VĚDECKÉ SPISY VYSOKÉHO UČENÍ TECHNICKÉHO V BRNĚ
Edice Habilitační a inaugurační spisy, sv. 523
ISSN 1213-418X

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ASYMPTOTIC PROPERTIES OF LINEAR DIFFERENTIAL AND DIFFERENCE EQUATIONS WITH DELAY

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Ústav matematiky

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ASYMPTOTICKÉ VLASTNOSTI LINEÁRNÍCH DIFERENCIÁLNÍCH A DIFERENČNÍCH ROVNIC SE ZPOŽDĚNÍM

ZKRÁCENÁ VERZE HABILITAČNÍ PRÁCE
APLIKOVANÁ MATEMATIKA

VUTIUM

BRNO 2015

KEYWORDS
delay differential equation, linear difference equation, dynamic equation on time scales, asymptotic stability, asymptotic properties.

KLÍČOVÁ SLOVA
diferenciální rovnice se zpožděním, lineární diferenční rovnice, dynamická rovnice na časových škálách, asymptotická stabilita, asymptotické vlastnosti.

MÍSTO ULOŽENÍ PRÁCE

Ústav matematiky
Fakulta strojního inženýrství
Vysoké učení technické v Brně
Technická 2896/2
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ISBN 978-80-214-5284-8
ISSN 1213-418X

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1 Introduction

The thesis is based on the author's research since the year 2003. There are four topics discussed in the text below: asymptotic estimates for linear delay differential equations, asymptotic estimates for linear difference equations, their asymptotic stability properties and finally asymptotic estimates for linear dynamic equations on time scales. The results introduced in the thesis are taken from the author's contributions [60–65] and papers, in which the author has participated, namely [21, 23–25, 45]. Since this is a shortened form of habilitation thesis, some assertions and proofs are omitted. But the main structure of the thesis remains preserved: within this introductory section, we are going to mention some applications of particular delay differential equations, which are a subject of the consequent discussions. Section 2 introduces two asymptotic estimates for linear differential equations with unbounded delays. Section 3 presents some asymptotic estimates for linear difference equations obtained via a discretization of those studied in Section 2. Section 4 recalls some existing stability criteria for delay differential equations with a constant delay. In Section 5, we consider their basic discretizations and introduce several types of necessary and sufficient stability conditions for these discrete equations. Section 6 unifies asymptotic estimates for differential and difference equations with unbounded delays in the frame of time scales theory.

We start with a brief introduction to linear delay differential equations, because all presented author's results are more or less connected to this topic. We illustrate some basic notions by the first order linear delay differential equation

$$\dot{x}(t) = ax(t) + bx(\tau(t)), \qquad t \in I = [t_0, \infty), \tag{1.1}$$

where $a, b \in \mathbb{R}$ and $\tau(t) \in C(I)$ is such that $\tau(t) < t$ for all $t > t_0$. Then $\tau(t)$ is referred to as *delayed argument* and the term $t - \tau(t)$ is called *delay* (*lag*). We distinguish two types of delay: if there exists $M > 0$ such that $t - \tau(t) < M$ for all $t > t_0$ then we have the *bounded delay* and if $t - \tau(t) \to \infty$ as $t \to \infty$ then we talk about the *unbounded delay.* A prototype of a bounded delay is the constant delay, i.e. $\tau(t) = t - \tau^*$, where $\tau^* \in \mathbb{R}^+$. On the contrary, an example of an unbounded delay is the proportional one, i.e. $\tau(t) = qt$, where $q \in (0, 1)$ is a real parameter.

We denote $I_0 = [t_{-1}, t_0]$, where $t_{-1} = \inf_{t \in I}\{\tau(t)\}$. As it is customary, the function $x(t)$ is called a solution of (1.1) if $x \in C(I_0) \cap C^1(I)$ and satisfies (1.1) for all $t \in I$. If we are given a continuous function $x_0(t)$ defined on I_0, then there is a unique solution $x(t)$ of (1.1) such that $x(t) \equiv x_0(t)$ on I_0. We call the homogeneous linear delay differential equation (1.1) *asymptotically stable* if all its solutions satisfy $x(t) \to 0$ as $t \to \infty$. For more fundamentals of related general and qualitative theory we refer to [66].

To describe the asymptotic behaviour of the studied delay differential equations, we utilize the standard asymptotic estimate notation

$$f(t) = O(g(t)) \quad \text{as} \quad t \to \infty,$$

where $g \in C(I)$. It means that there exist $K > 0$ and $t^\star \in \mathbb{R}$ such that the relation $|f(t)| \leq K|g(t)|$ is satisfied for all $t \geq t^\star$. Analogously we introduce this asymptotic symbol in the discrete case.

Motivation

Delay differential equations arise in many applications in medicine, biology, economy, social sciences and engineering (see e.g. Kolmanovskii and Myshkis [66] or Smith [90]). We sketch three problems which illustrate the usefulness and efficiency of the delay differential equation modelling. The first model describes the dynamics of the locomotive's pantograph moving nearby the support of a trolley wire. In this case a proportional (unbounded) delay is considered. The second application comes from population dynamics and it leads to the delay differential equation with a constant (bounded) delay. The third model deals with fuel combustion in small rockets.

Pantograph equation

In this part, we introduce the model which gave a name to the linear differential equation with a proportional delay - the *pantograph equation*. The first work dealing with this problem, which had arisen in British Railways Technical Centre in Derby, was published by Ockendon and Tayler (see [80]) in 1971. The authors had studied the dynamics of a current collection system for an electric locomotive, which lead to the vector delay differential equation

$$\dot{x}(t) = Ax(t) + Bx(qt), \qquad 0 < q < 1, \quad t \in [0, \infty), \tag{1.2}$$

where A and B are nonzero 4×4 complex matrices and q is a real scalar.

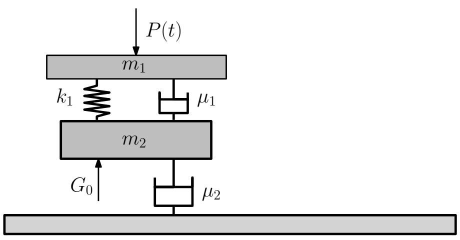

Figure 1: Scheme of the pantograph

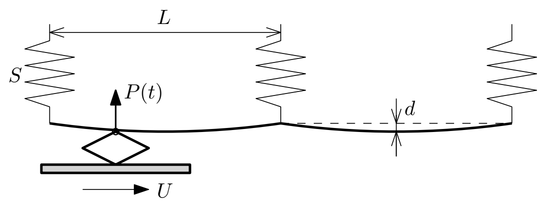

Figure 2: Scheme of the wire-pantograph system

The system can be described as follows (see Figures 1 and 2): The electric locomotive moves by a constant velocity U. The trolley wire is at constant tension and its supports are light stiff springs S, which are regularly spaced at a constant distance L apart. The pantograph is modelled by two masses (m_1 and m_2) connected by a spring (with spring constant k_1) and a velocity damper (with damping constant μ_1). The lower mass is connected with the roof of locomotive by a velocity damper (with damping constant μ_2) and a constant upward force G_0 is acting on it. This force is made by a stiff spring fastened between the lower arm of the pantograph and the locomotive roof.

The mathematical description of current collection system varies under diverse assumptions. Particularly, the so called inner problem is obtained considering a motion of the pantograph near a support provided neglecting the wire rigidity and damping. The inner problem is written in a vector form via equation (1.2) and some initial condition, which represents the continuity condition between situations before and after passing the support of the trolley wire. For a more precise formulation of the problem see [80].

Biosystem

To illustrate the usefulness of a constant delay case we mention a model from population dynamics, which was introduced in Baker, Bocharov and Rihan [7]. The population $V(t)$ is described by (1.1) with $\tau(t) = t - \xi$, $\xi \in \mathbb{R}^+$, i.e.

$$\dot{V}(t) = -\mu_1 V(t) + b_1 \, \mathrm{e}^{-\mu_0 \xi} \, V(t-\xi)\,.$$

The adults in the population die with rate $\mu_1 \in \mathbb{R}^+$ and produce offspring with rate $b_1 \in \mathbb{R}^+$. The offspring enter the equation only at the age of maturity ξ, but it is reduced by a factor $\exp(-\mu_0 \xi)$, which takes into account juvenile mortality.

Combustion in small rockets model

The fuel combustion model in small rockets is described by the system

$$\begin{gathered}\dot{x}_1(t) + (1-\gamma)x_1(t) + \gamma x_1(t-\tau) - x_2(t-\tau) = f_1(t),\\ \dot{x}_2(t) + \alpha x_1(t) + \alpha k x_2(t) = f_2(t),\end{gathered}$$

where $x_1(t)$ is the pressure in the combustion chamber, $x_2(t)$ is the fuel consumption, positive parameters α, γ, τ, k are given by the rockets construction and $f_1(t), f_2(t)$ are regulation inputs (see Kolmanovskii and Myshkis [66]).

There exist a lot of other models, which are described by delay differential equations and by their systems. The question of the solution properties, when the independent variable tends to infinity, is a common part of the solution analysis. There is mostly impossible to find the exact solution of the initial value problems for delay differential equations in explicit form, therefore the qualitative analysis of the solutions becomes very important. The analysis of appropriate numerical schemes also plays a key role with respect to the fact, that numerical solution is mostly the only way how the solutions of initial value problems can be obtained.

2 Asymptotics of differential equations with an unbounded delay

In this section, we introduce results derived in [62] and [18], respectively. The papers deal with asymptotic estimates for nonhomogeneous linear delay differential equations

1. $\dot{x}(t) = a(t)x(t) + b(t)x(\tau(t)) + f(t), \qquad t \in I,$

2. $\dot{x}(t) = -a(t)x(t) + \sum_{i=1}^{r} b_i(t)x(\tau_i(t)) + f(t), \quad t \in I.$

The main distinction between these equations consists in the sign of the coefficient at $x(t)$ (we assume $a(t) > 0$). We note that the asymptotic behaviour of the first equation has been analysed, under various assumptions, by many authors. We can mention some papers dealing with related problems, e.g. [6], [11], [32], [33], [67], [59] and others.

Particularly, Kato and McLeod [51] and Lim [70] investigated asymptotic behaviour of

$$\dot{x}(t) = ax(t) + bx(qt) + f(t), \qquad 0 < q < 1, \quad t \in [0,\infty)\,.$$

It was shown that for $a > 0$, $b \neq 0$ and $f(t) = O(\exp\{aqt\})$ as $t \to \infty$ either the solution $x(t)$ grows exponentially or it has a polynomial behaviour. These results have been generalized in [16] to equation

$$\dot{x}(t) = ax(t) + bx(\tau(t)) + f(t), \qquad a > 0, \quad b \neq 0, \quad t \in I, \tag{2.1}$$

where $\tau \in C^1(I)$, $\tau(t) < t$ for all $t \in I$, $\tau(t) \to \infty$ as $t \to \infty$, $0 < \dot{\tau}(t) \leq q < 1$ on I. Another generalization of (2.1) was investigated in [15], namely the equation

$$\dot{x}(t) = a(t)\left[x(t) - kx(\tau(t))\right] + f(t), \qquad k \neq 0, \quad t \in I,$$

where $a \in C(I)$ is positive and nondecreasing on I and $f(t)$ and $\tau(t)$ satisfy the same assumptions as in the case (2.1). Therefore both considered equations have an unbounded lag. In the sequel, we introduce the relevant asymptotic formulae including decreasing function $a(t)$ on I and we do not suppose explicitly that the delay is unbounded. Moreover, we show that in particular cases these formulae can be given in a more precise form.

Now we recall the result of Heard, which we utilize for comparison with the presented author's results throughout the thesis. Let us consider the equation

$$\dot{x}(t) = -ax(t) + bx(\tau(t)), \qquad t \in I. \tag{2.2}$$

Theorem 2.1 (Heard [43])**.** *Let $a > 0$, $b \neq 0$ be scalars, $\tau \in C^2(I)$ be such that $\dot{\tau}(t)$ is positive and nonincreasing on I and $q = \dot{\tau}(t_0) < 1$. Then for any solution $x(t)$ of (2.2) there exists a continuous periodic function ω of period $\log q^{-1}$ such that $x(t) = (\varphi(t))^{\alpha}\,\omega(\log\varphi(t)) + O\left((\varphi(t))^{\alpha_r - 1}\right)$ as $t \to \infty$, where $\varphi(t)$ is a solution of*

$$\varphi(\tau(t)) = q\varphi(t), \qquad t \in I,$$

$\alpha = \log(b/a)\big/\log q^{-1}$ and $\alpha_r = \Re(\alpha)$.

Remark 2.2. Particularly, it follows from Theorem 2.1 that considering the same assumptions it holds the estimate $x(t) = O(\psi(t))$ as $t \to \infty$ for any solution $x(t)$ of (2.2), where $\psi(t) = (\varphi(t))^{\alpha_r}$ is a solution of the functional equation

$$a\psi(t) = |b|\psi(\tau(t)), \qquad t \in I. \tag{2.3}$$

The equation $\dot{x}(t) = a(t)x(t) + b(t)x(\tau(t)) + f(t)$

The asymptotic estimate introduced in this part was published in [62]. We consider a nonhomogenous linear differential equation

$$\dot{x}(t) = a(t)x(t) + b(t)x(\tau(t)) + f(t), \qquad t \in I, \tag{2.4}$$

where $a, b, \tau, f \in C(I)$. Throughout the rest of this section, we assume that functions $a(t)$ and $b(t)$ are not identically zero on I.

Theorem 2.3. *Consider equation (2.4), where $a, b, f \in C(I)$, $\tau \in C^1(I)$, $a(t) > 0$, $\tau(t) < t$ for all $t \in I$, $\tau(t) \to \infty$ as $t \to \infty$, $f(t) = O(|b(t)|\exp\{\int_{t_0}^{\tau(t)} a(s)\mathrm{d}s\})$ as $t \to \infty$ and let exist a nonincreasing function $c \in C(I)$ such that $|b(t)| \leq c(t)a(t)$ for all $t \in I$. Further, assume that relation*

$$0 < a(\tau(t))\dot{\tau}(t) \leq \delta a(t)$$

is valid for a suitable real $0 < \delta < 1$ *and any* $t \in I$ *and let* $\varphi \in C(I)$ *be a positive solution of*

$$\varphi(t) = c(t)\varphi(\tau(t)), \qquad t \in I. \tag{2.5}$$

If $x(t)$ *is a solution of (2.4), then there exists* $L \in \mathbb{R}$ *such that*

$$x(t) = x_L(t) + O(\varphi(t)) \qquad as \qquad t \to \infty,$$

where $x_L(t)$ *is a solution of (2.4) such that*

$$\exp\{-\int_{t_0}^{t} a(s)\mathrm{d}s\}x_L(t) \to L \qquad as \qquad t \to \infty.$$

Remark 2.4. The existence of above mentioned solutions $x_L(t)$ and $\varphi(t)$ is guaranteed by auxiliary assertions introduced in the thesis. In a more general way, any solution $\varphi^*(t)$ of the inequality $\varphi^*(t) \geq c(t)\varphi^*(\tau(t))$, $t \in I$ can be considered instead of a solution $\varphi(t)$ of equation (2.5) occurring in Theorem 2.3.

Examples and remarks

First we show that Theorem 2.3 generalizes some known asymptotic results. In these particular cases we consider delays intersecting the identity function at the initial points. It is easy to see that the result of Theorem 2.3 holds for this case too.

If we put $a(t) \equiv a > 0$, $b(t) \equiv b \neq 0$ and $\tau(t) = qt$, $0 < q < 1$, then we get

Corollary 2.5. *Consider the equation*

$$\dot{x}(t) = ax(t) + bx(qt) + f(t), \qquad 0 < q < 1, \quad t \in [0, \infty), \tag{2.6}$$

where $a > 0$, $b \neq 0$ *are constants,* $f \in C([0, \infty))$ *and let* $f(t) = O(\exp\{aqt\})$ *as* $t \to \infty$. *Then for any* $L \in \mathbb{R}$ *there exists a solution* $x_L(t)$ *of (2.6) such that* $\exp\{-at\}x_L(t) \to L$ *as* $t \to \infty$. *Moreover, for any solution* $x(t)$ *of (2.6) there exists a suitable* $L \in \mathbb{R}$ *such that*

$$x(t) = x_L(t) + O(t^{\alpha}) \qquad as \quad t \to \infty, \quad \alpha = \frac{\log(a/|b|)}{\log q}.$$

Remark 2.6. This statement was published in a weaker form in Lim [70] and Kato and McLeod [51]. Its generalization to equation (2.1) with a more general form of a delay has been done in Čermák [16] and also in [18], where equation (2.4) with $b(t) = ka(t)$, $k \neq 0$ has been considered. We emphasize, that the key assumption in Čermák [15] is $a(t)$ nondecreasing. In the sequel, we demonstrate by an example that Theorem 2.3 is also applicable to the case of the decreasing function $a(t)$, particularly $a(t) = 1/t$.

Example 2.7. Consider the equation

$$\dot{x}(t) = \frac{1}{t}[x(t) - x(t^{1/2})] + f(t), \qquad t \in [1, \infty), \tag{2.7}$$

where $f \in C([1, \infty))$ and $f(t) = O(1/t^{1/2})$ as $t \to \infty$. It is easy to verify that all assumptions of Theorem 2.3 are fulfilled. Then for any $L \in \mathbb{R}$ there exists a solution $x_L(t)$ of (2.7) such that

$$x_L(t)/t \to L \qquad \text{as} \quad t \to \infty.$$

Moreover, for any solution $x(t)$ of (2.7) there exists $L \in \mathbb{R}$ such that

$$x(t) = x_L(t) + O(1) \qquad \text{as} \quad t \to \infty.$$

The following example demonstrates the case when the solving of the corresponding functional equation (2.5) is not trivial.

Example 2.8. Let us consider the equation

$$\dot{x}(t) = ax(t) + \frac{b}{t}x(qt) + f(t), \qquad 0 < q < 1, \quad t \in [1, \infty), \tag{2.8}$$

where $a > 0$, $b \neq 0$ are reals, $f \in C([1, \infty))$ and let $f(t) = O(t^{-1} \exp\{aqt\})$ as $t \to \infty$. Then in accordance with Theorem 2.3 for any $L \in \mathbb{R}$ there exists a solution $x_L(t)$ of (2.8) such that

$$\exp\{-at\}x_L(t) \to L \qquad \text{as} \quad t \to \infty.$$

Now it is easy to check that functional equation (2.5) turns to the form

$$\varphi(t) = \frac{|b|}{at}\varphi(qt)$$

and admits the solution

$$\varphi(t) = t^{\frac{\log(|b|/a)}{\log q^{-1}} - \frac{1}{2}(\frac{\log t}{\log q^{-1}} + 1)},$$

which is the function tending to zero as $t \to \infty$. Hence, by Theorem 2.3, for any solution $x(t)$ of (2.8) there exists $L \in \mathbb{R}$ such that

$$x(t) = x_L(t) + O(t^{\frac{\log(|b|/a)}{\log q^{-1}} - \frac{1}{2}(\frac{\log t}{\log q^{-1}} + 1)}) \qquad \text{as} \quad t \to \infty.$$

The equation $\dot{x}(t) = -a(t)x(t) + \sum_{i=1}^{r} b_i(t)x(\tau_i(t)) + f(t)$

The result introduced in this part was published in the paper [18], where a complementary case to the previous one, namely the differential equation (2.4) with a negative (non-constant) coefficient $a(t)$, was analysed. Since the utilized proof procedure is applicable also for corresponding equation with several delayed terms, we consider (2.4) in a more general form

$$\dot{x}(t) = -a(t)x(t) + \sum_{i=1}^{r} b_i(t)x(\tau_i(t)) + f(t), \quad t \in I, \tag{2.9}$$

where $a \in C(I)$ is a positive function on I, $b_i, f \in C(I)$, $\tau_i \in C^1(I)$, $i = 1, 2, \dots, r$ are functions which satisfy $\tau_i(t) < t$, $0 < \dot{\tau}_i(t) \le q_i < 1$ for all $t \in I$ and $\tau_i(t) \to \infty$ as $t \to \infty$, $i = 1, 2, \dots, r$.

Preliminaries

Let $t_{-1} = \inf\{\tau_i(t_0),\ i = 1, 2 \dots, r\}$ and $I^* = [t_{-1}, \infty)$. By a solution of (2.9) we understand a real-valued function $x \in C(I^*) \cap C^1(I)$ such that $x(t)$ satisfies (2.9) on I. In the sequel, we introduce the notion of embeddability of given functions into an iteration group. This property will be imposed on the set of delay arguments $\{\tau_1(t), \dots, \tau_r(t)\}$ throughout next considerations.

Definition 2.9. Let $\psi \in C^1(I^*)$, $\dot{\psi}(t) > 0$ on I^*. Say that $\{\tau_1(t), \dots, \tau_r(t)\}$ can be embedded into an iteration group $[\psi(t)]$ if for any $\tau_i(t)$ there exists a constant d_i such that

$$\tau_i(t) = \psi^{-1}(\psi(t) - d_i), \quad t \in I, \quad i = 1, 2, \dots, r. \tag{2.10}$$

Remark 2.10. The problem of embeddability of given functions $\{\tau_1(t), \dots, \tau_r(t)\}$ into an iteration group is closely related to the existence of a common solution $\psi(t)$ to the system of the simultaneous Abel equations

$$\psi(\tau_i(t)) = \psi(t) - d_i, \quad t \in I, \quad i = 1, 2, \dots, r.$$

The complete solution of these problems has been described by Neuman [79] and Zdun [93]. These papers contain conditions under which (2.10) holds for any $\tau_i(t)$, $i = 1, 2, \dots, r$ (see also Kuczma, Choczewski and Ger [58, Theorem 9.4.1]). We only note that the most important necessary condition is commutativity of any pair $\tau_i(t), \tau_j(t)$, $i, j = 1, 2, \dots, r$. We emphasize that if $\tau_i(t)$ are delays, then d_i must be positive.

Remark 2.11. Contrary to Theorem 2.3, we used here the apparatus of the Abel equation instead of the Schröder one. From the viewpoint of asymptotic properties at infinity, both approaches are essentially equivalent. While in Theorem 2.3 we followed the approach of Heard [43] based on the Schröder equation, the case of several delays is formally better provable via the Abel equations.

Asymptotic estimate

Next, we formulate the asymptotic estimates for equation (2.9) and its particular case $f(t) \equiv 0$ on I. The proofs of the assertions are omitted and can be found in [18].

Theorem 2.12. *Let $\{\tau_1(t), \dots, \tau_r(t)\}$ can be embedded into an iteration group $[\psi(t)]$. Let $x(t)$ be a solution of (2.9), where $a(t) \geq K \big/ \exp\{\eta\psi(t)\}$, $0 < \sum_{i=1}^{r} |b_i(t)| \leq Ma(t)$ for all $t \in I$ and suitable real constants $K > 0$, $M > 0$, $\eta < 1$. If $f(t) = O(\exp\{\nu\psi(t)\})$ as $t \to \infty$ for a suitable real ν, then*

$$x(t) = O(\exp\{\gamma\psi(t)\}) \quad \text{as } t \to \infty, \qquad \gamma > \max\left(\eta + \nu, \frac{\log M}{d_1}, \dots, \frac{\log M}{d_r}\right),$$

where d_i, $i = 1, 2, \dots, r$ are given by (2.10).

Theorem 2.13. *Let $\{\tau_1(t), \dots, \tau_r(t)\}$ can be embedded into an iteration group $[\psi(t)]$. Let $x(t)$ be a solution of (2.9) provided $f(t) \equiv 0$ on I, where $a(t) \geq K \big/ \exp\{\eta\psi(t)\}$, $0 < \sum_{i=1}^{r} |b_i(t)| \leq Ma(t)$ for all $t \in I$ and suitable real constants $K > 0$, $M > 0$, $\eta < 1$. Then*

$$x(t) = O(\exp\{\gamma\psi(t)\}) \quad \text{as } t \to \infty, \qquad \gamma = \max\left(\frac{\log M}{d_1}, \dots, \frac{\log M}{d_r}\right),$$

where d_i, $i = 1, 2, \dots, r$ are given by (2.10).

Examples and remarks

To recall some papers closely related to the presented estimate we mention Čermák [19], where the equation with power coefficients and a proportional delays have been studied.

In the sequel, we present several examples to demonstrate the usefulness of Theorem 2.12 and Theorem 2.13. First we introduce such a case of differential equation (2.9) that there are considered constant coefficients and one term with a power delay in the equation. Furthermore, the forcing term is omitted in this case:

Example 2.14. Consider the equation

$$\dot{x}(t) = -ax(t) + bx(t^q), \qquad 0 < q < 1, \quad t \in [1, \infty), \tag{2.11}$$

where $a > 0$, b are real constants. Then Theorem 2.13 gives the asymptotic estimate

$$x(t) = O(\log^\gamma t), \qquad \text{as} \quad t \to \infty, \ \gamma = \frac{\log \frac{|b|}{a}}{\log q^{-1}}.$$

The above asymptotic estimate coincides with the result of Heard [43], which can be rewritten for the case (2.11) as follows:

Corollary 2.15. *Let $a > 0$, $b \neq 0$ be real constants, $0 < q < 1$. Then for every solution $x(t)$ of (2.11) there exists a continuous periodic function ω with period $\log q^{-1}$ such that*

$$x(t) = (\log t)^\xi \omega(\log \log t) + O(\log^{\xi_r - 1} t) \qquad \text{as} \quad t \to \infty,$$

where ξ is a root of $bq^\xi - a = 0$ and $\xi_r = \Re(\xi)$.

Remark 2.16. The asymptotic estimate from Example 2.14 gives sufficient condition for asymptotic stability of (2.11) in the form $|b| < a$.

The next example illustrates the application of Theorem 2.12 to equation (2.9) with non-constant coefficients at power delayed terms.

Example 2.17. Consider the equation

$$\dot{x}(t) = -ax(t) + \frac{1}{t}\left[b_1 x(t^{q_1}) + b_2 x(t^{q_2})\right] + f(t), \tag{2.12}$$

where $a > 0$, b_1, b_2 are real constants, $0 < q_1 < q_2 < 1$ and $f(t) = O(\log^\nu t)$ as $t \to \infty$ for a suitable real ν. Then in accordance with Theorem 2.12 we obtain the asymptotic estimate of solution of (2.12) in the form

$$x(t) = O(\log^\gamma t) \quad \text{as} \quad t \to \infty, \text{ where} \quad \gamma > \max\left\{\nu, \frac{\log \frac{|b_1|+|b_2|}{a}}{\log q_1^{-1}}, \frac{\log \frac{|b_1|+|b_2|}{a}}{\log q_2^{-1}}\right\}.$$

Remark 2.18. If we consider $f(t) \equiv 0$ on I in the previous example, then the asymptotic estimate gives sufficient conditions for asymptotic stability of (2.12) in the form $|b_1| + |b_2| < a$ by Theorem 2.13.

3 Asymptotics of difference equations with an unbounded delay

In this section, we point our attention to asymptotic estimates for linear difference equations. Since all the difference equations within this section are studied as discretizations of particular delay differential equations, we utilize the notation with indices. In this section there are introduced asymptotic estimates of difference equations listed below:

1. $y_{n+1} - y_n = -ahy_n + bhy_{\tau_n}$,
2. $y_{n+1} - y_n = -ahy_n + bh((1-r_n)y_{\tau_n} + r_n y_{\tau_n+1})$,
3. $y_{n+1} - y_n = -ahy_n + h\sum_{i=1}^{r} b_i y_{\lfloor q_i n\rfloor}$,

where $n = n_0, n_0+1, n_0+2, \dots$, and the equation parameters will be specified in the appropriate place. The existence of unique solution of the above equations with appropriate initial conditions can be shown easily by the recurrent steps, since τ_n is going to be considered as a nondecreasing sequence of nonnegative integers such that $\tau_n \le n$. In this shortened version of habilitation thesis we introduce the result for the first equation. The asymptotic estimates for the second and the third equation can be found in the full version of thesis.

The equation $y_{n+1} - y_n = -ahy_n + bhy_{\tau_n}$

Note, that the following considerations can be found in [60]. We discuss the numerical discretization of the delay differential equation (2.2) in the form

$$y_{n+1} - y_n = -ahy_n + bhy_{\tau_n}, \tag{3.1}$$

where $a > 0$, $b \neq 0$ are reals, $\tau_n = \left\lfloor \frac{\tau(t_n)-t_0}{h} \right\rfloor$, $t_n = t_0 + nh$, $n = 0,1,2,\dots$, $h > 0$ is the stepsize and the symbol $\lfloor\ \rfloor$ is an integer part. Then y_n means the approximation of $x(t_n)$. Equation (3.1) is a difference equation obtained from (2.2) by the modified Euler method. It has been shown in [38] that numerical scheme (3.1) is convergent. Our aim is to describe some asymptotic properties of equation (3.1) (more precisely, to find conditions under which asymptotic behaviour of (2.2) and (3.1) is similar).

We especially investigate equations with unbounded lag. In the connection with the investigation of asymptotic properties of solutions of such equations we recall papers dealing with related problems, e.g. [17], [43], [48], [71], [51] and many others in the continuous case, and [40], [75], [82] and others in the discrete case.
In the sequel we summarize the assumptions necessary to formulate the result for the discrete case. First, let us denote (H) the assumptions on function $\tau(t)$:
(H): Let $\tau(t)$ be an increasing continuous function on I such that $\tau(t) < t$ for all $t \in I$ (the case $\tau(t_0) = t_0$ is also possible), $\tau(t+\tilde{h}) - \tau(t)$ is nonincreasing on I for arbitrary real $\tilde{h}$ fulfilling $0 < \tilde{h} \le h$ and let $\lim_{t\to\infty} \tau(t) = \infty$.

Further, throughout this section, we denote $T_{-1} = \tau(t_0)$ and $T_k = \tau^{-k}(t_0)$, $k = 0,1,2,\dots$, where $\tau^{-k}(t)$ means the k-th iteration of the inverse $\tau^{-1}(t)$. If we set $I_m = [T_{m-1}, T_m]$ for all $m = 0,1,2,\dots$, then function $\tau(t)$ is mapping I_{m+1} onto I_m.

Instead of functional equation (2.3) we consider the functional inequality

$$a\rho(t) \ge |b|\rho(t_0 + \left\lfloor \frac{\tau(t)-t_0}{h} \right\rfloor h), \qquad t \in I. \tag{3.2}$$

Now we can formulate the proposition ensuring some required properties of solutions of the inequality (3.2).

Proposition 3.1. *Consider the inequality (3.2), where $a > 0$, $b \neq 0$ are reals and let (H) be fulfilled.*

(i) *If $|b|/a \geq 1$, then there exists a positive continuous nondecreasing solution $\rho(t)$ of inequality (3.2).*

(ii) *If $|b|/a < 1$, then there exists a positive continuous decreasing solution $\rho(t)$ of inequality (3.2) such that $\rho(t+\tilde{h}) - \rho(t)$ is nondecreasing on I for arbitrary real $0 < \tilde{h} \leq h$.*

Asymptotic estimate

Theorem 3.2. *Let y_n, $n = 0, 1, 2, \ldots$ be a solution of (3.1), where $0 < ah < 1$, $b \neq 0$ are reals. Let (H) be fulfilled, let $\rho(t)$ be a positive solution of (3.2) with the properties guaranteed by Proposition 3.1 and let $\rho_n = \rho(t_n)$.*

(i) *If $|b|/a \geq 1$, then $y_n = O(\rho_n)$ as $n \to \infty$.*

(ii) *If $|b|/a < 1$ and moreover $\sum\limits_{k=1}^{\infty} \frac{\rho(T_{k-1}) - \rho(T_{k-1}+h)}{\rho(T_{k+1})} < \infty$, then $y_n = O(\rho_n)$ as $n \to \infty$.*

Remark 3.3. In the estimate concerning the case $|b|/a < 1$ it is also possible to take a solution $\psi(t)$ of functional equation (2.3) instead of a function $\rho(t)$ (which is a solution of (3.2)). Using the fact that the term $\rho(\tau(t)) - \rho(\tau(t) - h)$ is a positive nonincreasing function it could be shown that there exists a solution $\psi(t)$ of (2.3) such that $\psi(t) > \rho(t)$ for all $t > t_0$. In some cases the utilizing of $\psi(t)$ instead of $\rho(t)$ can be more applicable. The usage of the appropriate Abel equation in the proof procedure is also possible with respect to Remark 2.11.

Corollary 3.4. Consider the scalar pantograph equation

$$\dot{x}(t) = -ax(t) + bx(qt), \qquad t \in [0, \infty) \tag{3.3}$$

where $a > 0$, $b \neq 0$, $0 < q < 1$ are reals. Theorem 2.1 gives the estimate

$$x(t) = O(t^r), \qquad r = \frac{\log \frac{|b|}{a}}{\log q^{-1}}, \qquad \text{as} \quad t \to \infty \tag{3.4}$$

for equation (3.3). The corresponding difference equation is

$$y_{n+1} - y_n = -ahy_n + bhy_{\lfloor qn \rfloor}, \qquad n = 0, 1, 2, \ldots, \tag{3.5}$$

where the above assumptions on a, b, q are fulfilled and $0 < ah < 1$. Then the following estimate

$$y_n = O(n^r), \quad r = \frac{\log \frac{|b|}{a}}{\log q^{-1}} \qquad \text{as} \quad n \to \infty$$

holds for all solutions $\{y_n\}_{n=0}^{\infty}$ of difference equation (3.5).

Example 3.5. Consider the initial value problem:

$$\dot{x}(t) = -2x(t) + x(t/2), \quad x(0) = 1, \quad t \in [0, \infty).$$

In accordance with (3.4) we get the asymptotic estimate $x(t) = O(1/t)$ as $t \to \infty$. In the corresponding discrete case we consider formula (3.5) in the form

$$y_{n+1} - y_n = -2hy_n + hy_{\lfloor n/2 \rfloor}, \qquad y_0 = 1, \quad n = 0, 1, 2, \ldots.$$

Then $y_n = O(1/n)$ as $n \to \infty$ provided $h < 1/a$. If we violate the condition on stepsize, this asymptotic formula is not valid. Indeed, if $h = 1 > 1/a$, then the corresponding discrete equation $y_{n+1} = -y_n + y_{\lfloor n/2 \rfloor}$ admits solutions not tending to zero as $n \to \infty$.

It is obvious, that the assumption $0 < ah < 1$ has its relevance in the choice of suitable stepsize h to preserve the same behaviour of difference case as in the continuous case.

4 Asymptotic stability for differential equations with a constant delay

In this section, we survey some relevant results on asymptotic stability of differential equations with a constant delay. Curiously, this matter is more difficult then the problem of asymptotic stability of corresponding differential equations with unbounded delays. For example, the (optimal) asymptotic stability condition for

$$\dot{x}(t) = ax(t) + bx(qt), \qquad t \in I$$

has a very simple form

$$|b| < -a\,.$$

The same stability condition is valid also for

$$\dot{x}(t) = ax(t) + bx(\tau(t)), \qquad t \in I,$$

where $a, b \in \mathbb{R}$ and $\tau(t)$ fulfils the assumptions introduced in Theorem 2.3.

On the other hand, the asymptotic stability conditions for differential equations with constant delays depend not only on the values of coefficients, but also on the value of the delay. This can be observed already from the probably simplest differential equation with a constant delay,

$$\dot{x}(t) = bx(t-\tau), \qquad t \in I, \tag{4.1}$$

where $b, \tau \in \mathbb{R}$, $\tau > 0$. The asymptotic stability of all solutions of equation (4.1) is guaranteed for all pairs $(b, \tau) \in \mathbb{R} \times \mathbb{R}^+$ which satisfy the relation

$$-\frac{\pi}{2\tau} < b < 0 \tag{4.2}$$

(see Kolmanovskii and Myshkis [66]). We emphasize that the condition (4.2) is the necessary and sufficient one for asymptotic stability of (4.1) and it constructs an *asymptotic stability interval* $S_\tau = (-\pi/(2\tau), 0)$ for equation (4.1) with respect to its parameter b. As we can see the left bound of S_τ depends on the value of delay τ, therefore the stability interval is said to be *delay dependent*.

Now we consider the equation

$$\dot{x}(t) = ax(t) + bx(t-\tau), \qquad t \in I, \tag{4.3}$$

where $a, b, \tau \in \mathbb{R}$, $\tau > 0$. Then the set S_τ of all pairs $(a, b) \in \mathbb{R}^2$, for which equation (4.3) is asymptotically stable, is given by the relations (see e.g. Hayes [42])

$$a \le b < -a \tag{4.4}$$

and

$$|a| + b < 0, \qquad \tau < \frac{\arccos(-a/b)}{(b^2 - a^2)^{1/2}}\,. \tag{4.5}$$

As we can see the relation (4.4) is free of delay τ whereas condition (4.5) depends on the delay. Therefore the region given by the first relation is said to be *delay independent* and the second one *delay dependent.* The asymptotic stability analysis of (4.3) can be realized by zeros investigation of the quasi-polynomial

$$P(\lambda) = \lambda - a - b\exp(-\lambda\tau). \tag{4.6}$$

The asymptotic stability of (4.3) is then ensured if and only if the relation $\Re(\lambda) \le \delta < 0$ is valid for a suitable real δ and all zeros λ of (4.6). A similar technique can be used in the asymptotic stability investigation of some more general delay differential equations (e.g. delay differential equations of higher order, neutral type delay differential equations, etc.). The formulation of explicit necessary and sufficient asymptotic stability conditions for equations with several constant delays remains an open problem.

5 Asymptotic stability for difference equations with a constant delay

In this section there are introduced necessary and sufficient conditions ensuring that the following linear difference equations are asymptotically stable:

1. $y_{n+1} + \alpha y_n + \beta y_{n-k+1} + \gamma y_{n-k} = 0$,
2. $y_{n+2} + \alpha y_n + \beta y_{n-k+2} + \gamma y_{n-k} = 0$,
3. $y_{n+2} - \frac{1+ah}{1-ah} y_n - \frac{2bh}{1-ah} y_{n-k+1} = 0$,

where $n = 0, 1, 2, \ldots$. The other equation parameters will be specified later. Notice, that all three equations are of a fixed finite order ($k+1, k+2, k+1$, respectively) and unique solutions of these equations exist with respect to the appropriate initial conditions. In this shortened version of thesis the results for the above equations two and three are omitted, they can be found in the full version of thesis.

The above listed equations can be viewed as a numerical discretization of particular delay differential equations. The question of asymptotic stability properties of numerical schemes is a fundamental part of numerical analysis of differential equations. In addition, we note that the obtained results provide a contribution to qualitative theory of linear difference equations.

Let us consider the equation

$$y_{n+1} + \alpha_0 y_n + \alpha_1 y_{n-1} + \cdots + \alpha_k y_{n-k} = 0\,, \qquad n = 0, 1, 2, \ldots\,, \tag{5.1}$$

where $\alpha_0, \ldots, \alpha_k$ are real numbers and $k \ge 1$ is an integer. As it is customary, we call linear difference equation (5.1) *asymptotically stable*, if all its solutions satisfy $y_n \to 0$ as $n \to \infty$. Applying the Rouché's Theorem one can easily verify that the sufficient condition for asymptotic stability of (5.1) is

$$\sum_{j=0}^{k} |\alpha_j| < 1$$

(see Kocić and Ladas [57]). This condition is called the Cohn stability domain and provides a direct generalization of the well-known Clark's condition for a three-term linear difference equation (see Clark [27]). It is well known that problem of asymptotic stability of equation (5.1) is equivalent to the problem of location of all the zeros of the associated characteristic polynomial

$$P(\lambda) = \lambda^{k+1} + \alpha_0\lambda^k + \alpha_1\lambda^{k-1} + \cdots + \alpha_{k-1}\lambda + \alpha_k \tag{5.2}$$

inside the unit disk. In this connection, we can mention the Schur–Cohn criterion that provides necessary and sufficient conditions for the zeros of (5.2) to be inside the unit disk.

Now we introduce a general form of determinantal criterion, which enables us to decide whether or not all zeros of polynomial (5.2), where α_i, $i = 0, \dots, k$ are reals, lie inside the unit circle in the complex plane (see e.g. Elaydi [36] or Marden [76]).

Theorem 5.1 (The Schur-Cohn criterion)**.** *Polynomial* (5.2) *has all its zeros inside the unit circle if and only if it holds*

(i) $P(1) > 0$;

(ii) $(-1)^{k+1}P(-1) > 0$;

(iii) $k \times k$ *matrices*

$$B_k^{\pm} = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ \alpha_0 & \ddots & \ddots & & \vdots \\ \alpha_1 & \ddots & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & \\ \alpha_{k-3} & & & \ddots & 0 \\ \alpha_{k-2} & \alpha_{k-3} & \dots & \alpha_1 \quad \alpha_0 & 1 \end{bmatrix} \pm \begin{bmatrix} 0 & \cdots & & 0 & \alpha_k \\ \vdots & & & ⋰ & \alpha_{k-1} \\ & & ⋰ & ⋰ & \vdots \\ & ⋰ & ⋰ & & \vdots \\ 0 & ⋰ & & & \alpha_2 \\ \alpha_k & \alpha_{k-1} & \dots \quad \dots & \alpha_2 & \alpha_1 \end{bmatrix}$$

are positive innerwise.

However, the form of this criterion is applicable only for concrete choices of coefficients α_i and k, while it remains unsuitable to give explicit conditions in terms of α_i and k. As it is stated in many related papers, a formulation of such explicit (necessary and sufficient) conditions in the general case of (5.2) seems to be an extremely difficult matter, but turns out to be possible in some of its particular cases.

Levin and May [69] studied the three-term difference equation

$$y_{n+1} - y_n + \gamma y_{n-k} = 0\,, \qquad n = 0, 1, 2, \dots$$

and showed that the condition

$$0 < \gamma < 2\cos\frac{k\pi}{2k+1}$$

is necessary and sufficient for its asymptotic stability. This criterion has been later generalized by Kuruklis [68] to the difference equation

$$y_{n+1} + \alpha y_n + \gamma y_{n-k} = 0\,, \qquad n = 0, 1, 2, \dots\,. \tag{5.3}$$

We recall here his famous result whose original proof was based on tracing zeros of the corresponding characteristic polynomial with respect to changing equation's coefficients and a given circle.

Theorem 5.2. *Let $\alpha \neq 0$, γ be arbitrary reals and $k \geq 1$ be an integer. Equation (5.3) is asymptotically stable if and only if $|\alpha| < (k+1)/k$, and*

$$|\alpha| - 1 < \gamma < (\alpha^2 + 1 - 2|\alpha|\cos\phi)^{1/2} \qquad \text{for } k \text{ odd}\,,$$
$$|\alpha + \gamma| < 1 \quad \text{and} \quad |\gamma| < (\alpha^2 + 1 - 2|\alpha|\cos\phi)^{1/2} \qquad \text{for } k \text{ even}\,,$$

where $\phi \in (0, \pi/(k+1))$ is the solution of $\sin(kx)/\sin((k+1)x) = 1/|\alpha|$.

We note that when $\alpha = 0$, the necessary and sufficient condition for the asymptotic stability of (5.3) is $|\gamma| < 1$.

Another proof technique of this result was proposed by Papanicolaou [81], where the asymptotic stability region was constructed as a part of (α, γ)-plane. Further, Dannan and Elaydi [31] considered this equation in the advanced case

$$y_n + \alpha y_{n+1} + \gamma y_{n+k} = 0\,, \qquad n = 0, 1, 2, \ldots$$

and derived necessary and sufficient conditions for its asymptotic stability. A unification of these results was performed by Dannan [30], Cheng and Huang [26] and by Kipnis and Nigmatullin [52] in the study of the asymptotic stability of the equation

$$y_{n+m} + \alpha y_n + \gamma y_{n-k} = 0\,, \qquad n = 0, 1, 2, \ldots$$

with nonnegative integers m and k. Some extensions to complex or matrix coefficients α, γ have been obtained by Ivanov et al. [49] and by Kipnis and Malygina [55].

Considering linear difference equations with more than three terms, the situation changes. Kipnis and Levitskaya [54] showed that analytical studies of the stability domain of the four-term linear difference equation

$$y_{n+1} - y_n + \beta y_{n-\ell} + \gamma y_{n-k} = 0\,, \qquad n = 0, 1, 2, \ldots \tag{5.4}$$

with arbitrary reals β, γ and nonnegative integers ℓ, k are very complicated. Their statements on the behaviour of the stability domain are deduced from numerical experiments, rather than theoretical investigations. Other related papers discuss asymptotic stability conditions, which are either sufficient (but not necessary) or they are necessary and sufficient only under some other additional assumptions on equation's coefficients. In this connection, we can mention the result by Győri et al. [39] discussing the asymptotic stability condition for (5.4) under the assumption

$$\beta < 0, \qquad 0 < \gamma < \frac{k^k}{(k+1)^{k+1}}\,,$$

later generalized by Pituk [85] to a more general case of (5.1). For other related results providing stability conditions for (5.1) we refer to Kipnis and Komissarova [53], Kocić and Ladas [57], Liz [73] and Stević [91]. In the context with the previous discussion, the results introduced in this section generalize the previous ones and open a possible way to obtain efficient sets of necessary and sufficient asymptotic stability conditions for some another particular difference equations. There is realized a deeper analysis of asymptotic stability of particular numerical scheme in the full version of thesis, which provides an application of the presented results.

The equation $y_{n+1} + \alpha y_n + \beta y_{n-k+1} + \gamma y_{n-k} = 0$

The next considerations come from paper [23]. This section discusses two explicit forms of necessary and sufficient conditions for the asymptotic stability of the autonomous linear difference equation

$$y_{n+1} + \alpha y_n + \beta y_{n-k+1} + \gamma y_{n-k} = 0\,, \qquad n = 0, 1, 2, \dots\,. \tag{5.5}$$

where α, β, γ are real numbers and $k \geq 2$ is an integer. These conditions are derived by use of the Schur-Cohn criterion converted into a more applicable form. We compare our stability conditions with related results obtained by other authors and apply them to a problem of numerical discretization of delay differential equations. Indeed, difference equation (5.5) represents the application of Θ method to delay differential equation with a constant delay (4.3).

Note that although (5.5) is actually a four-term equation only for $k \geq 2$, the following results remain valid also for $k = 1$. The formulation of the main result of this section requires to distinguish the cases k odd and k even, and can be stated as follows.

Asymptotic stability conditions

Theorem 5.3. *Let α, β, γ be real constants and k be a positive odd integer. Then (5.5) is asymptotically stable if and only if*

$$|\alpha + \beta| < 1 + \gamma \tag{5.6}$$

and either

$$\gamma - 1 < |\alpha - \beta| \leq 1 - \gamma \tag{5.7}$$

or

$$|\alpha - \beta| > |1 - \gamma|, \qquad k < \arccos \frac{\alpha^2 - \beta^2 + \gamma^2 - 1}{2|\alpha\gamma - \beta|} \Big/ \arccos \frac{\alpha^2 - \beta^2 - \gamma^2 + 1}{2|\alpha - \beta\gamma|}\,. \tag{5.8}$$

Theorem 5.4. *Let α, β, γ be real constants and k be a positive even integer. Then (5.5) is asymptotically stable if and only if*

$$|\alpha + \gamma| < 1 + \beta \tag{5.9}$$

and either

$$\beta - 1 < |\alpha - \gamma| \leq 1 - \beta \tag{5.10}$$

or

$$|\alpha - \gamma| > |1 - \beta|, \qquad k < \arccos \frac{\alpha^2 - \beta^2 + \gamma^2 - 1}{2|\alpha\gamma - \beta|} \Big/ \arccos \frac{\alpha^2 - \beta^2 - \gamma^2 + 1}{2|\alpha - \beta\gamma|}\,. \tag{5.11}$$

In the sequel, we first mention three assertions following immediately from Theorem 5.3 and Theorem 5.4. Then we formulate an alternative system of necessary and sufficient conditions for the asymptotic stability of (5.5), which is, of course, equivalent to (5.6)–(5.8) for k odd, and (5.9)–(5.11) for k even, but whose applicability is quite different. Finally, we derive some consequences of this alternative system.

Corollary 5.5. *Let α, β, γ be real constants. Equation (5.5) is asymptotically stable for any integer $k \geq 1$ if and only if*

$$|\alpha + \beta| < 1 + \gamma, \qquad \gamma - 1 < \beta - \alpha \leq 1 - \gamma\,.$$

Now we put $\beta = 0$ and consider the three-term equation

$$y_{n+1} + \alpha y_n + \gamma y_{n-k} = 0\,, \qquad n = 0, 1, 2, \ldots. \tag{5.12}$$

Corollary 5.6. *Let α, γ be real constants and k be a positive integer.*

(i) *Let $\gamma(-\alpha)^{k+1} \leq 0$. Then (5.12) is asymptotically stable if and only if*

$$|\alpha| + |\gamma| < 1\,.$$

(ii) *Let $\gamma(-\alpha)^{k+1} > 0$. Then (5.12) is asymptotically stable if and only if either*

$$|\alpha| + |\gamma| \leq 1$$

or

$$|\alpha| + |\gamma| > 1, \quad ||\alpha| - |\gamma|| < 1, \quad k < \arccos \frac{\alpha^2 + \gamma^2 - 1}{2|\alpha\gamma|} \Big/ \arccos \frac{\alpha^2 - \gamma^2 + 1}{2|\alpha|}\,.$$

As another consequence, we show how our results imply the asymptotic stability criterion derived by Levin and May [69].

Corollary 5.7. *Let γ be a real constant and k be a positive integer. The difference equation*

$$y_{n+1} - y_n + \gamma y_{n-k} = 0\,, \qquad n = 0, 1, 2, \ldots$$

is asymptotically stable if and only if

$$0 < \gamma < 2 \cos \frac{k\pi}{2k+1}\,.$$

Remark 5.8. Our system of conditions, which are necessary and sufficient for the asymptotic stability of the three-term equation (5.12), is formally different from that derived in Kuruklis [68] (see Theorem 5.2). Note that all our conditions are given explicitly. In particular, no solving of a nonlinear equation is required. Such a system of necessary and sufficient conditions is convenient especially when we are given with fixed coefficients α, γ (and possibly β) and aim to find a critical order k_{cr} such that the equation (5.12) (possibly (5.5)) is asymptotically stable for all $k \leq k_{cr}$, whereas for $k > k_{cr}$ loses this property. On the other hand, the system of conditions derived in [68] enables us to discuss the asymptotic stability region for coefficients of the equation (5.12) as a part of (α, γ)-plane.

In this connection, it might be interesting to derive another system of necessary and sufficient conditions for the asymptotic stability of the four-term equation (5.5), which can be directly related to the system involved in Theorem 5.2. This system is introduced in the following assertion.

Theorem 5.9. *Let α, β, γ be arbitrary reals such that $\alpha - \beta\gamma \neq 0$ and $k \geq 2$ be an integer. Then (5.5) is asymptotically stable if and only if*

$$\begin{aligned} |\alpha+\beta| &< 1+\gamma, \qquad \textit{for } k \textit{ odd}, \\ |\alpha+\gamma| &< 1+\beta, \qquad \textit{for } k \textit{ even} \end{aligned}$$

and

$$\frac{\alpha^2-\beta^2}{|\alpha-\beta\gamma|} < \frac{k+1}{k}, \qquad 1+\alpha^2-\beta^2-\gamma^2 > 2|\alpha-\beta\gamma|\cos\phi,$$

where $\phi \in (0, \pi/k)$ is the solution of

$$\frac{\sin[(k+1)x]}{\sin(kx)} = \frac{\alpha^2-\beta^2}{|\alpha-\beta\gamma|}.$$

Remark 5.10. If $k = 1$, then (5.5) actually becomes the three-term difference equation

$$y_{n+1} + (\alpha+\beta)y_n + \gamma y_{n-1} = 0\,, \qquad n = 0, 1, 2, \dots.$$

Its asymptotic stability condition can be derived trivially in the form

$$|\alpha+\beta| < 1+\gamma < 2\,.$$

Next, we introduce two simple consequences of Theorem 5.9. If we put $\beta = -\alpha$ in (5.5), we obtain the difference equation

$$y_{n+1} + \alpha(y_n - y_{n-k+1}) + \gamma y_{n-k} = 0\,, \qquad n = 0, 1, 2, \dots. \tag{5.13}$$

It holds

Corollary 5.11. *Let α, γ be real constants and k be a positive integer.*

(i) *Let $(-\alpha)^{k+1} \leq 0$. Then (5.13) is asymptotically stable if and only if*

$$-1 < \gamma < 1 - 2|\alpha|\,.$$

(ii) *Let $(-\alpha)^{k+1} > 0$. Then (5.13) is asymptotically stable if and only if*

$$-1 < \gamma < 1 - 2|\alpha|\cos\frac{\pi}{k+1}\,.$$

Analogously we can deduce necessary and sufficient conditions for the asymptotic stability of the difference equation

$$y_{n+1} + \alpha(y_n + y_{n-k+1}) + \gamma y_{n-k} = 0\,, \qquad n = 0, 1, 2, \dots \tag{5.14}$$

obtained from (5.5) via the choice $\beta = \alpha$.

Corollary 5.12. *Let α, γ be real constants and k be a positive integer.*

(i) *Let $(-\alpha)^{k+1} \leq 0$. Then (5.14) is asymptotically stable if and only if*

$$-1 + 2|\alpha|\cos\frac{\pi}{k+1} < \gamma < 1\,.$$

(ii) *Let* $(-\alpha)^{k+1} > 0$. *Then (5.14) is asymptotically stable if and only if*

$$-1 + 2|\alpha| < \gamma < 1\,.$$

The studied difference equation (5.5) and its particular case (5.12) are of a great importance in numerical investigation of delay differential equations. Discussing the question of appropriate numerical discretizations of such equations, there are two basic approaches originating from the stability analysis of the underlying differential equations. The first approach consists in finding conditions guaranteeing the desired stability properties of a given test equation for all admissible delays (delay independent stability). The second approach discusses stability conditions for a fixed delay (delay dependent stability).

To make our next considerations as clear as possible, we recall a simple test equation (4.3), i.e.

$$\dot{x}(t) = ax(t) + bx(t-\tau)\,, \qquad t \in I\,,$$

where $a\,, b$ and $\tau > 0$ are real scalars. It is known that (4.3) is asymptotically stable if and only if (4.4) or (4.5) hold. The key problem in numerical analysis of (4.3) is to determine conditions under which a given discretization can retain these stability properties of (4.3).

A frequently used method of approximating (4.3) is the Θ-method. Let y_n mean the approximate value of $x(t_n)$, where $t_n = t_0 + nh$, $h > 0$ is the stepsize. For the sake of simplicity, we consider this stepsize under the constraint

$$\frac{\tau}{h} = k \in \mathbb{Z}^+ \tag{5.15}$$

which is usually employed in the frame of P-stability (see Bellen and Zennaro [8, p. 296]). Then applying the Θ-method ($0 \le \Theta \le 1$) to (4.3) we arrive just at the four-term difference equation (5.5) with

$$\alpha = -\frac{1 + (1-\Theta)ah}{1 - \Theta ah}, \quad \beta = -\frac{\Theta bh}{1 - \Theta ah}, \quad \gamma = -\frac{(1-\Theta)bh}{1 - \Theta ah} \tag{5.16}$$

and k being given by (5.15). Now the results of this section can yield the full answer to the above mentioned stability problem. In particular, Corollary 5.5 implies (after some simple calculations) that if

$$h \le \frac{1}{|a| + |b|}\,, \tag{5.17}$$

then the discretization (5.5), (5.16) is asymptotically stable for any positive integer k if and only if (4.5) holds. In other words, this necessary and sufficient condition is the same as that guaranteeing the asymptotic stability of (4.3) for all $\tau > 0$. Therefore, another interpretation of this result can be provided by

Corollary 5.13. *Let (5.17) hold. Then the differential equation (4.3) is asymptotically stable for all* $\tau > 0$ *if and only if its* Θ*-method discretization (5.5), (5.16) is asymptotically stable for all* $\tau > 0$ *and* $h > 0$ *satisfying (5.15) with arbitrary value of* $k \in \mathbb{Z}^+$.

Note that the restriction (5.17) on the stepsize h is independent of the parameter Θ. Considering a concrete Θ, it can be weakened to a less restrictive (unimprovable) form.

Similarly we can discuss the case of delay dependent stability by use of Theorem 5.3 and Theorem 5.4. Doing this, it is interesting to observe that both couples of the conditions

(5.6), $(5.8)_1$ and (5.9), $(5.11)_1$ are equivalent with $(4.5)_1$ under the stepsize restriction (5.17). Furthermore, the conditions $(5.8)_2$, $(5.11)_2$ become the inequality

$$k = \tau/h < \frac{\arccos p}{\arccos q}\,, \tag{5.18}$$

where

$$p = \frac{2a + (a^2+b^2)(1-2\theta)h}{2|b + ab(1-2\theta)h|}, \quad q = \frac{2 + 2a(1-2\theta)h + (a^2-b^2)(1-2\theta+2\theta^2)h^2}{2[1 + a(1-2\theta)h - (a^2-b^2)\theta(1-\theta)h^2]}\,,$$

which represents the necessary and sufficient condition for the asymptotic stability of the discretization (5.5), (5.16) in the delay dependent stability case $|a| + b < 0$. We can easily check that letting $h \to 0$ (with respect to (5.15)) the inequality (5.18) actually becomes $(4.5)_2$. Note also that if $\Theta = 0$ (the case of forward Euler method), then (5.5) with α, β, γ given by (5.16) is reduced to (5.12) with $\alpha = -1 - ah$ and $\gamma = -bh$. Similarly we can discuss the case of backward Euler method ($\Theta = 1$). Of course, these special choices of Θ significantly simplify all necessary calculations. For some other related results, concerning comparisons of stability and asymptotic properties of delay differential equations and their discrete analogues, we refer, e.g to Cooke and Győri [28], [21], and Čermák and Jánský [22].

6 Delay equations on time scales: essentials and asymptotics of the solutions

The content of this section comes from paper [21]. Our aim is to unify the asymptotic investigation of differential and difference equations with unbounded delay mentioned in Sections 2 and 3 in the frame of time scale calculus. Particularly, we introduce asymptotic estimate for dynamic equations with proportional delay on time scales. Notice, that asymptotic investigation of dynamic equations with a constant delay appears much more difficult (this fact is commented at the end of this section). Emphasize, that upto now we have strictly distinguished the solutions of differential and difference equations by terms $x(t)$ and y_n, respectively. In the sequel, unifying the approach we utilize $y(t)$ as a solution of appropriate dynamic equation on time scales and in its particular differential and difference case the symbol y will be preserved.

Next, we introduce and extend the notion of a delay dynamic equation on time scales $\mathbb{T}$, i.e. on arbitrary nonempty closed subsets of reals $\mathbb{R}$. The theory of time scales was introduced by Hilger [46] in order to unify and extend continuous and discrete analysis. For a general introduction to the basic time scale calculus we refer to the textbook by Bohner and Peterson [12, Sections 1-2], where all the necessary notions and properties utilized in this paper are involved. Using these tools it is possible, among others, jointly investigate the qualitative properties of differential and difference equations. In particular, if $\mathbb{T} = \mathbb{R}$, then dynamic equations become differential equations, while if $\mathbb{T} = h\mathbb{Z} = \{hk,\, k \in \mathbb{Z}\}$, $h > 0$ and $\mathbb{T} = \overline{q^{\mathbb{Z}}} = \{q^k,\, k \in \mathbb{Z}\} \cup \{0\}$, $q > 1$, they become h-difference and q-difference equations, respectively.

There are many papers illustrating conveniences of this approach (see, e.g. Akın-Bohner, Bohner and Saker [3], Bohner and Saker [14] or Dai and Tisdell [29]). Among those specialized to the asymptotic theory of dynamic equations we refer, e.g. to Bohner and Lutz [9, 10] or

Shi, Zhou and Yan [88]. However, only a few of them are concerned with dynamic equations, where time-dependent backward delays are present. We note that the qualitative (mostly oscillatory) investigations of these equations can be found in papers by Agarwal, Bohner and Saker [1], Anderson and Kenz [4], Bohner [13], Diblík and Vítovec [34], Mathsen, Wang and Wu [77], Pőtzsche [86], Zhang and Deng [94] or Čermák and Urbánek [20]. Our aim is to present asymptotic results for the autonomous dynamic equation with a proportional delay. Then under specific choices of $\mathbb{T}$ we obtain the asymptotic description of the solutions of the corresponding differential and difference equation. First of all we recall some basics of the time scales calculus.

Fundamentals of time scales calculus

Definition 6.1. Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T}, s > t\}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) = \sup\{s \in \mathbb{T}, s < t\}$. If t is a maximum of $\mathbb{T}$, we put $\sigma(t) = t$ and if t is a minimum of $\mathbb{T}$, we put $\rho(t) = t$. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$.

The jump operators $\sigma(t)$ and $\rho(t)$ allow the following classification of points of $\mathbb{T}$. If $\sigma(t) = t$ for $t \in \mathbb{T}$, $t < \sup \mathbb{T}$, then t is called right-dense, whereas if $\rho(t) = t$ for $t \in \mathbb{T}$, $t > \inf \mathbb{T}$, then t is called left-dense. Similarly, if $\sigma(t) > t$ for $t \in \mathbb{T}$, then t is called right-scattered, while if $\rho(t) < t$ for $t \in \mathbb{T}$, then t is called left-scattered.

Throughout this section the assumption is made that $\mathbb{T}$ is unbounded above (i.e. $\sup \mathbb{T} = \infty$) and has the topology and ordering inherited from real numbers. Let $a, b \in \mathbb{T}$, where $a < b$. Then we define the time scale interval $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T}, a \leq t \leq b\}$.

Definition 6.2. Assume that $y : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}$. Then we define $y^{\Delta}(t)$ to be the number (if it exists) such that

$$y^{\Delta}(t) = \lim_{\substack{s \to t \\ s \in \mathbb{T}}} \frac{y(\sigma(t)) - y(s)}{\sigma(t) - s}$$

and call it the delta derivative of y at t. If $y^{\Delta}(t)$ exists for all $t \in \mathbb{T}$, then we say that $y(t)$ is delta differentiable on $\mathbb{T}$.

Definition 6.3. If $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}$, then $F : \mathbb{T} \to \mathbb{R}$ is called the antiderivative of $f : \mathbb{T} \to \mathbb{R}$. Moreover, we define the Cauchy integral of $f(t)$ over $[a, b]_{\mathbb{T}}$ by

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

Instead of the usual continuity properties we introduce the notion of rd-continuity. This property guarantees the existence of the antiderivative of a given function.

Definition 6.4. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided $f(t)$ is continuous at right-dense points in $\mathbb{T}$ and its finite left-sided limits exist at left-dense points in $\mathbb{T}$. We write $f \in C_{rd}(\mathbb{T}, \mathbb{R})$ (or shortly $f \in C_{rd}(\mathbb{T})$).

Definition 6.5. A function $f : \mathbb{T} \to \mathbb{R}$ is called regressive provided

$$1 + \mu(t) f(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T}.$$

The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}(\mathbb{T}, \mathbb{R})$ (or shortly $\mathcal{R}(\mathbb{T})$).

Now we introduce the notion of a generalized exponential function $e_p(t, s)$, which is utilized in the subsequent asymptotic investigation.

Definition 6.6. If $p \in \mathcal{R}(\mathbb{T})$, then the generalized exponential function is defined by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(u)}(p(u)) \Delta u \right) \quad \text{for} \quad s,\, t \in \mathbb{T},$$

where $\xi_\mu : \{z \in \mathbb{C}, z \neq -\frac{1}{\mu}\} \to \{z \in \mathbb{C}, -\pi/\mu < \Im(z) \leq \pi/\mu\}$, $\mu > 0$ is the cylinder transformation given by

$$\xi_\mu(z) = \frac{1}{\mu} \mathrm{Log}(1 + z\mu)$$

and $\xi_0 : \mathbb{C} \to \mathbb{C}$ is the identity function.

The following property is often utilized as the alternative introduction of the generalized exponential function.

Proposition 6.7. *Consider the initial value problem*

$$y^\Delta = p(t) y, \qquad y(t_0) = 1, \tag{6.1}$$

where $p \in \mathcal{R}(\mathbb{T})$ and $t_0 \in \mathbb{T}$. Then the exponential function $e_p(\cdot, t_0)$ is the unique solution of (6.1).

The previous assertion can be extended to linear dynamic equations with a forcing term.

Proposition 6.8. *Consider the initial value problem*

$$y^\Delta = p(t) y + f(t), \qquad y(t_0) = 1, \tag{6.2}$$

where $p \in \mathcal{R}(\mathbb{T})$, $f \in C_{rd}(\mathbb{T})$ and $t_0 \in \mathbb{T}$. Then the unique solution of (6.2) *is given by*

$$y(t) = e_p(t, t_0) + \int_{t_0}^t e_p(t, \sigma(u)) f(u) \Delta u\,.$$

Among numerous properties of the generalized exponential function the following relations are useful in our investigation:

Proposition 6.9. *If $p \in \mathcal{R}(\mathbb{T})$, then*

$$\begin{aligned} e_p(t, s) &= \tfrac{1}{e_p(s, t)}, \\ e_p(t, s) e_p(s, r) &= e_p(t, r), \\ \left(\tfrac{1}{e_p(\cdot, s)} \right)^\Delta &= -\tfrac{p(t)}{e_p(\sigma(\cdot), s)} \end{aligned}$$

for any t, s, $r \in \mathbb{T}$.

Delay dynamic equations on time scales

Let $\tau : \mathbb{T} \to \mathbb{T}$ be an increasing and rd-continuous function satisfying $\tau(t) < t$ for all $t \in \mathbb{T}$. Let $a : \mathbb{T} \to \mathbb{R}$ be a regressive function and let $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ be a function with the property $f(\cdot, z(\cdot))$ is rd-continuous for any rd-continuous function $z : \mathbb{T} \to \mathbb{R}$. Then we consider the equation

$$y^{\Delta}(t) = a(t)y(t) + f(t, y(\tau(t))) \tag{6.3}$$

and call it a delay dynamic equation. In the sequel, we utilize the usual notations such as $\tau \in C_{rd}(I_{\mathbb{T}})$, $a \in \mathcal{R}(I_{\mathbb{T}})$ or $y \in C^1_{rd}(I_{\mathbb{T}})$ (which means that $y^{\Delta}(t)$ is rd-continuous on $I_{\mathbb{T}}$), where $I_{\mathbb{T}} \subset \mathbb{T}$ is a time scale interval.

For a given $t_0 \in \mathbb{T}$, a function $y(t)$ is said to be a solution of (6.3) provided $y \in C([\tau(t_0), \infty)_{\mathbb{T}}) \cap C^1_{rd}([t_0, \infty)_{\mathbb{T}})$ and $y(t)$ satisfies (6.3) for all $t \in [t_0, \infty)_{\mathbb{T}}$. If, moreover, we are given with an initial function $\phi \in C([\tau(t_0), t_0]_{\mathbb{T}})$ such that

$$y(t) = \phi(t) \qquad \text{for all } t \in [\tau(t_0), t_0]_{\mathbb{T}}\,, \tag{6.4}$$

then we say that $y(t)$ is a solution of the initial value problem (6.3), (6.4). It is easy to verify that under conditions placed on $a(t)$, $\tau(t)$, $f(\cdot, z(\cdot))$ and $\phi(t)$ there exists a unique solution $y(t)$ of the initial value problem (6.3), (6.4).

The assumption

$$\tau : \mathbb{T} \to \mathbb{T} \tag{H_1}$$

is quite natural and it is involved in all the above mentioned papers dealing with the qualitative theory of delay dynamic equations. On the other hand, it seems to be rather restrictive, especially in the asymptotic investigations of these equations. Its limitation consists in the relationship between the concrete form of τ and the type of $\mathbb{T}$. In particular, if $\tau(t) = t - h$, $h > 0$, then (H_1) is satisfied, e.g. for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = h\mathbb{Z}$, but not for $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ for any $q > 1$. Similarly, if $\tau(t) = t/q$, $q > 1$, then (H_1) is satisfied, e.g. for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, but not for $\mathbb{T} = h\mathbb{Z}$ for any $h > 0$. Our aim is to remove the assumption (H_1) from our further considerations on delay dynamic equations.

To approximate the term $y(\tau(t))$ for any $t \in \mathbb{T}$ we introduce the function

$$\tau^*(t) = \rho^*(\tau(t)), \qquad t \in \mathbb{T}\,,$$

where $\rho^* : \mathbb{R} \to \mathbb{T}$ is the extended backward jump operator defined for any $s \in \mathbb{R}$ by the relation

$$\rho^*(s) = \begin{cases} \sup\{u \in \mathbb{T}, u \le s\} & \text{for } s > \inf \mathbb{T}, \\ \inf \mathbb{T} & \text{for } s \le \inf \mathbb{T}. \end{cases}$$

Of course, $\tau^*(t) \le \tau(t)$ for all $t \in \mathbb{T}$ with the property $\tau(t) \ge \inf \mathbb{T}$ and, moreover, if $\tau(t) \in \mathbb{T}$ for some $t \in \mathbb{T}$, then $\tau^*(t) = \tau(t)$ for this t.

Now we introduce two equations extending the notion of the delay dynamic equation (6.3) to the case when the validity of (H_1) is not required. First, we approximate the term $y(\tau(t))$ in (6.3) utilizing the piecewise constant interpolation of the function $y(t)$ on $\mathbb{T}$. This leads us to the equation

$$y^{\Delta}(t) = a(t)y(t) + f(t, y(\tau^*(t))). \tag{6.5}$$

Similarly, considering the piecewise linear interpolation of $y(t)$ we obtain another extension of (6.3) in the form

$$y^{\Delta}(t) = a(t)y(t) + f\left(t, (1 - r(t))y(\tau^*(t)) + r(t)y(\sigma(\tau^*(t)))\right), \tag{6.6}$$

where

$$r(t) = \begin{cases} \frac{\tau(t)-\tau^*(t)}{\mu(\tau^*(t))} & \text{for } \mu(\tau^*(t)) \neq 0, \\ 0 & \text{for } \mu(\tau^*(t)) = 0. \end{cases}$$

To justify our approach we mention several basic properties of equations (6.5) and (6.6). The first property shows that the equations (6.5) and (6.6) are actually extensions of the equation (6.3). Its proof is obvious.

Proposition 6.10. *Let the hypothesis (H_1) be satisfied. Then both equations* (6.5) *and* (6.6) *become* (6.3).

Example 6.10. Let $\mathbb{T} = \mathbb{R}$. Then the equations (6.5) and (6.6) become the delay differential equation

$$y'(t) = a(t)y(t) + f(t, y(\tau(t))). \tag{6.7}$$

Example 6.10. Let $\mathbb{T} = h\mathbb{Z}$, $t_0 \in \mathbb{T}$, $t_n = t_0 + nh$, $y_n = y(t_n)$ and $a_n = a(t_n)$, $n = 0, 1, 2, \ldots$. Further, for any t_n we define $r_n = \frac{\tau(t_n)-t_0}{h} - \tau_n$ and $\tau_n = \left\lfloor \frac{\tau(t_n)-t_0}{h} \right\rfloor$. Then the equations (6.5) and (6.6) become the delay difference equations

$$y_{n+1} = (1 + a_n h)y_n + hf(t_n, y_{\tau_n})$$

and

$$y_{n+1} = (1 + a_n h)y_n + hf(t_n, (1 - r_n)y_{\tau_n} + r_n y_{\tau_n+1}),$$

respectively. Notice, that these formulae essentially represent the (convergent) Euler's schemes for numerical solving of the delay differential equation (6.7). Some of its particular cases have been studied in Section 3.

Now we discuss the question of the existence and uniqueness of solutions for the corresponding initial value problems. First we modify the relation (6.4) as

$$y(t) = \phi(t) \quad \text{for all } t \in [\tau^*(t_0), t_0]_{\mathbb{T}}. \tag{6.8}$$

To guarantee the existence and uniqueness for the initial value problem (6.5), (6.8) we introduce the following hypothesis:

If $\rho(t) = t = \sigma(t)$ and $\tau(t) \in \mathbb{T}$ for some $t \in \mathbb{T}$, then $\rho(\tau(t)) = \tau(t)$. $\quad (H_2)$

Under the assumption $\tau \in C_{rd}(\mathbb{T})$ the hypothesis (H_1) implies (H_2), whereas the converse does not hold.

Proposition 6.11. *Consider the delay dynamic equation* (6.5), *where $a \in \mathcal{R}(\mathbb{T})$, $f(\cdot, z(\cdot)) \in C_{rd}(\mathbb{T})$ for any $z \in C_{rd}(\mathbb{T})$ and $\tau \in C_{rd}(\mathbb{T})$ is an increasing function on $\mathbb{T}$ such that $\tau(t) < t$ for all $t \in \mathbb{T}$. Further, let the hypothesis (H_2) be satisfied, let $t_0 \in \mathbb{T}$ and let $\phi \in C([\tau^*(t_0), t_0]_{\mathbb{T}})$. Then there exists a unique solution $y(t)$ for the initial value problem* (6.5), (6.8).

Proposition 6.12. *Consider the delay dynamic equation* (6.6)*, where* $a \in \mathcal{R}(\mathbb{T})$*,* $f(\cdot, z(\cdot)) \in C_{rd}(\mathbb{T})$ *for any* $z \in C_{rd}(\mathbb{T})$ *and* $\tau \in C_{rd}(\mathbb{T})$ *is an increasing function on* $\mathbb{T}$ *such that* $\tau(t) < t$ *for all* $t \in \mathbb{T}$*. If* $t_0 \in \mathbb{T}$ *and* $\phi \in C([\tau^*(t_0), t_0]_{\mathbb{T}})$*, then there exists a unique solution* $y(t)$ *for the initial value problem* (6.6)*,* (6.8)*.*

Remark 6.13. The hypothesis (H_2), guaranteeing the existence and uniqueness for the initial value problem (6.5), (6.8), presents only a small restriction concerning the choice of $\mathbb{T}$ and $\tau(t)$. In particular, if $\mathbb{T}$ is a continuous or discrete time scale, then (H_2) is satisfied trivially regardless of the form of $\tau(t)$. Moreover, this hypothesis can be weakened provided we admit solutions $y(t)$ with the piecewise rd-continuous delta derivative on $\mathbb{T}$.

The equation $y^{\Delta}(t) = ay(t) + by(\rho^*(qt))$

In this section, we give the application of the above introduced extension of the notion of a delay dynamic equation. In particular, we discuss some qualitative properties of the solutions of the scalar pantograph equation

$$y'(t) = ay(t) + by(qt), \quad 0 < q < 1, \quad t \geq 0 \tag{6.9}$$

as well as its simple numerical discretization

$$y_{n+1} - y_n = ahy_n + bhy_{\lfloor qn \rfloor}, \quad 0 < q < 1, \quad h > 0, \quad n = 0, 1, 2, \dots . \tag{6.10}$$

From Theorem 2.1 we can conclude for (6.9) an asymptotic estimate

Corollary 6.14. *Consider the equation* (6.9)*, where* $a < 0$*,* $b \neq 0$ *are real scalars. If* $y(t)$ *is a solution of* (6.9)*, then*

$$y(t) = O(t^{\alpha}) \quad \text{as } t \to \infty, \qquad \alpha = \frac{\log |b|/(-a)}{\log q^{-1}} . \tag{6.11}$$

The qualitative theory for the delay difference equation (6.10) is less developed than in the corresponding continuous case. The analysis of this equation on the compact as well as unbounded domain is performed in papers [59], Liu [72] and Péics [82]. Further related results on the asymptotics of the solutions of delay difference equations can be found, e.g. in Elaydi and Győri [35] or Győri and Pituk [40].

Next, let us emphasize that the numerical formula (6.10) is not stable. In particular, there is a stepsize $h > 0$ such that the difference equation (6.10) admits an unbounded solution even if the exact equation (6.9) is asymptotically stable for all $0 < q < 1$ (for the precision of various types of stability of numerical methods for delay differential equations we refer to the book by Bellen and Zennaro [8]). Indeed, consider the equation

$$y'(t) = -2y(t) + y(t/2) . \tag{6.12}$$

By Corollary 6.14, parameters $a = -2$ and $b = 1$ satisfy the asymptotic stability condition $a + |b| < 0$ and this equation is asymptotic stable (more precisely, $y(t) = O(1/t)$ as $t \to \infty$ for any solution $y(t)$ of (6.12)). The corresponding difference equation (6.10) can be written in the form

$$y_{n+1} = (1 - 2h)y_n + hy_{\lfloor n/2 \rfloor}, \quad n = 0, 1, 2, \dots .$$

We show that (y_n) is unbounded as $n \to \infty$ provided $h = 1$ and $y_0 \neq 0$. Let $y_0 = 1$. Since

$$y_{2n+2} = -y_{2n+1} + y_n = y_{2n}\,,$$

we get $y_{2n} = 1$ for any $n = 0, 1, 2, \dots$. Obviously $y_1 = 0$, $y_3 = -1$ and using the induction principle we arrive at $y_{2^n-1} = -n + 1$ for any $n = 1, 2, 3, \dots$, hence the sequence (y_n) is unbounded as $n \to \infty$.

To describe the stability and asymptotic properties of the equations (6.9) and (6.10) from the unified viewpoint we apply the form (6.5) and introduce the dynamic pantograph equation

$$y^{\Delta}(t) = ay(t) + by(\rho^*(qt)), \quad 0 < q < 1, \quad t \in \mathbb{T}\,. \tag{6.13}$$

Note that this equation involves the exact pantograph equation (6.9) and its discretization (6.10) as particular cases via the choice $\mathbb{T} = \mathbb{R}^+ = [0, \infty)$ and $\mathbb{T} = h\mathbb{Z}^+ = \{hk, k = 0, 1, 2, \dots\}$, respectively. Indeed, if $\mathbb{T} = h\mathbb{Z}^+$, then $y^{\Delta}(t) = (y((n+1)h) - y(nh))/h$ at $t = nh$, $n = 0, 1, 2, \dots$ and adapting the difference equation notation with $y(nh)$ replaced by y_n we can rewrite (6.13) into the usual difference form (6.10).

Our aim is to extend the validity of the asymptotic estimate mentioned in Corollary 6.14 to the dynamic equation (6.13). To simplify our investigation we consider time scales $\mathbb{T}$ with a constant (zero or positive) graininess μ. Note that this assumption is consistent with the above declared unification of the qualitative investigation of equations (6.9) and (6.10) because both corresponding time scales $\mathbb{T} = \mathbb{R}^+$ and $\mathbb{T} = h\mathbb{Z}^+$ have the constant graininess.

To formulate the intended extension of Corollary 6.14, we introduce the scalar

$$\tilde{a} = \begin{cases} a & \text{for } 1 + a\mu \geq 0, \\ -a - \frac{2}{\mu} & \text{for } 1 + a\mu < 0 \end{cases} \tag{6.14}$$

as well as the auxiliary functional inequality

$$|b|\varphi(\rho^*(qt)) \leq -\tilde{a}\varphi(t), \quad t \in \mathbb{T}\,,$$

which plays the key role in our asymptotic analysis. Now we can present the main result formulating the upper bounds of solutions of the differential equation (6.9) (the case $\mu = 0$) as well as the difference equation (6.10) (the case $\mu = h > 0$) in the unified way.

Asymptotic estimate

Theorem 6.15. *Consider the equation* (6.13)*, where $a, b \in \mathbb{R}$, $b \neq 0$ and $\mathbb{T}$ is a time scale with a constant graininess μ and the property* $\inf \mathbb{T} \geq 0$*. Further, let $\tilde{a}$ be defined by* (6.14) *and assume that $\tilde{a} < 0$. Then*

$$y(t) = O(t^{\alpha}), \qquad \alpha = \frac{\log(|b|/(-\tilde{a}))}{\log q^{-1}} \tag{6.15}$$

for any solution $y(t)$ of (6.13)*.*

Remark 6.16. The possible generalization of Theorem 6.15 to time scales with variable graininess is straightforward, but it requires the introduction of some additional assumptions. In particular, the scalar $\tilde{a}$ is replaced by the function $\tilde{a} = \tilde{a}(t)$ defined by the relation (6.14) involving the variable graininess $\mu = \mu(t)$. Moreover, some restrictions on μ must be added to ensure the usability of Theorem 6.15 proof procedure.

Conclusions

The important feature of our previous considerations on delay dynamic equations is the omission of the hypothesis (H_2) representing along with a given delay $\tau(t)$ the significant restriction on the choice of $\mathbb{T}$. This enables us, among others, to apply our results to time scales $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=h\mathbb{Z}$ regardless of the form of the given delay and thus jointly investigate the corresponding differential equations and their numerical discretizations. To outline the usefulness of this approach we discuss the conclusion of Theorem 6.15 with respect to the choice of $\mathbb{T}$.

If $\mathbb{T}=\mathbb{R}^+$, then $\mu\equiv 0$, $\tilde{a}=a$ and the dynamic equation (6.13) becomes the differential equation (6.9). Hence, the estimate (6.11) holds for any solution $y(t)$ of the exact pantograph equation (6.9) with $a<0$ (thus we have generalized the classical result of Kato and McLeod [51] recalled in Corollary 6.14). However, if $\mathbb{T}=h\mathbb{Z}^+$, then $\mu\equiv h$, (6.13) becomes the difference equation (6.10) and the assumption $\tilde{a}<0$ implies that either $a<0$ provided $1+ah\geq 0$ or $2+ah>0$ provided $1+ah<0$. These relations essentially represent the restriction on the stepsize h of the corresponding discrete equation (6.10), namely $h<2/(-a)$, $a<0$. Theorem 6.15 now implies that if $h\leq 1/(-a)$, $a<0$, then the estimate

$$y_n=O(n^\alpha)\quad\text{as}\quad n\to\infty,\qquad \alpha=\frac{\log|b|/(-a)}{\log q^{-1}}$$

holds for any solution y_n of (6.10). If $1/(-a)<h<2/(-a)$, then

$$y_n=O(n^\beta)\quad\text{as}\quad n\to\infty,\qquad \beta=\frac{\log(|b|h/(2+ah))}{\log q^{-1}}\,.$$

In other words, the asymptotic estimates for the exact pantograph equation (6.9) and its discretization (6.10) coincide provided $h\leq 1/(-a)$.

As we observed earlier, these considerations are closely related to the discussions on the stability of the given numerical discretizations. By Corollary 6.14, the asymptotic stability property $\lim\limits_{t\to\infty} y(t)=0$ for the delay differential equation (6.9) can be read as $a+|b|<0$. The same condition remains preserved when we judge the asymptotic stability of the corresponding delay difference equation (6.10) with the stepsize $h\leq 1/(-a)$. If $1/(-a)<h<2/(-a)$, then by Theorem 6.15 this condition is modified as $a+2/h>|b|$. To summarize, if $h<2/(|b|-a)$, then the asymptotic stability property is guaranteed for the difference equation (6.10) under the same condition $a+|b|<0$ that ensures such a property for the corresponding exact equation (6.9). Of course, the similar considerations can be accomplished for other types of the stability (the contractivity property, etc.).

We note that the asymptotic stability analysis of dynamic equations with a constant delay

$$y^\Delta(t)=ay(t)+by(t-\tau),\quad 0<\tau<t,\quad t\in\mathbb{T}$$

represent a very difficult task. Indeed, if $\mathbb{T}=\mathbb{Z}^+$, then we need to analyse conditions under which all zeros of the characteristic polynomial are located inside the unit disk in the complex plane. On the other hand, stability analysis of the same dynamic equation for $\mathbb{T}=\mathbb{R}^+$ leads to the analysis of conditions ensuring that all zeros of appropriate characteristic quasi-polynomial are situated in the left half of the complex plane. It looks to be an extremely difficult matter to discover a bridge between these two approaches in the time scale calculus frame.

7 Summary and future directions

The thesis introduced several results dealing with asymptotic properties of special linear delay differential equations, corresponding linear difference equations and dynamic equations on time scales. After a brief motivation, Section 2 introduced asymptotic estimates for particular delay differential equations with unbounded delays. We showed that type of asymptotics of their solutions essentially depends on the sign of a coefficient function occurring at non-delayed term.

Section 3 formulated several asymptotic estimates for difference equations with unbounded delays. The studied difference equations can be obtained by a specific discretization from the differential equations studied in Section 2. A discussion of corresponding results was also included. In this connection, we note that the author's paper [59] concerning the convergence rate of solutions of difference equations with unbounded delay was referenced in Appleby and Buckwar [5] as the interesting one.

Section 4 presented a brief survey of results for asymptotic stability of linear delay differential equations with a constant delay. It was emphasized that the question of asymptotic stability is closely related to the location of the zeros of an appropriate quasi-polynomial.

In Section 5, the asymptotic stability of linear difference equations with real constant coefficients was discussed. There were obtained sets of explicit necessary and sufficient conditions for particular four-term difference equations. The results generalize (and improve) the fundamentals of the stability theory for difference equations. Since the analysis was based on the location of zeros of a polynomial, the results seem to be new also in polynomial theory. Beside the contribution to the fundamentals of difference equation theory, these results are found to be applicable in another fields. This presumption is supported by the utilization of our results by other researchers. For example, the results introduced in Theorem 5.3 and Theorem 5.4 were utilized in analysis of a numerical discretization of a partial delay differential equation by Pimenov [83]. Another citation of this result was made by Singh [89]. Since the results stated in this section are quite recent, we are awaiting another responses. The email correspondence with professor Kipnis [56] and his interest about these results encouraged us to the further asymptotic stability investigations.

Section 6 presented an asymptotic estimate for dynamic equation with proportional delay on time scales. In this section the connection with appropriate delay difference and differential equations was also illustrated. Moreover, a new approach was utilized in this place: to be the dynamic delay equation well posed for arbitrary time scale $\mathbb{T}$ and delay argument $\tau(t)$, the extended backward jump operator ρ^* was introduced. It enabled us to be no more restricted by the common assumption (H_1), i.e. $\tau : \mathbb{T} \to \mathbb{T}$. This approach inspired professors Agarwal, Anderson and Zafer, who in [2] commented our notion and introduced a complementary extended forward jump operator to generalize their result dealing with oscillatory properties of second order forced delay dynamic equation on time scales.

The author finds further research possibilities especially in the generalization of results in Section 5. Asymptotic stability conditions presented here in a compact and explicit form may be an useful pattern for reformulation of some existing results on asymptotic stability of other particular linear difference equations (see the papers by Dannan [30] and Matsunaga and Hajiri [78]) and, of course, for stability investigations of delay difference equations, which have not been considered yet. As an example may serve the equation

$$y_{n+m} + \alpha\, y_n + \beta\, y_{n-k+m} + \gamma\, y_{n-k} = 0, \qquad n = 0, 1, 2, \ldots$$

with positive integers $k > m$, which generalizes not only equations considered in the thesis, but all other types of scalar delay difference equations studied so far with respect to necessary and sufficient stability conditions. The investigation of these problems requires, among others, further developments of the proof method proposed in Section 5. The applicability of the asymptotic stability investigation can be documented by the author's paper [92], where the asymptotic stability conditions for particular full-term difference equation

$$y_{n+k} + a \sum_{s=1}^{k-1} y_{n+s} + b y_n = 0\,, \tag{7.1}$$

where a, b are nonzero real constants, are developed by the similar proof procedure. Equation (7.1) arises in the analysis of optimization process of multi-variable quadratic function $f(x_1, \dots, x_k) = x_1^2 + x_2^2 + \dots + x_k^2$ by the Nelder-Mead method. The convergence of the method is closely related to the asymptotic stability of (7.1) (see Han, Neumann and Xu [41]).

Finally, dealing with dynamic equations on time scales in Section 6 the technique utilized in the proof of Theorem 6.15 seems to be applicable also in more general cases. We can consider, e.g. the nonautonomous pantograph equation or even the pantograph equation in the nonlinear form (6.5), i.e. the equation

$$y^\Delta(t) = a(t)y(t) + f(t, y(\rho^*(qt))), \quad 0 < q < 1, \quad t \in \mathbb{T}\,,$$

where some growth conditions on functions $a(t)$ and $f(\cdot, z(\cdot))$ must be involved to guarantee the validity of the estimate (6.15) also for these equations. The other natural and interesting extension of our results is provided via the well known generalized pantograph equation having applications in many areas ranging from industrial problems to astrophysics. Modifying and applying the form (6.5) to this (originally differential) equation we arrive at the neutral dynamic equation

$$y^\Delta(t) = ay(t) + by(\rho^*(qt)) + cy^\Delta(\rho^*(\mu t)), \quad 0 < q,\, \mu < 1, \quad t \in \mathbb{T}\,.$$

Similarly, we can consider and investigate delay dynamic equations derived by means of the piecewise linear approximation of the delayed term (see formula (6.6)) or equations based on other numerical approximations (e.g. a trapezoidal rule).

Another following research direction seems can be the asymptotic stability analysis of linear differential equations of fractional order. The fractional calculus is a subject of wide recent research, which is supplied by many papers dealing with this topic. This research area is quite new and there arise many applications of differential equations of fractional order. Therefore, we are going to concentrate on this topic as well.

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List of symbols

$\mathbb{C}$	set of complex numbers
$\mathbb{N}$	set of natural numbers $\{1, 2, \dots\}$
$\mathbb{Z}$	set of integers
$\mathbb{N}_0$, $\mathbb{Z}^+$	set of all nonnegative integers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of all nonnegative real numbers
$\mathbb{T}$	time scale
$C(I)$	set of all continuous functions defined on interval I
$C^k(I)$, $k \in \mathbb{N}_0$	set of all continuous functions which have continuous all its derivatives up to the order k on I
$C_{rd}(\mathbb{T})$	set of all rd-continuous functions defined on $\mathbb{T}$
$\mathcal{R}(\mathbb{T})$	set of all regressive and rd-continuous functions on $\mathbb{T}$
$\Re(\lambda)$, $\Im(\lambda)$, $\lambda \in \mathbb{C}$	real and imaginary part of λ
$\lceil s \rceil$	the ceiling function applied to $s \in \mathbb{R}$
$\lfloor s \rfloor$	the floor function applied to $s \in \mathbb{R}$
$e_p(t, s)$	generalized exponential function
$y^{\Delta}(t)$	delta derivative of y at t

Abstract

The thesis deals with asymptotic properties of delay differential equations and their discretizations. Asymptotic estimates for particular cases of differential and difference equations with unbounded delays are introduced. Analysis of certain difference equations with constant delays led to introduction of efficient form of necessary and sufficient conditions for their asymptotic stability. These results can be applied in numerical analysis of delay differential equations, which fact is demonstrated by asymptotic stability discussion of certain numerical schemes. Finally, a unified approach is presented by formulation of asymptotic estimates for corresponding delay dynamic equation on time scales.