Gamma$, or by the horizontal subspaces $\chi(T_yY) \subset T_yY$ in the individual tangent spaces, i.e. by the horizontal distribution. But every horizontal subspace $\chi(T_yY)$ is complementary to the vertical subspace $V_yY$ and therefore it is canonically identified with a unique element $j^1_y s \in J^1_y Y$. On the other hand, each $j^1_y s \in J^1_y Y$ determines a subspace in $T_yY$ complementary to $V_yY$. This leads us to the equivalent definition, [12].
**Definition 2.10.** *A general connection* on the fibered manifold $(Y, p, M)$ is a section $\Gamma : Y \to J^1Y$ of the first jet prolongation $J^1Y \to Y$.
In local coordinates, a general connection $\Gamma$ is given by

$$\mathrm{d}y^p = F^p_i(x, y)\mathrm{d}x^i,$$

where $F^p_i(x, y)$ are smooth functions. Using this notation, curvature $C(\Gamma)$ of a connection $\Gamma$ has the following coordinate expression:

$$\begin{aligned} \mathrm{d}x^i &= 0 \\ \mathrm{d}y^p &= \left(\frac{\partial F^p_j}{\partial x^i} + \frac{\partial F^p_j}{\partial y^q} F^q_i\right) \mathrm{d}x^i \wedge \mathrm{d}x^j. \end{aligned} \tag{2.4}$$

Further, let $\widetilde{J}^rY \to M$ be the $r$-th nonholonomic jet prolongation of a fibered manifold $p : Y \to M$. In general, an *$r$-th order nonholonomic connection* on $Y$ is a section $\Gamma : Y \to \widetilde{J}^rY$. Such a connection is called semiholonomic or holonomic, if it has values in $\overline{J}^rY$ or in $J^rY$, respectively.

## 3 Prolongation of general connections

### 3.1 Foundations

In what follows, we recall some facts about the orders of bundle functors and several helpful results and observations. Let $p : Y \to M$ and $\overline{p} : \overline{Y} \to \overline{M}$ be two fibered manifolds and $s \geq r \leq q$ be three integers. We recall that two morphisms $f, g : Y \to \overline{Y}$ with the base maps $\underline{f}, \underline{g} : M \to \overline{M}$ determine the same $(r, s, q)$-jet $j^{r,s,q}_y f = j^{r,s,q}_y g$ at $y \in Y, p(y) = x$, if

$$j^r_y f = j^r_y g,\ j^s_y(f \mid Y_x) = j^s_y(g \mid Y_x),\ j^q_x \underline{f} = j^q_x \underline{g}.$$

Further, a bundle functor $G$ on $\mathcal{FM}$ is said to be of the order $(r, s, q)$ if $j^{r,s,q}_y f = j^{r,s,q}_y g$ implies $Gf \mid G_y Y = Gg \mid G_y Y$. Then the integer $q$ is called the base order, $s$ is called the fiber order and $r$ is called the total order of $G$.

It is well known that product preserving bundle functors can be expressed in the terms of Weil algebras. The most important result from this field is that each product preserving bundle functor $F$ on $\mathcal{M}f$ is a Weil functor $F = T^A$ determined by the Weil algebra $A$, [12]. Then the iteration $T^A \circ T^B$ of two Weil functors corresponds to the tensor product $A \otimes B$ of Weil algebras and natural transformations $T^A \to T^B$ are in bijection with algebra homomorphisms $A \to B$. In [15], the fiber product preserving bundle functors on $\mathcal{FM}_m$ are characterized in terms of Weil algebras.

Let $F$ be a natural bundle on $\mathcal{M}f_m$. The $F$-vertical functor is a bundle functor $V^F$ on $\mathcal{FM}_{m,n}$ defined by

$$V^F Y = \bigcup_{x \in M} F(Y_x),\ V^F f = \bigcup_{x \in M} F(f_x),$$

where $f_x$ is the restriction and corestriction of $f : Y \to \overline{Y}$ over $\underline{f} : M \to \overline{M}$ to the fibers $Y_x$ and $\overline{Y}_{\underline{f}(x)}$. Clearly, if the order of $F$ is $s$, then the order of $V^F$ is $(0, s, 0)$. For the tangent functor $F = T$ we obtain the classical vertical functor. M. Doupovec and W. M. Mikulski have recently proved, [2]

**Proposition 3.1.** *Let $G$ be a bundle functor on $\mathcal{FM}_{m,n}$. Then the following conditions are equivalent:*

- *(a) The order of $G$ is $(0, s, 0)$ for some $s$.*
- *(b) The base order of $G$ is zero.*
- *(c) $G$ is naturally equivalent to some $F$-vertical functor $V^F$.*
- *(d) There is an $\mathcal{FM}_{m,n}$-natural operator transforming connections on $Y \to M$ into connections on $GY \to M$.*

Further, the existence of the prolongation of higher order connections was studied by M. Doupovec and W. M. Mikulski in [5]. They proved that if $F : \mathcal{FM}_m \to \mathcal{FM}$ is a fiber product preserving bundle functor and $r' \leq r$ are two integers, then there exists a natural operator transforming the $r$-th order connections on $Y \to M$ into the

$r'$-th order connections on $FY \to M$ if and only if $F \cong V^A$ for some Weil algebra $A$.

It is well known that an arbitrary bundle functor $G$ on $\mathcal{FM}_{m,n}$ admits a natural operator transforming connections on $Y \to M$ into connections on $GY \to M$ by means of an auxiliary higher order linear connection on $M$, see [2], [12] and Section 3.4 below. By Proposition 3.1, if the base order of $G$ is not zero, then the use of such $r$-th order linear connection is unavoidable. Clearly, this is the case of all higher order jet functors. In the rest of this section we study the prolongation of connections into the higher order jet bundles, but we do not intend to present the complete classification of all natural operators of the given type. Let us note, that this problem was solved by I. Kolář for the first jet prolongation $J^1$. More precisely, he classified all natural operators transforming connections on $Y \to M$ and classical linear connections on the base manifold $M$ into connections on $J^1Y \to M$. The generalization of such classification to general $r$-th order jet prolongations is still an open question.

### 3.2 Vertical prolongation

Consider a connection $\Gamma : Y \to J^1Y$ on a fibered manifold $Y \to M$. If we apply the vertical functor $V$, we obtain a map $V\Gamma : VY \to VJ^1Y$. Let $i_Y : VJ^1Y \to J^1VY$ be the canonical involution constructed by H. Goldschmidt and S. Sternberg, see also [12]. Then the composition

$$\mathcal{V}\Gamma := i_Y \circ V\Gamma : VY \to J^1VY$$

is a connection on $VY \to M$, which will be called the *vertical prolongation* of $\Gamma$. Since this construction has geometrical character, $\mathcal{V}$ is an operator $J^1 \rightsquigarrow J^1V$ natural on the category $\mathcal{FM}_{m,n}$.

**Proposition 3.2.** *The vertical prolongation $\mathcal{V}$ is the only natural operator $J^1 \rightsquigarrow J^1V$.*

Let

$$\mathrm{d}y^p = F_i^p(x,y)\mathrm{d}x^i \tag{3.1}$$

be the coordinate expression of $\Gamma$ and let $Y^p = \mathrm{d}y^p$ be the additional coordinates on $VY$. Then the equations of $\mathcal{V}\Gamma$ are (3.1) and

$$\mathrm{d}Y^p = \frac{\partial F_i^p}{\partial y^q} Y^q \mathrm{d}x^i.$$

### 3.3 The operators $P(\Gamma, \Lambda)$ and $\mathcal{J}^r(\Gamma, \Lambda)$

We recall that a linear $r$-th order connection on $M$ is a linear base preserving morphism $\Lambda : TM \to J^rTM$ satisfying $\pi_0^r \circ \Gamma = \mathrm{id}_{TM}$, where $\pi_k^r$ denotes the canonical projection of $r$-jets onto $k$-jets. Clearly, for $r = 1$ this is the classical linear connection on $M$. We note that there is a bijection between the linear $r$-th order connections on $M$ and the principal connections on $P^rM$.

By [12], there are two well known geometric constructions transforming a connection $\Gamma : Y \to J^1Y$ and a classical linear connection $\Lambda : TM \to J^1TM$ into the connection on $J^1Y \to M$. First, let $\mathcal{V}\Gamma : VY \to J^1VY$ be the vertical prolongation of $\Gamma$ and let $\Lambda^* : T^*M \to J^1T^*M$ be the dual connection of $\Lambda$. Since $J^1Y \to Y$ is an affine bundle with the associated vector bundle $VY \otimes T^*M$, the section $\Gamma$ determines an identification $I_\Gamma : J^1Y \approx VY \otimes T^*M$. Then the composition

$$J^1Y \xrightarrow{I_\Gamma} VY \otimes T^*M \xrightarrow{\mathcal{V}\Gamma\otimes\Lambda^*} J^1VY \otimes J^1T^*M \xrightarrow{J^1(I_\Gamma)^{-1}} J^1J^1Y \tag{3.2}$$

determines a connection $P(\Gamma, \Lambda)$ on $J^1Y \to M$.

On the other hand, consider the $r$-th order linear connection $\Sigma : TM \to J^rTM$, the lifting map $\gamma : Y \times_M TM \to TY$ of $\Gamma$ and its $r$-th jet extension $J^r\gamma : J^rY \times_M J^rTM \to J^rTY$. Denoting by $\mu_r : J^rTY \to TJ^rY$ the flow natural transformation from [12], the composition

$$J^rY \times_M TM \xrightarrow{\mathrm{id}\times\Sigma} J^rY \times_M J^rTM \xrightarrow{J^r\gamma} J^rTY \xrightarrow{\mu_r} TJ^rY \tag{3.3}$$

is the lifting map of a connection on $J^rY \to M$, which will be denoted by $\mathcal{J}^r(\Gamma, \Sigma)$.

If $\widetilde{\Lambda} : TM \to J^1TM$ is the conjugate connection of $\Lambda$, then we have the following result, [12]:

**Proposition 3.3.** $P(\Gamma, \Lambda) = \mathcal{J}^1(\Gamma, \widetilde{\Lambda})$ *if and only if* $\Gamma$ *is curvature free.*

We remark, that curvature of the operator $\mathcal{J}^1(\Gamma, \Lambda)$ was studied by I. Kolář and A. Cabras in [1].

### 3.4 THE FLOW PROLONGATION $\mathcal{G}(\Gamma, \Sigma)$

The connection $\mathcal{J}^r(\Gamma, \Sigma)$ is a particular case of the following general construction. Let $G : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be an arbitrary bundle functor of the base order $q$. Then the couple of a connection $\Gamma : Y \to J^1Y$ and a $q$-th order linear connection $\Sigma : TM \to J^qTM$ induces a connection $\mathcal{G}(\Gamma, \Sigma)$ on $GY \to M$ in the following way. Consider a vector field $X$ on $M$ and denote by $\Gamma X : Y \to TY$ its $\Gamma$-lift. The flow prolongation $\mathcal{G}(\Gamma X)$ of such a vector field is defined by

$$\mathcal{G}(\Gamma X) := \frac{\partial}{\partial t}|_0 G(\mathrm{exp}t(\Gamma X)),$$

where $\mathrm{exp}t(\Gamma X)$ means the flow of $\Gamma X$. This vector field depends only on $q$-jets of the vector field $X$. This gives rise to a map

$$\mathcal{G}(\Gamma X) : GY \times_M J^qTM \to TGY,$$

which is linear in the second factor. Then the composition

$$\mathcal{G}(\Gamma X) \circ (\mathrm{id} \times_M \Sigma) : GY \times_M TM \to TGY$$

is the lifting map of a connection $\mathcal{G}(\Gamma, \Sigma)$ on $GY \to M$. In what follows the connection $\mathcal{G}(\Gamma, \Sigma)$ will be called the flow prolongation of $\Gamma$ by means of $\Sigma$.

Clearly, for $G = J^r$ we obtain the connection $\mathcal{J}^r(\Gamma, \Sigma)$ on $J^rY \to M$, which was constructed above and for $G = \widetilde{J}^r$ we get a connection $\widetilde{\mathcal{J}}^r(\Gamma, \Sigma)$ on $\widetilde{J}^rY \to M$. Obviously, the base order of a vertical functor $G = V^F$ is zero. Then the connection $\mathcal{G}(\Gamma, \Sigma)$ does not depend on $\Sigma$ and $\mathcal{G}(\Gamma, \Sigma) = \mathcal{V}^F\Gamma$ is exactly the $F$-vertical prolongation of $\Gamma$. In the simplest case $F = T$ we obtain in such a way the classical vertical prolongation $\mathcal{V}\Gamma := \mathcal{V}^T\Gamma$.

### 3.5 CLASSIFICATION PROBLEMS

The classification of all natural transformations $J^rJ^s \to J^sJ^r$ depending on a classical linear connection $\Lambda$ on the base manifold is a very difficult problem. Up to now this problem was solved only for $r = s = 1$, see [12]. By [12], the only two natural transformations $J^1J^1 \to J^1J^1$ depending on a symmetric linear connection $\Lambda$ on the base manifold are the identity of $J^1J^1$ and $\mathrm{ex}_\Lambda$. Further, I. Kolář and M. Doupovec have proved that the only natural transformation $J^rJ^s \to J^rJ^s$ is the identity.

The classification of all natural operators transforming connections on $Y \to M$ and classical linear connections on $M$ into connections on $J^rY \to M$ is also a complicated problem. It was solved only for $r = 1$, see [12]. In particular, all natural operators transforming a connection $\Gamma$ on $Y \to M$ and a symmetric linear connection $\Lambda$ on $M$ into a connection on $J^1Y \to M$ are of the form

$$(\Gamma, \Lambda) \mapsto k \cdot P(\Gamma, \Lambda) + (1 - k)\mathcal{J}^1(\Gamma, \Lambda), \quad k \in \mathbb{R}.$$

If $\Lambda$ is not symmetric, then the list of all natural operators contains some additional difference tensors, see [12] for more details.

### 3.6 ITERATION METHOD FOR HIGHER ORDER NONHOLONOMIC PROLONGATION

Obviously, the $r$-th nonholonomic prolongation $\widetilde{J}^rY$ is defined by iteration. The same method can be used to construct connections on $\widetilde{J}^rY \to M$. Consider a natural operator $A$ transforming a connection $\Gamma$ on $Y \to M$ and a linear connection $\Lambda$ on $M$ into a connection $A(\Gamma, \Lambda)$ on $J^1Y \to M$. Write

$$\begin{aligned} A_1(\Gamma, \Lambda) &= A(\Gamma, \Lambda) \\ A_2(\Gamma, \Lambda) &= A(A_1(\Gamma, \Lambda), \Lambda) \\ &\vdots \\ A_r(\Gamma, \Lambda) &= A(A_{r-1}(\Gamma, \Lambda), \Lambda). \end{aligned}$$

Then $A_r(\Gamma, \Lambda)$ is a connection on $\widetilde{J}^rY \to M$.

Let us now consider the case $r = 2$. Applying the above iteration process to the connections $P(\Gamma, \Lambda)$ and $\mathcal{J}^1(\Gamma, \Lambda)$ on $J^1Y \to M$, we obtain the following connections on $\widetilde{J}^2Y \to M$ :

$$P(P(\Gamma, \Lambda), \Lambda), \quad \mathcal{J}^1(\mathcal{J}^1(\Gamma, \Lambda), \Lambda), \quad P(\mathcal{J}^1(\Gamma, \Lambda), \Lambda) \text{ and } \mathcal{J}^1(P(\Gamma, \Lambda), \Lambda).$$

For example, to obtain the connection $P(P(\Gamma, \Lambda), \Lambda)$, the composition (3.2) should be replaced with

$$\widetilde{J}^2Y \xrightarrow{I_{P(\Gamma,\Lambda)}} VJ^1Y \otimes T^*M \xrightarrow{\mathcal{V}P(\Gamma,\Lambda)\otimes\Lambda^*} J^1VJ^1Y \otimes J^1T^*M \approx$$
$$\approx J^1(VJ^1Y \otimes T^*M) \xrightarrow{J^1(I_{P(\Gamma,\Lambda)})^{-1}} J^1\widetilde{J}^2Y, \quad (3.4)$$

where $I_{P(\Gamma,\Lambda)}$ is the identification of the affine bundle $\widetilde{J}^2Y \to J^1Y$ with the associated vector bundle $VJ^1Y \otimes T^*M$. Quite similarly to (3.3), the lifting map of $\mathcal{J}^1(\mathcal{J}^1(\Gamma, \Lambda), \Lambda)$ is of the form

$$\widetilde{J}^2Y \times_M TM \xrightarrow{\mathrm{id}\times\Lambda} \widetilde{J}^2Y \times_M J^1TM \xrightarrow{J^1\gamma_{\mathcal{J}^1(\Gamma,\Lambda)}} J^1TJ^1Y \xrightarrow{\mu} T\widetilde{J}^2Y, \quad (3.5)$$

where $\gamma_{\mathcal{J}^1(\Gamma,\Lambda)}$ is the lifting map of the connection $\mathcal{J}^1(\Gamma, \Lambda)$. Quite analogously we obtain the remaining mixed operators.

## 4 EHRESMANN PROLONGATION

Given two higher order connections $\Gamma : Y \to \widetilde{J}^rY$ and $\overline{\Gamma} : Y \to \widetilde{J}^sY$, the product of $\Gamma$ and $\overline{\Gamma}$ is the $(r+s)$-th order connection $\Gamma * \overline{\Gamma} : Y \to \widetilde{J}^{r+s}Y$ defined by

$$\Gamma * \overline{\Gamma} = \widetilde{J}^s\Gamma \circ \overline{\Gamma}.$$

If both $\Gamma$ and $\overline{\Gamma}$ are of the first order, then $\Gamma * \overline{\Gamma} : Y \to \widetilde{J}^2Y$ is semiholonomic if and only if $\Gamma = \overline{\Gamma}$ and $\Gamma * \overline{\Gamma}$ is holonomic if and only if $\Gamma$ is curvature-free, [8], [17].

Considering a connection $\Gamma : Y \to J^1Y$, we can define an $r$-th order connection $\Gamma^{(r-1)} : Y \to \widetilde{J}^rY$ by

$$\Gamma^{(1)} := \Gamma * \Gamma = J^1\Gamma \circ \Gamma, \qquad \Gamma^{(r-1)} := \Gamma^{(r-2)} * \Gamma = J^1\Gamma^{(r-2)} \circ \Gamma.$$

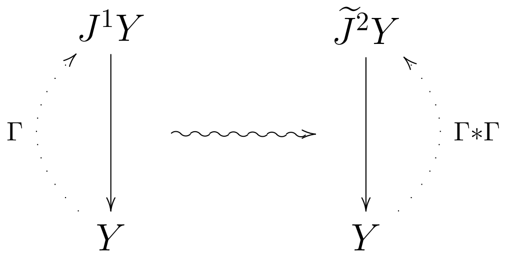
The connection $\Gamma^{(r-1)}$ is called the $(r-1)$-st prolongation of $\Gamma$ in the sense of Ehresmann, shortly $(r-1)$-st Ehresmann prolongation. By [8], the values of $\Gamma^{(r-1)}$ lie in the semiholonomic prolongation $\overline{J}^rY$ and $\Gamma^{(r-1)}$ is holonomic if and only if $\Gamma$ is curvature free, [17]. Let $y_i^p = F_i^p(x, y)$ be the coordinate expression of a connection $\Gamma : Y \to J^1Y$. Then the connection $\Gamma^{(1)} = \Gamma * \Gamma : Y \to \overline{J}^2Y$ has equations

$$y_i^p = F_i^p, \quad y_{ij}^p = \frac{\partial F_i^p}{\partial x^j} + \frac{\partial F_i^p}{\partial y^q} F_j^q.$$

For second order connections we have the following identification

**Proposition 4.1.** *Second order nonholonomic connections on $Y \to M$ are in bijection with triples $(\Gamma, \overline{\Gamma}, \Sigma)$, where $\Gamma, \overline{\Gamma} : Y \to J^1Y$ are first order connections on $Y \to M$ and $\Sigma : Y \to VY \otimes \overset{2}{\otimes} T^*M$ is a section.*

Now we come to the main result of this section. In particular, we find all natural operators transforming first order connections $\Gamma : Y \to J^1Y$ into second order semiholonomic connections $\Sigma : Y \to \overline{J}^2Y$. Taking into account the notation from the previous section, we set $G = \mathrm{id}_{\mathcal{FM}_{m,n}}$, $E = \overline{J}^2$ and $F = J^1$.




(4.1)

We remind the following property of $\overline{J}^2Y$, [12]. Given the local coordinates $(x^i, y^p, y^p_i, y^p_{ij})$ on $\overline{J}^2Y$, we have a natural map $e : \overline{J}^2Y \to \overline{J}^2Y$ with the coordinate expression

$$y^p_i = y^p_i, \qquad y^p_{ij} = y^p_{ji}.$$

**Remark.** J. Pradines introduced a natural map $\overline{J}^2Y \to \overline{J}^2Y$ with the same coordinate expression. We use the notation of [12], where the map $e$ is obtained from the natural exchange map $e_\Lambda : J^1J^1Y \to J^1J^1Y$ as a restriction to the subbundle $\overline{J}^2Y \subset J^1J^1Y$. Note that while $e_\Lambda$ depends on the linear connection $\Lambda$ on $M$, its restriction $e$ is independent of any auxiliary connection. We remark, that originally the map $e_\Lambda$ was introduced by M. Modugno.

I. Kolář and M. Modugno proved

**Proposition 4.2.** *All natural transformations $\overline{J}^2 \to \overline{J}^2$ form a one parametric family*

$$X \mapsto kX + (1-k)e(X), \qquad k \in \mathbb{R}.$$

Now we are ready to formulate a new result

**Proposition 4.3.** *All natural operators transforming first order connection $\Gamma : Y \to J^1Y$ into second order semiholonomic connection $Y \to \overline{J}^2Y$ form a one parametric family*

$$\Gamma \mapsto k \cdot (\Gamma * \Gamma) + (1-k) \cdot e(\Gamma * \Gamma), \qquad k \in \mathbb{R}. \tag{4.2}$$

**Remark.** In other words, all natural operators from Proposition 4.3 can be obtained from the Ehresmann prolongation $\Gamma * \Gamma$ by applying all natural transformations $\overline{J}^1 \to \overline{J}^2$ from Proposition 4.2.

Clearly, (4.2) can be written also as

$$\Gamma \mapsto (\Gamma * \Gamma) + t(\Gamma * \Gamma - e(\Gamma * \Gamma)), \qquad t \in \mathbb{R}. \tag{4.3}$$

We recall, that the difference tensor $\delta(U)$ of a semiholonomic 2-jet $U \subset \overline{J}^2 Y$ is the map $\delta : \overline{J}^2 Y \to VY \otimes \wedge^2 T^* M$ defined by $\delta(U) := U - e(U)$, in local coordinates

$$\delta(y^p_{ij}) = y^p_{ij} - y^p_{ji}.$$

Obviously, in our situation $\delta$ corresponds to the term $\Gamma * \Gamma - e(\Gamma * \Gamma)$ in (4.3).

Further, we can consider the connection $\Gamma * \Gamma$ as a section $Y \to \overline{J}^2 Y$. The bundle $\overline{J}^2 Y \to J^1 Y$ is an affine bundle with the associated vector bundle

$$VY \otimes \overset{2}{\otimes} T^* M = (VY \otimes S^2 T^* M) \oplus (VY \otimes \wedge^2 T^* M),$$

where the second part is determined by the values of the difference tensor $\delta$. The coordinate expression of (4.3) implies, that if $\Gamma$ is curvature free, then the difference tensor is zero and thus the associated vector bundle is reduced to the symmetric part. This corresponds to the subbundle $J^2 Y \to J^1 Y$. We showed above, that if $\Gamma$ is curvature free, the connection $\Gamma * \Gamma$ has values in holonomic jet prolongation $J^2 Y$, see also [17].

We remark that A. Cabras and I. Kolář have systematically studied the prolongation of second order connections to vertical Weil bundles $V^A Y \to M$. Further, M. Doupovec and W. M. Mikulski [5] have characterized all bundle functors $F$ on $\mathcal{FM}_m$, which admit natural operators transforming higher order connections on $Y \to M$ into higher order connections on $FY \to M$. The same authors have also introduced the prolongation of higher order connections to higher order jet bundles by means of some auxiliary linear connection $\Delta$ on the base manifold, [4].

It is interesting to pose a question whether the connection $P(\Gamma, \Lambda)$ on $J^1 Y \to M$ defined by the composition (3.2) could be generalized to some connection $P^r(\Gamma, \Lambda)$ on $J^r Y \to M$. Clearly, the composition (3.2) essentially depends on the identification $I_\Gamma : J^1 Y \to VY \otimes T^* M$ given by

$$Y^p_i = F^p_i - y^p_i.$$

It is well known that $J^r Y \to J^{r-1} Y$ is an affine bundle with the associated vector bundle $VY \otimes S^r T^* M$ over $J^{r-1} Y$. To generalize (3.2) to some connection $P^r(\Gamma, \Lambda)$ : $J^r Y \to J^1 J^r Y$, it is necessary to replace $I_\Gamma$ by some base preserving morphism

$$f : J^r Y \to VY \otimes S^r T^* M. \tag{4.4}$$

Denote by $C(\Gamma)$ the curvature of connection $\Gamma$. Clearly, $C(\Gamma)$ can be considered as a section $Y \to VY \otimes \wedge^2 T^* M$ with the coordinate expression

$$C(\Gamma) \equiv \left( \frac{\partial F^p_j}{\partial x^i} + \frac{\partial F^p_j}{\partial y^q} F^q_i \right) \frac{\partial}{\partial y^p} \otimes (\mathrm{d}x^i \wedge \mathrm{d}x^j).$$

By [9] all natural operators transforming connections $\Gamma : Y \to J^1Y$ into sections $Y \to VY \otimes \overset{2}{\otimes} T^*M$ are of the form

$$\Gamma \mapsto k \cdot C(\Gamma),\ k \in \mathbb{R}.$$

If we denote by $i$ the canonical projection $J^2Y \to Y$ and by $iC(\Gamma)$ the composition $C(\Gamma) \circ i : J^2Y \to VY \otimes \overset{2}{\otimes} T^*M$, we have

**Proposition 4.4.** *All natural operators transforming connections $\Gamma : Y \to J^1Y$ into base preserving morphisms $J^2Y \to VY \otimes \overset{2}{\otimes} T^*M$, are of the form*

$$\Gamma \mapsto k \cdot iC(\Gamma),\ k \in \mathbb{R}.$$

As $C(\Gamma)$ has values in $VY \otimes \wedge^2 T^*M \subset VY \otimes \overset{2}{\otimes} T^*M$, we have

**Corollary 4.1.** *The only natural operator transforming connections $\Gamma : Y \to J^1Y$ into the base preserving morphisms $J^2Y \to VY \otimes S^2T^*M$ is the zero one.*

So $I_\Gamma$ has no analogy (4.4) for $r = 2$. On the other hand we verify easily

**Proposition 4.5.** *The only natural operator transforming connections $\Gamma : Y \to J^1Y$ into the base preserving morphisms $J^1Y \to VY \otimes T^*M$ is*

$$\Gamma \mapsto I_\Gamma.$$

This proves that there is no analogue to the operator $P(\Gamma, \Lambda)$ for higher order jet prolongations.

## 5 PRINCIPAL PROLONGATION

Given a principal bundle $P \to M$ with a structure group $G$, one can define nonholonomic principal $r$-th order connections on $P$ as $G$-invariant sections $P \to \widetilde{J}^rP$, [17]. Let dim$M = m$. The $r$-th order principal prolongation $W^rP$ of a principal bundle $P \to M$ is defined as the space of all $r$-jets at $(0, e) \in \mathbb{R}^m \times G$ of all local principal bundle isomorphisms $\mathbb{R}^m \times G \to P$, where $e \in G$ denotes the unit, [12]. Denoting by $P^rM$ the $r$-th order frame bundle, we have the natural identification

$$W^rP = P^rM \times_M J^rP. \tag{5.1}$$

Further, $W^rP \to M$ is a principal bundle with the structure group $W^r_mG = J^r_{(0,e)}(\mathbb{R}^m \times G, \mathbb{R}^m \times G)_{(0,-)}$, i.e.

$$W^r_mG = G^r_m \times T^r_mG \tag{5.2}$$

as a set. Here $G^r_m = \text{inv}J^r_0(\mathbb{R}^m, \mathbb{R}^m)_0$ and $T^r_mG = J^r_0(\mathbb{R}^m, G)$. For any $(A, B), (A', B') \in G^r_m \times T^r_mG$, the multiplication $\mu : W^r_mG \times W^r_mG \to W^r_mG$ is given by

$$\mu\left((A, B), (A', B')\right) = \left(A \circ A', (B \circ A').B'\right), \tag{5.3}$$

where the dot is the multiplication in the Lie group $T^r_m G$ and $\circ$ is the composition of jets, see [12]. This defines on $W^r_m G$ the structure of semidirect product

$$W^r_m G = G^r_m \rtimes T^r_m G. \tag{5.4}$$

If we replace holonomic jets by nonholonomic or semiholonomic ones, we obtain the nonholonomic or semiholonomic principal prolongations $\widetilde{W}^r P$ and $\overline{W}^r P$, respectively. Quite analogously to (5.1), (5.4) we have

$$\widetilde{W}^r P = \widetilde{P}^r M \times_M \widetilde{J}^r P, \qquad \overline{W}^r P = \overline{P}^r M \times_M \overline{J}^r P$$

and

$$\widetilde{W}^r_m G = \widetilde{G}^r_m \rtimes \widetilde{T}^r_m G, \qquad \overline{W}^r_m G = \overline{G}^r_m \rtimes \overline{T}^r_m G.$$

Moreover, we have a natural identification

$$\widetilde{W}^r(\widetilde{W}^s P) = \widetilde{W}^{r+s} P$$

of principal bundle structures with corresponding structure groups.
In what follows all connections on the principal bundle $P \to M$ are supposed to be principal.

## 6 Connections on principal prolongations

We remark that the following part of the thesis can be found in [16].

Let $\Gamma : P \to J^1 P$ be a connection on $P \to M$ and $\Lambda : P^1 M \to J^1 P^1 M$ be a linear connection. By [13], $\Gamma$ and $\Lambda$ induce the connection $p(\Gamma, \Lambda)$ on $W^1 P \to M$, which is defined in the following way. First, we define a subspace

$$R(\Gamma) := P^1 M \times_M \Gamma(P) \subset P^1 M \times_M J^1 P = W^1 P.$$

This is a reduction of the principal bundle $W^1 P \to M$ to the subgroup $G^1_m \rtimes i(G) \subset W^1_m G$, where $i$ is an injection $G \hookrightarrow T^1_m G$. Therefore $R(\Gamma)$ can be identified with $P^1 M \times_M P$ and the product connection $\Lambda \times \Gamma$ on $P^1 M \times_M P$ can be identified with a connection in $R(\Gamma)$. Finally, this connection can be uniquely extended into a connection $p(\Gamma, \Lambda)$ in $W^1 P$. Clearly, $p(\Gamma, \Lambda)$ is fully determined by its value in

$$(u, \Gamma(v)) \in R(\Gamma) \subset W^1 P.$$

Let

$$\Delta(u) = j^1_x \lambda \in J^1 P^1 M, \quad \lambda : M \to P^1 M$$

and

$$\Gamma(v) = j^1_x \varphi \in J^1 P, \ \varphi : M \to P.$$

Then we have

$$(\Gamma * \Gamma)(v) = j^1_x (\Gamma \circ \varphi)$$

so that

$$p(\Gamma,\Lambda)(u,\Gamma(v)) = (\Lambda(u),(\Gamma*\Gamma)(v)) = j^1_x(\lambda(y),\Gamma(\varphi(y))).$$

This formula can be used to obtain the coordinate expression of $p(\Gamma,\Lambda)$. If we denote the local coordinates on $W^1P$ by

$$(x^i, x^i_j, y^p, y^p_i),$$

the equations of $p(\Gamma,\Lambda)$ are

$$\mathrm{d}y^p = \Gamma^p_i(x,y)\mathrm{d}x^i \tag{6.1}$$

$$\mathrm{d}x^i_j = \Lambda^i_{lk}x^l_j\mathrm{d}x^k \tag{6.2}$$

$$\mathrm{d}y^p_i = \left(\frac{\partial\Gamma^p_k}{\partial x^j}x^k_i + \frac{\partial\Gamma^p_k}{\partial y^q}\Gamma^q_j x^k_i + \Gamma^p_l\Lambda^l_{kj}x^k_i\right)\mathrm{d}x^j, \tag{6.3}$$

[7], where (6.1) and (6.2) are the coordinate expressions of the connections $\Gamma$ and $\Lambda$ respectively.

On the other hand, the couple $(\Gamma,\Lambda)$ induces the connection $\mathcal{W}^1(\Gamma,\Lambda)$ on $W^1P\to M$ by means of the flow prolongation with the equations (6.1), (6.2) and

$$\mathrm{d}y^p_i = \left(\frac{\partial\Gamma^p_j}{\partial x^k}x^k_i + \frac{\partial\Gamma^p_j}{\partial y^q}y^q_i + \Gamma^p_k\Lambda^k_{lj}x^l_i\right)\mathrm{d}x^j, \tag{6.4}$$

see [7]. By Proposition 3.3, there is an interesting relation between general connections $P(\Gamma,\Lambda)$ and $\mathcal{J}^1(\Gamma,\Lambda)$ on $J^1\to Y$. Now we present a similar relation between principal connections $p(\Gamma,\Lambda)$ and $\mathcal{W}^1(\Gamma,\widetilde{\Lambda})$ on $W^1P$. We first recall the concept of the curvature $C(\Gamma)$ of principal connection $\Gamma$ on $P\to M$. To deduce the coordinate expression we have to use the structure equations of $\Gamma$, see [12]. Let the equations of $\Gamma$ be of the form $\mathrm{d}y^p = \Gamma^p_i\mathrm{d}x^i$. Further, denote by $c^p_{qr}$ the structure constants of the Lie group $G$ and by $R^p_{ij}$ the curvature tensor. Then $C(\Gamma)$ is determined by

$$R^p_{ij} = \Gamma^p_{[ij]} + c^p_{qr}\Gamma^q_i\Gamma^r_j,$$

where $\Gamma^p_{[ij]}$ means an antisymmetrization of $\Gamma^p_{ij}$ in the subscripts, see [12] for details. Now if $\widetilde{\Lambda}$ is the conjugate connection of $\Lambda$, I. Kolář and G. Virsik in [13] proved

**Proposition 6.1.** $p(\Gamma,\Lambda) - \mathcal{W}^1(\Gamma,\widetilde{\Lambda}) = C(\Gamma)$.

**Remark.** For $W^rP$, $\overline{W}^rP$ and $\widetilde{W}^rP$, we have not found a similar construction to that of the operator $p(\Gamma,\Lambda)$, yet. Thus the generalization of Proposition 6.1 to higher order principal prolongations is still an open question.

### 6.2 CONSTRUCTION OF CONNECTIONS ON NONHOLONOMIC PRINCIPAL PROLONGATIONS $\widetilde{W}^rP$

**I.** First, given two connections $\Gamma : P\to\widetilde{J}^{r+1}P$ and $\Lambda:\widetilde{P}^rM\to J^1\widetilde{P}^rM$, we can construct a connection

$$\varkappa_{r+1}(\Gamma,\Lambda):\widetilde{W}^rP\to J^1\widetilde{W}^rP, \tag{6.5}$$

for the groupoid form see also [7]. Let us denote by $\Gamma_1 := \pi^{r+1}_r \Gamma : P \to \widetilde{J}^r P$ the underlying connection of order $r-1$, where $\pi^{r+1}_r : \widetilde{J}^{r+1} P \to \widetilde{J}^r P$ is the jet projection. Write

$$R(\Gamma_1) := \widetilde{P}^r M \times_M \Gamma_1(P) \subset \widetilde{P}^r M \times_M \widetilde{J}^r P = \widetilde{W}^r P.$$

One finds easily, that $R(\Gamma_1)$ is a reduction of the principal bundle $\widetilde{W}^r P$ to the subgroup $\widetilde{G}^r_m \times i(G) \subset \widetilde{W}^r_m G$, where $i : G \to \widetilde{T}^r_m G$ is an injection. As every $\Gamma(v) \in \widetilde{J}^{r+1} P$ can be considered as an element of $J^1 \widetilde{J}^r P$ over $\Gamma_1(v)$, we obtain in such a way a map

$$\varphi : R(\Gamma_1) \to J^1 \widetilde{P}^r M \times_M J^1 \widetilde{J}^r P = J^1(\widetilde{P}^r M \times_M \widetilde{J}^r P) = J^1 \widetilde{W}^r P$$

defined by

$$\varphi(u, \Gamma_1(v)) = (\Lambda(u), \Gamma(v)) \text{ for } (u, v) \in \widetilde{P}^r M \times_M P.$$

Then $\varphi$ is right invariant and thus it can be extended into the connection on $\widetilde{W}^r P$, which will be denoted by $\varkappa_{r+1}(\Gamma, \Lambda)$.

**II.** Now let $\Gamma : P \to J^1 P$ be a connection on $P \to M$ and $\Lambda : \widetilde{P}^r M \to J^1 \widetilde{P}^r M$ be a connection on $\widetilde{P}^r M$. Using (6.5) and the Ehresmann prolongation $\Gamma^{(r)} : P \to \widetilde{J}^{r+1} P$ of $\Gamma$, we have the connection

$$\widetilde{p}_r(\Gamma, \Lambda) := \varkappa_{r+1}(\Gamma^{(r)}, \Lambda) \tag{6.6}$$

on $\widetilde{W}^r P \to M$. Denoting by $p(\Gamma, \Lambda)$ the connection from previous section, we have

$$p(\Gamma, \Lambda) = p_1(\Gamma, \Lambda) = \varkappa_2(\Gamma * \Gamma, \Lambda).$$

**III.** Suppose we have a connection $\Gamma : P \to J^1 P$ and a connection $\Lambda : \widetilde{P}^r M \to J^1 \widetilde{P}^r M$. Let us note that $\Lambda$ can be interpreted as an $r$-th order linear connection denoted by the same symbol $\Lambda : TM \to \widetilde{J}^r TM$. Then the flow prolongation of $\Gamma$ with respect to $\Lambda$ induces the connection $\widetilde{\mathcal{W}}^r(\Gamma, \Lambda)$ on $\widetilde{W}^r P \to M$.

**IV.** Further, quite analogously to the nonholonomic jet prolongation $\widetilde{J}^r Y \to M$, we can construct connections on $\widetilde{W}^r P \to M$ by means of iteration. Indeed, we have $\widetilde{W}^1 P = W^1 P$ and $\widetilde{W}^r P = W^1(\widetilde{W}^{r-1} P)$. For example, starting from connections $\Gamma : P \to J^1 P$, $\Lambda : P^1 M \to J^1 P^1 M$ and using the basic operators $p(\Gamma, \Lambda)$ and $\mathcal{W}^1(\Gamma, \Lambda)$ on $\widetilde{W}^1 P$, we have the following connections on $\widetilde{W}^2 P$ :

$$p(p(\Gamma, \Lambda), \Lambda), \quad p(\mathcal{W}^1(\Gamma, \Lambda), \Lambda), \quad \mathcal{W}^1(p(\Gamma, \Lambda), \Lambda) \quad \text{and} \quad \mathcal{W}^1(\mathcal{W}^1(\Gamma, \Lambda), \Lambda).$$

Obviously, such an iteration process can be applied for an arbitrary order $r$.

## 6.3 CONSTRUCTION OF CONNECTIONS ON SEMIHOLONOMIC PRINCIPAL PROLONGATIONS $\overline{W}^r P$

I. Given two principal connections $\Gamma : P \to J^1 P$ and $\Lambda : \overline{P}^r M \to J^1 \overline{P}^r M$ we construct a connection

$$\overline{p}_r(\Gamma, \Lambda) : \overline{W}^r P \to J^1 \overline{W}^r P. \tag{6.7}$$

Denote by $\Gamma^{(r-1)} : P \to \widetilde{J}^r P$ the Ehresmann prolongation of $\Gamma$. By Section 3.1, this connection has values in $\overline{J}^r P$. Further, $\Gamma^{(r)} : P \to \widetilde{J}^{r+1} P$ is of the form

$$\Gamma^{(r)} = \Gamma^{(r-1)} * \Gamma = J^1 \Gamma^{(r-1)} \circ \Gamma : P \to \overline{J}^{r+1} P.$$

This yields that for $v \in P$, $\Gamma^{(r)}(v) \in \overline{J}^{r+1} P$ is the element of $J^1 \overline{J}^r P$ over $\Gamma^{(r-1)}(v) \in \overline{J}^r P$. Write

$$R(\Gamma^{(r-1)}) := \overline{P}^r M \times_M \Gamma^{(r-1)}(P) \subset \overline{P}^r M \times_M \overline{J}^r P = \overline{W}^r P.$$

Quite analogously to the nonholonomic principal prolongation we prove that $R(\Gamma^{(r-1)})$ is a reduction of $\overline{W}^r P$ to the subgroup $\overline{G}^r_m \times i(G) \subset \overline{W}^r_m G$, where $i$ is the injection of $G$ into $\overline{T}^r_m G$. Then we can define a map

$$\varphi : R(\Gamma^{(r-1)}) \to J^1 \overline{P}^r M \times_M J^1 \overline{J}^r P = J^1 \overline{W}^r P$$

by

$$\varphi(u, \Gamma^{(r-1)}(v)) = (\Lambda(u), \Gamma^{(r)}(v)).$$

This defines a connection $\overline{p}_r(\Gamma, \Lambda)$ on $\overline{W}^r P \to M$.

II. Let $\Gamma : P \to J^1 P$ be a connection and $\Lambda : TM \to J^r TM$ be an $r$-th order linear connection on $M$. Using the flow prolongation of $\Gamma$ with respect to $\Lambda$, we have the connection $\overline{\mathcal{W}}^r(\Gamma, \Lambda)$ on $\overline{W}^r P \to M$.

## 6.4 CONSTRUCTION OF CONNECTIONS ON HOLONOMIC PRINCIPAL PROLONGATIONS $W^r P$

I. Let $\Gamma : P \to J^1 P$ and $\Lambda : P^r M \to J^1 P^r M$ be principal connections and suppose that $\Gamma$ is curvature-free. By [17], the Ehresmann prolongation $\Gamma^{(r-1)} : P \to \widetilde{J}^r P$ is holonomic. Quite analogously to the connection (6.7) from 6.3 we can construct the connection $p_r(\Gamma, \Lambda) : W^r P \to J^1 W^r P$. For example, $p_2(\Gamma, \Lambda) : W^2 P \to J^1 W^2 P$ is of the form

$$p_2(\Gamma, \Lambda) = \varkappa_3(\Gamma^{(2)}, \Lambda).$$

II. The flow prolongation of $\Gamma : P \to J^1 P$ with respect to an $r$-th order linear connection $\Lambda : TM \to J^r TM$ defines the connection $\mathcal{W}^r(\Gamma, \Lambda)$ on $W^r P \to M$.

## 7 Identifications of connections

In this section we focus on the description of connections on higher order principal prolongations by means of the couples of connections, one on the higher order frame bundle of the base manifold and the other being the $(r+1)$-st order principal connection on $P$. We discuss the connections on all three principal prolongations $\widetilde{W}^r P$, $\overline{W}^r P$ and $W^r P$, respectively. We start with the identification of the connections on $\widetilde{W}^1 P = \overline{W}^1 P = W^1 P$. Let $\Delta : W^1 P \to J^1(W^1 P)$ be a connection on $W^1 P$. Further let $p_1 : W^1 P \to P^1 M$ and $p_2 : W^1 P \to P$ be two canonical principal bundle projections. Clearly, $p_1 \Delta$ is the connection on $P^1 M$. If we set $\Gamma = p_2 \Delta$, then we can construct

$$\mathrm{pr}_2 \Delta(u, \Gamma(v)) \in J^1 J^1 P \quad \text{for } (u,v) \in P^1 M \times_M P, \tag{7.1}$$

where $\mathrm{pr}_2$ means the projection onto the second argument. By [6], (7.1) lies in $\overline{J}^2 P$ and is independent of $u$. Thus (7.1) defines the second order connection on $P$ denoted by $\mu(\Delta) : P \to \overline{J}^2 P$. Using this notation I. Kolář and G. Virsik in [13] proved

**Proposition 7.1.** *The map $\Delta \mapsto (\mu(\Delta), p_1 \Delta)$ establishes a bijection between connections on $W^1 P$ and pairs consisting of a second order semiholonomic connection on $P$ and a classical linear connection on $M$.*

Now we present the generalization of this proposition, see also [16]. For the grupoid version of the following properties see [11].

**Proposition 7.2.** *There is a bijection between the connections on $\widetilde{W}^r P$ and the couples consisting of the connection on $\widetilde{P}^r M$ and the nonholonomic connection of order $(r+1)$ on $P$.*

*Proof.* First, given a couple of connections $(\Lambda, \Gamma)$, where $\Lambda : \widetilde{P}^r M \to J^1 \widetilde{P}^r M$ and $\Gamma : P \to \widetilde{J}^{r+1} P$, we have constructed a connection $\varkappa_{r+1}(\Gamma, \Lambda)$ on $\widetilde{W}^r P$, see Section 6.2. On the other hand, let $\Omega : \widetilde{W}^r P \to J^1 \widetilde{W}^r P$ be a connection. We are going to find a couple $(\Lambda, \Gamma)$, which corresponds to $\Omega$. Write $p_1 : \widetilde{W}^r P \to \widetilde{P}^r M$ and $p_2 : \widetilde{W}^r P \to P$ for the projections and set

$$\Lambda := p_1 \Omega, \ \Gamma := p_2 \Omega.$$

Then $\Lambda$ is a connection on $\widetilde{P}^r M$ and $\Gamma$ is a connection on $P$. Furthermore, consider the Ehresmann prolongation $\Gamma^{(r-1)} : P \to \widetilde{J}^r P$. For $(u,v) \in \widetilde{P}^r M \times_M P$ we have

$$\Omega(u, \Gamma^{(r-1)}(v)) \in J^1 \widetilde{W}^r P = J^1 \widetilde{P}^r M \times_M J^1 \widetilde{J}^r P.$$

The second projection $pr_2$ yields

$$\mathrm{pr}_2 \Omega(u, \Gamma^{(r-1)}(v)) \in J^1 \widetilde{J}^r P = \widetilde{J}^{r+1} P. \tag{7.2}$$

One verifies directly that this is independent of $u$. Hence (7.2) determines a map

$$\Omega^* : P \to \widetilde{J}^{r+1}P. \tag{7.3}$$

Obviously, this map is $G$-invariant, so it is an $(r+1)$-st order principal connection on $P$. Finally, the mapping $\Omega \mapsto (p_1\Omega, \Omega^*)$ determines the requuired bijection. □

Using the notation (7.3) from the previous proof, we obtain directly

**Corollary 7.1.** *Let $\widetilde{p}_r(\Gamma, \Lambda)$ be the connection* (6.6) *on $\widetilde{W}^r P$. Then we have $(\widetilde{p}_r(\Gamma, \Lambda))^* = \Gamma^{(r)}$. In particular, for $r = 1$ we obtain $(p(\Gamma, \Lambda))^* = \Gamma * \Gamma$.*

Next we describe connections on $\overline{W}^r P$ by means of a couple of connections $(\Lambda, \Sigma)$, where $\Lambda : \overline{P}^r M \to J^1\overline{P}^r M$ is a connection on $\overline{P}^r M$ and $\Sigma : P \to \overline{J}^{r+1}P$ is a semiholonomic $(r+1)$-st order connection on $P$.

**Proposition 7.3.** *There is a bijection between the connections on $\overline{W}^r P$ and couples consisting of a connection on $\overline{P}^r M$ and the $(r+1)$-st order semiholonomic connection on $P$.*

**Remark.** The same proof as that used in the Thesis is not possible for the description of connections on the holonomic principal prolongation $W^r P$ because of the additional assumption on the connection $\Gamma$ on $P$ to be curvature free.

## 8 Conclusions

We showed several generalizations concerning prolongations of connections. We first treated the general connections and described the construction of a connection on $r$-th order jet prolongation of a fibered manifold by means of first order general connection and a linear connection on the base manifold. We also determined all natural operators transforming a first order connection into the $r$-th order general connection by means of the Ehresmann prolongation. We used these results to describe the connections on the higher order principal prolongations and showed some useful identifications. Some of these results were published in [16]. We also distinguished the holonomic, semiholonomic and nonholonomic principal or jet prolongations, respectively. Yet, the complete description of the connections on $r$-th order principal prolongations is still an open question.

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# 10 Přehled

Teorie kalibračně přirozených operátorů je úzce spojena s prodlužováním hlavních bandlů ve smyslu prací C. Ehresmanna a I. Koláře. Ti totiž zavedli hlavní prodloužení $W^r_m G$ Lieovy grupy $G$ a hlavní prodloužení $W^r P$ hlavního bandlu $P \to M$ pomocí terminologie jetů. D. J. Eck přitom dokázal, že každý kalibračně přirozený bandl může být zkonstruován tímto způsobem. Zcela zásadním se pro fyzikální účely stává kalibračně přirozený bandl prvního hlavního prodloužení $W^1 P$. Je to hlavní bandl, který lze vyjádřit ve tvaru $W^1 P = P^1 M \times_M J^1 P$, se strukturní grupou $W^1_m G = G^1_m \rtimes T^1_m G$, kde $G$ je strukturní grupa hlavního bandlu $P \to M$, $G^1_m$ je diferenciální grupa řádu 1 dimenze $m = \dim M$ a $T^1_m$ je funktor $m$- dimenzionálních rychlostí prvního řádu.

V poslední době se ovšem ukazuje, že ke studiu některých otázek teoretické fyziky týkajících se teorie polí je potřeba studovat i hlavní prodloužení obecně $r$-tého řádu. Zde je však situace složitější, protože hlavní prodloužení vyššího řádu, stejně jako jetové prodloužení fibrovaných variet, se rozpadá na případy neholonomního, semiholonomního a anholonomního hlavního prodloužení $\widetilde{W}^r P$, $\overline{W}^r P$ a $W^r P$, které je potřeba studovat zvlášť. Dodejme, že pro fyzikální účely je nejdůležitější holonomní případ. Základní otázkou jsou pak samozřejmě konstrukce konexí na těchto kalibračně přirozených bandlech, které budou využívat hlavní konexi na $P$ a konexi na bandlu reperů vyššího řádu $P^r M$. Tento proces se pak nazývá prodlužování hlavních konexí. Známy jsou i konstrukce využívající hlavní konexi na $P$ a konexi na $P^1 M$.

Prodlužování hlavních konexí vychází z technik známých při prodlužování konexí obecných. Souhrnně jsou tyto procesy popsány v [12]. Ukazuje se například, že zásadním přirozeným operátorem převádějícím konexi $\Gamma : Y \to J^1 Y$ na fibrované varietě $Y \to M$ na konexi $r$-tého řádu $\Gamma^{(r)} : Y \to \overline{J}^r Y$ je tzv. Ehresmannovo prodloužení konexe $\Gamma$. Dále jsou známy postupy pro konstrukci konexí na jetovém prodloužení $J^1 Y$ fibrované variety $Y \to M$ pomocí konexe na $Y$ a lineární konexe na $M$. Jedním z nich je tokové prodloužení označované $\mathcal{J}^1$. Dalším je konstrukce, kterou zavedl I. Kolář v [9]. Ta využívá afinní strukturu bandlu $J^1 J^1 Y \to J^1 Y$ a přirozené zobrazení $J^1 T Y \to T J^1 Y$, které zavedli L. Mangiarotti a M. Modugno. Pomocí těchto faktů lze pak z konexe $\Gamma$ na $Y$ a lineární konexe $\Lambda$ na $M$ sestrojit konexi $P(\Gamma, \Lambda)$ na $J^1 Y$. Ukazuje se, že právě operátory $P$ a $\mathcal{J}^1$ hrají zásadní roli při zkoumání vlastností hlavních konexí pomocí asociovaných bandlů.

V této disertační práci jsou ukázány konstrukce operátorů prodlužujících obecné konexe na jetové prodloužení $r$-tého řádu. Dále je zde dokázáno, že Ehresmannovo prodloužení $\Gamma^{(r)}$ obecné konexe $\Gamma$ na fibrované varietě $Y \to M$ je jediný přirozený operátor převádějící obecnou konexi prvního řádu na semiholonomní konexi druhého řádu na $Y$. Tento fakt je pak použit při hledání konstrukcí konexí na hlavních prodlouženích vyššího řádu. Je zde uveden také souřadnicový popis takových operátorů pro $r = 2$. Nakonec jsou tyto postupy aplikovány při prodlužování hlavních konexí.