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Luděk Nechvátal

HOMOGENIZATION OF PARTIAL DIFFERENTIAL EQUATIONS WITH UNCERTAIN INPUT DATA

VYSOKÉ UČENÍ TECHNICKÉ V BRNĚ
Fakulta strojního inženýrství
Ústav matematiky

Ing. Luděk Nechvátal, Ph.D.

HOMOGENIZATION OF PARTIAL DIFFERENTIAL EQUATIONS WITH UNCERTAIN INPUT DATA

HOMOGENIZACE PARCIÁLNÍCH DIFERENCIÁLNÍCH ROVNIC S NEJISTÝMI VSTUPNÍMI DATY

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Ústav matematiky
Fakulta strojního inženýrství
Vysoké učení technické v Brně
Technická 2896/2
616 69 Brno

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Curriculum Vitae

Ing. Luděk Nechvátal, Ph.D.

Personal data

Date and place of birth:	December 24th, 1975, Brno
Citizenship:	Czech republic
Nationality:	Czech
Marital status:	unmarried
Domicile:	Lerchova 322/32, 602 00 Brno
E-mail:	`nechvatal@fme.vutbr.cz`

Education and academic qualification

1982–1990: Elementary school Lerchova, Brno (primary), Elementary school Sirotkova, Brno (secondary) – mathematics focused class

1990–1994: Gymnasium tř. kpt. Jaroše, Brno – mathematics–physics focused class

1994–1999: FME BUT, study branch: Mathematical engineering, master's degree Ing.

1999–2003: Internal doctoral study at the Institute of Mathematics, FME BUT, academic degree Ph.D.

Career overview

1999–2003: lecturer at the Institute of mathematics FME BUT

2004–until now: senior lecturer at the Institute of mathematics FME BUT

Pedagogic activities

- Teaching at FME BUT:
  - seminars: Mathematics I, II, III, III-B, Numerical methods I
  - lectures: Mathematics III, Numerical methods I
- Lectures in English language from Mathematics III for ERASMUS students

Scientific activities

- Modelling of heterogeneous structures (homogenization of partial differential equations)
- Problems with uncertain input data
- Fractional calculus and its applications

Academic interships abroad

- Two-weeks stay at Université Pierre et Marie Curie, Paris 6, France (ERASMUS teacher's mobility in 2004)
- Two times one-week stay at Chalmers University of Technology, Göteborg, Sweden (ERASMUS teacher's mobility in 2006 and 2010)

Non-University activities

- Series of lectures for Honeywell Aerospace from Dynamics of flight and control (in 2008)
- Member of the Mathematical Olympiad committee for South Moravian region

Projects

- Coworker on four grants GAČR (in 2000–2002, 2003–2005, 2008–2010 and 2011–until now)
- Coworker on a research plan MŠMT (in 2007–2011)
- Coworker on a project FRVŠ (in 2007)
- Coworker on a project OPVK (2011–until now)

Appreciation

- Prize of the Josef Hlávka foundation in the year 1999

1 Introduction

This is a shorten version of the author's habilitation thesis. It is based on the research during the years 2002–2011. The main part of the text is devoted to the homogenization of partial differential equations with respect to uncertain input data. A summary of results obtained in the field of discrete fractional calculus and dynamic difference equations of fractional orders is presented as well.

Mathematical modelling of problems set in a highly heterogeneous medium brings some difficulties, especially from the numerical viewpoint (to capture the structure we need very fine meshes and thus the resulting number of equations can exceed the computational capabilities). One of the useful mathematical methods designed to overcome this shortcoming is the homogenization theory (for introduction we refer, e.g. to [17]) following the idea of replacement of a heterogeneous environment by a homogeneous one having the same properties on the macroscopic level. Although the homogenization provides quite a powerful tool, its practical use is restricted to the case of periodic structures. Unfortunately, in the real world, we can meet rather almost periodic and sometimes even completely stochastic structures so that some uncertainty in the spatial distribution should be taken into account. Another aspect of treating with real environments is an uncertainty in the sense of knowledge of the physical/material constants such as the modulus of elasticity, heat conductivity, etc. These parameters are usually obtained by measurements and consequently by (numerical) solving of an inverse (identification) problem. Both of these steps exhibit errors, therefore the physical constants vary with some extent around the nominal values. The overall error can be superimposed in the case of a highly heterogeneous medium.

The work focuses on mathematical modelling of problems with heterogeneous structures using the homogenization of partial differential equations. Contrary to the usual approach, we assume input data (namely the coefficients of the studied equations) to be uncertain in some sense.

We adopt a deterministic approach to the problem with uncertainties, the so-called worst scenario method (or method of reliable solution) introduced by Hlaváček (for a comprehensive guide see [38]). As the name of the method suggests, the main idea is to locate the input data that are critical from a certain point of view. In other words, the method searches for dangerous states. As a criterion evaluating which data are "bad" or "good", respectively, a suitable functional has to be chosen.

The presented results are based on the author's papers [55], [56], [57], [58], [59], [60] and on the collaborative papers [20], [21], [23] and [31]. The text is organised as follows. Basic concepts used in the main parts of the text are introduced in Section 2. The linear problems are discussed in Section 3. Several worst scenarios focused on finding some critical ranges of effective (homogenized) properties of the studied heterogeneous medium are introduced and analysed here. Section 4 deals with a nonlinear analogue of the problem introduced in Section 3. This nonlinearity is assumed to be of the strongly monotone type. Both the uncertainties in the values of the coefficients as well as some shape uncertainties of the heterogeneous structure are investigated. Some functional analysis tools designed to simplify the key steps in the homogenization procedure are discussed in Section 5. Additional comments and further research perspectives are presented in Section 6. Finally, Section 7 motivates the study of differential equations of fractional orders and presents our results concerning discretised analogues of these equations.

Let us start with some preliminaries. The symbol $(\cdot,\cdot)$ stands for the scalar product in $\mathbb{R}^d$, $d = 2, 3, \ldots$, and $|\cdot| = \sqrt{(\cdot,\cdot)}$ is the Euclidean norm. The row and column vectors will be not distinguished. By Ω we denote a domain in $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$ and

ν being its unit outward normal vector. The closure of Ω is denoted by $\overline{\Omega}$. If $A \in \mathbb{R}^{d\times d}$ is a matrix, then A^T and A_j denotes its transpose and jth row, respectively (if another subscript is required, we shall write, e.g. $A_{n,j}$ for the jth row of the matrix A_n). A matrix $A \in \mathbb{R}^{d\times d}$ is called symmetric if $A = A^T$ and the space of all symmetric matrices is denoted by $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ (with dimension $d(d+1)/2$).

The symbols $C(\Omega)$, $C(\overline{\Omega})$, $C^k(\overline{\Omega})$, $k = 1, 2, \dots$, and $C_0^\infty(\Omega)$ stand for the space of functions continuous on Ω, the space of functions from $C(\Omega)$ continuously extendable on $\overline{\Omega}$, the space of functions from $C(\overline{\Omega})$ having the partial derivatives up to order k in $C(\overline{\Omega})$ and the space of infinitely differentiable (smooth) functions with compact support in Ω, respectively. The Lebesgue spaces $L^p(\Omega)$, $1 \le p \le \infty$, of integrable functions on Ω are equipped with the standard norms. We shall employ also its vector valued analogues for $p = 2, \infty$ with the norms

$$\|u\|_{L^2(\Omega;\mathbb{R}^d)} = \left(\int_\Omega |u|^2 \,\mathrm{d}x\right)^{1/2} = \left(\sum_{i=1}^d \|u_i\|^2_{L^2(\Omega)}\right)^{1/2}, \quad \|u\|_{L^\infty(\Omega;\mathbb{R}^d)} = \operatorname*{ess\,sup}_{x\in\Omega} \max_{i\in\{1,\dots,d\}} |u_i(x)|.$$

Dual spaces to $L^p(\Omega)$, $1 \le p < \infty$, are represented by $L^{p'}(\Omega)$, where $p' = p/(p-1)$ is the dual exponent. The Lebesgue measure of a measurable set $S \subset \mathbb{R}^d$ is denoted by $\operatorname{meas}_d S$ and the integral mean value $(\operatorname{meas}_d S)^{-1} \int_S f(x)\,\mathrm{d}x$ of a function $f \in L^1(S)$ is denoted by $\mathfrak{M}_S f$. The Sobolev space $H^1(\Omega)$ of functions with integrable (distributive) derivatives is equipped with the norm $\|u\|_{H^1(\Omega)} = \big(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega;\mathbb{R}^d)}\big)^{1/2}$ and its subspace $H^1_0(\Omega)$ of functions with zero trace is equipped with the norm $\|u\|_{H^1_0(\Omega)} = \|\nabla u\|_{L^2(\Omega;\mathbb{R}^d)}$ which is possible due to the Poincaré inequality (see, e.g. [17, Prop. 3.35]). The dual space to $H^1_0(\Omega)$ is denoted by $H^{-1}(\Omega)$. For any $F \in H^{-1}(\Omega)$ there exist $d+1$ functions $f_0, f_1, \dots, f_d$ in $L^2(\Omega)$ such that, formally,

$$F = f_0 + \sum_{i=1}^d \frac{\partial f_i}{\partial x_i} \tag{1.1}$$

and $\|F\|^2_{H^{-1}(\Omega)} = \inf \sum_{i=0}^d \|f_i\|^2_{L^2(\Omega)}$, where the infimum is taken over all $(d+1)$-tuples $f_0, f_1, \dots,$ f_d such that (1.1) holds.

Let $Y = (0, \overline{y}_1) \times (0, \overline{y}_2) \times \dots \times (0, \overline{y}_d)$, where $\overline{y}_1, \dots, \overline{y}_d$ are given positive numbers, be the so-called reference cell. Then a function $f(y)$ defined a.e. on $\mathbb{R}^d$ is called Y-periodic iff $f(y+k\overline{y}_i e_i) = f(y)$ a.e. on $\mathbb{R}^d$, $\forall k \in \mathbb{Z}$, $i = 1, \dots, d$, where e_i is the unit orthogonal basis vector of $\mathbb{R}^d$. If the function f has another variable (say x), it is called Y-periodic in y. Function spaces of Y-periodic functions are denoted by the subscript $\#$. A function v is in $X_\#(Y)$ iff v is Y-periodic and $v \in X(Q)$ for each compact subset $Q \subset \mathbb{R}^d$. The space $X_\#(Y)$ can be "smaller" than the space $X(Y)$ of functions extended by Y-periodicity, e.g. while $L^p_\#(Y)$ can be identified with $L^p(Y)$, $C_\#(Y)$ is a closed subspace of $C(\overline{Y})$, since the elements of $C_\#(Y)$ have the same values on the opposite faces of Y. Similarly, by $H^1_\#(Y)$ we denote the space of Y-periodic functions from $H^1(Y)$ having the same traces on the opposite faces of Y and, in addition, having the zero integral mean value over Y. The norm on $H^1_\#(Y)$ is introduced as $\|v\|_{H^1_\#(Y)} = \|\nabla v\|_{L^2(Y;\mathbb{R}^d)}$ which is possible due to the Poincaré-Wirtinger inequality (see, e.g. [17, Prop. 3.38]).

We shall also use spaces of abstract functions $v : \mathcal{O} \to X$, where $\mathcal{O}$ is either Ω or Y and X is a function space. In particular, the norm of $L^p(\Omega; C_\#(Y))$, $1 \le p < \infty$, is introduced as $\|v\|_{L^p(\Omega,C_\#(Y))} = \big(\int_\Omega \big(\sup_{y\in Y} |v(x,y)|\big)^p \mathrm{d}x\big)^{1/p}$.

If X is a Banach space and X' its dual, then the duality pairing is denoted by $\langle\cdot,\cdot\rangle_{X',X}$. The convergences are denoted as usual, " $\to$ " means the strong convergence (in norm), " $\rightharpoonup$ " the weak convergence, " $\overset{*}{\rightharpoonup}$ " the $*$-weak convergence and finally, the uniform convergence is denoted by "$\rightrightarrows$". The notation $\varepsilon \to 0+$ stands for a sequence of small positive parameters

ε_n converging to zero as $n \to \infty$ (the same holds for the sequences $\{h_n\}$ and $\{k_n\}$ used as discretization parameters in Section 3).

2 Basic principles

2.1 Homogenization

A typical model problem serving for demonstrating the homogenization procedure is a boundary value problem for linear second order elliptic partial differential equation. This type of problem models many physical phenomena, e.g. the stationary heat conduction, diffusion, electrical circuit, etc. For simplicity, let us consider the homogeneous Dirichlet boundary value problem

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) = f \quad \text{in } \Omega\,, \\ u = 0 \quad \text{on } \partial\Omega\,. \end{cases} \tag{2.1}$$

Suppose that a heterogeneous structure occupies the domain Ω such that the heterogeneities are very small compared to the size of Ω and are evenly distributed. For an easier mathematical description, we can consider this “nice” distribution to be periodic. A typical example of such heterogenous structure is a composite material composed of two (or more) finely mixed components. In general, the composite materials have “better” properties compared to the properties of particular components and are nowdays widely used in many branches of industry. Obviously, the material properties (such as the heat conductivity, modulus of elasticity, etc.) oscillate between two (or more) different values, so that the matrix of coefficients $A(x)$ is discontinuous. The discontinuities cause that the model is difficult to treat, especially from the numerical point of view – a discretization of fine discontinuous structure is very extensive resulting into very large numbers of equations. At this place it is also worth mentioning that a simple averaging of properties of the particular components does not provide a good description of the composite's macroscopic behaviour.

The periodic description is stored in the matrix of coefficients $A(x)$ and can be characterized by a positive real parameter ε, i.e. we have $A_\varepsilon(x)$. The smaller ε means the finer structure. This suggests to investigate the behaviour of material in a sequence when $\varepsilon \to 0+$, i.e. when the period is diminishing, see Figure 1 in the full version of the thesis. Mathematically, the homogenization of (2.1) means the analysis of a sequence of problems of the type (2.1) for $\varepsilon \to 0+$ (one term of this sequence is supposed to be the original one). Of course, there are some natural questions: is there any u_0 to which u_ε converge? If so, what is the limit problem of which u_0 is the solution? How well does u_0 approximate u_ε? The aim of the mathematical homogenization theory is to answer these questions.

Taking the reference period Y and assuming that the matrix $A(y)$ is Y-periodic, the matrix $A_\varepsilon(x)$ can be defined in a natural way by $A_\varepsilon(x) = A(y)|_{y=x/\varepsilon}$, see Figure 2 in the full version of the thesis. Replacing $A(x)$ in (2.1) by $A_\varepsilon(x)$, $\varepsilon \to 0+$, we obtain the sequence of problems

$$\begin{cases} \mathcal{A}_\varepsilon u_\varepsilon := -\mathrm{div}(A_\varepsilon(x)\nabla u_\varepsilon) = f \quad \text{in } \Omega\,, \\ u_\varepsilon = 0 \quad \text{on } \partial\Omega\,. \end{cases} \tag{2.2}$$

Since $A_\varepsilon(x)$ is discontinuous, (2.2) can not be taken in the classical sense, we proceed to the weak formulation

$$\begin{cases} \text{Find } u_\varepsilon \in H_0^1(\Omega) \text{ such that} \\ \int_\Omega (A_\varepsilon(x)\nabla u_\varepsilon, \nabla v)\,\mathrm{d}x = \langle f, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,, \quad \forall v \in H_0^1(\Omega)\,. \end{cases} \tag{2.3}$$

The solvability of the problem is based on the Lax-Milgram lemma (see, e.g. [17, Thm. 4.6]). To fulfill all its assumptions, for any $0 < \alpha < \beta$ and any open set $\mathcal{O} \subset \mathbb{R}^d$, we introduce the set $\mathcal{M}(\alpha, \beta, \mathcal{O})$ of all matrix valued functions $M = (m_{ij})_{i,j=1}^{d} \in L^\infty(\mathcal{O}; \mathbb{R}^{d\times d})$ satisfying

$$(M(x)\xi, \xi) \geq \alpha|\xi|^2 \,, \tag{2.4}$$

$$|M(x)\xi| \leq \beta|\xi| \,, \tag{2.5}$$

$\forall \xi \in \mathbb{R}^d$ and a.e. on $\mathcal{O}$. Assuming $A \in \mathcal{M}(\alpha, \beta, Y)$ is Y-periodic, we also have $A_\varepsilon \in \mathcal{M}(\alpha, \beta, \Omega)$ and therefore we can state:

Theorem 2.1. *Let $A \in \mathcal{M}(\alpha, \beta, Y)$ be Y-periodic and $f \in H^{-1}(\Omega)$. Then there exists a unique solution $u_\varepsilon \in H_0^1(\Omega)$ of (2.3). Moreover, we have the estimate*

$$\|u_\varepsilon\|_{H_0^1(\Omega)} \leq \frac{1}{\alpha}\|f\|_{H^{-1}(\Omega)} \,. \tag{2.6}$$

A complete proof can be found, e.g. in [17, Thm. 4.16].

We wish to pass to the limit in (2.3). Since the sequence of solutions $\{u_\varepsilon\}$, $\varepsilon \to 0+$, is bounded in $H_0^1(\Omega)$, there exists an element $u_0 \in H_0^1(\Omega)$ such that, up to a subsequence, $u_{\varepsilon'} \rightharpoonup u_0$ in $H_0^1(\Omega)$ as $\varepsilon' \to 0+$. On the other hand, denoting $\zeta_\varepsilon = A_\varepsilon \nabla u_\varepsilon$, it is easy to check that $\|\zeta_\varepsilon\|_{L^2(\Omega;\mathbb{R}^d)} \leq \alpha^{-1}\beta\|f\|_{H^{-1}(\Omega)}$ due to the Cauchy-Schwarz inequality, (2.5) and (2.6). It means that there exists a subsequence (still denoted by $\{\zeta_{\varepsilon'}\}$) such that $\zeta_{\varepsilon'} \rightharpoonup \zeta_0$ in $L^2(\Omega; \mathbb{R}^d)$ as $\varepsilon' \to 0+$. This implies the limit of (2.3) in the form $\int_\Omega (\zeta_0, v)\,\mathrm{d}x = \langle f, v\rangle_{H^{-1}(\Omega), H^1(\Omega)}$. The crucial step is to identify ζ_0 in terms of ∇u_0. The main difficulty is caused by the fact that ζ_ε contains the product of two weakly convergent sequences which, in general, does not converge (weakly) to the product of their limits. It is known that for a sequence of functions $g_\varepsilon(x) = g(y)|_{y=x/\varepsilon}$, $\varepsilon \to 0+$, where $g \in L^p_\#(Y)$, $1 \leq p \leq \infty$, it holds

$$g_\varepsilon \rightharpoonup \mathfrak{M}_Y g \quad \text{in } L^p(\Omega) \,, \quad 1 \leq p < \infty \,, \quad \text{and} \quad g_\varepsilon \overset{*}{\rightharpoonup} \mathfrak{M}_Y g \quad \text{in } L^\infty(\Omega) \,. \tag{2.7}$$

Hence, we have $\zeta_0 \neq A_0 \nabla u_0$, where $A_0 = \mathfrak{M}_Y A$. Similarly, (2.7) holds if we take $g_\varepsilon(x) = g(x, y)|_{y=x/\varepsilon}$ (assuming g is Y-periodic in y and regular enough).

To overcome the mentioned difficulty, several concepts have been introduced so far. First, the form of the homogenized problem can be derived (in a rather heuristic way) using the asymptotic expansion method, where the solution $u_\varepsilon(x)$ is assumed to possess the two-scale expansion (one variable is "slow" representing the global behaviour while the second variable is "fast" representing the local behaviour)

$$u_\varepsilon(x) = u_0(x, y) + \varepsilon u_1(x, y) + \varepsilon^2 u_2(x, y) + \ldots |_{y=x/\varepsilon} \,. \tag{2.8}$$

Substituting (2.8) into (2.2) and equating the terms with the same powers of ε, we obtain an infinite system of equations which enables to compute consecutively the functions u_0, u_1, etc., for details see, e.g. [17, Chap. 7]. Here we only note that the first three equations of the mentioned system determine the form of the homogenized problem with $u_0(x, y) = u_0(x)$ being the homogenized solution.

The classical method due to Tartar (called the energy method) is based on a special choice of the test functions designed so that passing to the limit in (2.3) is possible (we remind that (2.3) contains the product of two weakly convergent sequences), see, e.g. [17, Chap. 8].

Study of problems of the type (2.3) with (in general) a non-periodic symmetric matrix A led to the notion of G-convergence introduced by Spagnolo. It was later extended by Tartar and Murat also for the non-symmetric case under the notion of H-convergence, see [51].

For the case of variational formulations of the mentioned problems, the Γ-convergence of functionals was proposed as a related tool, see, e.g. [24].

Finally, the most powerful tool in the periodic homogenization, called the two-scale convergence, was introduced at the end of 80's by Nguetseng and later popularized by Allaire, see [61], [4]. Only recently, this concept was extended under the notion of Σ-convergence to non-periodic cases, see [62].

To summarise the above considerations, we can state:

Theorem 2.2. *Let $A \in \mathcal{M}(\alpha, \beta, Y)$ be Y-periodic, u_ε be the solution of (2.3) with $A_\varepsilon(x) = A(x/\varepsilon)$, $\varepsilon \to 0+$, and $f \in H^{-1}(\Omega)$. Then $u_\varepsilon \rightharpoonup u_0$ in $H_0^1(\Omega)$ and $A_\varepsilon \nabla u_\varepsilon \rightharpoonup B \nabla u_0$ in $L^2(\Omega; \mathbb{R}^d)$, where u_0 is the unique solution of the so-called homogenized problem:*

$$\begin{cases} \text{Find } u_0 \in H_0^1(\Omega) \text{ such that} \\ \int_\Omega (B\nabla u_0, \nabla v)\,\mathrm{d}x = \langle f, v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}\,, \quad \forall v \in H_0^1(\Omega)\,. \end{cases} \tag{2.9}$$

The constant matrix $B = (b_{ij})_{i,j=1}^d \in \mathbb{R}^{d\times d}$ is given by $b_{ij} = \mathfrak{M}_Y\big(a_{ij}(y) - (A_i(y), \nabla w_j(y))\big)$ and the auxiliary function $w = (w_1, \ldots, w_d)$ is the unique solution of the local problem

$$\begin{cases} \text{Find } w_j \in H_\#^1(Y),\ j = 1, \ldots, d\,, \text{ such that} \\ \int_Y (A(y)\nabla w_j, \nabla \phi)\,\mathrm{d}y = \int_Y (A_j^T(y), \nabla\phi)\,\mathrm{d}y\,, \quad \forall \phi \in H_\#^1(Y)\,. \end{cases} \tag{2.10}$$

Moreover, $B \in \mathcal{M}(\alpha, \beta^2/\alpha, \Omega)$ (in the symmetric case $A = A^T$ even $B \in \mathcal{M}(\alpha, \beta, \Omega)$).

Remark 2.3. By the weak convergence $u_\varepsilon \rightharpoonup u_0$ in $H_0^1(\Omega)$, we have also $u_\varepsilon \to u_0$ in $L^2(\Omega)$ due to the Rellich theorem on compact embedding. However, as for the gradient, we have $\nabla u_\varepsilon \rightharpoonup \nabla u_0$ in $L^2(\Omega; \mathbb{R}^d)$ only – the gradient ∇u_ε is discontinuous on the interfaces of particular phases of the periodic heterogeneous structure, i.e. it contains a periodic component. Hence, we can expect the behaviour in the sense of convergence (2.7).

From now on, the reference cell is taken (without a loss of generality) as the unit cube, i.e. $Y = (0, 1)^d$.

2.2 Worst scenario method

Worst scenario method is a deterministic method designed to solve problems containing uncertain inputs such as the coefficients in the equation, right-hand side, functions from boundary conditions, geometry of the domain, etc. The key step is ability to determine a set of the so-called admissible data on which a suitable functional can be defined. This functional evaluates a state (physical quantity) of the model problem from certain point of view, hence it serves as criterion saying which data are "bad" or "good", respectively. Then the worst scenario is obtained by looking for maxima of the functional. Although this approach can be sometimes too pessimistic (especially in the cases, when the probability of occurrence of the "bad" data is small), it does not require any probabilistic information on the data distribution.

This subsection recalls the results of [38, Chap. II] concerning the general abstract scheme of the method.

Let us consider a state problem $\mathcal{P}(A; u)$, where A denotes input data and u is the state variable. Let U^{ad} denote a given set of admissible data and assume $A \in U^{ad} \subset U$, where U is a Banach space and $u \in W$, where W is a reflexive Banach space. Further, assume that

- (A1) a unique solution $u(A)$ of $\mathcal{P}(A; u)$ exists for any $A \in \widetilde{U}^{ad}$, where $U^{ad} \subset \widetilde{U}^{ad} \subset U$,
- (A2) the sets U^{ad} and $\widetilde{U}^{ad}$ are compact in U,
- (A3) if $A_n \in U^{ad}$ and $A_n \to A$ in U as $n \to \infty$, then $u(A_n) \rightharpoonup u(A)$ in W,
- (A4) a criterion functional $\Phi : \widetilde{U}^{ad} \times W \to \mathbb{R}$ is given such that: if $A_n \in \widetilde{U}^{ad}$, $A_n \to A$ in U and $v_n \rightharpoonup v$ in W as $n \to \infty$, then $\limsup_{n\to\infty} \Phi(A_n, v_n) \le \Phi(A, v)$.

The goal is to solve the worst scenario maximization problem

$$\begin{cases} \text{Find } A^\blacktriangle \in U^{ad} \text{ such that for all } A \in U^{ad} \text{ we have} \\ \Phi(A, u(A)) \le \Phi(A^\blacktriangle, u(A^\blacktriangle))\,. \end{cases} \tag{2.11}$$

Theorem 2.4. *Let (A1)–(A4) be fulfilled. Then (2.11) has at least one solution.*

For the proof of the theorem see [38, Thm. 3.1].

Remark 2.5. The weak convergences $u(A_n) \rightharpoonup u(A)$ in (A3) and $v_n \rightharpoonup v$ in (A4), respectively, can be replaced by the strong convergences.

A modification for the case of nonunique solution can be found in [38, Remark 3.1 and Thm. 3.2].

Remark 2.6. The next step (as it is usual) consists in dealing with a finite dimensional approximation of the above introduced worst scenario problem due to the numerical reasons. Compared to the classical theory with exactly given input data, a discretization of problems with uncertain inputs is a more difficult task, because usually both the set of admissible data and the state space are required to be discretised. The introduction of approximate worst scenario method including the corresponding convergence analysis to the infinite dimensional problem is covered in [38, Sect. 3.2 and 3.3]. Here we only note that the original paper [37] devoted to the worst scenario method dealt with a model problem for the quasi-linear equation. This kind of problem exhibits a well-known phenomenon, where the uniqueness of a solution is guaranteed while the uniqueness of an approximate solution is not. This fact is taken into account in the mentioned paragraphs.

3 Homogenization of linear elliptic problems with uncertain input coefficients

This section summarises the results published in [57]. A suitable initial value problem for the linear elliptic problem set in a highly heterogeneous (periodic) medium is studied by means of the homogenization method. The matrix of the coefficients in the equation is considered to be uncertain in some sense. Several worst scenario problems related to finding the critical values of the homogenized coefficients are investigated. The analysis is accompanied by a few numerical experiments.

3.1 Model problem

For convenience, we consider a mixed initial boundary value problem with Dirichlet and Neumann condition in the form

$$\begin{cases} -\operatorname{div}(A_\varepsilon(x)\nabla u_\varepsilon) = f \quad \text{in } \Omega\,, \\ u_\varepsilon = 0 \quad \text{on } \Gamma_D\,, \\ (A_\varepsilon(x)\nabla u_\varepsilon, \nu) = g \quad \text{on } \Gamma_N\,, \end{cases} \tag{3.1}$$

where $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ ($\text{meas}_{d-1}\Gamma_D > 0$), $A_\varepsilon(x) = A(y)|_{y=x/\varepsilon}$ and $A(y)$ is a Y-periodic matrix function.

The reference cell Y is supposed to consist of a finite number of disjoint subdomains $Y_k \subset Y$, $k = 1, \dots, m$ and their complement Y_0 in Y. The uncertain matrix function $A(y)$ is considered to be constant on each of these subdomains with the values known in given intervals only and such that it is symmetric and positive definite, i.e. $A = A^T$, $(A(y)\xi, \xi) > 0$ $\forall \xi \neq 0$ and a.e. on Y. More precisely, let us introduce the set of admissible data U^{ad} as

$$U^{ad} = \{A \in L^\infty_\#(Y; \mathbb{R}^{d\times d}_{\text{sym}}) : A \in \mathbb{R}^{d\times d}_{\text{sym}} \text{ on } Y_k\,, A|_{Y_k} \in [C^\ell_k, C^u_k],\ k = 0, \dots, m\}\,,$$

where $C^\ell_k = (c^\ell_{ij,k})^d_{i,j=1} \in \mathbb{R}^{d\times d}_{\text{sym}}$ and $C^u_k = (c^u_{ij,k})^d_{i,j=1} \in \mathbb{R}^{d\times d}_{\text{sym}}$ are given matrices of lower and upper bounds such that the positive definiteness of the quadratic form $(A\xi, \xi)$ is not violated for any $A \in U^{ad}$.

Remark 3.1. The positive definiteness of the form $(A\xi, \xi)$ is equivalent with the ellipticity condition, i.e. there exists $\alpha > 0$ such that $(A\xi,\xi) > 0 \Leftrightarrow (A\xi,\xi) \geq \alpha|\xi|^2$, $\forall \xi \neq 0$. In the case of symmetric interval matrices, a sufficient condition for positive definiteness of the form $(A\xi,\xi)$ was introduced by Rohn, see [66]. In our notation, it can be formulated: a quadratic form $(A\xi,\xi)$, $A \in U^{ad}$, is positive definite if $\lambda_{\min}\left(2^{-1}(C_k^\ell + C_k^u)\right) - \rho\left(2^{-1}(C_k^u - C_k^\ell)\right) > 0$, $k = 0, 1, \ldots, m$, where $\lambda_{\min}(M)$ and $\rho(M)$ are the minimal eigenvalue and the spectral radius of the matrix M.

Finally, directly from the construction of U^{ad}, it follows that there exist constants α_{ad}, β_{ad} such that

$$A \in \mathcal{M}(\alpha_{ad}, \beta_{ad}, Y)\,, \quad \forall A \in U^{ad}\,. \tag{3.2}$$

Remark 3.2. In the case of an anisotropic medium in principal direction, the matrix A is diagonal and in the case of isotropic medium the diagonal elements coincide.

Introducing the subspace V of $H^1(\Omega)$ as $V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$ (in the sense of traces) with the norm $\|v\|_V = \|\nabla v\|_{L^2(\Omega;\mathbb{R}^d)}$ (this is possible due to the Poincaré-Friedrichs inequality, see, e.g. [17, Prop. 3.36]), the weak formulation of (3.1) reads:

$$\begin{cases} \text{Find } u_\varepsilon^A \in V \text{ such that} \\ \int_\Omega (A_\varepsilon(x)\nabla u_\varepsilon^A, \nabla v)\,\mathrm{d}x = \int_\Omega f v\,\mathrm{d}x + \int_{\Gamma_N} g v\,\mathrm{d}s\,, \quad \forall v \in V\,. \end{cases} \tag{3.3}$$

The solvability of (3.3) is again a consequence of Lax-Milgram lemma. More precisely, we have

Theorem 3.3. *Let $A \in U^{ad}$, $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_N)$. Then there exists a unique solution of (3.3) satisfying $\|u_\varepsilon^A\|_V \leq C$, where the constant C does not depend on ε and A.*

3.2 Homogenized linear elliptic operator

Following the ideas presented in Subsection 2.1, it can be guessed that the boundary conditions do not affect the homogenized operator. Hence, the homogenized (limit) problem to (3.3) reads:

$$\begin{cases} \text{Find } u^A \in V \text{ such that} \\ \int_\Omega (B_A \nabla u^A, \nabla v)\,\mathrm{d}x = \int_\Omega f v\,\mathrm{d}x + \int_{\Gamma_N} g v\,\mathrm{d}s\,, \quad \forall v \in V\,, \end{cases} \tag{3.4}$$

where the constant matrix $B_A = (b_{ij}^A)_{i,j=1}^d \in \mathbb{R}^{d\times d}$ is given by

$$b_{ij}^A = \mathfrak{M}_Y(a_{ij} - (A_i, \nabla w_j^A)) \tag{3.5}$$

and $w^A = (w_1^A, \ldots, w_d^A)$ is the solution of the local problem

$$\begin{cases} \text{Find } w_j^A \in H^1_\#(Y),\ j = 1, \ldots, d\,, \text{ such that} \\ \int_Y (A\nabla w_j^A, \nabla\phi)\,\mathrm{d}y = \int_Y (A_j^T, \nabla\phi)\,\mathrm{d}y\,, \quad \forall \phi \in H^1_\#(Y)\,. \end{cases} \tag{3.6}$$

Remark 3.4. Since the original matrix $A \in \mathcal{M}(\alpha_{ad}, \beta_{ad}, Y)$ is assumed to be symmetric, we also have $B_A \in \mathcal{M}(\alpha_{ad}, \beta_{ad}, Y)$. This and the assumptions on f and g guarantee existence of a unique solution u^A due to the Lax-Milgram lemma. Moreover, B_A is also symmetric, for details see, e.g. [25].

If A is diagonal with elements that are even functions with respect to the planes of symmetry $y_j = 1/2$, $j = 1, \ldots, d$, then B_A is also diagonal. It is a direct consequence of (3.5) taking into account that the solution w_j^A of (3.6) is an even function with respect to the plane $y_j = 1/2$ and it is an odd function with respect to the planes $y_i = 1/2$, $i = 1, \ldots, j-1, j+1, \ldots, d$.

Correctors

As we have observed in the previous sections, the homogenized solution u^A approximates the original "heterogeneous" solution u_ε^A in the sense of weak convergence $u_\varepsilon^A \rightharpoonup u^A$ in $H_0^1(\Omega)$. It implies (due to the compact embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ stated by Rellich theorem) that

$u_\varepsilon^A \to u^A$ in $L^2(\Omega)$ and $\nabla u_\varepsilon^A \rightharpoonup \nabla u^A$ in $L^2(\Omega;\mathbb{R}^d)$. In order to improve the approximation of the gradient, let us define the function

$$ {}_C u_\varepsilon^A(x) := u^A(x) - \varepsilon(w^A(x/\varepsilon), \nabla u^A(x))\,, \tag{3.7}$$

where u^A is the solution of (3.4) and w^A is the Y-periodic extension of the solution to (3.6). This function is called the homogenized solution with corrector.

Theorem 3.5. *Let u^A solve (3.4) such that $u^A \in C^2(\overline{\Omega})$. Then $\|u_\varepsilon^A - {}_C u_\varepsilon^A\|_{H^1(\Omega)} \to 0$ as $\varepsilon \to 0+$.*

The proof can be found in [29, p. 38].

Remark 3.6. The homogenized solution with corrector improves approximation of the gradient ∇u_ε^A, however, it violates the Dirichlet boundary condition, i.e. ${}_C u_\varepsilon^A \neq 0$ on Γ_D. It can be "repaired" if we multiply the corrector by the so-called cut-off function $m_\varepsilon : \overline{\Omega} \to [0;1]$, for details see, e.g. [29, p. 30].

3.3 Choice of the criterion functionals

In this subsection we introduce a few criteria and prove existence of a solution to the corresponding worst scenario problems.

Homogenized coefficients

The first problem deals with the range of the homogenized coefficients. A natural question is whether the extremal values of the original discontinuous matrix A yield also the extremal values of the homogenized coefficients. Therefore, in view of (3.5), we define the functionals $\Phi_{ij} : U^{ad} \times H^1_\#(Y) \to \mathbb{R}$ as

$$\Phi_{ij}(A,\phi) = \mathfrak{M}_Y(a_{ij} - (A_i, \nabla\phi))\,, \quad A \in U^{ad},\ \phi \in H^1_\#(Y)\,, \quad i,j = 1,\dots,d\,. \tag{3.8}$$

Let us set $J_{ij}(A) = \Phi_{ij}(A, w_j^A)$, where w_j^A is the unique solution of (3.6), and consider the following worst scenario problems:

$$\begin{cases} \text{Find } A^\blacktriangle \in U^{ad} \text{ and } A^\blacktriangledown \in U^{ad} \text{ such that} \\ J_{ij}(A) \le J_{ij}(A^\blacktriangle)\,, \quad \forall A \in U^{ad}, \quad i,j = 1,\dots,d \quad \text{and} \\ J_{ij}(A^\blacktriangledown) \le J_{ij}(A)\,, \quad \forall A \in U^{ad}, \quad i,j = 1,\dots,d\,, \text{ respectively.} \end{cases} \tag{3.9}$$

The solvability of (3.9) is based on the following compactness and continuity properties (the proofs can be found in the full version of the thesis).

Lemma 3.7. *Every sequence $\{A_n\} \subset U^{ad}$, $n \to \infty$, contains a subsequence converging to an element $A \in U^{ad}$ in $L^\infty_\#(Y;\mathbb{R}^{d\times d}_{\mathrm{sym}})$.*

Lemma 3.8. *Let $A_n, A \in U^{ad}$ such that $A_n \to A$ in $L^\infty_\#(Y;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ as $n \to \infty$. Then $w_j^{A_n} \to w_j^A$ in $H^1_\#(Y)$, $j = 1,\dots,d$.*

These two properties are crucial in the proof of the following assertion, see the full version of the thesis.

Theorem 3.9. *Problem (3.9) has at least one solution.*

Generalized gradient

The second problem focuses on the auxiliary function w^A. As we have observed, this function plays an essential role in the solution with corrector ${}_C u_\varepsilon^A$ which (under some assumptions) improves the approximation of the homogenized solution in the sense of strong convergence $\|u_\varepsilon^A - {}_C u_\varepsilon^A\|_{H^1(\Omega)} \to 0$. In other words, ${}_C u_\varepsilon^A$ approximates both the values of u_ε^A as well as the gradient ∇u_ε^A. In technical applications, the so-called generalized gradient of the solution plays an important role (it can represent, e.g. the heat flow vector). Since A_ε is bounded in $L^\infty(\Omega)$, we have also the convergence $\|A_\varepsilon \nabla u_\varepsilon^A - A_\varepsilon\, {}_C\nabla u_\varepsilon^A\|_{L^2(\Omega;\mathbb{R}^d)} \to 0$ as $\varepsilon \to 0+$ and thus, for ε small enough, the term $A_\varepsilon\, {}_C\nabla u_\varepsilon^A$ represents a reasonable approximation of the generalized gradient $A_\varepsilon \nabla u_\varepsilon$. By (3.7) we have

$$
\begin{aligned}
&A_\varepsilon(x)\, {}_C\nabla u_\varepsilon^A(x) = A(x/\varepsilon)\nabla \left(u^A(x) - \varepsilon(w^A(x/\varepsilon), \nabla u^A(x))\right) \\
=&A(x/\varepsilon)\left(\nabla u^A(x) - \varepsilon(\varepsilon^{-1}\nabla_y w^A(x/\varepsilon), \nabla u^A(x)) - \varepsilon(w^A(x/\varepsilon), \nabla\nabla u^A(x))\right) .
\end{aligned}
$$

Neglecting the term $\varepsilon(w^A(x/\varepsilon), \nabla\nabla u^A(x))$, we introduce the vector $t^A(x,y) = A(y)(\nabla u^A(x) - (\nabla_y w^A(y), \nabla u^A(x))$. Let us eliminate the influence of the global function ∇u^A by the constraint condition $|\nabla u^A| = 1$. Since we are interested in finding maximal (critical) values of the generalized gradient, we define the criterion functional as

$$
J(A) = (\mathrm{meas}_d \widetilde{Y})^{-1} \left(\int_{\widetilde{Y}} |\hat{t}^A(y)|^2 \mathrm{d}y \right)^{1/2} , \tag{3.10}
$$

where $\hat{t}^A$ is taken as $\hat{t}^A = \max_{|\nabla u^A| \le 1} t^A$ and $\widetilde{Y}$ is a suitable subset of the reference cell Y (typically at places of "sharp changes" of the composite components, where the "peaks" of derivatives are high). The ith component of t^A is a linear function in the variable $\xi = \nabla u^A$ of the form (c^i, ξ), where the vector $c^i = (c_1^i, \ldots, c_d^i)$ is given by $c_j^i = (-A_i, \nabla w_j^A - e_j)$ and e_j is the standard unit orthonormal basis vector. By the method of Lagrange multipliers we can observe that this linear function has the maximal value $|c_i|$ on the unit disk $|\xi| \le 1$. Hence, J can be expressed as $J(A) = (\mathrm{meas}_d \widetilde{Y})^{-1} (\int_{\widetilde{Y}} \sum_{i=1}^d \sum_{j=1}^d (-A_i, \nabla w_j^A - e_j)^2 \,\mathrm{d}y)^{1/2}$.

Remark 3.10. Obviously, the "smooth" gradient ∇u^A does not affect the values of the generalized gradient t as strongly as the rapidly oscillating matrix $\nabla w^A(x/\varepsilon)$. Thus, we have eliminated its influence to obtain the microstructure description only (in the variable y). Since we look for the maximal values on the set $|\nabla u^A| \le 1$, we get an upper estimate of t^A for a.a. $x \in \Omega$. The elimination by the constrained condition $|\nabla u^A| = 1$ is carried out for each component of t^A separately. It would be more natural to use this constrained condition for the (squared) length of the gradient $|t^A|^2$. However, it leads to a much more complicated form.

The corresponding worst scenario problem reads:

$$
\begin{cases}
\text{Find } A^\blacktriangle \in U^{ad} \text{ such that} \\
J(A) \le J(A^\blacktriangle) , \quad \forall A \in U^{ad} ,
\end{cases} \tag{3.11}
$$

where $J(A)$ is given by (3.10).

Theorem 3.11. *Problem (3.11) has at least one solution.*

Proof. See the full version of the thesis. □

Homogenized solution

Now, let us define the functional J by the relation

$$
J(A) = (\mathrm{meas}_d \widetilde{\Omega})^{-1} \int_{\widetilde{\Omega}} u^A(x) \,\mathrm{d}x , \tag{3.12}
$$

where $\widetilde{\Omega}$ is a suitably chosen subset of Ω and u^A is the solution of (3.4). It represents the average value of the homogenized solution (e.g. temperature) over $\widetilde{\Omega}$. In other words, we are interested in the impact of the coefficients matrix A on u^A at some (critical) places of material.

The corresponding worst scenario problem reads:

$$\begin{cases}\text{Find } A^{\blacktriangle} \in U^{ad} \text{ such that} \\ J(A) \le J(A^{\blacktriangle}), \quad \forall A \in U^{ad}, \end{cases} \tag{3.13}$$

where J is now given by (3.12).

Lemma 3.12. *Let $A_n \to A$ in $L^\infty_\#(Y;\mathbb{R}^{d\times d}_{\mathrm{sym}})$. Then $B_{A_n} \to B_A$ in $\mathbb{R}^{d\times d}_{\mathrm{sym}}$.*

Theorem 3.13. *Problem (3.13) has at least one solution.*

The proofs of the statements can be found in the full version of the thesis.

3.4 Finite dimensional approximation of the problems

This subsection deals with approximate solutions to the problems discussed above. Let $H^1_{\#,h}(Y)$ be a finite dimensional subspace of $H^1_\#(Y)$ (h is a discretization parameter, e.g. the one from the finite element method). The Galerkin approximation of (3.6) reads:

$$\begin{cases}\text{Find } w^A_{h,j} \in H^1_{\#,h}(Y),\ j = 1,\dots,d, \text{ such that} \\ \int_Y (A\nabla w^A_{h,j}, \nabla\phi_h)\,\mathrm{d}y = \int_Y (A^T_j, \nabla\phi_h)\,\mathrm{d}y, \quad \forall \phi_h \in H^1_{\#,h}(Y). \end{cases} \tag{3.14}$$

Theorem 3.14. *There exists a unique solution of (3.14). Moreover, there exits a sequence of subspaces $\{H^1_{\#,h}(Y)\}$ such that $w^A_{h,j} \to w^A_j$ in $H^1_\#(Y)$, $j = 1,\dots,d$, as $h \to 0+$.*

Denoting $J^h_{ij}(A) = \Phi_{ij}(A, w^A_{h,j})$, where Φ_{ij} are given by (3.8) and w^A_h is the vector of solutions to (3.14), the finite dimensional approximation of (3.9) reads:

$$\begin{cases}\text{Find } A^{\blacktriangle}_h \in U^{ad} \text{ and } A^{\blacktriangledown}_h \in U^{ad} \text{ such that} \\ J^h_{ij}(A) \le J^h_{ij}(A^{\blacktriangle}_h), \quad \forall A \in U^{ad},\ i,j = 1,\dots,d \quad \text{and} \\ J^h_{ij}(A^{\blacktriangledown}_h) \le J^h_{ij}(A), \quad \forall A \in U^{ad},\ i,j = 1,\dots,d, \text{ respectively.} \end{cases} \tag{3.15}$$

Theorem 3.15. *Problem (3.15) has at least one solution.*

Lemma 3.16. *Let $\{A^{\blacktriangle}_h\}$, $h \to 0+$, be a sequence of approximate solutions to (3.15) such that $A^{\blacktriangle}_h \to A$ in $L^\infty_\#(Y;\mathbb{R}^{d\times d}_{\mathrm{sym}})$, let w^A_h and w^A be the vectors of solutions to (3.14) and (3.6), respectively, and let $\{H^1_{\#,h}(Y)\}, h \to 0+$, be a sequence approximating $H^1_\#(Y)$. Then $w_h^{A^{\blacktriangle}_h} \to w^A$ in $H^1_\#(Y;\mathbb{R}^d)$ as $h \to 0+$.*

Theorem 3.17. *Let $\{A^{\blacktriangle}_h\}$ and $\{A^{\blacktriangledown}_h\}$, $h \to 0+$, be sequences of solutions to (3.15), let $A^{\blacktriangle}$ and $A^{\blacktriangledown}$ be solutions of (3.9) and let a sequence of finite-dimensional subspaces $\{H^1_{\#,h}(Y)\}$ approximate $H^1_\#(Y)$. Then there exist extracted subsequences $\{A^{\blacktriangle}_{h'}\}$ and $\{A^{\blacktriangledown}_{h'}\}$ such that $J^{h'}_{ij}(A^{\blacktriangle}_{h'}) \to J_{ij}(A^{\blacktriangle})$ as $h' \to 0+$ and $J^{h'}_{ij}(A^{\blacktriangledown}_{h'}) \to J_{ij}(A^{\blacktriangledown})$ as $h' \to 0+$, respectively.*

Remark 3.18. In general, $\widetilde{A} \neq A^{\blacktriangle}$, since the uniqueness of $A^{\blacktriangle}_h$ and $A^{\blacktriangle}$ is not guaranteed.

Denoting by $J_h(A)$ the functional defined by (3.10) with w^A replaced by w^A_h, the approximation of (3.11) reads:

$$\begin{cases}\text{Find } A^{\blacktriangle}_h \in U^{ad} \text{ such that} \\ J_h(A) \le J_h(A^{\blacktriangle}_h), \quad \forall A \in U^{ad}. \end{cases} \tag{3.16}$$

Theorem 3.19. *Problem (3.16) has at least one solution.*

Theorem 3.20. *Let $\{A_h^\blacktriangle\}$, $h \to 0+$, be a sequence of solutions to (3.16), let $A^\blacktriangle$ be the solution of (3.11) and let a sequence of finite-dimensional subspaces $\{H^1_{\#,h}(Y)\}$ approximate $H^1_\#(Y)$. Then there exists a subsequence $\{A_h^\blacktriangle\}$ such that $J_{h'}(A_{h'}^\blacktriangle) \to J(A^\blacktriangle)$ as $h' \to 0+$, where J is given by (3.10).*

The proof relies on the same steps as the proof of Theorem 3.17.

An approximation of (3.13) requires the introduction of two parameters: h being the discretization parameter of $H^1_\#(Y)$ and k being the discretization parameter of V. Let us denote $J_{h,k}(A) = (\mathrm{meas}_d\,\widetilde{\Omega})^{-1} \int_{\widetilde{\Omega}} u^A_{h,k}(x)\,\mathrm{d}x$. Here, $\widetilde{\Omega}$ has the same meaning as in (3.12) and $u^A_{h,k}$ is the solution of the following discrete problem:

$$\begin{cases} \text{Find } u^A_{h,k} \in V_k \text{ such that} \\ \int_\Omega (B^h_A \nabla u^A_{h,k}, \nabla v_k)\,\mathrm{d}x = \int_\Omega f v_k\,\mathrm{d}x + \int_{\Gamma_N} g v_k\,\mathrm{d}s\,, \quad \forall v_k \in V_k\,, \end{cases} \tag{3.17}$$

where B^h_A is given by (3.5) with w^A replaced by w^A_h being the solution of (3.14) and V_k is a finite dimensional subset of V.

Remark 3.21. It can be proved that $B^h_A \in \mathcal{M}(\alpha_{ad}, \beta_{ad}, Y)$ for all $A \in U^{ad}$ and therefore existence and uniqueness of the solution to (3.17) is guaranteed by the Lax-Milgram lemma.

The finite dimensional approximation of (3.13) reads:

$$\begin{cases} \text{Find } A^\blacktriangle_{h,k} \in U^{ad} \text{ such that} \\ J_{h,k}(A) \le J_{h,k}(A^\blacktriangle_{h,k})\,, \quad \forall A \in U^{ad}\,. \end{cases} \tag{3.18}$$

Theorem 3.22. *Problem (3.18) has at least one solution.*

Theorem 3.23. *Let $\{A^\blacktriangle_{h,k}\}$, $(h,k) \to (0+,0+)$, be a sequence of solutions to (3.18) and let $A^\blacktriangle$ be a solution of (3.13). Moreover, let $\{H^1_{\#,h}(Y)\}$ and $\{V_k\}$ be sequences approximating $H^1_\#(Y)$ and V, respectively. Then there exists a subsequence $\{A^\blacktriangle_{h_n,k_n}\}$ such that $J_{h_n,k_n}(A^\blacktriangle_{h_n,k_n}) \to J(A^\blacktriangle)$ as $n \to \infty$, where J is defined by (3.12).*

Some comments to the proofs of this subsection statements can be found in the full version of the thesis.

3.5 Numerical experiments

In this subsection we show a few 2D examples demonstrating the above considerations. The input parameters are not real, they have an illustrative character only.

Methods of computations

All algorithms were programmed under the MATLAB environment with help of the PDE toolbox and the NAG toolbox routine `E04JAF`.

The solution $w^A_h = (w^A_{h,1}, w^A_{h,2})$ of (3.14) is computed by the finite element method (using the linear triangular elements). The algorithm is slightly modified for the requirements of periodic solutions. The periodic boundary condition involves the values of w^A being almost everywhere the same on the opposite faces of Y. This means that the triangulation nodes correspond on the opposite faces, i.e. they are positioned on the same levels and have the same prescribed values. This correspondence can be ensured by the same numbering of two opposite nodes. The resulting system of linear equations has a linearly dependent row, since the "position" of solution is not fixed, so that we add the condition of zero mean value of w^A into the stiffness matrix.

The homogenized coefficients and the generalized gradient are obtained by means of numerical integration.

The approximate homogenized solution $u^A_{h,k}$ is computed by the finite element method (again with the linear triangular elements). In this case, the MATLAB routine `assempde` included in the PDE toolbox is used, see [64].

Finally, the maximum (or minimum) of the criterion functional is obtained in the following way. Since the matrix of the coefficients A can be represented by $3(m+1)$ values (we remind that m is the number of subsets of the reference cell Y, see Subsection 3.1), finding extremes of the functional reduces to finding extremes of $3(m+1)$ variable function on a compact set. These extremes are obtained by use of the NAG `E04JAF` iterative routine based on a quasi-Newton method which suitably approximates the Hess matrix from the function values, see [52].

Examples
See the full version of the thesis.

4 Homogenization of monotone problems with uncertain coefficients

This section summarises the results published in [31], [58] and [59] (we refer also to a very introductory paper [60] intended for readers that are not familiar with the topic). We shall deal with the homogenization of a nonlinear boundary value problem in the form

$$\begin{cases} \mathcal{A}(u) := -\mathrm{div}(a(x, \nabla u)) = f \quad \text{in } \Omega\,, \\ u = 0 \quad \text{on } \partial\Omega\,. \end{cases} \tag{4.1}$$

This kind of problem represents a nonlinear conservation law. The coefficients of the operator $\mathcal{A}$ describing a periodic structure are considered to be uncertain in the values, known in some bounds only, but still satisfying certain continuity and monotonicity conditions.

The weak formulation of (4.1) considered in a sequence for $\varepsilon \to 0+$ takes the form

$$\begin{cases} \int_\Omega (a_\varepsilon(x, \nabla u^a_\varepsilon), \nabla v)\,\mathrm{d}x = \langle f, v\rangle_{H^{-1}(\Omega), H^1_0(\Omega)}\,, \quad \forall v \in H^1_0(\Omega)\,, \\ u^a_\varepsilon \in H^1_0(\Omega)\,, \end{cases} \tag{4.2}$$

where $a_\varepsilon(x,\xi) := a(y,\xi)|_{y=x/\varepsilon}$ and $a(y,\xi)$ is an uncertain function from the set of admissible data U^{ad}.

Let us describe U^{ad} in details. Let Y consist of a finite number of subdomains Y_ℓ occupied by different components of a composite, i.e. $\overline{Y} = \bigcup_{\ell=1}^m \overline{Y}_\ell$, $Y_j \cap Y_\ell = \emptyset$, $\forall j \neq \ell$, and $\mathrm{meas}_{\,d}\, Y_\ell > 0$. Each coefficient $a_i(y,\xi)$, $i = 1,\dots,d$, is supposed to be a function Y-periodic in y, constant in y on each Y_k and in the variable ξ dependent on ξ_i only, i.e. $a_i(y,\xi) = a^\ell_i(\xi_i)$ for $y \in Y_\ell$, where $a^\ell_i : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous and strongly monotone inside a fixed interval I_i and linear outside of it. More precisely, let $I_i = [r_i, s_i]$, $i = 1,\dots,d$, $r_i < s_i$, be fixed closed intervals and let each function a^ℓ_i satisfy for all $\ell = 1,\dots,m$:

$$\begin{aligned} &|a^\ell_i(\xi_i) - a^\ell_i(\eta_i)| \le \beta^\ell_i |\xi_i - \eta_i|\,, \quad \forall \xi_i, \eta_i \in I_i\,, \\ &(a^\ell_i(\xi_i) - a^\ell_i(\eta_i)) \cdot (\xi_i - \eta_i) \ge \alpha^\ell_i (\xi_i - \eta_i)^2\,, \quad \forall \xi_i, \eta_i \in I_i\,, \\ &a^\ell_i(\xi_i) = a^\ell_i(r_i) - c^\ell_i(r_i - \xi_i)\,, \quad \forall \xi_i < r_i\,, \\ &a^\ell_i(\xi_i) = a^\ell_i(s_i) + c^\ell_i(\xi_i - s_i)\,, \quad \forall \xi_i > s_i\,, \end{aligned}$$

where $\alpha^\ell_i, \beta^\ell_i, c^\ell_i$ are fixed positive constants such that $\alpha^\ell_i < \beta^\ell_i < c^\ell_i$. Let $S_i(\alpha^\ell_i, \beta^\ell_i, c^\ell_i)$ denote the set of all functions $a_i(y,\xi)$ satisfying the conditions listed above. Now we can define the admissible set U^{ad}_i for the ith coefficient $a_i(y,\xi)$, $i = 1,\dots,d$, as $U^{ad}_i = \{a_i \in S_i(\alpha^\ell_i, \beta^\ell_i, c^\ell_i) : a^{\min}_i(y,\xi) \le a_i(y,\xi) \le a^{\max}_i(y,\xi)\}$, where $a^{\min}_i, a^{\max}_i$ are given functions from $S_i(\alpha^\ell_i, \beta^\ell_i, c^\ell_i)$. The entire set U^{ad} is defined as $U^{ad} = U_1^{ad} \times \cdots \times U_d^{ad}$.

The solvability of (4.2) results from the following abstract theorem known from the theory of monotone operators.

Theorem 4.1. *Let V be a Hilbert space and $A : V \to V'$ an operator satisfying for some $\alpha_0, \beta_0 > 0$ and for all $u_1, u_2 \in V$:*

$$\|A(u_1) - A(u_2)\|_{V'} \leq \beta_0 \|u_1 - u_2\|_V \qquad \textit{(Lipschitz continuity)}. \tag{4.3}$$

$$\langle A(u_1) - A(u_2), u_1 - u_2 \rangle_{V',V} \geq \alpha_0 \|u_1 - u_2\|_V^2 \qquad \textit{(strong monotonicity)}. \tag{4.4}$$

Then the operator equation $A(u) = f$ has a unique solution for each $f \in V'$.

This theorem can be proved by means of the Banach fixed point theorem. The function u is a solution of the equation $A(u) = f$ iff it is the fixed point of the mapping $T_\theta(u) = u - \theta D^{-1}(A(u) - f)$, where $D : V \to V'$ is the duality map of V and $\theta > 0$. It can be shown that for $0 < \theta < 2\alpha_0/\beta_0^2$ the mapping $T_\theta : V \to V$ is contractive and thus there exists a fixed point. Details can be found, e.g. in [30, Sect. 4], [67, Sect. 25.4].

In our problem, the construction of U^{ad} implies existence of positive constants α_{ad} and β_{ad} such that every function $a \in U^{ad}$ satisfies the estimates

$$|a(y,\xi) - a(y,\eta)| \leq \beta_{ad} |\xi - \eta| \,, \quad \forall y, \xi, \eta \in \mathbb{R}^d \,, \tag{4.5}$$

$$(a(y,\xi) - a(y,\eta), \xi - \eta) \geq \alpha_{ad} |\xi - \eta|^2 \,, \quad \forall y, \xi, \eta \in \mathbb{R}^d \,. \tag{4.6}$$

Then, taking $V = H_0^1(\Omega)$, it is not difficult to verify that the operator $\mathcal{A}_\varepsilon(u) := -\mathrm{div}(a_\varepsilon(x, \nabla u))$ from (4.2) satisfies (4.3) and (4.4) with $\beta_0 = \beta_{ad}$ and $\alpha_0 = \alpha_{ad}$.

To summarise the above considerations we can state

Theorem 4.2. *Let $a \in U^{ad}$. Then there exists a unique solution u_ε^a of (4.2) for every $f \in H^{-1}(\Omega)$ and every $\varepsilon > 0$ fixed.*

Although existence and uniqueness of the solution can be obtained also under weaker monotonicity and continuity assumptions (see, e.g. [30]), we shall need the introduced properties in the following subsections.

4.1 Homogenized operator of the monotone type

A nonlinear analogy of Theorem 2.2 corresponding to (4.2) reads:

Theorem 4.3. *Let $a \in U^{ad}$, u_ε^a be the solution of (4.2) with $f \in H^{-1}(\Omega)$ and $\varepsilon \to 0+$. Then $u_\varepsilon^a \rightharpoonup u^a$ in $H_0^1(\Omega)$ and $a_\varepsilon(x, \nabla u_\varepsilon^a) \rightharpoonup b_a(\nabla u^a)$ in $L^2(\Omega; \mathbb{R}^d)$, where u^a is the unique solution of the so-called homogenized problem*

$$\left\{ \begin{array}{l} \displaystyle\int_\Omega (b_a(\nabla u^a), \nabla v)\,\mathrm{d}x = \langle f, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \,, \quad \forall v \in H_0^1(\Omega) \,, \\ u^a \in H_0^1(\Omega) \,. \end{array} \right. \tag{4.7}$$

The coefficient $b_a : \mathbb{R}^d \to \mathbb{R}^d$ is defined as

$$b_a(\xi) = \mathfrak{M}_Y a(y, \xi + \nabla w_\xi^a(y)) \,, \tag{4.8}$$

where w_ξ^a is the unique solution of the so-called local problem

$$\left\{ \begin{array}{l} \int_Y (a(y, \xi + \nabla w_\xi^a), \nabla \phi)\,\mathrm{d}x = 0 \,, \quad \forall \phi \in H_\#^1(Y) \,, \\ w_\xi^a \in H_\#^1(Y) \,. \end{array} \right. \tag{4.9}$$

Moreover, $b_a : \mathbb{R}^d \to \mathbb{R}^d$ satisfies the following estimates

$$|b_a(\xi) - b_a(\eta)| \leq \widetilde{\beta}|\xi - \eta|\,, \quad \forall \xi, \eta \in \mathbb{R}^d\,, \tag{4.10}$$

$$(b_a(\xi) - b_a(\eta), \xi - \eta) \geq \alpha_{ad}|\xi - \eta|^2\,, \quad \forall \xi, \eta \in \mathbb{R}^d\,, \tag{4.11}$$

where the constant $\widetilde{\beta}$ depends on α_{ad}, β_{ad} and the bound of the coefficient $a(y, \xi)$ at the point $\xi = 0$.

A detailed proof of the theorem can be found in [25, Thm. 5.3] – the procedure is sketched in the full version of this thesis.

The same homogenization result for the case of Sobolev space $H_0^{1,p}(\Omega)$, $p \neq 2$, under analogous hypothesis on $a(y, \xi)$, was first presented in [33]. Let us note that some other variants of monotonicity and continuity assumptions also have been studied. The most general result on homogenization of monotone operators was formulated in [14] covering also the case of multivalued mappings.

4.2 Worst scenario

In this subsection we introduce the criterion functional over the set U^{ad}, formulate the corresponding worst scenario problem and prove its solvability. Although we have fairly enough freedom with the choice of this functional based on the aim of interest and expert decisions, in view of Subsection 2.2 certain continuity assumptions must be satisfied. For our purposes the following definition is satisfactory.

Definition 4.4. The criterion functional is a functional $\Phi : U^{ad} \times H_0^1(\Omega) \to \mathbb{R}$ satisfying: if $a_n, a \in U^{ad}$, $a_n \rightrightarrows a$ on $Y \times \mathbb{R}^d$ and $v_n \to v$ in $H_0^1(\Omega)$ as $n \to \infty$, then $\Phi(a_n, v_n) \to \Phi(a, v)$.

In our problem, Φ can be given, e.g. by $\Phi(a, v) = (\text{meas}_d\, \widetilde{\Omega})^{-1} \int_{\widetilde{\Omega}} v \,\mathrm{d}x$, where $\widetilde{\Omega}$ is a subset of Ω of a positive measure. This choice is motivated by having interest in finding some critical values of the homogenized solution (representing, e.g. temperature) in some crucial places of the material (e.g. where measurements take place). Since the solution need not be continuous and thus the maximum need not exist, the integral mean value is used. Similarly, the solution u^a in Φ can be replaced, e.g. by its gradient or the generalized gradient.

Once the set of admissible data and the criterion functional are given, we can formulate the worst scenario problem:

$$\begin{cases} \text{Find } a^{\blacktriangle} \in U^{ad} \text{ such that} \\ J(a) \leq J(a^{\blacktriangle})\,,\ \forall a \in U^{ad}\,, \end{cases} \tag{4.12}$$

where $J(a) := \Phi(a, u^a)$, u^a is the solution of the homogenized problem (4.7) and Φ is a criterion functional.

Solvability of (4.12) is ensured by the following assertion.

Theorem 4.5. *There exists at least one solution of (4.12).*

The proof of this theorem relies on the following auxiliary results (proofs can be found in the full version of the thesis).

Lemma 4.6. *The set U^{ad} is compact in the following sense: each sequence of functions $\{a_n\} \subset U^{ad}$ contains a uniformly convergent subsequence $\{a_{n'}\}$ of $\{a_n\}$, i.e. there exists an element $a \in U^{ad}$ such that $a_{n'} \rightrightarrows a$ on $Y \times \mathbb{R}^d$.*

Lemma 4.7. *Let $a_n, a \in U^{ad}$ be such that $a_n \rightrightarrows a$ on $Y \times \mathbb{R}^d$ as $n \to \infty$. Then $b_{a_n} \rightrightarrows b_a$ on $\mathbb{R}^d$, where b_{a_n} and b_a are defined by (4.8) with the integrand a_n and a, respectively.*

Lemma 4.8. *Let $a_n, a \in U^{ad}$ be such that $a_n \rightrightarrows a$ on $Y \times \mathbb{R}^d$ as $n \to \infty$. Then $u^{a_n} \to u^a$ in $H_0^1(\Omega)$, where u^{a_n} and u^a are the solutions of (4.7) with the coefficient b_{a_n} and b_a, respectively.*

4.3 Shape uncertainties

A slightly different problem was considered in [59]. We assume that the domain Ω is occupied by a two-phase composite (composed of an inclusion and a matrix) with a periodic structure such that the periodicity cell contains one fibre of the inclusion only. The geometry (shape) of the fibre is assumed to be uncertain in the sense that it can vary with a vector of parameters p, where some bounds on p are given (e.g. the cylinder can vary with $p = (p_1, p_2)$, where p_1 denotes the radius of the base and p_2 denotes the height).

More precisely, let $m \geq 1$ be an integer. Then we define $U^{ad} = [p_1^\ell, p_1^u] \times [p_2^\ell, p_2^u] \times \cdots \times [p_m^\ell, p_m^u]$, where $0 < p_i^\ell < p_i^u$, $i = 1, \dots, m$ are suitable real constants (constraints). Let $p \in U^{ad}$ represent the geometrical parameters of a fibre and let Y_p denote the occupied set. Moreover, we assume that Y_p is a domain in $\mathbb{R}^d$ (i.e. an open and simply connected set) satisfying $\overline{Y}_p \subset \overline{Y}, \forall p \in U^{ad}$.

Further, we introduce a function $a_p(y,\xi) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ with the following properties:

$$
\begin{aligned}
&a_p(y,\xi) = a_1(\xi) \text{ on } Y_p\,, \quad a_p(y,\xi) = a_2(\xi) \text{ on } Y \setminus Y_p\,,\ a_1(\xi) \neq a_2(\xi)\,,\\
&a_p(y+k,\xi) = a_p(y,\xi)\,, \quad \forall y \in Y,\ \forall k \in \mathbb{Z}^d,\ \forall \xi \in \mathbb{R}^d\,,\\
&|a_i(\xi) - a_i(\eta)| \leq \beta_i |\xi - \eta|\,, \quad \forall \xi,\eta \in \mathbb{R}^d, i = 1,2\,,\\
&(a_i(\xi) - a_i(\eta), \xi - \eta) \geq \alpha_i |\xi - \eta|^2\,, \quad \forall \xi,\eta \in \mathbb{R}^d, i = 1,2\,,
\end{aligned}
$$

where $\alpha_i < \beta_i$ are positive constants. Hence, the function a_p is constant in the first variable y on both Y_p and $Y \setminus Y_p$, it is Y-periodic in y and satisfies the strong monotonicity and Lipschitz continuity conditions in the second variable ξ, i.e. we have

$$
\begin{aligned}
&|a_p(y,\xi) - a_p(y,\eta)| \leq \beta |\xi - \eta|\,, \quad \forall y \in Y,\ \forall \xi,\eta \in \mathbb{R}^d\,,\\
&(a_p(y,\xi) - a_p(y,\eta), \xi - \eta) \geq \alpha |\xi - \eta|^2\,, \quad \forall y \in Y,\ \forall \xi,\eta \in \mathbb{R}^d\,,
\end{aligned}
$$

where $\alpha = \min_i \alpha_i$ and $\beta = \max_i \beta_i$.

For any $p \in U^{ad}$ and $\varepsilon \to 0+$ let us consider the following sequence of problems:

$$
\begin{cases}
\text{Find } u_\varepsilon^p \in H_0^1(\Omega) \text{ such that}\\
\int_\Omega (a_p(x/\varepsilon, \nabla u_\varepsilon^p), \nabla v)\,\mathrm{d}x = \langle f, v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}\,, \quad \forall v \in H_0^1(\Omega)\,.
\end{cases}
\tag{4.13}
$$

Solvability of (4.13) is again based on Theorem 4.1. More precisely, we have

Theorem 4.9. *Let $p \in U^{ad}$. Then there exists a unique solution u_ε^p of (4.13) for every $f \in H^{-1}(\Omega)$ and every $\varepsilon > 0$ fixed.*

The homogenized problem to (4.13) reads:

$$
\begin{cases}
\text{Find } u^p \in H_0^1(\Omega) \text{ such that}\\
\int_\Omega (b_p(\nabla u^p), \nabla v)\,\mathrm{d}x = \langle f, v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}\,, \quad \forall v \in H_0^1(\Omega)\,,
\end{cases}
\tag{4.14}
$$

where the coefficient $b_p : \mathbb{R}^d \to \mathbb{R}^d$ is given by $b_p(\xi) = \mathfrak{M}_Y a_p(y, \xi + \nabla w_\xi^p(y))$ and the function w_ξ^p is the solution of the local problem

$$
\begin{cases}
\text{Find } w_\xi^p \in H_\#^1(Y) \text{ such that}\\
\int_Y (a_p(y, \xi + \nabla w_\xi^p(y)), \nabla \phi)\,\mathrm{d}y = 0\,, \quad \forall \phi \in H_\#^1(Y)\,.
\end{cases}
$$

In accordance with Definition 4.4, a functional $\Phi : U^{ad} \times H_0^1(\Omega) \to \mathbb{R}$ is called criterion if the following convergence holds: taking arbitrary sequences $\{p_n\} \subset U^{ad}$, $\{v_n\} \subset H_0^1(\Omega)$ such that $p_n \to p$ in $\mathbb{R}^m$ (the limit p is in U^{ad} since U^{ad} is a compact set in $\mathbb{R}^m$) and $v_n \to v$ in $H_0^1(\Omega)$ as $n \to \infty$, we have $\Phi(p_n, v_n) \to \Phi(p, v)$. Having the set of admissible data and the criterion functional, the worst scenario problem reads:

$$\begin{cases} \text{Find } p^{\blacktriangle} \in U^{ad} \text{ such that} \\ J(p) \leq J(p^{\blacktriangle}), \quad \forall p \in U^{ad}, \end{cases} \tag{4.15}$$

where $J(p) = \Phi(p, u^p)$ and u^p is the solution of (4.14).

The arguments related to solvability of (4.15) are very similar to those provided in Subsection 4.2, hence, we can state:

Theorem 4.10. *There exists at least one solution of (4.15).*

5 Alternative approaches to the two-scale convergence

The results of this section are based on the author's works [55], [56]. It surveys the two-scale convergence concept and discusses some alternative approaches to it.

The two-scale convergence is a special type of the weak convergence. It was developed for the homogenization theory in order to simplify proofs. It overcomes difficulties resulting from properties of weakly converging sequences of periodic functions. In such sequences the weak limit does not keep the "information on oscillations" of the original functions. In some cases, the two-scale limit is able to conserve this information and thus, it makes limit procedures possible. It stands between the usual strong and weak convergences. The concept was first introduced by Nguetseng [61] and later developed by Allaire in [4] in early 90's (see also the survey papers [16] and [46]). In the definition of two-scale convergence, the special so-called admissible test function is used. The widest set of these functions is not clear and thus it motivates alternative approaches. One of them is based on a two-scale transform which changes a sequence of one variable functions into a sequence of two-variable functions. This transform was first used by the authors in the homogenization of some problems set in porous media (see [6]) and it is a suitable tool for an alternative definition of the two-scale convergence, see also [13] and[15]. Another definition is based on the so-called inverse two-scale transform which defines a sequence of one-variable functions from the test function $\psi(x, y)$.

If not specified, the use of L^p spaces is restricted to $1 < p < \infty$ in this section. The proofs or comments to this subsection statements can be found in the full version of the thesis.

5.1 Definitions

Let us begin with the classical definition by Nguetseng and Allaire (which was introduced for the case of L^2):

Definition 5.1. We say that a sequence $\{u_\varepsilon(x)\} \subset L^p(\Omega)$ two-scale converges to a limit $u_0(x, y) \in L^p(\Omega \times Y)$ iff

$$\lim_{\varepsilon \to 0+} \int_\Omega u_\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_\Omega \int_Y u_0(x, y) \psi(x, y) \, \mathrm{d}x \mathrm{d}y \tag{5.1}$$

holds for each $\psi(x, y) \in L^{p'}(\Omega; C_\#(Y))$. If, in addition, $\lim_{\varepsilon \to 0+} \|u_\varepsilon(x) - u_0(x, x/\varepsilon)\|_{L^p(\Omega)} = 0$, we say that $\{u_\varepsilon\}$ two-scale converges strongly to u_0.

Remark 5.2. The strong two-scale convergence is also called the corrector type result in the vocabulary of homogenization. Assumptions making two-scale convergence strong are discussed in Subsection 5.2. Although $\{u_\varepsilon\}$ is a sequence of d-variable functions $x_1, \ldots, x_d$, the limit is a function of $2d$ variables $x_1, \ldots, x_d, y_1, \ldots, y_d$. It enables us to describe the oscillatory behaviour of u_ε better. Usually it is not emphasised, but the definition is connected with a fixed sequence of periods ε_n, i.e. for an extracted subsequence $\{u_{\varepsilon'}\}$ the same extracted subsequence ε' of periods must be considered also in the test function, see examples in Subsection 5.4.

Let us introduce an alternative definition. In [6], the authors dealt with a homogenization technique which was used for the description of a porous media. It is suitable for an alternative approach to the two-scale convergence. The idea is based on the so-called two-scale transform which changes a sequence of functions $u_\varepsilon(x)$ into a sequence of double-variable functions $\widehat{u}_\varepsilon(x,y)$. For each ε let us consider the small non-overlapping cubes $C_\varepsilon^k = \varepsilon Y + \varepsilon k$, $k \in \mathbb{Z}^d$. Here, for the sake of simplicity, we restrict ourselves to the domains that can be decomposed into these cubes, i.e. $\overline{\Omega} = \bigcup_k \overline{C}_\varepsilon^k$. The sequence $\{\widehat{u}_\varepsilon(x,y)\}$ is defined by the relation

$$\widehat{u}_\varepsilon(x,y) = u_\varepsilon\left(\varepsilon\left\lfloor \frac{x}{\varepsilon}\right\rfloor + \varepsilon y\right)\,, \quad x \in \Omega\,,\ y \in Y, \tag{5.2}$$

where $\lfloor x \rfloor$ is the floor function (the greatest integer less or equal to x). On each cube $C_\varepsilon^k \times Y$, the function $\widehat{u}_\varepsilon$ is constant in the variable x, and as a function of y it is the function $u_\varepsilon(x)$ on C_ε^k transformed onto the unit cube Y. The alternative definition reads:

Definition 5.3. We say that a sequence $\{u_\varepsilon(x)\} \subset L^p(\Omega)$ two-scale converges to a limit $u_0(x,y) \in L^p(\Omega \times Y)$ iff $\widehat{u}_\varepsilon \rightharpoonup u_0$ in $L^p(\Omega \times Y)$. Moreover, if $\widehat{u}_\varepsilon \to u_0$ in $L^p(\Omega \times Y)$, we say that $\{u_\varepsilon\}$ two-scale converges strongly.

This approach is also called the periodic unfolding method – this term was first used in [15].

The second alternative approach is based on the so-called inverse two-scale transform which makes a sequence $\{\overline{\psi}_\varepsilon(x)\}$ from a two-variable function $\psi(x,y)$. The functions $\overline{\psi}_\varepsilon$ are constructed as follows. Similarly, as in the previous transform, we consider non-overlapping cubes C_ε^k that cover the domain Ω (here the domain Ω need not be the union of the cubes C_ε^k, i.e. $\overline{\Omega} \subseteq \bigcup_k \overline{C}_\varepsilon^k$). Outside the domain Ω we put $\psi(x,y) = 0$. Let us average the extented function $\psi(x,y)$ in the first variable

$$\overline{\psi}_\varepsilon(x) = \frac{1}{\varepsilon^d}\int_{C_\varepsilon^k} \psi\Big(\xi, \frac{x}{\varepsilon}\Big)\,\mathrm{d}\xi\,, \quad x \in C_\varepsilon^k\,. \tag{5.3}$$

Definition 5.4. We say that a sequence $\{u_\varepsilon(x)\} \subset L^p(\Omega)$ two-scale converges to a limit $u_0(x,y) \in L^p(\Omega \times Y)$ iff

$$\lim_{\varepsilon\to 0+} \int_\Omega u_\varepsilon(x)\,\overline{\psi}_\varepsilon(x)\,\mathrm{d}x = \int_\Omega\int_Y u_0(x,y)\psi(x,y)\,\mathrm{d}x\mathrm{d}y \tag{5.4}$$

holds for each $\psi(x,y) \in L^{p'}(\Omega \times Y)$. Moreover, if $\lim_{\varepsilon\to 0+} \|u_\varepsilon(x) - \overline{u}_0^\varepsilon(x)\|_{L^p(\Omega)} = 0$, we say that $\{u_\varepsilon\}$ two-scale converges strongly to u_0.

Remark 5.5. The two-scale transform used in Definition 5.3 enables to define the two-scale and strong two-scale convergence more naturally with help of the weak convergence in L^p spaces. On the other hand, Definition 5.4 is similar to the classical definition by Nguetseng and Allaire, but it differs from Definition 5.1 by the choice of test function on the left-hand side of integral identity.

Why do we look for alternative approaches to the original one? In the definition we want to test the convergence with functions from a space as small as possible, in applications the largest class is desirable. In Definition 5.1 we can not take the test function $\psi(x,y)$ from the whole space $L^{p'}(\Omega \times Y)$, since it is not defined correctly on the zero-measure set $\{[x,y] \in \Omega \times Y : y = x/\varepsilon\}$ and thus the measurability of the composed function $\psi(x, x/\varepsilon)$ is not guaranteed. Moreover, the test functions must satisfy some convergence which is not always obvious. On the other hand, the space of test functions can not be too small. The class of suitable test functions is discussed at the top of the following subsection. Such functions are called admissible. Further, we will see that the two mentioned alternative definitions avoid the described problems.

5.2 Comparison of the definitions

The main goal of this subsection is to prove the equivalence of definitions in the sense that each of them yields the same two-scale limits.

First, we proceed with the notion of admissible test function mentioned above. Since the widest set of suitable test functions ψ is not clear (we do not know the minimal conditions making these functions regular enough), the following characterization is useful.

Definition 5.6. A Y-periodic (in y) test function $\psi(x,y) \in L^{p'}(\Omega\times Y)$ is said to be admissible iff

$$\lim_{\varepsilon\to 0+} \left\| \psi\Big(x, \frac{x}{\varepsilon}\Big) \right\|_{L^{p'}(\Omega)} = \|\psi(x,y)\|_{L^{p'}(\Omega\times Y)} \tag{5.5}$$

and for a separable subspace $X \subseteq L^{p'}(\Omega \times Y)$

$$\left\| \psi\Big(x, \frac{x}{\varepsilon}\Big) \right\|_{L^{p'}(\Omega)} \leq \|\psi(x,y)\|_{X} \,. \tag{5.6}$$

Let us emphasise that there exist Y-periodic functions in y which do not satisfy the convergence (5.5) even if the measurability of the composed function $\psi(x, x/\varepsilon)$ is guaranteed, see [4]. Allaire showed (for $p = 2$) that $L^{p'}(\Omega; C_\#(Y))$ but also, e.g. $L^{p'}_\#(Y; C(\overline{\Omega}))$ are composed of the admissible functions. All these spaces are separable and their elements are functions continuous at least in one variable. Such functions are Carathéodory which is a sufficient condition for measurability of the composed function $\psi(x, x/\varepsilon)$ and moreover, they satisfy (5.5) and (5.6). These two properties are needful in the proof of the compactness property, see Subsection 5.3. If ψ belongs to these spaces, we also have

$$\lim_{\varepsilon\to 0+} \int_\Omega \psi\Big(x, \frac{x}{\varepsilon}\Big)\, \mathrm{d}x = \int_\Omega \int_Y \psi(x,y)\, \mathrm{d}x\mathrm{d}y \,, \tag{5.7}$$

for details, see [4]. This relation is natural: taking the stationary sequence $\{u_\varepsilon = 1\}$, (5.1) and (5.7) yield two-scale limit $u_0(x,y) = 1$. As mentioned above, the space of admissible functions can not be too small. Taking, e.g. $C_0^\infty(\Omega; C_\#^\infty(Y))$, Definition 5.1 admits even two-scale converging sequences that are unbounded in $L^p(\Omega)$.

In Definition 5.3, the situation is simplified, since the two-scale convergence follows from the L^p-theory – the weak convergence can be tested also by smooth functions from $C_0^\infty(\Omega\times Y)$ due to the density property.

The special choice of test function in Definition 5.4 is also motivated by the effort to enlarge the class of test functions compared to Definition 5.1. The following lemma justifies the use of the whole $L^{p'}(\Omega \times Y)$ for testing the convergence (5.4).

Lemma 5.7. *Let $\psi \in L^{p'}(\Omega \times Y)$ be a Y-periodic function and $\overline{\psi}_\varepsilon$ be defined by (5.3). Then we have*

$$\|\overline{\psi}_\varepsilon(x)\|_{L^{p'}(\Omega)} \leq \|\psi(x,y)\|_{L^{p'}(\Omega\times Y)} \,, \tag{5.8}$$

$$\lim_{\varepsilon\to 0+} \|\overline{\psi}_\varepsilon(x)\|_{L^{p'}(\Omega)} = \|\psi(x,y)\|_{L^{p'}(\Omega\times Y)} \,. \tag{5.9}$$

Remark 5.8. If the union of the cubes gives the entire domain Ω, i.e. $\overline{\Omega} = \bigcup_k \overline{C}^k_\varepsilon$, then we have $\|\overline{\psi}_\varepsilon(x)\|_{L^{p'}(\Omega)} = \|\psi(x,y)\|_{L^{p'}(\Omega\times Y)}$. Lemma 5.7 says that every test function $\psi \in L^{p'}(\Omega \times Y)$ can be called admissible (compare with the properties in Definition 5.6).

The following lemma specifies properties of the functions $\widehat{u}_\varepsilon$ used in the two-scale transform approach (we remind that Ω is considered to be a union of the cubes C^k_ε).

Lemma 5.9. *Let $u_\varepsilon \in L^p(\Omega)$ and $\widehat{u}_\varepsilon$ be defined by (5.2). Then $\widehat{u}_\varepsilon \in L^p(\Omega\times Y)$ and $\|u_\varepsilon\|_{L^p(\Omega)} = \|\widehat{u}_\varepsilon(x,y)\|_{L^p(\Omega\times Y)}$.*

Remark 5.10. The situation is more complicated in the case of cubes exceeding the domain Ω. The two-scale transform defined by (5.2) works well on the cubes C_ε^k, i.e. a function u_ε defined on C_ε^k is transformed into a function $\widehat{u}_\varepsilon$ defined on $C_\varepsilon^k \times Y$. Near the boundary, where $C_\varepsilon^k \cap \Omega \neq C_\varepsilon^k$, it can cause difficulties. Analogously to the proof of Lemma 5.7, let us consider the minimal number of the cubes C_ε^k covering Ω. The union $S_\varepsilon = (\bigcup_k \overline{C}_\varepsilon^k) \backslash \Omega$ is of a positive measure. In the case of a "good" boundary, $\text{meas}_d S_\varepsilon \to 0$ (as $\varepsilon \to 0+$), but $\|\widehat{u}_\varepsilon\|$ can not be estimated by $\|u_\varepsilon\|$ as the following example shows:

Let us take $\Omega = (0, a)$, $a \in \mathbb{R}$, and the sequence of periods ε, such that the interval $(0, a)$ can not be expressed as the union of the small intervals $I_\varepsilon^k = (\varepsilon k, \varepsilon(k+1))$, $k \in \mathbb{Z}$. We define the sequence $\{u_\varepsilon\} \subset L^1(\Omega)$ by

$$u_\varepsilon(x) = \begin{cases} 0\,, & x \in (0, a - \varepsilon^2) \\ \varepsilon^{-2}\,, & x \in (a - \varepsilon^2, a) \end{cases} .$$

Thus, the intervals I_ε^k exceed the interval $(0, a)$ by $\varepsilon - \varepsilon^2$ (on this small part we put $u_\varepsilon = 0$). Obviously, $\|u_\varepsilon\|_{L^1(\Omega)} = 1$ while $\|\widehat{u}_\varepsilon\|_{L^1(\Omega \times Y)} = \varepsilon^2$, i.e. $\|u_\varepsilon\| \not\approx \|\widehat{u}_\varepsilon\|$ (we have $\|u_\varepsilon\| \geq \|\widehat{u}_\varepsilon\|$ only).

By the transform we want to preserve the norms of u_ε and $\widehat{u}_\varepsilon$ even if the cubes exceed Ω. It is not difficult in 1D, since it is sufficient to re-scale with the actual length of the boundary segment $C_\varepsilon^k \cap \Omega$. In a higher dimension it is more difficult.

In [16], the authors made an attempt to solve the mentioned problem with boundary cells by splitting the domain Ω into Ω_ε containing the "complete" cubes C_ε^k and the remainder part Λ_ε containing "uncomplete" cubes and putting $\widetilde{u}_\varepsilon = 0$ on Λ_ε. The two-scale transform (unfolding) is then well defined on Ω, however, it does not satisfy the equality of integrals

$$\int_\Omega u_\varepsilon(x)\, \mathrm{d}x = \int_\Omega \int_Y \widehat{u}_\varepsilon(x, y)\, \mathrm{d}x\mathrm{d}y\,. \tag{5.10}$$

Therefore, the additional condition was introduced: a sequence of functions $f_\varepsilon(x)$ is said to satisfy the unfolding criterion if $\lim_{\varepsilon \to 0+} \int_{\Lambda_\varepsilon} f_\varepsilon(x) \mathrm{d}x = 0$. This property guarantees that (5.10) holds in the limit.

A further improvement was done recently in [32], where the authors introduced a modified two-scale transform extended by identity on Λ_ε, i.e.

$$\widehat{u}_\varepsilon(x, y) = \begin{cases} u_\varepsilon\left(\varepsilon \left\lfloor \dfrac{x}{\varepsilon} \right\rfloor + \varepsilon y\right) & \text{for } x \in \Omega_\varepsilon\,, \\ u_\varepsilon(x) & \text{for } x \in \Lambda_\varepsilon\,. \end{cases}$$

This modification satisfies (5.10) which implies the isometry $\|u_\varepsilon\|_{L^p(\Omega)} = \|\widehat{u}_\varepsilon\|_{L^p(\Omega \times Y)}$. We also note that a slightly different notation for the two-scale transform is used in [32] (more convenient for possible extensions to non-periodic cases).

Theorem 5.11. *Let us assume $\overline{\Omega} = \bigcup_k \overline{C}_\varepsilon^k$ and let $\{u_\varepsilon\} \subset L^p(\Omega)$ two-scale converge (in the Nguetseng-Allaire sense) to $u_0 \in L^p(\Omega \times Y)$. Then $\{u_\varepsilon\}$ two-scale converges to u_0 also in the sense of Definition 5.3 and Definition 5.4.*

Remark 5.12. It holds even a stronger property. Under the assumption of Theorem 5.11 we have $\int_\Omega u_\varepsilon(x) \psi\big(x, \frac{x}{\varepsilon}\big)\, \mathrm{d}x = \int_\Omega \int_Y \widehat{u}_\varepsilon(x, y) \psi(x, y)\, \mathrm{d}x\mathrm{d}y$.

Let us continue with the strong two-scale convergence. The usual weak convergence $u_\varepsilon \rightharpoonup u$ equipped with the additional condition $\|u_\varepsilon\| \to \|u\|$ is also strong, i.e. it holds $\|u_\varepsilon - u\| \to 0$. The following theorem introduces the similar additional assumptions that strengthen the two-scale convergence into the strong one (in the case of Nguetseng-Allaire definition).

Theorem 5.13. *A sequence $\{u_\varepsilon\} \subset L^p(\Omega)$ two-scale converges strongly to a limit u_0 if $\{u_\varepsilon\}$ two-scale converges to u_0 and the relations $\|u_\varepsilon\|_{L^p(\Omega)} \to \|u_0\|_{L^p(\Omega \times Y)}$, $\|u_0(x, x/\varepsilon)\|_{L^p(\Omega)} \to \|u_0\|_{L^p(\Omega \times Y)}$ hold.*

More details and the proof can be found in [4] and [39].

Remark 5.14. In the two-scale transform approach, the weak convergence of a sequence $\{\widehat{u}_\varepsilon\}$ plays the role of the two-scale convergence. Hence, $\{u_\varepsilon\}$ two-scale converges strongly if $\{\widehat{u}_\varepsilon\}$ converges weakly to u_0 and $\|\widehat{u}_\varepsilon\|_{L^p(\Omega\times Y)} \to \|u_0\|_{L^p(\Omega\times Y)}$.

The similar result holds for the inverse two-scale transform approach. As in Theorem 5.13, the additional assumptions $\|u_\varepsilon\|_{L^p(\Omega)} \to \|u_0\|_{L^p(\Omega\times Y)}$, $\|\overline{u}_0^\varepsilon\|_{L^p(\Omega)} \to \|u_0\|_{L^p(\Omega\times Y)}$ strengthen two-scale convergence into strong one. On the other hand, due to Lemma 5.7, each function $u_0 \in L^p(\Omega \times Y)$ satisfies the convergence $\|\overline{u}_0^\varepsilon\|_{L^p(\Omega)} \to \|u_0\|_{L^p(\Omega\times Y)}$. Thus, we have

Lemma 5.15. *A sequence $\{u_\varepsilon\}$ two-scale converges strongly (in the sense of Definition 5.4) if it two-scale converges to u_0 and $\|u_\varepsilon\|_{L^p(\Omega)} \to \|u_0\|_{L^p(\Omega\times Y)}$.*

Theorem 5.16. *Let $\overline{\Omega} = \bigcup_k \overline{C}_\varepsilon^k$ and let a sequence $\{u_\varepsilon\} \subset L^p(\Omega)$ two-scale converge strongly to a limit u_0 according to Nguetseng-Allaire's definition. Then $\{u_\varepsilon\}$ also two-scale converges strongly to u_0 in the sense of Definition 5.1 and Definition 5.3.*

Remark 5.17. Theorems 5.11, 5.16 together show the equivalence of the used definitions, it means that all of them yield the same limits.

5.3 Compactness

The two-scale convergence can be used in applications due to the following compactness property.

Theorem 5.18. *A bounded sequence $\{u_\varepsilon\} \subset L^p(\Omega)$ is compact with respect to the two-scale convergence, i.e. there exists an extracted subsequence two-scale converging to a function $u_0 \in L^p(\Omega \times Y)$.*

Allaire's proof in [4] is carried out for the admissible test functions from $L^2(\Omega; C_\#(Y))$. It is based on properties of the dual space to $L^2(\Omega; C_\#(Y))$. This space is not so transparent, since it is represented by $L^2(\Omega; M_\#(Y))$, where $M_\#(Y)$ is the space of Y-periodic Radon measures.

In the alternative approach based on the two-scale transform, the situation is more straightforward, since the two-scale compactness follows directly by bounded sequences in $L^p(\Omega \times Y)$ (a closed ball is compact with respect to the weak convergence).

A modification of the theorem for the case of two-scale convergence based on the inverse two-scale transform is proved in the full version of the thesis.

5.4 Examples and properties

Let us discuss a few typical examples of two-scale convergent sequences.

Example 5.19. (i) Let $a(y)$ be a Y-periodic bounded function such that $\mathfrak{M}_Y a = 0$ and $b_1(x)$, $b_2(x)$ be arbitrary functions from $L^p(\Omega)$. Then the sequence $\{u_\varepsilon\}$ defined as $u_\varepsilon(x) = b_1(x)a(x/\varepsilon) + b_2(x)$ converges weakly to $b_2(x)$ in $L^p(\Omega)$ and it two-scale converges (strongly) to $b_1(x)a(y) + b_2(x)$. We can see that the weak limit is the function b_2 only. It says nothing on the periodic behaviour of u_ε. On the other hand, in the two scale limit the information on "oscillations" is kept. This loss of information in the weak limit causes some "unpleasant" properties mentioned in Subsection 2.1, e.g. taking two weakly converging sequences $u_\varepsilon \rightharpoonup u$, $v_\varepsilon \rightharpoonup v$ does not imply $u_\varepsilon v_\varepsilon \rightharpoonup uv$, etc. The example shows that the weak limit is the average of two-scale limit with respect to y. This is a direct consequence of the definition if we take test function ψ depending on the variable x only.

(ii) Let us consider the same functions $a(y)$, $b_1(x)$, $b_2(x)$, but the other sequence $\{v_\varepsilon\}$ defined by $v_\varepsilon(x) = b_1(x)a(x/\varepsilon^2) + b_2(x)$. Then the two-scale and weak limit coincide, it means that the two-scale limit is constant in the variable y. In this case, the information on oscillations is not kept. Similarly, taking a sequence $w_n(x) = b_1(x)a(cx\varepsilon) + b_2(x)$ with c irrational, the two-scale limit equals to b_2 only. It is a consequence of the fact that diminishing of the periods is not in a resonance with the periods in the test function.

The sequences from Example 5.19 point out an interesting fact. An extracted subsequence of a weakly converging sequence converges to the same limit as the entire sequence. In the case of two-scale convergence we must consider the convergence also with respect to the subsequence of periods. Otherwise the limits may differ (e.g. the sequence $\{v_\varepsilon\}$ from Example 5.19 can be considered as an extracted subsequence from $\{u_\varepsilon\}$).

Example 5.20. Let $\{u_\varepsilon\} \subset L^p(\Omega)$ be a sequence satisfying $\|u_\varepsilon\|_{L^p(\Omega)} \to \|u_0\|_{L^p(\Omega\times Y)}$. Then $\{u_\varepsilon\}$ need not two-scale converge to u_0. Let $u(y)$ be a Y-periodic function and let us consider functions $u_0(x,y) = u(y)$, $\tilde{u}_0(x,y) = u(y-1/2)$. Since $u(y)$ is periodic, we have $\|u_0\|_{L^p(\Omega\times Y)} = \|\tilde{u}_0\|_{L^p(\Omega\times Y)}$. Taking $u_\varepsilon(x) = \tilde{u}_0(x/\varepsilon)$, then $\|u_\varepsilon\|_{L^p(\Omega)} \to \|u_0\|_{L^p(\Omega\times Y)}$, but $\{u_\varepsilon\}$ two-scale converges to $\tilde{u}_0$.

Theorem 5.21. *Let $\{u_\varepsilon\} \subset L^p(\Omega)$ two-scale converge to $u_0 \in L^p(\Omega\times Y)$ and converge weakly to u in $L^p(\Omega)$. Then*

$$\liminf_{\varepsilon\to 0+} \|u_\varepsilon\|_{L^p(\Omega)} \geq \|u_0\|_{L^p(\Omega\times Y)} \geq \|u\|_{L^p(\Omega)}\,. \tag{5.11}$$

The first inequality (5.11) can be proved with help of the Young inequality and the definition of two-scale convergence. The second inequality follows the Hölder inequality and it can be interpreted: two-scale limit conserves more information on a periodic behaviour of $\{u_\varepsilon\}$ than the usual weak limit.

Example 5.22. Let us consider the sequences $\{u_\varepsilon\}$ and $\{v_\varepsilon\}$ from Example 5.19. Denoting the two-scale limit by u_0 and the weak limit by u, we have $\lim \|u_\varepsilon\|_{L^p(\Omega)} = \|u_0\|_{L^p(\Omega\times Y)} > \|u\|_{L^p(\Omega)}$, $\lim \|v_\varepsilon\|_{L^p(\Omega)} > \|u_0\|_{L^p(\Omega\times Y)} = \|u\|_{L^p(\Omega)}$ and finally, the sum $u_\varepsilon + v_\varepsilon$ yields the sharp inequalities.

Theorem 5.21 further implies: every sequence $\{u_\varepsilon\}$ strongly convergent to a function u in $L^p(\Omega)$ also two-scale converges to $u_0(x,y) = u(x)$. The following convergence theorem is meaningful in applications.

Theorem 5.23. *Let $\{u^1_\varepsilon\} \subset L^{p_1}(\Omega), \dots, \{u^m_\varepsilon\} \subset L^{p_m}(\Omega)$, $1 \leq p_i < \infty$, $i = 1,\dots,m$, be two-scale converging strongly to $u^1_0(x,y),\dots,u^m_0(x,y)$ and let $f(x,\xi_1,\dots,\xi_m)$ be a Carathéodory function satisfying the growth condition $|f(x,\xi_1,\dots,\xi_m)| \leq g(x) + c\sum_{i=1}^m |\xi_i|^{p_i/r}$, where c is a positive constant and $g \in L^r(\Omega)$, $1 \leq r < \infty$. Then $f(x,u^1_\varepsilon(x),\dots,u^m_\varepsilon(x)) \to \int_Y f(x,u^1_0(x,y),\dots,u^m_0(x,y))\,\mathrm{d}y$ in $L^r(\Omega)$.*

Remark 5.24. A special case of this theorem is the convergence $u^1_\varepsilon \dots u^m_\varepsilon \to \mathfrak{M}_Y(u^1_0 \dots u^m_0)$ in $L^r(\Omega)$ which often occurs in proofs. This convergence changes into the weak one as one of $\{u^i_\varepsilon\}$ two-scale converges (not strongly) only.

6 Additional comments and further perspectives

We have surveyed some phenomena appearing when problems set in a highly heterogeneous medium are studied. Contrary to the traditional approach, some uncertainties in the inputs (coefficients in the equation) of the model problem were taken into account. Occurrence of uncertain inputs is natural, besides the mentioned experimental detection of the tabular values, the data can vary with time, there can be a difference between the laboratory and the manufactured material properties, etc. Hence, a certain amount of errors should be expected. We have been motivated by the effort to obtain reliable solutions with respect to the uncertain set of inputs. For this kind of problems, the worst scenario method is very convenient. The principle of the method can be paraphrased as "what is the worst that can happen on the given set of input data". Knowledge of the worst states (and of the data under which these states arise) can serve as a feedback to make some adjustments in the model/technological process. Of course, different configurations can be expected depending on the choice of the criterion functional.

We have focused on several linear as well as nonlinear problems of the monotone type with uncertainties in the coefficients of the equation. In the linear problems, the numerical experiments suggest the behaviour in the sense "higher values of particular components imply higher values of effective (homogenized) parameters and vice versa", however we should be careful in the case of strongly anisotropic media. Moreover, the experiments were performed for two-phased composites only.

One of the keystones of the worst scenario method is the compactness of the set of admissible functions U^{ad} in a suitable topology. In the case of nonlinear monotone problem discussed in Section 4, we have been successful due to the two restrictions. First, the ith component of the coefficient $a(y,\xi)$ is considered to be constant in the variable ξ except the ith component ξ_i, i.e. the problem is not treated in its full generality. Second, the uncertain coefficients were restricted to intervals of finite lengths so that the Arzelà-Ascoli theorem could be applied (it is not a significant limitation in practical problems, since these intervals can be arbitrarily large). A possible generalization and relaxation of the introduced properties is a subject for further research. In this context, we refer also to [35] and [36], where the worst scenario problem for a 1D and 2D monotone type state problem (with a coefficient in the form $a(x,\xi) = \tilde{A}(|\xi|^2)\xi$ and the matrix $\tilde{A}$ being uncertain) is analysed.

Further note that the second key step in the proofs of solvability of the worst scenario problems was the continuity of the solutions of the state problem on the input data (coefficients of equation).

Although several homogenization concepts considering also a non-periodic structure were developed, its practical use still remains restricted to the case of periodic structure, since we have not an easy analogy of the local (cell) problem as in the case of periodic structure. The question, whether the worst scenario method could be applied in some sense also to non-periodic heterogenous structures, remains an open problem and is a subject matter of further research.

Possible directions of further research include also a sensitivity analysis with respect to input data. It is based on investigating the stationary points of the criterion functional gradient, hence the critical data can be located better. Some other types of non-linear problems are also planned to be analysed.

A progress in development of mathematical models and methods of their solving involves also modelling by the so-called fractional differential equations, i.e. differential equations containing derivatives of a non-integer order. The study of such equations is also our aim of interest. The first results in this area have been already obtained and are surveyed in the following section.

7 Fractional differential and difference equations

The fractional calculus is a branch of mathematics extending the classical calculus to non-integer orders. Its origins fall to the end of 17th century, when l'Hospital and Leibniz in their correspondence opened the question of meaning of half order derivatives. Since then many famous names contributed to the theory, however, first applications appeared just before a few decades. At present, the fractional calculus is a well accepted theory and is a subject matter of huge interest of mathematicians as well as engineers, finding its applications in many areas such as viscoelasticity, fluid mechanics, control theory and others. The fundamentals of fractional calculus and a guide to applications can be found, e.g. in the monographs [65], [50], [63], [42].

A typical example serving as a motivation for the study of fractional differential equations is the well-known problem of diffusion. Roughly speaking, diffusion is a process of spreading particles through random motion from regions of higher concentration to regions of lower concentration. In a typical diffusion process, the mean squared displacement of a particle is proportional to time, i.e. $\langle x^2(t)\rangle \sim t$. It leads to the (parabolic) linear partial differential equation

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + f(x,t) \quad x \in I \subset \mathbb{R}\,,\; t > 0 \tag{7.1}$$

equipped with suitable initial and boundary conditions. However, in the case of anomalous diffusion, where the mean squared displacement is described by a power law $\langle x^2(t)\rangle \sim t^\alpha$, the modelling by (7.1) does not correspond to reality well. If $\alpha > 1$, the process is called super (or fast) diffusion and if $\alpha < 1$, we talk about sub (or slow) diffusion. These types of diffusion appear, e.g. in plasma physics, porous media or cellular processes. Traditionally, when the anomalous diffusion is considered, a nonlinearity appears in (7.1). Nevertheless, this phenomenon can be modelled with help of the fractional calculus too. Several models have been proposed so far including the replacement of both the time derivative as well as the spatial derivative in (7.1) by operators of a fractional order, see [63], [2], [26], [43] and the references therein.

Let us start with an introduction of (ordinary) fractional derivatives. Although many different definitions appeared in the past (see, e.g. [63]), the modern theory is usually based on the Riemann-Liouville and Caputo differential operators. In both cases, the Cauchy formula for repeated integration is a keystone in these introductions.

Theorem 7.1. (Cauchy formula) *Let $f(t)$ be integrable on an interval (a,b). Then, for any $n \in \mathbb{Z}^+$, we have*

$$\int_a^t \Big(\int_a^{\tau_n} \dots \Big(\int_a^{t_2} f(\tau_1)\,\mathrm{d}\tau_1 \Big)\,\mathrm{d}\tau_2 \Big) \dots \Big)\,\mathrm{d}\tau_n = \frac{1}{(n-1)!} \int_a^t (t-\tau)^{n-1} f(\tau)\,\mathrm{d}\tau\,, \quad t \in (a,b)\,.$$

The assertion can be quite easily proved by the induction principle. Using $(n-1)! = \Gamma(n)$, we can see that the right-hand side makes sense also for non-integer values of n, hence, the αth integral is defined as

$$I_a^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau\,, \quad \alpha \in \mathbb{R}^+\,.$$

Then the Riemann-Liouville and Caputo derivatives of order $\alpha \in \mathbb{R}^+$ are introduced as

$${}_{RL}D_a^\alpha f(t) := \frac{\mathrm{d}^{\lceil\alpha\rceil}}{\mathrm{d}t^{\lceil\alpha\rceil}} I_a^{\lceil\alpha\rceil-\alpha} f(t) \quad \text{and} \quad {}_{C}D_a^\alpha f(t) := I_a^{\lceil\alpha\rceil-\alpha} \frac{\mathrm{d}^{\lceil\alpha\rceil} f}{\mathrm{d}t^{\lceil\alpha\rceil}}(t)\,,$$

respectively, where $\lceil x\rceil$ is the ceiling function (the smallest integer greater or equal to x). Note that the fractional derivatives depend on the lower terminal a as the integral does. In other words, its value at t depends on the history – it is not a local operator.

Remark 7.2. Fractional partial derivatives are approached analogously to the classical case of integer order partial derivatives. Considering a two-variable function $f(t_1,t_2)$ and denoting by $I_{a_i}^{\alpha_i} f(t_1,t_2)$ the fractional integral of order α_i with respect to the variable t_i, $i = 1,2$, the

Riemann-Liouville fractional partial derivative with respect to t_i is introduced as

$$ {}_{RL}\partial_{a_i}^{\alpha_i} f(t_1,t_2) := \frac{\partial^{\lceil\alpha_i\rceil}}{\partial t_i^{\lceil\alpha_i\rceil}} I_{a_i}^{\lceil\alpha_i\rceil-\alpha_i} f(t_1,t_2)\,, \quad i=1,2 $$

and the mixed Riemann-Liouville fractional partial derivative is introduced as

$$ \begin{aligned} &{}_{RL}\partial_{a_1,a_2}^{\alpha_1+\alpha_2} f(t_1,t_2) := {}_{RL}\partial_{a_2}^{\alpha_2}\left({}_{RL}\partial_{a_1}^{\alpha_1} f(t_1,t_2)\right) \\ =&\frac{\partial^{\alpha_1+\alpha_2}}{\partial t_1^{\alpha_1}\partial t_2^{\alpha_2}} \int_{a_1}^{t_1}\int_{a_2}^{t_2} \frac{(t_1-\tau_1)^{\lceil\alpha_1\rceil-\alpha_1-1}(t_2-\tau_2)^{\lceil\alpha_2\rceil-\alpha_2-1}}{\Gamma(\lceil\alpha_1\rceil-\alpha_1)\Gamma(\lceil\alpha_2\rceil-\alpha_2)} f(\tau_1,\tau_2)\,\mathrm{d}\tau_1\mathrm{d}\tau_2\,. \end{aligned} $$

Similarly, we can proceed with the Caputo partial derivatives.

One of the classical methods of (numerical) solving of (7.1) is based on a discretization in one variable which results into a system of ordinary equations in the second variable. If we replace, e.g. the time derivative by a fractional derivative and discretize the equation in the spatial variable, this system becomes a fractional order differential system (in time) that can be (as in the classical case) further discretized. It motivates the study of ordinary fractional difference equations and their systems which is our main subject of interest. Surprisingly, a discrete counterpart of the fractional calculus is much less developed. The pioneering works in this field are contained in the papers [1], [3], [27], [34] and [49]. Building the theory on discrete sets actually requires the same key tools as in the continuous case. In particular, we can succeed if a discrete analogy of the Cauchy formula will be available and if its right-hand side allows an extension to non-integer orders.

Subsection 7.1 follows this approach and introduces discrete fractional sums and differences on a special two-parametric discrete set. Also an explanation of preference of the backward schemes to forward ones is contained here. In Subsection 7.2 we consider a certain linear initial value problem of fractional order and discuss its solvability as well as the structure of solution. Restricting to a special (two-term) equation we are able to find a closed form of the solution with help of a discrete Mittag-Leffler function. Possible directions of further research in the filed of fractional difference equations are outlined in Subsection 7.3.

7.1 On (q,h)-analogue of fractional calculus

In this subsection we summarise and comment the results of [23]. Our considerations are embedded in the so-called (q,h)-calculus framework. For certain reasons explained later, we prefer discretizations based on backward differences. We shall take advantage of the time scales theory and its notation.

By a time scale $\mathbb{T}$ we understand any non-empty and closed subset of reals. For any $t\in\mathbb{T}$ the backward jump operator is introduced as $\rho(t) := \sup\{s\in\mathbb{T},\, s<t\}$ and the backward graininess function $\nu(t) := t-\rho(t)$. For a function $f:\mathbb{T}\to\mathbb{R}$ we can define the so-called nabla derivative $f^{\nabla}(t)$, see [10] and [11]. This definition coincides with the classical derivative in the case of $\mathbb{T}=\mathbb{R}$. If $\mathbb{T}$ is a discrete time scale, i.e. such that $\nu(t)\neq 0$ for $t\in\mathbb{T}$, then $f^{\nabla}(t)$ exists for all $t\in\mathbb{T}$ (except the leftmost point of $\mathbb{T}$) and it is given by

$$ f^{\nabla}(t) = \frac{f(t)-f(\rho(t))}{\nu(t)}\,. \tag{7.2} $$

The nabla integral of f over the time scale interval $[a,b] := \{t\in\mathbb{T},\, a\le t\le b\}$, $a,\, b\in\mathbb{T}$ is defined by $\int_a^b f(t)\nabla t := F(b)-F(a)$, where F is an antiderivative of f, i.e. the function satisfying $F^{\nabla}=f$ on $\mathbb{T}$. If $a,\, b\in\mathbb{T}$ and $a>b$, then $\int_a^b f(t)\nabla t := -\int_b^a f(t)\nabla t$ and we put $\int_a^a f(t)\nabla t := 0$. It is known that considering discrete time scales, this nabla integral exists and can be calculated (provided $a<b$) via the formula $\int_a^b f(t)\nabla t = \sum_{t\in\mathbb{T}\cap(a,b]} f(t)\nu(t)$.

The most important discrete time scales are those originating from the arithmetic and geometric sequence of real numbers, namely $\mathbb{T}_h^{t_0} := \{t_0 + hk,\ k \in \mathbb{Z}\}$, $h > 0$ and $\mathbb{T}_q^{t_0} := \{t_0 q^k,\ k \in \mathbb{Z}\} \cup \{0\}$, $q > 1$, respectively, where $t_0 \in \mathbb{R}$. These sets form the basis for the study of h-calculus and q-calculus. Note that the standard definitions of nabla h-derivative (backward h-difference) and nabla q-derivative of f coincide with the general formula (7.2) via the choice $\rho(t) = t - h$ and $\rho(t) = q^{-1}t$ (provided $t_0 > 0$). Both these time scales are characterized by linearity of the backward jump operator, hence, the natural unification and extension of these discrete settings is enabled by the time scale with the backward jump operator $\rho(t) = q^{-1}(t-h)$. Denoting $[x]_q := \frac{q^x-1}{q-1}$, $q > 0$, we can observe that $\rho^k(t) = q^{-k}(t-[k]_q h) = q^{-k}t + [-k]_q h, \quad k \in \mathbb{Z}^+$, where the symbol ρ^k means the kth iterate of ρ. If we admit also non-positive integers in the previous formula, then $\rho^0(t) = t$, $\sigma(t) := \rho^{-1}(t) = t + h$ is the forward jump operator and $\sigma^k(t) := \rho^{-k}(t)$ is its kth iterate. Then, for a given $t_0 \in \mathbb{R}^+$, we define

$$\mathbb{T}_{(q,h)}^{t_0} := \{t_0 q^{-k} + [-k]_q h,\ k \in \mathbb{Z}\} \cup \Big\{\frac{h}{1-q}\Big\}, \qquad q \geq 1\,, h \geq 0\,, q + h > 1\,. \tag{7.3}$$

Obviously, $\mathbb{T}_{1,h}^{t_0} = \mathbb{T}_h^{t_0}$ (in this case we put $h/(1-q) := -\infty$) and $\mathbb{T}_{q,0}^{t_0} = \mathbb{T}_q^{t_0}$.

Further we introduce the q-factorial $[m]_q! := [m]_q[m-1]_q \ldots [1]_q$, $m \in \mathbb{Z}^+$, $q > 0$, and the (q,h)-power function $(t-s)_{(q,h)}^{(m)} := \prod_{j=0}^{m-1}(t - \rho^j(s))$, $m \in \mathbb{Z}_0^+$, $q \geq 1$. An extension of the nabla integral and derivative on $\mathbb{T}_{(q,h)}^{t_0}$ to fractional orders is based on the following analogue of Cauchy formula:

Theorem 7.3. (Nabla (q,h)-Cauchy formula) *Let $n \in \mathbb{Z}^+$, $f : \mathbb{T}_{(q,h)}^{t_0} \to \mathbb{R}$ and $a, t \in \mathbb{T}_{(q,h)}^{t_0}$. Then*

$${}_a\nabla_{(q,h)}^{-n} f(t) := \int_a^t \Big(\int_a^{\tau_n} \ldots \Big(\int_a^{\tau_2} f(\tau_1)\nabla\tau_1\Big) \ldots \nabla\tau_{n-1}\Big)\nabla\tau_n = \frac{1}{[n-1]_{q^{-1}}!}\int_a^t (t-\rho(\tau))_{(q,h)}^{(n-1)} f(\tau)\nabla\tau\,. \tag{7.4}$$

The assertion can be proved by the induction principle using the property $\nabla_{(q,h)}\int_a^t g(t,s)\nabla s = \int_a^t \nabla_{(q,h)} g(t,s)\nabla s + g(\rho(t),t)$ valid on any time scale $\mathbb{T}$.

Remark 7.4. To clarify the equality (7.4), we present also the form for $q = 1$ and $t_0 = h$ (expressed by sums), because of the importance of the time scale $\mathbb{T}_h^h = h\mathbb{Z}$ in numerical analysis. Let $n \in \mathbb{Z}^+$, $f : \mathbb{T}_h^h \to \mathbb{R}$ and $a, t \in \mathbb{T}_h^h$. Then

$$h^n \sum_{\tau_n = \frac{a}{h}+1}^{\frac{t}{h}} \ \sum_{\tau_{n-1} = \frac{a}{h}+1}^{\frac{\tau_n}{h}} \cdots \sum_{\tau_1 = \frac{a}{h}+1}^{\frac{\tau_2}{h}} f(h\tau_1) = h \sum_{\tau = \frac{a}{h}+1}^{\frac{t}{h}} \frac{\prod_{j=1}^{n-1}(t - h\tau + hj)}{(n-1)!} f(h\tau)\,.$$

Remark 7.5. The equality (7.4) can be viewed also as the formula for the nth (q,h)-integral of the constant function $f(t) = 1$ on $\mathbb{T}_{(q,h)}^{t_0}$. This suggests that the nabla Cauchy formula could be written on any discrete $\mathbb{T}$ if the repeated integration of a constant function is available, i.e. if we have an explicitly given system of monomials $\hat{h}_n(t,s)$ (with $\hat{h}_0(t,s) = 1$) such that $(\hat{h}_n(t,s))^\nabla = \hat{h}_{n-1}(t,s)$. However, the calculation of $\hat{h}_n(t,s)$ for $n > 1$ is a difficult task which seems to be solvable only in some particular cases.

The q-factorial on the right-hand side of (7.4) can be extended to non-integer values by the q-Gamma function defined

$$\Gamma_{q^{-1}}(x) = \frac{(q^{-1}, q^{-1})_\infty (1-q^{-1})^{1-x}}{(q^{-x}, q^{-1})_\infty}\,, \quad q > 1\,,\ x \in \mathbb{R} \setminus \{0, -1, \ldots\}\,, \tag{7.5}$$

where $(a, q^{-1})_\infty := \prod_{j=0}^{\infty}(1 - aq^{-j})$.

Remark 7.6. The q-gamma function is usually introduced for $0 < q < 1$ (it can be introduced also for $q > 1$, however, the resulting definition is not – in terms of properties – fully equivalent to the case $0 < q < 1$, for details see [48]). Since the definition of $\mathbb{T}^{t_0}_{(q,h)}$ assumes $q \geq 1$, the reciprocal value q^{-1} is used in (7.5). The q-gamma function retains the q-factorial property $\Gamma_{q^{-1}}(x+1) = [x]_{q^{-1}}\Gamma_{q^{-1}}(x)$ and it becomes the standard Euler gamma function if $q \to 1+$.

Similarly, the (q,h)-power function can be extended to non-integer values as

$$(t-s)^{\alpha}_{(q,h)} := \begin{cases} [t]^{\alpha} \dfrac{([s]/[t], q^{-1})_{\infty}}{(q^{-\alpha}[s]/[t], q^{-1})_{\infty}} & \text{for } q > 1\,,\ h \geq 0\,, \\ h^{\alpha} \dfrac{\Gamma((t-s)/h + \alpha)}{\Gamma((t-s)/h)} & \text{for } q = 1\,,\ h > 0\,, \end{cases}$$

where $[t] = t + hq^{-1}/(1-q^{-1})$ and $[s] = s + hq^{-1}/(1-q^{-1})$, for details see [23] and [20]. Now we are in a position, where the nabla (q,h)-fractional integral (backward (q,h)-fractional sum) can be introduced.

Definition 7.7. Let $\alpha \in \mathbb{R}^+$, $f : \mathbb{T}^{t_0}_{(q,h)} \to \mathbb{R}$ and let $a, t \in \mathbb{T}^{t_0}_{(q,h)}$. Then we define the nabla (q,h)-fractional integral of f at t by

$${}_a\nabla^{-\alpha}_{(q,h)} f(t) := \frac{1}{\Gamma_{q^{-1}}(\alpha)} \int_a^t (t - \rho(\tau))^{\alpha-1}_{(q,h)} f(\tau) \nabla(\tau)\,. \tag{7.6}$$

In accordance with the continuous case, the nabla (q,h)-fractional derivative (backward (q,h)-fractional difference) in the Riemann-Liouville sense is introduced as follows.

Definition 7.8. Let $\alpha \in \mathbb{R}^+$, $f : \mathbb{T}^{t_0}_{(q,h)} \to \mathbb{R}$ and let $a, t \in \mathbb{T}^{t_0}_{(q,h)}$. Then we define the nabla (q,h)-fractional derivative of f at t by

$${}_a\nabla^{\alpha}_{(q,h)} f(t) := \nabla^{\lceil \alpha \rceil}_{(q,h)}\, {}_a\nabla^{-(\lceil \alpha \rceil - \alpha)}_{(q,h)} f(t)\,, \tag{7.7}$$

where $\nabla^m_{(q,h)}$ is the mth (q,h)-derivative (given iteratively $\nabla^m_{(q,h)} := \nabla_{(q,h)} \nabla^{m-1}_{(q,h)}$).

From the practical viewpoint the following form of (7.6) is more convenient:

$${}_a\nabla^{-\alpha}_{(q,h)} f(t) := \nu^{\alpha}(t) \sum_{k=0}^{n-1} (-1)^k \begin{bmatrix} -\alpha \\ k \end{bmatrix}_{q^{-1}} q^{-\binom{k}{2} - k(1+\alpha)} f(\rho^k(t))\,,$$

where $n \in \mathbb{Z}^+$ is given by $a = \rho^n(t)$ and

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q^{-1}} = \frac{\Gamma_{q^{-1}}(\alpha+1)}{\Gamma_{q^{-1}}(\beta+1)\Gamma_{q^{-1}}(\alpha-\beta+1)}$$

is the q-binomial coefficient. Similarly, expanding (7.7) yields

$${}_a\nabla^{\alpha}_{(q,h)} f(t) = \begin{cases} \nu^{-\alpha}(t) \displaystyle\sum_{k=0}^{n-1} (-1)^k \begin{bmatrix} \alpha \\ k \end{bmatrix}_{q^{-1}} q^{-\binom{k}{2} - k(1+\alpha)} f(\rho^k(t))\,, & \alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+\,, \\ \nu^{-\alpha}(t) \displaystyle\sum_{k=0}^{\alpha} (-1)^k \begin{bmatrix} \alpha \\ k \end{bmatrix}_{q^{-1}} q^{-\binom{k}{2} - k(1+\alpha)} f(\rho^k(t))\,, & \alpha \in \mathbb{Z}^+\,. \end{cases} \tag{7.8}$$

Remark 7.9. Although the delta (forward) (q,h)-fractional counterpart can be built in the same way as in the nabla case, we have to deal with one inconvenience. The right-hand side of delta (q,h)-Cauchy formula is in the form

$$\int_a^{\rho^{n-1}(t)} g_{n-1}(t,\tau)\, \nabla\tau\,,$$

where g_{n-1} contains again the (q,h)-power function and the q-factorial. It means that, in addition to the nabla case, we have to extend the term $\rho^{n-1}(t)$ in the upper terminal of integration to non-integer values. The quantity $\rho^m(t) = q^{-m}t + [-m]_q h$, $m \in \mathbb{Z}$, makes sense also if we replace the integer m by any real value, this is closely related to the problem of continuous iterations, for details see [23]. On the other hand, using the operator $\rho^{\alpha-1}(t)$, $\alpha \in \mathbb{R}$, in the upper terminal causes the fact that the resulting domain of fractional integration differs form $\mathbb{T}^{t_0}_{(q,h)}$, hence fractional discretizations based on nabla (backward) differences are preferred.

The paper [23] further discusses basic properties of (q,h)-fractional integrals and derivatives such as the composition rule. While the composition rule is valid for the (q,h)-fractional integrals, it is generally not true for the (q,h)-fractional derivatives, see Remark 7.20 below for a counterexample. It corresponds well to the continuous case.

7.2 Discrete Mittag-Leffler functions in linear fractional difference equations

In this subsection we survey the results published in [20]. We shall discuss solvability of certain fractional difference equation on $\mathbb{T}^{t_0}_{(q,h)}$. Let $a \in \mathbb{T}^{t_0}_{(q,h)}$ such that $a > h/(1-q)$ and let $\widetilde{\mathbb{T}}^{\sigma^m(a)}_{(q,h)}$ denote the restriction of $\mathbb{T}^{t_0}_{(q,h)}$ given by $\widetilde{\mathbb{T}}^{\sigma^m(a)}_{(q,h)} := \{t \in \mathbb{T}^{t_0}_{(q,h)} : t \geq \sigma^m(a), m = 0, 1, \dots\}$ (we remind that $\sigma(t) = qt + h$ is the forward jump operator and $\sigma^m(t)$ is its mth iterate). For any $\alpha \in \mathbb{R}^+$ let us consider the following initial value problem:

$$\sum_{j=1}^{\lceil\alpha\rceil} p_{\lceil\alpha\rceil-j+1}(t)\ {}_a\nabla^{\alpha-j+1}_{(q,h)} y(t) + p_0(t)\, y(t) = 0\,, \quad t \in \widetilde{\mathbb{T}}^{\sigma^{\lceil\alpha\rceil+1}(a)}_{(q,h)}\,, \tag{7.9}$$

$${}_a\nabla^{\alpha-j}_{(q,h)} y(t)\big|_{t=\sigma^{\lceil\alpha\rceil}(a)} = y_{\alpha-j}\,, \quad j = 1, 2, \dots, \lceil\alpha\rceil\,, \tag{7.10}$$

where $p_j(t)$ are arbitrary (real) functions on $\widetilde{\mathbb{T}}^{\sigma^{\lceil\alpha\rceil+1}(a)}_{(q,h)}$, $j = 1, \dots, \lceil\alpha\rceil - 1$, $p_{\lceil\alpha\rceil}(t) = 1$ on $\widetilde{\mathbb{T}}^{\sigma^{\lceil\alpha\rceil+1}(a)}_{(q,h)}$ and $y_{\alpha-j}$, $j = 1, \dots, \lceil\alpha\rceil$, are arbitrary real scalars.

Considering some regularity condition on coefficients $p_j(t)$ we want to show that (7.9), (7.10) possesses a unique solution. This result as well as the structure of solution are well-known from literature if α is a positive integer. If α is not an integer, then expanding the definition of nabla (q,h)-fractional derivative (see (7.7) and (7.8)) we can observe that the equation (7.9) is of the general form $\sum_{i=0}^{n-1} a_i(t) y(\rho^i(t)) = 0$, $t \in \widetilde{\mathbb{T}}^{\sigma^{\lceil\alpha\rceil+1}(a)}_{(q,h)}$, n being such that $t = \sigma^n(a)$ which is usually called the equation of Volterra type. If such equation has two different solutions, then their values differ at least at one of the points $\sigma(a), \sigma^2(a), \dots, \sigma^{\lceil\alpha\rceil}(a)$. In particular, if $a_0(t) \neq 0$ for all $t \in \widetilde{\mathbb{T}}^{\sigma^{\lceil\alpha\rceil+1}(a)}_{(q,h)}$, then arbitrary values of $y(\sigma(a)), y(\sigma^2(a)), \dots, y(\sigma^{\lceil\alpha\rceil}(a))$ determine uniquely the solution $y(t)$ for all $t \in \widetilde{\mathbb{T}}^{\sigma^{\lceil\alpha\rceil+1}(a)}_{(q,h)}$. The following proposition (for the proof see [20]) expresses that the values $y_{\alpha-1}$, $y_{\alpha-2}$, $\dots$, $y_{\alpha-\lceil\alpha\rceil}$ introduced by (7.10) keep the same property.

Proposition 7.10. *Let $y : \widetilde{\mathbb{T}}^{\sigma(a)}_{(q,h)} \to \mathbb{R}$ be a function. Then (7.10) represents a one-to-one mapping between $(y(\sigma(a)), y(\sigma^2(a)), \dots, y(\sigma^{\lceil\alpha\rceil}(a)))$ and $(y_{\alpha-1}, y_{\alpha-2}, \dots, y_{\alpha-\lceil\alpha\rceil})$.*

The solvability of (7.9), (7.10) is based on the notion of ν-regresivity. The general notion of ν-regressivity (on arbitrary time scale) of a matrix function and a corresponding linear nabla dynamic system can be found in [11]. Considering a higher order linear difference equation, the notion of ν-regressivity for such an equation can be introduced by means of its transformation to the corresponding first order linear dynamic system. We are going to follow this approach and generalize the ν-regressivity for the linear fractional difference equation (7.9).

Definition 7.11. Let $\alpha \in \mathbb{R}^+$. Then the equation (7.9) is called ν^α-regressive provided the matrix

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -\frac{p_0(t)}{\nu^{\lceil\alpha\rceil-\alpha}(t)} & -p_1(t) & \cdots & -p_{\lceil\alpha\rceil-2}(t) & -p_{\lceil\alpha\rceil-1}(t) \end{pmatrix} \tag{7.11}$$

satisfies $\det(I-\nu(t)A(t)) \neq 0$ for all $t \in \widetilde{\mathbb{T}}_{(q,h)}^{\sigma^{\lceil\alpha\rceil+1}(a)}$.

Remark 7.12. The explicit expression of the ν^α-regressivity property for (7.9) can be read as $1+\sum_{j=1}^{\lceil\alpha\rceil-1} p_{\lceil\alpha\rceil-j}(t)(\nu(t))^j + p_0(t)(\nu(t))^\alpha \neq 0$ for all $t \in \widetilde{\mathbb{T}}_{(q,h)}^{\sigma^{\lceil\alpha\rceil+1}(a)}$. If α is a positive integer, then both these introductions agree with the definition of ν-regressivity of a higher order linear difference equation presented in [11].

Theorem 7.13. *Let (7.9) be ν^α-regressive. Then the problem (7.9)–(7.10) has a unique solution defined for all $t \in \widetilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$.*

The proof is based on the following steps. Proposition 7.10 allows us to determine the values $y(\sigma(a)), y(\sigma^2(a)), \ldots, y(\sigma^{\lceil\alpha\rceil}(a))$. The substitution $z_j(t) = {}_a\nabla_{(q,h)}^{\alpha-\lceil\alpha\rceil+j-1} y(t)$, $t \in \widetilde{\mathbb{T}}_{(q,h)}^{\sigma^j(a)}$, $j = 1,2,\ldots,\lceil\alpha\rceil$ transforms (after some rearrangement of the equation $z_1(t) = {}_a\nabla_{(q,h)}^{\alpha-\lceil\alpha\rceil} y(t)$) the problem (7.9), (7.10) into the vector from

$$\nabla_{(q,h)} z(t) = A(t)z(t) + b(t)\,, \quad z(\sigma^{\lceil\alpha\rceil}(a)) = (y_{\alpha-\lceil\alpha\rceil}, \ldots, y_{\alpha-1})^T\,, \quad t \in \widetilde{\mathbb{T}}_{(q,h)}^{\sigma^{\lceil\alpha\rceil+1}(a)}\,,$$

where $A(t)$ is given by (7.11). The ν^α-regressivity of the matrix $A(t)$ enables to write

$$z(t) = (I-\nu(t)A(t))^{-1}(z(\rho(t)) + \nu(t)b(t))\,, \quad t \in \widetilde{\mathbb{T}}_{(q,h)}^{\sigma^{\lceil\alpha\rceil+1}(a)}\,,$$

hence, using the value of $z(\sigma^{\lceil\alpha\rceil}(a))$, we can solve this system by the step method starting from $t = \sigma^{\lceil\alpha\rceil+1}(a)$.

Similarly to the classical case of differential/difference equation of nth order, a linear independence of solutions to (7.9) plays an essential role. We start with the following notion.

Definition 7.14. Let $\gamma \in \mathbb{R}$, $0 \le \gamma < 1$. For m functions $y_j : \widetilde{\mathbb{T}}_{(q,h)}^a \to \mathbb{R}$, $j = 1,2,\ldots,m$, we define the γ-Wronskian $W_\gamma(y_1,\ldots,y_m)(t)$ as determinant of the matrix

$$V_\gamma(y_1,\ldots,y_m)(t) = \begin{pmatrix} {}_a\nabla_{(q,h)}^{-\gamma} y_1(t) & {}_a\nabla_{(q,h)}^{-\gamma} y_2(t) & \cdots & {}_a\nabla_{(q,h)}^{-\gamma} y_m(t) \\ {}_a\nabla_{(q,h)}^{1-\gamma} y_1(t) & {}_a\nabla_{(q,h)}^{1-\gamma} y_2(t) & \cdots & {}_a\nabla_{(q,h)}^{1-\gamma} y_m(t) \\ \vdots & \vdots & \ddots & \vdots \\ {}_a\nabla_{(q,h)}^{m-1-\gamma} y_1(t) & {}_a\nabla_{(q,h)}^{m-1-\gamma} y_2(t) & \cdots & {}_a\nabla_{(q,h)}^{m-1-\gamma} y_m(t) \end{pmatrix}, \quad t \in \widetilde{\mathbb{T}}_{(q,h)}^{\sigma^m(a)}.$$

Note that $W_\gamma(y_1,\ldots,y_m)(t)$ coincides for $\gamma = 0$ with the classical definition of the Wronskian (see [10]). Moreover, $W_\gamma(y_1,\ldots,y_m)(t) = W_0({}_a\nabla_{(q,h)}^{-\gamma} y_1, \ldots, {}_a\nabla_{(q,h)}^{-\gamma} y_m)(t)$.

The structure of solution to (7.9) is described by the following assertion.

Theorem 7.15. *Let functions $y_1(t),\ldots,y_{\lceil\alpha\rceil}(t)$ be solutions of the ν-regressive equation (7.9) and let $W_{\lceil\alpha\rceil-\alpha}(y_1,\ldots,y_{\lceil\alpha\rceil})(\sigma^{\lceil\alpha\rceil}(a)) \neq 0$. Then any solution $y(t)$ of (7.9) can be written in the form $y(t) = \sum_{k=1}^{\lceil\alpha\rceil} c_k y_k(t)$, $t \in \widetilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$, where $c_1,\ldots,c_{\lceil\alpha\rceil}$ are real constants.*

To prove the statement let us take any solution $y(t)$ of (7.9). By Proposition 7.10 there exist constants $y_{\alpha-1},\dots,y_{\alpha-\lceil\alpha\rceil}$ such that $y(t)$ is satisfying (7.10). Consider now $u(t)=\sum_{k=1}^{\lceil\alpha\rceil} c_k y_k(t)$, where the $\lceil\alpha\rceil$-tuple $(c_1,\dots,c_{\lceil\alpha\rceil})$ is the unique solution of

$$W_{\lceil\alpha\rceil-\alpha}(y_1,\dots,y_{\lceil\alpha\rceil})(\sigma^{\lceil\alpha\rceil}(a))\cdot\begin{pmatrix} c_1\\ c_2\\ \vdots\\ c_{\lceil\alpha\rceil}\end{pmatrix}=\begin{pmatrix} y_{\alpha-\lceil\alpha\rceil}\\ y_{\alpha-\lceil\alpha\rceil+1}\\ \vdots\\ y_{\alpha-1}\end{pmatrix}.$$

The linearity of (7.9) implies that $u(t)$ has to be its solution. Moreover, ${}_a\nabla^{\alpha-j}_{(q,h)}u(t)\big|_{t=\sigma^{\lceil\alpha\rceil}(a)}=y_{\alpha-j}$, $j=1,2,\dots,\lceil\alpha\rceil$, hence $u(t)$ is a solution of the initial value problem (7.9), (7.10) and by Theorem 7.13, it must be $y(t)=u(t)$ and the proof is complete.

Remark 7.16. The $\lceil\alpha\rceil$-tuple of solutions to (7.9) having a non-zero Wronskian surely exists – it is sufficient to take, e.g. the starting vectors e_i, $i=1,\dots,\lceil\alpha\rceil$, in (7.10), where e_i is the unit orthogonal basis vector.

Two-term equation and (q,h)-Mittag-Leffler function

Now, let us turn our attention to eigenfunctions of the fractional operator ${}_a\nabla^{\alpha}_{(q,h)}$, $\alpha\in\mathbb{R}^+$. In other words, we are going to solve the equation (7.9) in the special form

$${}_a\nabla^{\alpha}_{(q,h)}y(t)=\lambda y(t)\,,\qquad \lambda\in\mathbb{R}\,,\qquad t\in\widetilde{\mathbb{T}}^{\sigma^{\lceil\alpha\rceil+1}(a)}_{(q,h)}\,. \tag{7.12}$$

We assume that the ν^{α}-regressivity condition for (7.12) is ensured, i.e. $\lambda(\nu(t))^{\alpha}\neq 1$. A development of methods of solving fractional difference equations is just at the beginning. Some techniques how to explicitly solve these equations (at least in particular cases) are shown, e.g. in [7], [8] and [47], where a discrete analogue of the Laplace transform turns out to be the most developed method. Here, we present the technique originating from the role played by the Mittag-Leffler function in the continuous fractional calculus (see, e.g. [65]). In particular, we introduce the notion of a discrete Mittag-Leffler function in a setting formed by the time scale $\widetilde{\mathbb{T}}^{a}_{(q,h)}$ and demonstrate its significance with respect to eigenfunctions of the operator ${}_a\nabla^{\alpha}_{(q,h)}$. These our results generalize and extend those derived in [54] and [19].

The Mittag-Leffler function is a generalization of the exponential function and its two-parameteric form (more convenient in the fractional calculus) can be introduced for $\mathbb{T}=\mathbb{R}$ by the series expansion

$$E_{\alpha,\beta}(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+\beta)}\,,\qquad \alpha,\,\beta\in\mathbb{R}^+,\quad t\in\mathbb{R}\,. \tag{7.13}$$

The fractional calculus frequently employs (7.13), because the function

$$t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})=\sum_{k=0}^{\infty}\lambda^k\frac{t^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)} \tag{7.14}$$

(a modified Mittag-Leffler function, see [65]) satisfies, under special choices of β, a continuous (differential) analogy of (7.12). Some extensions of the definition formula (7.13) and their utilization in special fractional calculus operators can be found in [40] and [41].

Considering a discrete calculus, the form (7.14) seems to be much more convenient for discrete extensions than the form (7.13) which requires, among others, the validity of the law of exponents. The following introduction extends the discrete Mittag-Leffler function defined and studied in [53].

Definition 7.17. Let α, β, $\lambda\in\mathbb{R}$. We introduce the (q,h)-Mittag-Leffler function $E^{s,\lambda}_{\alpha,\beta}(t)$ by the series expansion

$$E^{s,\lambda}_{\alpha,\beta}(t)=\sum_{k=0}^{\infty}\lambda^k\frac{(t-s)^{(\alpha k+\beta-1)}_{(q,h)}}{\Gamma_{q^{-1}}(\alpha k+\beta)}\,,\quad s,t\in\widetilde{\mathbb{T}}^{a}_{(q,h)},\ t\geq s\,.$$

It is easy to check that the series on the right-hand side converges (absolutely) if $|\lambda|(\nu(t))^\alpha<1$. As might be expected, the particular (q,h)-Mittag-Leffler function $E^{a,\lambda}_{1,1}(t)=\prod_{k=0}^{n-1}\frac{1}{1-\lambda\nu(\rho^k(t))}$, where $n\in\mathbb{Z}^+$ satisfies $t=\sigma^n(a)$, is the solution of the initial value problem

$$\nabla_{(q,h)}y(t)=\lambda y(t),\quad y(a)=1,\quad t\in\widetilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}\,,\tag{7.15}$$

i.e. it is a discrete (q,h)-analogue of the exponential function.

Remark 7.18. Setting $q=1$, (7.15) becomes the implicit Euler method for numerical solving of the classical testing initial value problem $y'=\lambda y$, $y(a)=1$, where the approximate solution is in the form $E^{a,\lambda}_{1,1}(t)=(1-\lambda h)^{(t-a)/h}$.

The main property of the (q,h)-Mittag-Leffler function is described by the following assertion.

Theorem 7.19. *Let $\eta\in\mathbb{R}^+$ and $t\in\widetilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$. Then ${}_a\nabla^{-\eta}_{(q,h)}E^{a,\lambda}_{\alpha,\beta}(t)=E^{a,\lambda}_{\alpha,\beta+\eta}(t)$. If moreover $\alpha k+\beta-1\notin\{0,-1,\dots,-\lceil\eta\rceil+1\}$ for all $k\in\mathbb{Z}^+$ and $t\in\widetilde{\mathbb{T}}^{\sigma^{\lceil\eta\rceil+1}(a)}_{(q,h)}$ then*

$${}_a\nabla^{\eta}_{(q,h)}E^{a,\lambda}_{\alpha,\beta}(t)=\begin{cases}E^{a,\lambda}_{\alpha,\beta-\eta}(t)\,, & \beta-\eta\notin\{0,-1,\dots,-\lceil\eta\rceil+1\}\,,\\ \lambda E^{a,\lambda}_{\alpha,\beta-\eta+\alpha}(t)\,, & \beta-\eta\in\{0,-1,\dots,-\lceil\eta\rceil+1\}\,.\end{cases}$$

The proof is based on the key properties of the (q,h)-power function: for any $\alpha\in\mathbb{R}^+$, $\beta\in\mathbb{R}$ and $t\in\widetilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$ we have

$${}_a\nabla^{-\alpha}_{(q,h)}\frac{(t-a)^{(\beta)}_{(q,h)}}{\Gamma_{q^{-1}}(\beta+1)}=\frac{(t-a)^{(\alpha+\beta)}_{(q,h)}}{\Gamma_{q^{-1}}(\alpha+\beta+1)}\,,$$

and, if moreover $t\in\widetilde{\mathbb{T}}^{\sigma^{\lceil\alpha\rceil+1}(a)}_{(q,h)}$, then

$${}_a\nabla^{\alpha}_{(q,h)}\frac{(t-a)^{(\beta)}_{(q,h)}}{\Gamma_{q^{-1}}(\beta+1)}=\begin{cases}\dfrac{(t-a)^{(\beta-\alpha)}_{(q,h)}}{\Gamma_{q^{-1}}(\beta-\alpha+1)}\,, & \beta-\alpha\notin\{-1,\dots,-\lceil\alpha\rceil\}\,,\\ 0\,, & \beta-\alpha\in\{-1,\dots,-\lceil\alpha\rceil\}\,.\end{cases}\tag{7.16}$$

These properties can be proved with help of the q-Vandermonde identity.

Remark 7.20. The relation (7.16) can serve as a counterexample demonstrating that the composition rule is (in general) not valid in the case of fractional differences. Indeed, taking any $\alpha,\beta\in\mathbb{R}^+\setminus Z^+$ such that $\beta-\alpha\in\{-1,\dots,-\lceil\alpha\rceil\}$, then for any $\gamma\in\mathbb{R}^+$ such that $\beta-(\alpha+\gamma)\notin\{-1,\dots,-\lceil\alpha+\gamma\rceil\}$ we have ${}_a\nabla^{\gamma}_{(q,h)}\,{}_a\nabla^{\alpha}_{(q,h)}\neq{}_a\nabla^{\alpha+\gamma}_{(q,h)}$.

By Theorem 7.19 we immediately have

Corollary 7.21. *Let $\alpha\in\mathbb{R}^+$. Then the functions $E^{a,\lambda}_{\alpha,\beta}(t)$, $\beta=\alpha-\lceil\alpha\rceil+1,\dots,\alpha-1,\alpha$ define eigenfunctions of the operator ${}_a\nabla^{\alpha}_{(q,h)}$ on each set $[\sigma(a),b]\cap\widetilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$, where $b\in\widetilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$ is satisfying $|\lambda|(\nu(b))^\alpha<1$.*

It can be shown that the Wronskian $W_{\lceil\alpha\rceil-\alpha}(E^{a,\lambda}_{\alpha,\alpha-\lceil\alpha\rceil+1},E^{a,\lambda}_{\alpha,\alpha-\lceil\alpha\rceil+2},\dots,E^{a,\lambda}_{\alpha,\alpha})(\sigma^{\lceil\alpha\rceil}(a))=\prod_{k=1}^{\lceil\alpha\rceil}\frac{1}{1-\lambda(\nu(\sigma^k(a)))^\alpha}\neq0$, hence, as a consequence of Theorem 7.15, we have:

Theorem 7.22. *Let $y(t)$ be any solution of (7.12) defined on $[\sigma(a),b]\cap\widetilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$, where $b\in\widetilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$ is satisfying $|\lambda|(\nu(b))^\alpha<1$. Then $y(t)=\sum_{j=1}^{\lceil\alpha\rceil}c_jE^{a,\lambda}_{\alpha,\alpha-\lceil\alpha\rceil+j}(t)$, where $c_1,\dots,c_{\lceil\alpha\rceil}$ are real constants.*

7.3 Further research

Our further research is directed towards investigation of qualitative properties of fractional difference equations. In the paper [21] we have considered (7.12) restricted to the case $0 < \alpha \leq 1$ and $q = h = 1$. This equation can be understood as a fractional analogue of the backward difference scheme $\nabla y(t) = \lambda y(t)$, hence it is a good starting point for a qualitative analysis. In particular, we have obtained some stability and asymptotic results of (7.12) under considerations. Our method is based on converting the equation onto a Volterra difference equation of convolution type. The theory of Volterra difference equations of convolution type is discussed in [28] utilizing the Z-transform as a basic tool. In addition, we have answered some open questions related to stability of this type of equations.

An extension of (7.12) to the vector valued case in the setting formed by $\mathbb{T}_h^{t_0}$ has been studied in the forthcoming paper [22]. Contrary to the scalar case, the backward h-Laplace transform is employed in the study of stability properties. The backward h-Laplace transform of a function $f : \mathbb{T}_h^{t_0} \to \mathbb{R}$ is given by a power series with coefficients $f(t_n)$, where $t_n = t_0 + nh$, $n = 1, 2, \dots$ and the center at h^{-1}. The idea is to locate the singular points of the h-Laplace image of the system (that depend on the eigenvalues of system's matrix). This provides an information on the radius of convergence of the transform and consequently on the limit behaviour of the solution.

The results obtained in [21] and [22] represent a tool for numerical analysis of the fractional differential equation ${}_{RL}D^{\alpha}y(t) = \lambda y(t)$, $0 < \alpha < 1$, $\lambda \in \mathbb{R}$, and the fractional differential system ${}_{RL}D^{\alpha}y(t) = Ay(t)$, $0 < \alpha < 1$, $A \in \mathbb{R}^{d\times d}$, respectively. In particular, they serve for a discussion on stability of basic numerical schemes of these equations. It is consequently useful in analysis of a fractional analogue of the diffusion equation (7.1).

Going back to the main theme of this thesis, possible directions of further research include also the homogenization of fractional differential operators. To the author's knowledge, this approach is not much developed yet. In this direction, we refer, e.g. to [5], [12], [45], where the homogenization of fractional diffusion equation (under some special domain and initial conditions settings) has been studied.

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Abstract

This thesis deals with the homogenization of certain partial differential equations with respect to uncertain input parameters in the equation's coefficients. As a basic tool for treating such the uncertainties, the so-called worst scenario method is used, looking for the "dangerous" states. Several linear as well as nonlinear (of the strongly monotone type) problems are discussed. A summary of results in the field of fractional calculus is also presented.