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SECOND ORDER DISCRETE BOUNDARY VALUE PROBLEMS

VYSOKÉ UČENÍ TECHNICKÉ V BRNĚ
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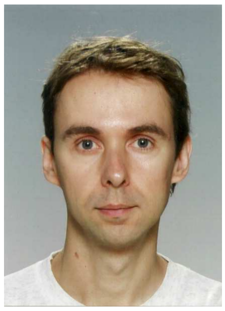

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Během svého působení na Univerzitě Tomáše Bati ve Zlíně vedl přednášky a cvičení z algebry, geometrie a informatiky a cvičení z matematické analýzy. Na Univerzitě Palackého v Olomouci vedl přednášky a cvičení předmětů Aplikace deskriptivní geometrie 1 a 2 s využitím programu DesignCAD a v současnosti zabezpečuje výuku předmětů Zobrazovací metody 1 a 2. Dále vybudoval a učí předměty Výpočetní geometrie 1 a 2 a Počítačová grafika 1 a 4 zaměřené na prostorové modelování, typografii, zobrazování grafů funkcí, technické rýsování a vytváření a úpravu digitální grafiky. Byl spoluřešitelem grantu na zřízení této učebny a je správcem její unixové části. Vedl tři bakalářské diplomové práce z geometrie a počítačové grafiky. Je autorem, resp. spoluautorem, tří učebních textů, z nichž dva jsou z matematiky a jeden z počítačové typografie. Podílel se na vypracování projektu FRVŠ pro vybavení počítačové učebny. Byl členem kolektivu, který připravil a zpracoval podklady pro studijní plán nového studijního oboru zaměřeného na počítačovou grafiku a grafický design.

V rámci doktorského studia pod vedením prof. Mikeše vypracoval a obhájil dizertační práci z diferenciální geometrie. V této oblasti je spoluautorem šesti vědeckých článků o torzoformních polích v semisymetrických prostorech. O publikovaných výsledcích referoval na seminářích a konferencích. Byl členem řešitelského kolektivu grantového projektu Riemannova a afinní geometrie podporovaná počítačem, 201/05/2707 (odpovědný řešitel prof. RNDr. Josef Mikeš, DrSc.). V současnosti svůj výzkum zaměřuje na diskrétní matematiku, speciálně na diskrétní dynamické systémy. Je autorem jednoho a spoluautorem devíti článků o řešitelnosti diferenčních rovnic, které byly publikovány v impaktovaných matematických časopisech. Je autorem nebo spoluautorem celkem 21 publikací, z toho 10 v impaktovaných časopisech (např. Jounal of Difference Equations and Applications, Nonlinear Analysis, Abstract and Applied Analysis). Je recenzentem mezinárodního časopisu Applications of Mathematics.

I. Introduction

This habilitation dissertation is devoted to second order discrete boundary value problems. Essentially, this is a collection of the papers [26]–[34], which provides new or more general existence results for regular discrete problems. Singular discrete problems studied here have not been solved before.

Chapter II of the dissertation consists of Sections 1–5 and is a compilation of the papers [26] and [27]. Sections 1 and 2 deal with the discrete Dirichlet problem. In particular, known existence results by Y. Li [23] are generalized. A characterization of regular and singular boundary value problems is given in Section 3. Sections 4 and 5 study the discrete mixed problem with the p-Laplacian $\phi_p(y) = |y|^{p-2}y$, $p > 1$, both in regular and in singular cases. The existence results proved here extend and generalize the previous ones for the regular case by Z. He [15]. The singular case is new.

Chapter III of the thesis is formed by Sections 6–13 and contains results of the papers [28]–[30]. The solvability of difference and differential boundary value problems on compact intervals and their connections are discussed there. In Section 6, the lower and upper functions method is extended for singular difference equations. The main result of Section 7 provides conditions for the solvability of a sequence of singular discrete mixed problems. In addition, solutions of these problems give a sequence of approximate functions locally uniformly converging for $n \to \infty$ to a solution of the corresponding singular differential mixed problem. Section 7 is completed by two examples demonstrating the theory. In Section 8 we derive a mathematical model describing a rescaled radial stress of shallow membrane caps. Such model can be transformed to a singular difference mixed problem and we provide conditions for its solvability. In Sections 8 and 9 we also prove the lower and upper functions method as well as the approximation principle for more general cases. Using these results, we find conditions for the solvability of the generalized membrane problems in Section 10. Further generalization of singular mixed problems of previous sections is proved in Sections 11–13, which are based on the paper [30]. Here we consider equations admitting discontinuities not only at a boundary, but also inside a given region.

Difference and differential equations on the half-line are considered in Chapter IV of the dissertation, which is based on the papers [31]–[34] and which has Sections 14–19. Section 14 shows a construction of a mathematical model describing a density profile of a gas in bubbles in some liquid. Such model leads to the differential problem having so-called bubble-type solutions. If we properly extend the equation on the real line, bubble-type solutions become homoclinic solutions of the extended equation. Section 15 (by the paper [31]) investigates an autonomous discrete case of the problem and provides the existence of homoclinic points of the corresponding autonomous difference equation. A non-autonomous discrete case of the problem is studied in Sections 16–19. Section 16 describes four types of possible solutions of this equation: escape, homoclinic, damped and non-monotonous solutions. Section 17 provides the first existence result about the existence of non-monotonous or damped solutions. This is contained in the paper [32]. Section 18 (by the paper [33]) gives the second existence result about the existence of escape solutions. Section 19 (by the paper [34]) presents the third existence result about the existence of homoclinic solutions. This is the main result of Chapter IV. Examples with numerical simulations illustrate the results of Chapter IV.

Motivation of this work

Difference equations and their associated operators occupy a central and growing area in modern applicable analysis. We refer to the monographs [1], [5], [6], [12], [19], [22], [36], [37], [40]. Linear and nonlinear difference equations appear as direct mathematical models in almost all areas of science, engineering and technology where discrete phenomena abound, but also provide the field of numerical analysis with powerful tools. From the advent and rise of computers, where differential equations are solved by employing their approximative difference formulations, the interest in studying difference equations has been increasing even more. Difference equations usually describe the evolution of certain phenomena over the course of time. This description allows to compute a sequence of values recursively from a given set of values. The following examples have been chosen to illustrate the diversity of the uses and types of difference equations.

- In some bottom-feeding fish populations, recruitment appears to be essentially unaffected by fishing. These species have very high fertility rates and very low survivorship to adulthood. This leads to the Beverton and Holt population model (1957)

  $$x(n+1) = \frac{ax(n)}{1+bx(n)}$$

  with positive constants a, b, which is equivalent to the Verhulst difference equation. ([8], p. 72)

- It was observed that some species of fish, including salmon, habitually cannibalize their eggs and young. Such population can be described be the Ricker metered model (1958) which has a form of the nonlinear first order difference equation

  $$x(n+1) = \alpha x(n)\mathrm{e}^{-\beta x(n)},$$

  where α, β are positive constants. ([8], p. 73)

- The difference equation of an order $k+1$

  $$x(n+1) = ax(n) + F(x(n-k))$$

is often used to study whale populations. Here $x(n)$ represents the adult breeding population, $a \in [0,1]$ is the survival coefficient, and $F(x(n-k))$ is the recruitment to the adult stage with a delay of k years. ([8], p. 77)

- In the study of the effects of selection and mutation in genetics a nonlinear difference equation of the form

$$p(n+1) = (1-\mu)\frac{p(n)}{1-s+2sp(n)-sp^2(n)}$$

has been developed. Here one particular gene which has two alleles, A and a, is studied, and $p(n)$ is the fraction of A-alleles among the adults of generation n. It is assumed that all individuals with at least one A-allele reach adulthood, but that only a certain fraction, say $1-s$, of the a-homozygotes reach adulthood. Further assumption is that the alleles mutate from A- to a-alleles and μ is the mutation rate (the fraction of A-alleles that mutate). ([36], p. 143)

- Samuelson's business cycle model is built on the following assumptions. Current consumption $C(t)$ is a linear increasing function of previous period's income $Y(t-1)$ and current investment $I(t)$ rises with rising consumption $C(t) - C(t-1)$. More precisely

$$C(t) = cY(t-1), \quad I(t) = v(C(t) - C(t-1)),$$

where $c \in (0,1)$, $v > 0$ are parameters and $Y(t) = C(t) + I(t) + G(t)$ is a national accounting identity and $G(t) = 1$ is a fixed government expenditure level. Substitution gives a typical second order difference equation for the national income $Y(t)$

$$Y(t) - c(1+v)Y(t-1) + cvY(t-2) = 1,$$

where $Y(0)$, $Y(1)$ are given. ([40], p. 54)

- The classical Hansen-Samuelson's accelerator-multiplier model is given by the same second order difference equation as before

$$y(n) = cy(n-1) + \alpha(y(n-1) - y(n-2)) + A_0,$$

where the constant $A_0 = C_0 + I_0 + G_0$ represents the sum of the minimum consumption, the autonomous investment and the fixed government spending, and $y(n)$ is the national income in period n. The coefficient $\alpha > 0$ is the accelerator and the constant $c \in (0,1)$ represents Keynes' marginal propensity to consume, while $\frac{1}{1-c}$ is Keynes' muiltiplier. ([37], p. 243)

- International macroeconomic data exhibit substantial and persistent fluctuations in the real exchange rates for currencies. In the model proposed by S. Chen in 1999, the real exchange rate $x(n)$ (defined as the ratio of the domestic currency price to the price of domestic output) of a given country is determined according to the second order nonlinear difference equation

$$x(n+1) = x(n) + \frac{1}{\mu}(bx^3(n-1) - ax^2(n) + cx(n) - d),$$

where a, b, c, d, μ are positive constants. ([37], p. 345)

- The system

$$x(n+1) = ax(n)\mathrm{e}^{-by(n)}, \quad y(n+1) = cx(n)(1 - \mathrm{e}^{by(n)})$$

is known as the Nicholson and Bailey model (1935) for a host-parasite system; $x(n)$ denotes the number of hosts and $y(n)$ the number of parasites. ([8], p. 79)

- A special case of the business cycles model by Botomazava and Touzé (1998) is given by the second order equation

$$k(n+1) = (1-\tau)\beta k(n) - (1-\tau)\frac{\beta}{\alpha}k^2(n-1),$$

where $k(n)$ is the capital per household in period n, the constants α, β satisfy $\alpha, \beta > 0$ and a payroll tax rate τ belongs to $[0, 1)$. ([37], p. 352)

- Consider the simplest example of the Fermi acceleration model: a point mass moving between a fixed wall and an oscillating wall. We simplify the setup of the model by assuming that the amplitude a of the wall oscillation is very small compared to the distance L of the walls. When the particle with a velocity v is reflected elastically from the massive wall moving with a velocity V, it rebounds with a velocity $\tilde{v} = -v + 2V$. In the case considered here, we register the particle velocity just before the n-th impact as $v_n > 0$. The particle hits the oscillating wall at a phase Φ_n and is reflected with a velocity $-v_n + 2V(\Phi_n)$, where V is a velocity of the oscillating wall. Then it moves to the fixed wall, is reflected and returns with a velocity

$$v_{n+1} = v_n - 2V(\Phi_n),$$

after a time $2L/|v_{n+1}|$. If the wall velocity oscillates with period T, the phase of the wall oscillation at the next impact is given by

$$\Phi_{n+1} = \Phi_n + \frac{2L}{Tv_{n+1}}.$$

This system can be transformed to the second order difference equation

$$y(n+1) = y(n) + (y(n) - y(n-1))\frac{1}{1 - \frac{T}{L}V(y(n))}.$$

([21], p. 137)

- Finally, two models arising in the theory of shallow membrane caps and in the hydrodynamics, which have forms of nonlinear second order difference equations, are derived and investigated in details in Chapter III and Chapter IV of the habilitation dissertation, respectively.

Main objectives of this work

1. To prove new existence results for Dirichlet problems and for singular mixed problems with p-Laplacian.
2. To investigate the connection between singular difference and differential boundary value problems on compact intervals and to derive an approximation principle.
3. To construct a mathematical model describing a radial stress of shallow membrane caps and to find conditions for its solvability and for the solvability of its generalization.
4. To construct a mathematical model describing a density profile of a gas in bubbles in some liquid and to find conditions which guarantee the existence of homoclinic solutions of the model.

II. Discrete boundary value problems on compact intervals

In many areas, for example in the study of solid state physics, chemical reaction or population dynamics, we can find difference equations subject to some boundary conditions. Such problems are called discrete boundary value problems. They have been investigated in several monographs. See R. P. Agarwal [1], R. P. Agarwal, D. O'Regan, P. J. Y. Wong [5], R. P. Agarwal, P. J. Y. Wong [6] and W. G. Kelley, A. C. Peterson [19]. We can find also a lot of papers dealing with discrete boundary value problems. Here we investigate Dirichlet and singular mixed discrete problems.

1 Dirichlet problem

For fixed $T \in \mathbb{N}$ we define the discrete interval $[1, T] = \{1, 2, \dots, T\}$ and study a difference equation of the form

$$\Delta(p(t)\Delta u(t-1)) + f(t, u(t)) = g(t), \quad t \in [1, T] \tag{1.1}$$

subject to the boundary conditions

$$u(0) = 0, \quad u(T+1) = 0. \tag{1.2}$$

The discrete boundary value problem (1.1), (1.2) is called the *Dirichlet problem.* Here

$$\left.\begin{array}{c} p\colon [1, T+1] \to \mathbb{R} \text{ is positive}, \quad g\colon [1, T] \to \mathbb{R} \\ f\colon [1, T] \times \mathbb{R} \to \mathbb{R} \text{ is continuous.} \end{array}\right\} \tag{1.3}$$

Recall that $f(t, x)$ is continuous on $[1, T] \times \mathbb{R}$ if for each $t \in [1, T]$, $f(t, x)$ is a continuous function of x.

Definition 1.1 By *a solution* u of problem (1.1), (1.2) we mean $u\colon [0, T+1] \to \mathbb{R}$ such that u satisfies the difference equation (1.1) on $[1, T]$ and the boundary conditions (1.2).

The following results are based on the paper [27]. We were motivated by the paper [23] by Yongjin Li, where he used the variational approach and proved the existence result for problem (1.1), (1.2). Here we use a completely different approach based on the lower and upper functions method. By means of this we generalized the result in [23].

Theorem 1.2 (Existence)
Assume that (1.3) *and*

$$\exists r > 0 \quad \text{such that} \quad x f(t, x) \le 0 \quad \text{for } t \in [1, T] \text{ and } |x| \ge r, \tag{1.4}$$

hold. Then problem (1.1), (1.2) *has at least one solution.*

Corollary 1.3 *Assume that* (1.3) *holds. Let*

$$g(t) < 0, \quad f(t, 0) \ge 0 \quad \text{for } t \in [1, T], \tag{1.5}$$

$$\exists r > 0 \quad \text{such that} \quad f(t, x) \le 0 \quad \text{for } t \in [1, T] \text{ and } x \ge r. \tag{1.6}$$

Then problem (1.1), (1.2) *has a solution* u *such that*

$$u(t) > 0 \quad \text{for } t \in [1, T]. \tag{1.7}$$

Corollary 1.4 *Assume that* (1.3) *holds. Let*

$$g(t) > 0, \quad f(t, 0) \le 0 \quad \text{for } t \in [1, T], \tag{1.8}$$

$$\exists r > 0 \quad \text{such that} \quad f(t, x) \ge 0 \quad \text{for } t \in [1, T] \text{ and } x \le -r. \tag{1.9}$$

Then problem (1.1), (1.2) *has a solution* u *such that*

$$u(t) < 0 \quad \text{for } t \in [1, T]. \tag{1.10}$$

2 Singular mixed problem with p-Laplacian

Problem (1.1), (1.2) studied in the previous sections is regular, because function f in equation (1.1) is continuous (see (1.3)). Otherwise the corresponding problem is singular (see e.g. (2.3)). Most existence results in literature concern regular problems. Singular discrete problems have received less attention. We refer to [3], [4] and [5] where the solvability of the singular Dirichlet discrete problem was studied. Existence theorems for singular higher order discrete problems can be found in [6].

Let $T \in \mathbb{N}$ be fixed. We define the discrete interval $[1, T+1] = \{1, 2, \ldots, T+1\}$ and consider a singular mixed problem which has a form of the following second order difference equation with the p-Laplacian

$$\Delta\Big(\phi_p(\Delta u(t-1))\Big) + f(t, u(t), \Delta u(t-1)) = 0, \quad t \in [1, T+1] \tag{2.1}$$

subject to the mixed boundary conditions

$$\Delta u(0) = 0, \quad u(T+2) = 0. \tag{2.2}$$

We denote $\phi_p(y) = |y|^{p-2}y$, $p > 1$ and investigate the solvability of problem (2.1), (2.2).

Definition 2.1 By *a solution* u of the mixed problem (2.1), (2.2) we mean $u\colon [0, T+2] \to \mathbb{R}$ such that u satisfies the difference equation (2.1) on $[1, T+1]$ and the boundary conditions (2.2). If $u(t) > 0$ for $t \in [1, T+1]$, we say that u is *a positive solution* of problem (2.1), (2.2).

Let $\mathcal{D} \subset \mathbb{R}^2$. We say that f is continuous on $[1, T+1] \times \mathcal{D}$, if $f(t, \cdot, \cdot)$ is continuous on $\mathcal{D}$ for each $t \in [1, T+1]$.

If $\mathcal{D} = \mathbb{R}^2$, problem (2.1), (2.2) is regular. If $\mathcal{D} \neq \mathbb{R}^2$ and f has singularities on $\partial\mathcal{D}$, then problem (2.1), (2.2) is singular.

We will assume that

$$\left.\begin{array}{l} \mathcal{D} = (0, \infty) \times \mathbb{R},\ f \text{ is continuous on } [1, T+1] \times \mathcal{D} \\ \text{and } f \text{ has a singularity at } x = 0, \text{ i.e.} \\ \limsup\limits_{x \to 0+} |f(t, x, y)| = \infty \text{ for each } t \in [1, T+1] \\ \text{and for some } y \in \mathbb{R}. \end{array}\right\} \tag{2.3}$$

Problem (2.1), (2.2) where f is continuous and has the form $f(t, x) = a(t)g(x)$ has been investigated by Zhimin He in [15]. Here we extend the existence results of [15] onto the singular problem (2.1), (2.2) where f depends both on u and on Δu. These results have been published in [26].

The next theorem provides sufficient conditions for the solvability of the singular problem (2.1), (2.2). The proof is based on the construction of a sequence of approximating auxiliary regular problems and on the lower and upper functions method.

Theorem 2.2 (Existence)
Assume (2.3) *and let the following conditions hold:*

$$\text{there exists } c \in (0,\infty) \text{ such that } f(t,c,0) \le 0 \text{ for } t \in [1,T+1], \tag{2.4}$$

$$f \text{ is nonincreasing in } y \text{ for } t \in [1,T+1],\ x \in (0,c], \tag{2.5}$$

$$\lim_{x\to 0+} f(t,x,y) = \infty \text{ for } t \in [1,T+1],\ y \in [-c,c]. \tag{2.6}$$

Then problem (2.1), (2.2) *has a solution u satisfying*

$$0 < u(t) \le c \text{ for } t \in [0,T+1]. \tag{2.7}$$

III. Discretization of differential boundary value problems

In this chapter, which is based on the papers [28]–[30], we study a connection between discrete (difference) and continuous (differential) boundary value problems. Particular significance of our investigation lies in the fact that strange and interesting distinctions can occur between the theory of differential equations and the theory of difference equations. For example [2], properties such as existence, uniqueness and multiplicity of solutions may not be shared between the theory of differential equations and the theory of difference equation, even though the right hand side of the equations under consideration may be the same. Questions about difference problems associated with regular differential problems have been also discussed for example in [13], [16], [19], [38], [39].

3 Discrete and continuous singular mixed problems

We investigate discrete mixed problems as approximations of continuous mixed problems. Therefore we choose $T \in (0,\infty)$, $n \in \mathbb{N}$, $n \ge 2$ and introduce a step $h = \frac{T}{n}$. Using this notation we consider the singular discrete mixed boundary value problem

$$\frac{1}{h^2}\Delta^2 u_{k-1} + f(t_k, u_k) = 0, \quad k = 1,\dots,n-1, \tag{3.1}$$

$$\Delta u_0 = 0, \quad u_n = 0, \tag{3.2}$$

where $f\colon [0,T] \times (0,\infty) \to \mathbb{R}$ is continuous and $f(t,x)$ has a singularity at $x = 0$, i.e. we assume

$$f \in C([0,T] \times (0,\infty)), \quad \limsup_{x\to 0+} |f(t,x)| = \infty \quad \text{for each } t \in (0,T). \tag{3.3}$$

Here

$$t_0 = 0, \quad t_k = hk, \quad \Delta u_{k-1} = u_k - u_{k-1} \quad \text{for } k = 1, \dots, n. \tag{3.4}$$

Definition 3.1 A vector $(u_0, \dots, u_n) \in \mathbb{R}^{n+1}$ satisfying equation (3.1) and the mixed boundary conditions (3.2) is called *a solution* of problem (3.1), (3.2). If $u_k > 0$ for $k = 0, \dots, n-1$, the solution is called *positive*.

The continuous version of problem (3.1), (3.2) has the form

$$y''(t) + f(t, y(t)) = 0, \tag{3.5}$$

$$y'(0) = 0, \quad y(T) = 0. \tag{3.6}$$

Definition 3.2 A function $y \in C[0,T] \cap C^2[0,T)$ satisfying equation (3.5) for $t \in [0,T)$ and fulfilling the mixed boundary conditions (3.6) is called *a solution* of problem (3.5), (3.6). If $y(t) > 0$ for $t \in [0,T)$, the solution is called *positive*.

Following the paper [28], we assume that there exist functions

$$\alpha, \beta \in C[0,T], \quad 0 < \alpha(t) \le \beta(t) \quad \text{for } t \in (0,T) \tag{3.7}$$

and denote

$$\alpha_k = \alpha(t_k), \quad \beta_k = \beta(t_k), \quad k = 0, \dots, n. \tag{3.8}$$

Using this we provide conditions which imply that for each $n \in \mathbb{N}$, $n \ge 2$, the singular discrete problem (3.1), (3.2) has a positive solution $(u_0, \dots, u_n)$ in the sense of Definition 3.1. Then we construct an approximate function $S^{[n]} \in C[0,T]$ by

$$S^{[n]}(t) = u_k + v_k(t - t_k), \quad t \in [t_k, t_{k+1}], \quad k = 0, \dots, n-1, \tag{3.9}$$

where $hv_k = \Delta u_k$. Having the sequence $\{S^{[n]}\}$ we prove that, under proper conditions, there is a subsequence $\{S^{[m]}\}$ locally uniformly converging on $[0,T)$ for $m \to \infty$ to a positive solution y of the singular continuous problem (3.5), (3.6). This is so-called *Approximation principle*, which is contained in Theorem 3.3.

Theorem 3.3 (Approximation principle)
Assume that conditions (3.3), (3.7) *and* (3.8) *hold. Let for each $n \ge 2$ the vectors $(\alpha_0, \dots, \alpha_n)$ and $(\beta_0, \dots, \beta_n)$ be a lower function and an upper function of problem* (3.1), (3.2) *and let $\alpha_0 > 0$, $\beta_n = 0$. Then for each $n \ge 2$ problem* (3.1), (3.2) *has a solution $(u_0, \dots, u_n)$, a sequence $\{S^{[n]}\}$ can be given by* (3.9) *and there exists a subsequence $\{S^{[m]}\} \subset \{S^{[n]}\}$ which converges locally uniformly on $[0,T)$ to a solution $y \in C[0,T] \cap C^2[0,T)$ of problem* (3.5), (3.6).

If, in addition,

$$|f(t,x)| \le g_0(t,x) + g_1(t,x) \quad \textit{for } t \in [0,T],\ x \in (0,\infty), \tag{3.10}$$

where $g_0 \in C([0,T] \times (0,\infty))$ is nonincreasing in its second variable with

$$\int_0^T g_0(t, \alpha(t))\mathrm{d}t < \infty \tag{3.11}$$

and $g_1 \in C([0,T] \times [0,\infty))$, then moreover $y \in C^1[0,T]$.

Remark 3.4 The singular discrete problem (3.1), (3.2) has not been studied before. Assumptions (3.10) and (3.11) are forced from a singularity of this problem. We point out that these assumptions are needed neither for the solvability of problem (3.1), (3.2) nor for the convergence of the sequence $\{S^{[m]}\}$ to a solution $y \in C[0,T] \cap C^2[0,T)$ of problem (3.5), (3.6). They are used just in order to prove that $y'(t)$ is continuous also at $t = T$.

4 Problems arising in the theory of shallow membrane caps

Consider a shallow membrane cap which is rotationally symmetric in its undeformed state and its shape is described in cylindrical coordinated by $z = C(1-r^\gamma)$, where $r \in [0,1]$ and $\gamma > 1$. The undeformed radius is $r = 1$ and $C > 0$ is the height at the center of the cap. When a radial stress is applied on the boundary and a small uniform vertical pressure P is applied to the membrane, the shape that the cap takes is described by a nonlinear model. Baxley and Robinson [7], Dickey [11] and Johnson [17] showed that under the assumptions of small strain and small constant vertical pressure, if the deformed membrane is also rotationally symmetric, the rescaled radial stress S_r on a membrane satisfies the differential equation

$$r^2 S_r'' + 3rS_r' = \frac{\lambda^2 r^{2\gamma-2}}{2} + \frac{\beta\nu r^2}{S_r} - \frac{r^2}{8S_r^2}. \tag{4.1}$$

Here $\nu \in [0, 0.5)$ is the Poisson ratio while λ and β are positive constants depending on the pressure P, the thickness of the membrane and Young's modulus. Dickey focused his attention on boundary conditions

$$\lim_{r\to 0+} r^3 S_r'(r) = 0, \quad S_r(1) = A > 0,$$

where the first (regularity) condition follows from the radial symmetry and the second condition means that the stress boundary is specified. Motivated by 4.1 we investigate the solvability of the singular discrete mixed boundary value problem

$$\frac{1}{h^2}\Delta(t_k^3 \Delta u_{k-1}) + t_k^3 \left(\frac{1}{8u_k^2} - \frac{a_0}{u_k} - b_0 t_k^{2\gamma-4}\right) = 0, \quad k = 1, \dots, n-1, \tag{4.2}$$

$$\Delta u_0 = 0, \quad u_n = 0, \tag{4.3}$$

where $a_0 \geq 0$, $b_0 > 0$, $\gamma > 1$. The continuous version of problem (4.2), (4.3) has the form

$$(t^3 y')' + t^3 \left(\frac{1}{8y^2} - \frac{a_0}{y} - b_0 t^{2\gamma - 4} \right) = 0, \tag{4.4}$$

$$\lim_{t \to 0+} t^3 y'(t) = 0, \quad y(T) = 0, \tag{4.5}$$

and it was studied for example in [18] and [25]. The next theorem provide the existence result for the discrete membrane problem (4.2), (4.3) and its continuous version (4.4), (4.5), which is contained in the paper [29].

Theorem 4.1 (Solvability of the membrane problem)
Let $a_0 \geq 0$, $b_0 > 0$, $\gamma > 1$. Then for each $n \geq 2$, the difference problem (4.2), (4.3) *has a positive solution $(u_0, \ldots, u_n)$. Let $y^{[n]} \in C[0,T]$ be a piece-wise linear function with $y^{[n]}(t_k) = u_k$, $k = 0, \ldots, n$. Then there exists a subsequence $\{y^{[m]}\} \subset \{y^{[n]}\}$ such that*

$$\lim_{m \to \infty} y^{[m]}(t) = y(t) \quad \textit{locally uniformly on } (0,T),$$

and $y \in C[0,T] \cap C^2(0,T)$ is a positive solution of the differential problem (4.4), (4.5).

Further generalization of singular mixed problems is proved in Sections 11–13 of the dissertation. These results are based on the paper [30].

IV. Boundary value problems arising in hydrodynamics

This chapter is based on the papers [31]–[34] and is devoted to the study of difference and differential equations on the half-line. This study has been inspired by some models arising in hydrodynamics.

5 Construction of bubble-models

We investigate singular boundary value problems originating from the Cahn-Hilliard theory, which is used in hydrodynamics to study a behavior of non-homogeneous fluids. The state of a non-homogeneous fluid is described by the following system of partial differential equations (see [14], [20], [24])

$$\varrho_t + \operatorname{div}(\varrho \vec{v}) = 0 \tag{5.1}$$

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} + \nabla(\mu(\varrho) - \gamma\Delta\varrho) = 0, \tag{5.2}$$

where ϱ is the density of the fluid, $\vec{v}$ denotes the vector-velocity of the particles of the fluid, $\mu(\varrho)$ is the chemical potential of the fluid and γ is a constant parameter. By considering the case where the motion of the fluid is zero, system (5.1), (5.2) is reduced to a single equation of the form

$$\gamma\Delta\varrho = \mu(\varrho) - \mu_0, \tag{5.3}$$

where μ_0 is a constant depending of the state of the fluid. Equation (5.3) can describe the formation of microscopical bubbles in a non-homogeneous fluid, in particular, vapor inside a liquid. When we introduce the spherical coordinate system in $\mathbb{R}^3$ and search for radially symmetric strictly increasing solutions depending only on the radial variable r, then equation (5.3) leads to an ordinary differential equation of the form

$$\gamma\left(\varrho'' + \frac{2}{r}\varrho'\right) = \mu(\varrho) - \mu(\varrho_\ell), \tag{5.4}$$

with the boundary conditions

$$\varrho'(0) = 0, \quad \lim_{r\to\infty} \varrho(r) = \varrho_\ell > 0. \tag{5.5}$$

The first condition in (5.5) follows from the central symmetry and it is also necessary for the smoothness of solution $\varrho(r)$ of equation (5.4) at $r = 0$. The second condition in (5.5) means that the bubble is surrounded by an external liquid with the density ϱ_ℓ. If there exists a strictly increasing solution of problem (5.4), (5.5) for some $\varrho(0) = \varrho_0$ with $0 < \varrho_0 < \varrho_\ell$, then ϱ_0 is the density of the gas at the center of the bubble and the solution $\varrho(r)$ determines an increasing mass density in the bubble. Such solution are called bubble-type solutions and equation (5.4) is known as the density profile equation. If the bubble-type solutions exist, many important physical properties of the non-homogeneous fluid depend on them, [10], [14], [24]. Note that boundary value problems of the same kind arise in the nonlinear field theory, in particular, when describing bubbles generated by scalar fields of the Higgs type in the Minkowski spaces [35], which can be treated as the classical pattern of elementary particles [9].

6 Four types of solutions of non-autonomous difference equations

In the simplest models for non-homogeneous fluids, the chemical potential μ in equation (5.4) is a third degree polynomial with three distinct real roots. After some substitution (see [24]), problem (5.4), (5.5) is reduced to the form

$$(t^2u')' = 4\lambda^2t^2(u+1)u(u-\xi), \tag{6.1}$$

$$u'(0) = 0, \quad u(\infty) = \xi, \tag{6.2}$$

where $\lambda \in (0, \infty)$ and $\xi \in (0, 1)$ are parameters.

Now, we construct a discretization of problem (6.1), (6.2). Consider $h > 0$ and a sequence $\{t_n\}_{n=0}^{\infty} \subset [0, \infty)$ such that

$$t_0 = 0, \quad t_n = nh, \; n \in \mathbb{N}. \tag{6.3}$$

Then the discrete analogy of problem (6.1), (6.2) has the form of the following difference problem

$$\frac{1}{h^2}\Delta(t_n^2 \Delta x(n-1)) = 4\lambda^2 t_n^2 (x(n)+1)x(n)(x(n)-\xi), \; n \in \mathbb{N}, \tag{6.4}$$

$$\Delta x(0) = 0, \quad \lim_{n\to\infty} x(n) = \xi. \tag{6.5}$$

Here $t_n = hn$, $n \in \mathbb{N}$ and problem (6.4), (6.5) has an equivalent form

$$\begin{aligned} x(n+1) = x(n) + \left(\tfrac{n}{n+1}\right)^2 \Big(& x(n) - x(n-1) + \\ & + 4\lambda^2 h^2 (x(n)+1)x(n)(x(n)-\xi)\Big), \quad n \in \mathbb{N}, \end{aligned} \tag{6.6}$$

$$x(0) = x(1), \quad \lim_{n\to\infty} x(n) = \xi. \tag{6.7}$$

We will investigate the following generalization of the non-autonomous equation (6.6)

$$x(n+1) = x(n) + \left(\frac{n}{n+1}\right)^2 \Big(x(n) - x(n-1) + h^2 f(x(n))\Big), \quad n \in \mathbb{N}, \tag{6.8}$$

where f is supposed to fulfil

$$L_0 < 0 < L, \quad f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}), \quad f(L_0) = f(0) = f(L) = 0, \tag{6.9}$$

$$x f(x) < 0 \text{ for } x \in (L_0, L) \setminus \{0\}, \tag{6.10}$$

$$\exists \bar{B} \in [L_0, 0) \text{ such that } \int_{\bar{B}}^{L} f(z)\,\mathrm{d}z = 0. \tag{6.11}$$

A sequence $\{x(n)\}_{n=0}^{\infty}$ which satisfies (6.8) is called a solution of equation (6.8). We will work with the initial condition

$$x(0) = B, \quad x(1) = B, \quad B \in (L_0, 0). \tag{6.12}$$

Definition 6.1 Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (6.8), (6.12) such that

$$\{x(n)\}_{n=1}^{\infty} \text{ is increasing}, \quad \lim_{n\to\infty} x(n) = 0. \tag{6.13}$$

Then $\{x(n)\}_{n=0}^{\infty}$ is called *a damped solution.*

Definition 6.2 Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (6.8), (6.12) which fulfils

$$\{x(n)\}_{n=1}^{\infty} \text{ is increasing,} \quad \lim_{n\to\infty} x(n) = L. \tag{6.14}$$

Then $\{x(n)\}_{n=0}^{\infty}$ is called *a homoclinic solution.*

Definition 6.3 Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (6.8), (6.12). Assume that there exists $b \in \mathbb{N}$, such that $\{x(n)\}_{n=1}^{b+1}$ is increasing and

$$x(b) \le L < x(b+1). \tag{6.15}$$

Then $\{x(n)\}_{n=0}^{\infty}$ is called *an escape solution.*

Definition 6.4 Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (6.8), (6.12). Assume that there exists $b \in \mathbb{N}$, $b > 1$, such that $\{x(n)\}_{n=1}^{b}$ is increasing and

$$0 < x(b) < L, \quad x(b+1) \le x(b). \tag{6.16}$$

Then $\{x(n)\}_{n=0}^{\infty}$ is called *a non-monotonous solution.*

Theorem 6.5 (On four types of solutions)
Assume that (6.9) *and* (6.10) *hold. Let* $\{x(n)\}_{n=0}^{\infty}$ *be a solution of problem* (6.8), (6.12). *Then* $\{x(n)\}_{n=0}^{\infty}$ *is just one of the following four types:*

(I) $\{x(n)\}_{n=0}^{\infty}$ *is an escape solution;*

(II) $\{x(n)\}_{n=0}^{\infty}$ *is a homoclinic solution;*

(III) $\{x(n)\}_{n=0}^{\infty}$ *is a damped solution;*

(IV) $\{x(n)\}_{n=0}^{\infty}$ *is a non-monotonous solution.*

Theorem 6.6 (On the existence of non-monotonous or damped solutions)
Assume that (6.9), (6.10) *and* (6.11) *hold. Let* $B \in (\bar{B}, 0)$. *Then there exists* $h_B > 0$ *such that if* $h \in (0, h_B]$, *then the corresponding solution* $\{x(n)\}_{n=0}^{\infty}$ *of problem* (6.8), (6.12) *is non-monotonous or damped.*

Theorem 6.7 (On the existence of escape solutions)
There exists $h^* > 0$ *such that for any* $h \in (0, h^*]$ *the initial value problem* (6.8), (6.12) *has an escape solution for some* $B \in (L_0, \bar{B})$.

Theorem 6.8 (On the existence of homoclinic solutions)
There exists $h^* > 0$ *such that for any* $h \in (0, h^*]$ *there exists a homoclinic solution* $\{x^*(n)\}_{n=0}^{\infty}$ *of problem* (6.8), (6.12), *that is* $\{x^*(n)\}_{n=1}^{\infty}$ *is increasing and* $\lim_{n\to\infty} x^*(n) = L$.

Theorems 6.5–6.8 have been proved in sections 16–19 of the habilitation. In Section 15 of the habilitation the autonomous case of problem (6.8), (6.12) has been investigated.

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Abstract

This habilitation dissertation is devoted to second order discrete boundary value problems. Essentially, this is a collection of the papers [26]–[34], which provides new or more general existence results for regular discrete problems. Singular discrete problems studied here have not been solved before.

Chapter II of the dissertation deals with the discrete Dirichlet problem. In particular, known existence results by Y. Li [23] are generalized. Then the discrete mixed problem with the p-Laplacian, both in regular and in singular cases, is investigated. The existence results proved here extend and generalize the previous ones for the regular case by Z. He [15]. The singular case is new.

Chapter III of the thesis contains new results about the solvability of singular difference and differential boundary value problems on compact intervals and about their connections. The application on mathematical models describing a radial stress of shallow membrane caps is given.

Chapter IV investigates difference and differential equations on the half-line and provides new existence results of four types of their possible solutions: escape, homoclinic, damped and non-monotonous solutions. The application on a mathematical model describing a density profile of a gas in bubbles in some liquid is given. Examples with numerical simulations illustrate the results of Chapter IV.