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PROJECTIVE GEOMETRIES OVER CLIFFORD ALGEBRAS

VYSOKÉ UČENÍ TECHNICKÉ V BRNĚ
Fakulta strojního inženýrství
Ústav matematiky

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PROJECTIVE GEOMETRIES OVER CLIFFORD ALGEBRAS

PROJEKTIVNÍ GEOMETRIE NAD CLIFFORDOVÝMI ALGEBRAMI

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2007–2008: lecturer, Support Center for Students with Special Needs, MU Brno.

2008–until now: senior lecturer at the Institute of mathematics FME BUT

Pedagogic activities

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- Differential geometry (geometry structures, parabolic geometries, planar curves)
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1 Introduction

In modern differential geometry [45, 42] affected by Klein's Erlangen programme [63, 40, 41], the theory of Cartan connections [40] together with the theory of semisimple Lie algebra representation [19, 56, 61] makes the foundations of the theory of parabolic geometries [71]. Shortly, Cartan geometries can be understood as curved versions of homogeneous bundles $G \to G/H$ and parabolic geometries as Cartan geometries with G a semisimple group, $\mathfrak{g}$ a $|k|$–graded algebra and the adjoint action of H preserving the filtration $\mathfrak{g}$. In our considerations, the fact that parabolic geometries are endowed with a particular class of Weyl connections is crucial, because it makes the research of projective transformations with respect to this class sensible.

The core of the presented thesis is the continuation of the research handled in the PhD thesis [24] and papers [32, 23], in which we established the notions of A–structures, A–planar curves and proved that under certain additional conditions, the morphisms preserving the appropriate A–structures are exactly the morphisms preserving the class of A–planar curves for fixed connection with covariant derivative preserving A. For example, almost quaternionic geometries, which were also our motivation, have this property and as it is also a parabolic geometry, we proved that the only possible choice of corresponding Connection class is that of Weyl connections. But as the class of Weyl connections share the same A–planar curves, no additional choice is needed and the morphisms preserving the A–planar curves are the morphisms of almost quaternionic geometry.

Among parabolic geometries there exist some other structures based on affinors. An example is almost complex projective structure (M, J, ∇) with affinor J s.t. $J^2 = -\mathrm{id}_M = -E$ and an almost product projective structure (M, J, ∇) with affinor J s.t. $J^2 = E$ for more details see our papers [25, 31]. While almost quaternionic structure is the parabolic geometry, an almost complex and almost product are not parabolic and become parabolic geometries just after the choice of a special class of connections [39, 6, 79]. Thus, in certain sense, we obtain the projective versions of both structures called almost complex projective and almost product projective. Using this definition, our structures can be seen as parabolic geometries and, similarly to almost quaternionic after normalization, they are endowed with a class of Weyl connections playing the role of the special class of connections. As our main result, we have shown that the morphisms of these projective structures are exactly the morphisms preserving the class of geodesics of Weyl connections, i.e. the generalized projective transformations with respect to the class of Weyl connections $[\nabla]$. Proofs of these claims for almost quaternionic structure can be found in [32, 23], for almost complex structure in [25] and for almost product structure in [31]. Note that important fact for our calculations is that our examples are $|1|$–graded (irreducible) parabolic geometries. To see the impact of our results one can find the papers [44, 22].

Next group of new results is described in our papers [30, 28, 26, 27], where the theory of A–structures is developed and used for classification of some less known structures such as almost quaternionic of the second kind (para–quaternionic) and more generally the triple structures containing apart from the almost quaternionic and almost para–quaternionic one also their analogues for commuting affinors. We partially discussed even the case of distributions which can be defined also by means of affinors. All these cases share the property that the subbundle A is locally isomorphic to an algebra. In this case we proved that the additional conditions can be significantly simplified. For these geometries we characterized the classes of projective transformations with respect to a given class of connections.

The notion of A–planar curves, in other words the notion of generalized geodesics, is widely studied for structures based on generally one affinor [46, 47, 48, 50, 49] and is known as F–

planar curves, in case of almost hypercomplex structure [52, 51, 23, 22] this is referred to as 4–planar or $\mathbb{H}$–planar curves. Let us note that in case that A is a commutative algebra, we obtain so–called spaces over algebras [64].

As a next step in our research we focused on the A–structures where A is locally isomorphic to a Clifford algebra. This leads to the definition of a subclass of A–structures called Cliffordian structures. All important structures such as almost quaternionic (based on Clifford algebra $\mathcal{C}l(0,2)$), almost para–quaternionic (based on Clifford algebra $\mathcal{C}l(2,0)$) or almost complex projective (based on Clifford algebra $\mathcal{C}l(0,1)$) are Cliffordian. If A is finitely generated algebra, the A–structures and thus also the Cliffordian structures, are G–structures, where the structure group G is the group of automorphisms preserving the subbundle A. From the theory of G–structures we know that there exists a class of distinguished connections sharing the same torsion. Such connections are called $\mathcal{D}$–connections and their uniqueness is given by G–invariant decomposition of the appropriate cohomologic component. Motivated by the fact that the choice of the projective structure corresponds to the choice of the class of connections sharing geodesics and the choice of "complex" projective structure corresponds to the choice of connections sharing the J–planar curves, we define the projective A–structures as A–structures together with the choice of connection class sharing A–planar curves. If A is a Clifford algebra, we obtain a class of so–called Cliffordian projective structures. In our papers [35, 29, 33] we proved that every Cliffordian structure admits Cliffordian projective structure and we found explicit description of the appropriate class of $\mathcal{D}$–connections preserving planar curves and described several properties.

2 Almost quaternionic manifolds

In this chapter, we provide basic definitions and results which form the foundations of the theory of A–structures, and we use the case of almost quaternionic geometry to interpret them. Almost quaternionic geometry [60, 14] is widely studied geometric structure based on quaternions $\mathbb{H}$. Let us note that from our point of view the almost quaternionic structure is determined by a subbundle $Q \subset \Gamma(TM \otimes T^*M)$, which is locally isomorphic to quaternions $\mathbb{H}$. Historically, the notion of almost quaternionic structure is used even for a smooth manifold endowed with three affinors F, G and H, which, together with the identity, satisfy the properties of quaternions. In modern literature, this is strictly referred to as almost hypercomplex and we use this notation, too. Almost quaternionic structure is determined by the following definition.

Definition 2.1. Let M be a manifold of dimension $n = 4m$, and assume that there is a 3–dimensional vector bundle Q consisting of affinors (tensors of type $(1,1)$) over M satisfying the following condition: On any coordinate neighbourhood of $x \in M$, there is a local basis $\{F, G, H\}$ of Q such that $F^2 = G^2 = H^2 = -E$, $FG = -GF = H$, $GH = -HG = F$, $HF = -FH = G$, where we denote by E the identity affinor on M. Then the bundle $Q \to M$ is called an *almost quaternionic structure (or geometry)* on M and (M, Q) is called an *almost quaternionic manifold* .

Note that almost quaternionic geometry is both a G–structure, an A–structure, Cliffordian and parabolic geometry and that is why we refer to it in all following chapters.

2.1 Q–connections

The theory of Q–connections was established by pioneer work of Mario Obata [54, 53, 55]. The first article [54] was published in 1958 in Journal of the Mathematical society of Japan. The theory was developed by plenty of mathematicians during next twenty years, for example in articles [43, 68, 37, 38] and finally in the work of Shigeyoshi Fujimora which was published in papers [14, 15, 16, 17] and work of Vasile Oproiu in papers [57, 58]. By Q-connections we understand special classes of connections complying with quaternionic geometry and, as we find out later, the automorphisms preserving their geodesics are exactly the morphisms of quaternionic geometry. Let us recall the basic facts.

Definition 2.2. Let Γ be an affine connection on an almost quaternionic manifold (M, Q) which will be called a *Q–connection* if it satisfies the following conditions:

$$\nabla F = \alpha \otimes G - \beta \otimes H,$$
$$\nabla G = -\alpha \otimes F + \gamma \otimes H,$$
$$\nabla H = \beta \otimes F - \gamma \otimes G,$$

where ∇ denotes the operator of covariant differentiation with respect to Γ and α, β, γ are certain 1–forms and $\{F, G, H\}$ is a local basis of Q.

The Definition 2.2 does not depend on the choice of the basis F, G and H of Q and the expression of terms on the right hand side came from nature of Q (for more details see [57]). Note that Γ is a Q–connection if and only if it preserves cross-sections φ of bundle $Q \to M$, which is equivalent to the condition that $(2, 2)$–tensor

$$A = I \otimes I - F \otimes F - G \otimes G - H \otimes H$$

is covariantly constant with respect to Γ, see [14].

Let F, G and H be a basis of Q. A Q–connection is called a V–connection if $\alpha = \beta = \gamma = 0$, i.e. affinors F, G and H are covariantly constant with respect to an affine connection Γ. This suggest that the V–connection should rather be thought of as the property of hypercomplex structures a quaternionic.

In particular, M. Obata proved that on an almost quaternionic manifold with an affine connection, there always exists a V–connection (which is called *Obata connection*), i.e. there always exists a Q–connection with $\alpha = \beta = \gamma = 0$. In fact, for an affine connection Γ and arbitrary 1–forms α, β and γ, if we put in coordinates

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} - \frac{1}{4}(F^h_a F^a_{i;j} + G^h_a G^a_{i;j} + H^h_a H^a_{i;j}) + \frac{1}{2}(\gamma_j F^h_i + \beta_j G^h_i + \alpha_j F^h_i),$$

then $\bar{\Gamma}$ is a V–connection. It is not hard to see the existence of Q–connection including V–connection on an almost quaternionic manifold with a connection.

Finally on an almost quaternionic manifold with a connection Γ, a curve $x^h = x^h(t)$ is called a *Q–planar curve* if it satisfies the system of the ordinary differential equations

$$\frac{\partial^2 x^h}{dt^2} + \Gamma^h_{ab}\frac{\partial x^a}{\partial t}\frac{\partial x^b}{\partial t} = (\varphi_1(t) I^h_a + \varphi_2(t) F^h_a + \varphi_3(t) G^h_a + \varphi_4(t) H^h_a)\frac{\partial x^a}{\partial t},$$

where $\varphi_s(t)$ are certain functions of the parameter t.

One can find the following theorems on Q–plannar curves, Q–plannar connections, their properties and transformations in [14]. In fact, we will see that in [32], more general results were presented.

Theorem 2.3 ([15])**.** *On an almost quaternionic manifold, affine connections Γ and $\bar{\Gamma}$ share the same Q–planar curves if and only if there exist 1–forms η, λ, μ and ν satisfying*

$$\frac{1}{2}S^h_{(ij)} = \eta_{(i} I^h_{j)} + \lambda_{(i} F^h_{j)} + \mu_{(i} G^h_{j)} + \nu_{(i} H^h_{j)},$$

where $S^h_{ij} = \bar{\Gamma} - \Gamma$ and $T_{(ij)} = \frac{1}{2}(T_{ij} + T_{ji})$ for an arbitrary tensor field T.

Now, we will say that affine connections Γ and $\bar{\Gamma}$ are *Q–projectively related* if and only if there exist such 1–forms as in Theorem 2.3.

Theorem 2.4 ([14])**.** *Symmetric Q–connections Γ and $\bar{\Gamma}$ are Q–projectively related if and only if there exists a 1–form η such that $\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + A^{ha}_{(ij)}\eta_a$.*

It is easy to see that Q–projectively related symmetric connection share the same torsion and in the case of a symmetric connection the class of Q–projectively related connections is parametrized by one–forms. A symmetric Q–connection may not exist on any almost quaternionic manifold but we later show that on almost quaternionic manifold, there exists a class of connections with, in certain sense, minimal torsion. This class is determined uniquely and is parametrized by one–forms, too. We shall use exactly such class and show that Theorem 2.4 is a consequence of our results.

If a transformation f of M to itself leaves the bundle invariant, then f is called *Q–transformation* of (M, Q). S. Ishihara proved [37, 38] that f is a Q–transformation if and only if it preserves the tensor field A. If f maps any Q–planar curve with respect to Γ into another one with respect to Γ then f is called *Q–projective transformation.* The following theorem compares the Q–transformations and Q–projective transformations.

Theorem 2.5 ([15])**.** *Let (M, Q) be an almost quaternionic manifold of dimension $m > 4$ with an affine connection Γ. Then, a transformation f of M onto itself is a Q–projective transformation of (M, Q) with respect to Γ if and only if*

1. *f is Q–transformation of (M, Q).*
2. *Γ and the affine connection induced by f from Γ are Q–projectively related.*

The above Q–projective transformations are exactly those transformations of our interest and we will call them Q–planar. In fact, one of our results is a modification of Theorem 2.5.

2.2 Integrability of almost quaternionic structures

Let us say a few words about integrability of an almost quaternionic structures. If a manifold M can be covered by a system of coordinate neighbourhoods in which the components of F, G and H are all constant, we say that the almost quaternionic structure $\{F, G, H\}$ is *integrable* and call it a *quaternionic structure.* The results on integrability of an almost quaternionic structure are based on Kentaro Yano and Mitsue Ako papers, mainly the paper [80] which was published in Hokkaido mathematical journal in 1972.

Concerning integrability, the proper tool in case of quaternionic geometry is the Nijenhuis tensor. Let P and Q be two affinors on a differentiable manifold. The expression

$$\begin{aligned} N(P, Q)(X, Y) = {} & [PX, PY] - P[QX, Y] - Q[X, PY] \\ & + [QX, PY] - Q[PX, Y] - P[X, QY] + (PQ + QP)[X, Y], \end{aligned}$$

where X and Y are arbitrary tensor fields of suitable type, defines a tensor field of type $(2, 1)$ and is called a *Nijenhuis tensor* of P and Q.

Theorem 2.6 ([80])**.** *Let $Q = \langle F, G, H\rangle$ be an almost quaternionic structure. If two of six Nijenhuis tensors*

$$N(F, F), N(G, G), N(H, H), N(G, H), N(H, F), N(F, G)$$

vanish, then the others vanish too.

Note that the torsion of the Obata connection is given by the *structure tensor*

$$T = \frac{1}{12}(N(F, F) + N(G, G) + N(H, H))$$

and then it is easy to see the following theorem.

Theorem 2.7 ([80]). *Let (M, Q) be an almost quaternionic manifold. There exists a symmetric affine connection ∇ on M such that $\nabla F = 0$, $\nabla G = 0$, $\nabla H = 0$ if and only if two of six Nijenhuis tensors:*

$$N(F, F), N(G, G), N(H, H), N(G, H), N(H, F), N(F, G)$$

vanish.

As we show later, in our paper [29] we generalized these properties for the classes of some geometries.

2.3 Almost para–quaternionic manifolds

An almost quaternionic geometry is the best known example of geometry based on affinors, but there exists a plenty of other structures based on one or more affinors. For example, an almost complex geometry or an almost para–quaternionic geometry. We will shortly discuss the almost para–quaternionic one.

Definition 2.8. Let M be a manifold of dimension $n = 4m$, and assume that there is a 3–dimensional vector bundle Q consisting of affinors over M satisfying the following condition: On any coordinate neighbourhood of $x \in M$, there is a local basis $\{F, G, H\}$ of Q such that $F^2 = -E$, $G^2 = H^2 = E$ and $F = -GH = HG$, $G = HF = -FH$, $H = FG = -GF$, where we denote by E the identity affinor on M. Then the bundle $Q \to M$ is called an *almost para–quaternionic structure* and the differentiable manifold with an almost para–quaternionic structure an *almost para–quaternionic manifold.*

Theorem 2.9 ([81]). *In order that there exists, on an almost para–quaternionic manifold $(M, Q = \langle F, G, H\rangle)$, a symmetric affine connection ∇ such that $\nabla F = 0$, $\nabla G = 0$, $\nabla H = 0$, it is necessary and sufficient that two of six Nijenhuis tensors $N(F, F)$, $N(G, G)$, $N(H, H)$, $N(G, H)$, $N(H, F)$, $N(F, G)$ vanish.*

Note that the proof of Theorem 2.9 is based on special connection, such that torsion of this connection is given by the *structure tensor*

$$T = \frac{1}{6}([F, F] + [G, G] + [H, H]).$$

Remark 2.10. Similar results for almost product structures, i.e. differential manifold with one affinor P, such that $P^2 = E$, can be found in papers [82, 36, 18]. More generally in Walker's papers [76, 77] the author claims that for any system of distributions there exists an affine connection with respect to which the distributions are parallel and which is symmetric if the system is integrable. Walker's work is summarized in paper [78]. If $D, \bar{D}$ form a complete system of distributions (i.e. they are disjoint and $D + \bar{D} = TM$) then there are two affinors $P, \bar{P}$ associated with them such that

$$P^2 = P,\ \ \bar{P}^2 = \bar{P},\ \ P\bar{P} = \bar{P}P = 0 \text{ and } P + \bar{P} = E,$$

where rank $P = r$ and rank $\bar{P} = \bar{r}$.

The representation of distributions by affinors can by extended to any complete system D_i such that the affinors P_i satisfy the properties

$$P_i^2 = P_i,\ \ P_iP_j = 0 \text{ for } i \neq j, \text{ and } \sum_i P_i = E.$$

Let Γ^i_{jk} be Christoffel symbols of a symmetric connection ∇ on M then the connection $\bar{\nabla}$ based on Christoffel symbols $\bar{\Gamma}^i_{jk}$

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} - P^p_k \nabla_j P^i_j - P^p_j \nabla_k P^i_p + P^p_k P^q_j \nabla_q P^i_p$$

makes D parallel [78].

2.4 $\mathcal{D}$–connections

An almost quaternionic geometry can be understood as a G–structure and on any G–structure there is a possibility of choosing the class of $\mathcal{D}$–connections, i.e. those connections with, in certain sense, distinguished torsion. Furthermore, as almost quaternionic geometry is parabolic, it is possible to choose the class of $\mathcal{D}$–connections canonically. In terms of parabolic geometries, these are called Weyl connections (see chapter 4). Let us remind some basic facts from the theory of G–structures.

The theory of *G–structures* belongs to the modern differential geometry [40, 41, 63] which is based on so called Felix Klein's Erlangen Programme [63]. From the definition, *G–structures* are reductions of the bundle of frames P^1M with respect to the structure group G, for more details see [40, 41, 63]. In our example, it is very well known (see [60]) that the structure group of an almost quaternionic geometry is

$$GL(n, \mathbb{H}) \times_{\mathbb{Z}_2} Sp(1),$$

where $Sp(1)$ are the unit quaternions in $GL(1, \mathbb{H})$.

In fact, we are interested in structure preserving connections and classes of connections on G–structures. In D. V. Alekseevsky and S. Marchiafava paper [2] *Quaternionic structures on a manifold and subordinated structures* which was published in Annali di Mathematica pura ed applicata at 1996, the theory of so called *$\mathcal{D}$–connections* was developed. The class of $\mathcal{D}$–connections preserves the structure, shares the same torsion and the connections are parametrized by the first prolongation of Lie algebra $\mathfrak{g}$.

The structures of our interest can be understood as G–structures. We shall study the generalized projective transformations and thus the classes of geodesics of suitable connections, i.e. connections with the same (in certain sense distinguished) torsion. In case of G–structures, $\mathcal{D}$–connections are such sensible choice, but the choice of the complement $\mathcal{D}$ is needed.

In Chapter 4 we will see that concerning irreducible parabolic geometries this class coincides with the unique class of connections which are called Weyl connections. But not all geometries of our interest are parabolic.

Briefly from structural theory, let $G \subset GL(\mathbb{V})$ be a linear reductive group with Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(\mathbb{V}) = \mathbb{V} \otimes \mathbb{V}^*$. If we fix a G–invariant complement $\mathcal{D}$ to the subspace $\partial(\mathfrak{g} \otimes \mathbb{V}^\star)$ in $\mathbb{V} \otimes \wedge^2 \mathbb{V}^\star$, where ∂ is the Spencer operator of alternations, then we shall investigate the theory of *$\mathcal{D}$-connections.*

More precisely, the vertical bundle $VP := \ker\,(Tp)$ is trivialized as a vector bundle G–structure $p : P \to M$ by the principal action. So

$$\omega(X_u) := T_e(r_u^{-1})\Psi(X_u) \in \mathfrak{g}, \tag{2.1}$$

where the principal connection $\Psi : TP \to VP$ is a fibre projection viewed as a 1-form in $\Psi \in \Omega(P, TP)$ and in this way we get a $\mathfrak{g}$-valued 1-form $\omega \in \Omega(P, \mathfrak{g})$, which is called *connection form* of the principal connection Ψ.

Definition 2.11. Let $\mathbb{V} = \mathbb{R}^n$ and G be a Lie group of linear transformations of $\mathbb{V}$, and the $\mathfrak{g}$ be a Lie algebra of G. The *first prolongation* $\mathfrak{g}^{(1)}$ of $\mathfrak{g}$ is a space of symmetric bilinear mappings $t : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ such that, for each fixed $v_1 \in \mathbb{V}$, the mapping $v \in \mathbb{V} \mapsto t(v, v_1) \in \mathbb{V}$ is in $\mathfrak{g}$.

Definition 2.12. We say that the connected linear Lie group G with Lie algebra $\mathfrak{g}$ is of type k if its k-th prolongation vanishes, i.e. $\mathfrak{g}^{(k)} = 0$ and $\mathfrak{g}^{(k-1)} \neq 0$. In this case, any G-structure is called to be a *structure of type k.*

From the theory of G–structures, we can define a principal bundle $\bar{P} \to P$ in such way that the sections of this bundle are in 1–1 correspondence with connections ω of G-structure. This bundle is called the bundle of the *derivation of the G-structure*. On the bundle $\bar{P} \to P$ there is a canonical $\mathbb{V} \otimes \wedge^2 \mathbb{V}^\star$–valued *torsion function* on the manifold $\bar{P}$ of horizontal subspaces:

$$t : \bar{P} \to \mathbb{V} \otimes \wedge^2 \mathbb{V}^\star.$$

We construct a torsion tensor and subbundle of bundle of derivation which includes only connections with special torsion. This subbundle is called the *first prolongation*, for more detail see [2].

Definition 2.13. Let $\pi : P \to M$ be a G-structure and let $\mathcal{D}$ be a G–invariant complement to the subspace $\partial(\mathfrak{g} \otimes \mathbb{V}^\star)$ in $\mathbb{V} \otimes \wedge^2 \mathbb{V}^\star$, where ∂ is the Spencer operator of alternations. A connection ω is called a *$\mathcal{D}$–connection* if its torsion function

$$t^\omega = t \circ s_\omega : P \to \bar{P} \to \partial(\mathfrak{g} \otimes \mathbb{V}^\star) \oplus \mathcal{D} \subset \mathbb{V} \otimes \wedge^2 \mathbb{V}^\star,$$

where $s_\omega : p \mapsto H_p$ is a section of the bundle $\bar{P} \to P$ that defines connection ω, has values in $\mathcal{D}$.

Theorem 2.14 ([1, 2])**.** *Let $\pi : P \to M$ be a G-structure and let $\mathcal{D}$ be a G–invariant complement to the subspace $\partial(\mathfrak{g} \otimes \mathbb{V}^\star)$ in $\mathbb{V} \otimes \wedge^2 \mathbb{V}^\star$.*

1. *G-structure $\pi : P \to M$ admits a $\mathcal{D}$-connection ∇.*

2. *Let $\omega, \bar{\omega}$, be $\mathcal{D}$-connections. Then the corresponding operators of covariant derivative $\nabla, \bar{\nabla}$ are related by*

$$\bar{\nabla} = \nabla + S$$

where S is a tensor field such that for any $x \in M, S_x$ belongs to the first prolongation $\mathfrak{g}_x^{(1)}$ of the Lie algebra $\mathfrak{g}_x \subset \mathfrak{gl}(T_x M)$.

Theorem 2.14 reads that the class of appropriate $\mathcal{D}$–connections is parametrized by the first prolongation of the Lie algebra. Classical result (see [3]) reads that the Lie algebra of an almost quaternionic structure is isomorphic to $\mathfrak{gl}(n, \mathbb{H}) \oplus \mathfrak{sp}_1$ and the first prolongation is $\mathfrak{g}^{(1)} \cong \mathbb{V}^\star$ with respect to the identification $\mathbb{V}^\star \ni \xi \mapsto S^\xi \in \mathfrak{g}^{(1)}$ described by

$$S^\xi = 2\ \mathrm{Sym}\Big[\xi \otimes 1 - (\xi \circ I) \otimes I - (\xi \circ J) \otimes J - (\xi \circ K) \otimes K\Big]. \tag{2.2}$$

Thus it is a structure of type 2 in the sense of Definition 2.12. Furthermore, we see that the difference of two $\mathcal{D}$–connections in (2.2) is the same as the difference of two Q–related symmetric connections, see Theorem 2.4. We will see in Chapter 4 that in the case of parabolic geometries there exists a canonical choice of such torsion that the mentioned Theorem 2.4 can be generalized.

Finally note that the Lie algebra of an almost hypercomplex structure is $\mathfrak{gl}(n, \mathbb{H})$ and it is easy to show that $\mathfrak{g}^{(1)} = 0$, i.e. an almost hypercomplex structure is a structure of type 1.

3 A–structures

The notion of an A–structure was established in [32] as a natural generalization of almost quaternionic structure, which is determined by a subbundle $Q \subset \Gamma(TM \otimes T^*M)$ (see Definition 2.1). An A–structure is determined by a choice of a subbundle $A \subset \Gamma(TM \otimes T^*M)$, which admits the definition of generalized geodesics, so called A–planar curves as a natural generalization of Q–planar curves. As an analogy of Q–related connections we establish A–related connections, i.e. connections sharing the class of A–planar curves, and we study the transformations preserving the A–planar curves of the connection in question. We prove that, with some additional conditions, the morphisms preserving the geodesics of the class of A–related connections $[\nabla]_A$ are exactly the morphisms of the appropriate A–structure. If we restrict to the case of A being an algebra, we can analyse the appropriate structure quite effectively. We present results known in several special geometries in a general formulation.

3.1 General theory

The general theory of A–structures was developed in our articles [30, 28, 26, 27, 24]. The A–structures applications in almost complex projective geometry one can find in paper [25]. The A–structures applications in almost product geometry one can find in paper [31].

Definition 3.1. Assume that we have given a smooth manifold M, such that $\dim(M) = m$. Let A be a smooth ℓ-dimensional ($\ell < m$) vector subbundle in $T^*M \otimes TM$, such that the identity affinor $E = id_{TM}$ restricted to T_xM belongs to $A_xM \subset T_x^*M \otimes T_xM$ at each point $x \in M$. We say that M is equipped with an ℓ-dimensional *A–structure*.

Definition 3.2. Let M be a smooth manifold equipped with an A–structure and ∇ be a linear connection preserving A. We define the class of *A–related connections* :

$$[\nabla]_A = \{\nabla + \Upsilon_1 \odot F_1 + \cdots + \Upsilon_\ell \odot F_\ell\}, \tag{3.1}$$

where $\langle F_1, \ldots, F_\ell \rangle = A$ as a vector space and Υ_i are one forms on M.

By the choice of the connection ∇ in such way that $\nabla A \subset A$, it turns out that $\bar{\nabla} A \subset A$ for all $\bar{\nabla} \in [\nabla]_A$ and the whole connection class shares the same torsion. To be able to treat A–structures effectively, it is useful to require an additional condition for the A–structure to have generic rank, see Definition 3.3. It is a technical condition forced by the proofs in [32]. In Theorem 3.4 we show that, if we restrict to structures with A being an algebra, this property is, in appropriate dimension, a consequence of weak generic rank with obvious geometric meaning, and the cases, where A is an algebra, cover all structures of our interest.

Definition 3.3. Let (M, A) be a smooth manifold M equipped with an ℓ–rank A–structure. We say that the A–structure has

1. *generic rank* ℓ if for each $x \in M$ the subset of vectors $(X, Y) \in T_xM \oplus T_xM$, such that the A–hulls

$$A(X) = \{F(X), F \in A\} \text{ and } A(Y) = \{F(Y), F \in A\}$$

generate a vector subspace $A(X) \oplus A(Y)$ of dimension 2ℓ, is open and dense in $T_xM \oplus T_xM$.

2. *weak generic rank* ℓ if for each $x \in M$ the subset of vectors

$$\mathcal{V} := \{X \in T_xM |\, \dim A(X) = \ell\}$$

is open and dense in T_xM.

One immediately checks that any A–structure which has generic rank ℓ has weak generic rank ℓ. Indeed, if $U \subset T_xM$ is an open subset of vectors X with $A(X)$ of dimension lower than ℓ, then $U \times U$ is an open subset with too low dimension, too.

The following Theorem contains the properties of generic and weak generic rank. Results (3), (4) are quite obvious, (1) reads that in appropriate dimension and A being an algebra, generic rank is a consequence of weak generic rank. Result (2) shows an effective way of checking the property of weak generic rank if A is an algebra. The following result summarizes them both. Proofs of all claims can be found in [30]. Finally, the results (1) and (2) lead to Corollary 3.5.

Theorem 3.4 ([30])**.** *Let M be a smooth manifold of dimension at least two.*

1. *Let (M, A) be a smooth manifold of dimension m equipped with A–structure of rank ℓ, such that $2\ell \leq m$. If A_x is an algebra (i.e. for all $f, g \in A_x$, $fg := f \circ g \in A_x$) for all $x \in M$, and the A–structure has weak generic rank ℓ then the A–structure has generic rank ℓ.*

2. *Let (M, A) be a smooth manifold of dimension m equipped with A–structure of rank ℓ, such that $\ell \leq m$. Let $X_1, \ldots, X_m$ be a basis of A–module T_xM. If there exists $X \in T_xM$ such that $\dim(A(X)) = n$, for all $x \in M$, then the A–structure has weak generic rank at least n.*

3. *Let F be an affinor and let (M, A) be a smooth manifold of dimension $m \geq 2$ equipped with A–structure, where $A = \langle E, F \rangle$. If $F \neq qE$, $q \in \mathbb{R}$ then the A–structure has weak generic rank 2.*

4. *Let (M, A) be a smooth manifold with A–structure of rank ℓ, such that $\ell \leq \dim M$. If $A_x \subset T_x^\star M \otimes T_xM$ is an algebra with inversion, for all $x \in M$, then A has weak generic rank. Moreover, if $2\ell \leq \dim M$ than A has generic rank ℓ.*

Let us note that the result (3) from Theorem 3.4 led to rich theory of *F–planar curves* and *F–planar transformations*. One can see papers [46, 47, 48, 50, 49] for more details.

Corollary 3.5 ([30])**.** *Let (M, A) be a smooth manifold with A–structure of rank ℓ, such that $2\ell \leq \dim M$ and let A be an algebra. If there exists $X \in T_xM$ such that $\dim(A(X)) = n$ then the A–structure has generic rank at least n.*

In this section, assume that $(M, A, [\nabla]_A)$ is an A–structure with generic rank ℓ, where $A = \langle F_1, \ldots, F_\ell \rangle$ is an algebra on m–dimensional manifold M, where $2\ell \leq m$, equipped with the class of A-related connections $[\nabla]_A$ (see Definition 3.2).

On a smooth manifold endowed with an A–structure and a linear connection, we can generalize the notions of Q–planar curves , Q–transformations and a Q–projective transformation known for almost quaternionic structure. We talk about so–called A–planar curves, A–transformations and A–planar transformations (A–projective transformations).

Definition 3.6. Let (M, A) be a smooth manifold equipped with an A–structure of generic rank ℓ and a linear connection ∇.

- A smooth curve $c : \mathbb{R} \to M$ is said to be *A–planar curve* if $\nabla_{\dot{c}}\dot{c} \in A(\dot{c})$.

- If a diffeomorphism f of M to itself leaves the bundle A invariant, then f is called an *A–transformation* of (M, A).

- A diffeomorphism $f : M \to M$ is called *A–planar map* if each A–planar curve on M is mapped onto the A–planar curve $f_\star c$ on M.

- Let $\bar{M}$ be another manifold with a linear connection $\bar{\nabla}$ and B–structure. A diffeomorphism $f : M \to \bar{M}$ is called *(A, B)–planar map* if each A–planar curve c on M is mapped onto the B–planar curve $f_\star c$ on M. In the special case, where A is the trivial structure given by $\langle E \rangle$, we talk about *B–planar maps.*

The following Theorem shows that if the class of A–related connections is chosen properly, the class contains the connections sharing the A–planar curves and the geodesics of all connections are exactly the A–planar curves of every one of them. Instead of A–planar curves, we refer to them as to the geodesics $[\nabla]_A$. The next Theorem shows that all connections with this property and having the same torsion lie already in $[\nabla]_A$.

Theorem 3.7 ([26, 24, 32]). *Let $(M, A, [\nabla]_A)$ be an A–structure of rank ℓ, such that $2\ell \leq \dim M$ equipped with the class of A–related connections $[\nabla]_A$ preserving A. A curve $c : \mathbb{R} \to M$ is A–planar with respect to at least one connection $\tilde{\nabla} \in [\nabla]_A$ on M if and only if $c : \mathbb{R} \to M$ is a geodesic of some connection from $[\nabla]_A$. Moreover this happens if and only if c is A–planar with respect to all connections from $[\nabla]_A$.*

Theorem 3.8 ([26, 24, 32]). *Let $(M, A, [\nabla]_A)$ be an A–structure of rank ℓ, such that $2\ell \leq \dim M$ equipped with the class of A–related connections $[\nabla]_A$ preserving A. Let $\bar{\nabla}$ be a linear connection on M preserving A, such that ∇ and $\bar{\nabla}$ have the same torsion. If any geodesic of $\bar{\nabla}$ is A–planar for any $\tilde{\nabla} \in [\nabla]_A$, then $\bar{\nabla}$ lies in the class of connections $[\nabla]_A$.*

The following claims summon the results obtained for the A–structure morphisms, see e.g. papers [32, 26]. In the sequel, we will find the following results quite useful. Indeed, Theorem 3.9 from [32] describes the morphisms of A–structures generally and the last Corollary 3.10 of this chapter translates this result into the terms of generalized projective A–structures.

Theorem 3.9 ([26, 24, 32]). *Let (M, A, ∇), (M', A', ∇') be smooth manifolds of dimension m equipped with A–structure and A'–structure of the same generic rank ℓ, such that $2\ell < m$, and with a linear connection. Assume that the A–structure satisfies the property*

$$\forall X \in T_xM,\ \forall F \in A, \exists c_X \mid \dot{c}_X = X,\ \nabla_{\dot{c}_X}\dot{c}_X = \beta(X)F(X), \tag{3.2}$$

where $\beta(X) \neq 0$. If $f : M \to M'$ is an (A, A')–planar mapping, then f is a morphism of the A–structures, i.e $f^*A' = A$.

The following Corollary 3.10 deals with A–projective transformations, i.e. projective transformation maps with respect to all connection from the class of A–related connections $[\nabla]_A$.

Corollary 3.10 ([26, 24, 32]). *Let $(M, A, [\nabla]_A)$ be a smooth manifold of dimension m equipped with A–structure of the generic rank ℓ, such that $2\ell < m$, together with class of connections $[\nabla]_A$, such that $[\nabla]_A$ preserves A and satisfies the property*

$$\forall X \in T_xM,\ \forall F \in A, \exists c_X \mid \dot{c}_X = X,\ \nabla_{\dot{c}_X}\dot{c}_X = \beta(X)F(X), \tag{3.3}$$

where $\beta(X) \neq 0$. Then a diffeomorphism $f : M \to M$ is an A–transformation of (M, A) if and only if it preserves the class of unparametrized geodesics of all A–related connections $[\nabla]_A$ on M .

But the connection class $[\nabla]_A$ is parametrized by the space of one–forms on M. In case of almost quaternionic geometry we can prove directly that for the corollary to hold it is enough to use the connection class parametrized by one–forms.

3.2 Cliffordian projective geometries

In the sequel we focus on the class of A–structures where A is an algebra. For checking the generic rank we can thus use Corollary 3.5 and for the description of the appropriate projective transformations Corollary 3.10. All considerations are made for algebras over $\mathbb{R}$ and thus in case of commutative algebra A it is possible to diagonalize A and apply the theory easily, some of examples one can see for example in papers [13, 27]. The case of anticommutative algebra A

is more interesting. In this chapter we focus on the case of A–structures, where A is a Clifford algebra. As shown in the previous chapter, A being an algebra leaves us to check easily the property of a generic rank and use the notion of A–planar curves for the description of the geometry morphisms. General results for arbitrary Clifford algebra can be derived inductively by means of Bott periodicity.

We shall call such structures almost Cliffordian in order to distinguish them from almost Clifford structures, which are determined locally by the choice of the basis of A. This is the same difference as between the almost quaternionic and almost hypercomplex structure. Our aim is to define Clifford projective structure as a generalization of the quaternionic one. This is done by the choice of a class of A–related connections, i.e. connections sharing the A–planar curves and torsion.

It is the existence of the connection class that leads us to the claim that the homogeneous model of almost quaternionic geometry is a *quaternionic projective space* $\mathbb{P}^n\mathbb{H}$, while on contrary, the homogeneous model of almost hypercomplex geometry is the space $\mathbb{H}^n$.

Motivated by these facts, we described some of these connections and their properties for almost Cliffordian G–structures based on *Clifford algebra* $\mathcal{C}l(3,0)$ in paper [33] which was motivated by paper [8], and Clifford algebras $\mathcal{C}l(s,t)$ generally in papers [35, 29]. As we shall see an almost Clifford structures and almost Cliffordian structures are G–structures based on Clifford algebras and we can work with the class of $\mathcal{D}$–connections.

We use the following notation. We focus on the Clifford algebra $\mathcal{O} := \mathcal{C}l(s,t)$, i.e. the free unitary anti commutative algebra generated by elements $I_i, i = 1, \dots t$ (called complex unities), and elements $J_j,\ j = 1, \dots, s$ (called product unities), which are anti commuting, i.e. $I_i^2 = -E$, $J_j^2 = E$ and $K_iK_j = -K_jK_i,\ K_i \neq K_j$, where $K \in \{I_i, J_j\}$. On the other hand, we can see the Clifford algebra $\mathcal{O}$ as a 2^{s+t}–dimensional vector space and the algebra $\mathcal{O}$ is generated by elements $F_i,\ i = 1, \dots, k$ as a vector space. We choose a basis $F_i,\ i = 1, \dots, k = 2^{t+s}$, such that $F_1 = E,\ F_i = I_{i-1}$ for $i = 2, \dots, t+1,\ F_j = J_{j-t-1}$ for $j = t+2, \dots, s+t+1$ and by all different multiples of I_i and J_j of length $2, ..., s+t$. Let us note that both complex and product unities can be found among these multiple generators.

The following Lemma 3.11 deals with so–called full matrix representation of a Clifford algebra, i.e. special representation of the appropriate Clifford algebra $\mathcal{C}l(s,t)$ on a vector space $\mathbb{R}^{(2^{s+t})p}$ for $p \in \mathbb{N}$. The construction of such representation can be found in [35]. On the other hand, one can see that there is a matrix representation of any Clifford algebra $\mathcal{O}$ on the vector space $\mathbb{R}^{(2^{s+t})p}$, where $p \in \mathbb{N}$, such that the following lemma holds. We will work with this representation in sequel.

Lemma 3.11 ([35]). *Let $F_1, \dots, F_k$ denote the $k = 2^{s+t}$ elements of the full matrix representation of Clifford algebra $\mathcal{C}l(s,t)$ on $\mathbb{R}^{kp}$, where $p \in \mathbb{N}$. Then there exists a real vector $X \in \mathbb{R}^{kp}$ such that the dimension of a linear span $\langle F_iX | i = 1, \dots, k\rangle$ equals to k.*

Definition 3.12. Let $\mathcal{O} = \mathcal{C}l(s,t)$ be a Clifford algebra. If M is an km–dimensional manifold, where $k = 2^{s+t}$ and $m \in \mathbb{N}$ then an *almost Clifford manifold* is given by a reduction of the structure group $GL(km, \mathbb{R})$ of the principal frame bundle of M to

$$GL(m, \mathcal{O}) := \{A \in GL(km, \mathbb{R}) | AI_i = I_iA, AJ_j = J_jA\},$$

where $\mathcal{O}$ is an arbitrary Clifford algebra and $I_i, i = 1, \dots, t,\ I_i^2 = -E$ and $J_j,\ j = 1, \dots, s,\ J_j^2 = E$ is the set of anticommuting affinors such that the free associative unitary algebra generated by $\langle I_i, J_j, E\rangle$ is isomorphically equivalent to $\mathcal{O}$.

It is easy to see that an almost Clifford structure is not an A–structure, because the basis $F_0, \dots, F_\ell \in \mathcal{O}$ has to be chosen.

Definition 3.13. The A–structure where A is isomorphically equivalent to a Clifford algebra $\mathcal{O}$ is called an *almost Cliffordian manifold.*

The following Theorem and Corollary, for more details see paper [29], generalizes classical properties of Nijenhuis tensor for geometries based on Clifford algebras. By a choice of suitable coordinate system for almost quaternionic geometry, it is possible to introduce $\mathcal{D}$–connections in such way that the torsion can be expressed in the form of a linear combination of Nijenhuis tensors. If $N(I,I)$ and $N(J,J)$ are vanished, a torsion free connection is induced (see section 2.2). We intend to formulate to similar results in our future research. Nevertheless, the geometry of Nijenhuis tensors is an important part of any geometry based on affinors and thus we are going to mention several properties.

Theorem 3.14 ([29])**.** *Let $\mathcal{O}$ be a Clifford algebra $\mathcal{C}l(s,t)$ and let $F,G \in \mathcal{O}$ such that $F \neq G$. If the Nijenhus tensors $N(F,F)$ and $N(G,G)$ vanish, then $N(FG,FG)$ vanishes.*

Corollary 3.15 ([29])**.** *Let $\mathcal{O}$ be a Clifford algebra $\mathcal{C}l(s,t)$. If the Nijenhuis tensors $N(I_i,I_i)$ vanish, where I_i are the algebra generators of $\mathcal{O}$, then*

$$N(F_i,F_j)=0, \quad \text{where } F_i \text{ are vector space generators.}$$

In order to define also generalized projective geometry we have to choose an appropriate class of connections. One can easily check that the class of A–related connections, where $A=\mathcal{O}$

$$[\nabla]_{\mathcal{O}} = \nabla + \sum_{i=1}^{\dim A} \Upsilon_i \odot F_i, \tag{3.4}$$

where Υ_i are one forms on M, share the same class of A–planar curves, but we have to describe them more carefully for Cliffordian manifolds. As any A–structure is also a G–structure, we can find the corresponding structure group and its Lie algebra with its first prolongation.

One can see that an almost Cliffordian manifold M is given as a G–structure provided that there is a reduction of the structure group of the principal frame bundle of M to

$$G := GL(m,\mathcal{O})GL(1,\mathcal{O}) = GL(m,\mathcal{O}) \times_{Z(GL(1,\mathcal{O}))} GL(1,\mathcal{O}),$$

where $Z(G)$ is a center of G. The action of G on T_xM looks like

$$QXq, \text{ where } Q \in GL(m,\mathcal{O}), q \in GL(1,\mathcal{O}),$$

where the right action of $GL(1,\mathcal{O})$ is blockwise. In this case the tensor fields in the form $F_1,\dots,F_k$ can be defined only locally. It is easy to see that the Lie algebra $\mathfrak{gl}(m,\mathcal{O})$ of a Lie group $GL(m,\mathcal{O})$ is of the form

$$\mathfrak{gl}(m,\mathcal{O}) = \{A \in \mathfrak{gl}(km,\mathbb{R}) | AI_i = I_iA, AJ_j = J_jA\}$$

and the Lie algebra $\mathfrak{g}$ of a Lie group $GL(m,\mathcal{O})GL(1,\mathcal{O})$ is of the form

$$\mathfrak{g} = \mathfrak{gl}(m,\mathcal{O}) \oplus \mathfrak{gl}(1,\mathcal{O}).$$

Let us note that the case of $\mathcal{C}l(0,3)$ was studied in a detailed way in [7] and the case of $\mathcal{C}l(0,2)$ for example in [60].

We are looking for a class of A–related $\mathcal{D}$–connections. The difference of two such connections then lies in the first prolongation of the appropriate Lie algebra $\mathfrak{g}^{(1)}$. Theorem 3.16 describes the class of tensors belonging to the first prolongation of an arbitrary Clifford algebra. A particular choice of coefficients ϵ_i and the proof that such coefficients always exist and that the chosen connection class is in one–one correspondence with one–forms ξ on $\mathbb{V}$, can be found in [35]. These considerations lead us to Definition 3.17.

Theorem 3.16 ([35]). *Let $\mathcal{O}$ be the Clifford algebra $\mathcal{C}l(s,t)$. For any one–form ξ on $\mathbb{V}$ and any $X, Y \in \mathbb{V}$, the elements of the form*

$$S_X^\xi(Y) = \sum_{i=1}^{k} \epsilon_i(\xi(F_iX)F_iY + \xi(F_iY)F_iX), \ k = 2^{s+t},$$

where the coefficients ϵ_i depend on the type of $\mathcal{O}$, belong to the first prolongation $\mathfrak{g}^{(1)}$ of the Lie algebra $\mathfrak{g}$ of the Lie group $GL(m,\mathcal{O})GL(1,\mathcal{O})$.

Definition 3.17. A *Cliffordian projective structure (geometry)* $(M, A, [\nabla]_\mathcal{O})$ is given by the Cliffordian structure (M, A) and equivalence class of affine connections

$$[\nabla]_\mathcal{O} = \nabla + \sum_{i=1}^{k} \epsilon_i(\Upsilon \circ F_i) \odot F_i, \tag{3.5}$$

where coefficients ϵ_i are from Theorem 3.16.

Finally, note that traditionally a projective structure is the set of all unparametrized geodesics of a given affine connection. In fact, there are many affine connections with the same unparametrized geodesics and the projective structure is often alternatively defined as an equivalence class of affine connections with the same unparametrized geodesics (see [4, 5]). In almost complex geometry (M, J), a generalized *complex geodesics* is a map $\varphi : \mathbb{R} \to M$ such that $\nabla_{\dot\varphi}\dot\varphi \in \langle\dot\varphi, J(\dot\varphi)\rangle$ and a complex projective structure is given by the complex structure J and equivalence class of affine connections with the same generalized complex geodetics.

As we shall see later, in case of almost quaternionic geometry the class of these connections coincides with the class of so–called Weyl connections and thus the class of almost quaternionic geometries is projective. Indeed, as one can find in [60], the homogeneous model of an almost quaternionic geometry is nothing but a quaternionically projective space. Now, we can describe projective transformation of Cliffordian structures as transformations of Cliffordian projective structures.

Theorem 3.18 ([35]). *Let $(M, A, [\nabla]_\mathcal{O})$ be a Cliffordian projective structure, such that there is a local basis $\langle F_i\rangle = \mathcal{O}$ such that $\nabla F_i = 0$ for some $\nabla \in [\nabla]_\mathcal{O}$. Then a diffeomorphism $f : M \to M$ is a morphism of Cliffordian projective structures if and only if it preserves the class of unparametrized geodesics of all connections $[\nabla]_\mathcal{O}$ on M.*

To complete the description, let us show the first prolongation of Lie algebra of Clifford structure.

Lemma 3.19 ([35]). *Let M be a (km)–dimensional Clifford manifold based on Clifford algebra $\mathcal{O} = \mathcal{C}l(s,t)$, $k = 2^{s+t}$, $s + t > 1$, $m \in \mathbb{N}$, i.e. manifold equipped with G–structure, where*

$$G = GL(m,\mathcal{O}) = \{B \in GL(km,\mathbb{R}) | BI_i = I_iB, \ BJ_j = J_jB\},$$

where I_i and J_j are algebra generators of $\mathcal{O}$. Then the first prolongation $\mathfrak{g}^{(1)}$ of Lie algebra $\mathfrak{g}$ of Lie group G vanishes.

4 Parabolic geometries

Some of the above structures are also parabolic geometries. This holds for some cases of our interest: almost quaternionic, almost Cliffordian projective where $\mathcal{O} = \mathcal{C}l(1,0)$ i.e. almost projective product and almost Cliffordian projective where $\mathcal{O} = \mathcal{C}l(0,1)$, i.e. almost projective complex. A homogeneous model of all these geometries is an appropriate projective space and as we shall see later, projective geometry itself is also parabolic. For our considerations, it is fundamental that there exists a canonical G–invariant complement necessary for the choice of the class of $\mathcal{D}$–connections and this class is determined uniquely. In the terms of parabolic geometries this class is called the Weyl connections. We shall recall basic notions.

The *homogeneous model* (Klein geometry [63, 71, 40, 41]) is a homogeneous space $M \cong G/P$ together with a transitive action on M by a Lie group G, which acts as the symmetry group of the geometry. *Cartan's generalized spaces* [71] are curved analogues of the homogeneous spaces G/P defined by means of an absolute parallelism on a principal P-bundle. Let $P \subset G$ be a Lie subgroup in a Lie group G, and $\mathfrak{g}$ be the Lie algebra of G. A *Cartan geometry* of type (G, P) on a manifold M is a principal fiber bundle $p : \mathcal{G} \to M$ with structure group P which is endowed with a $\mathfrak{g}$-valued one-form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, called the *Cartan connection* such that ω is P-equvariant, ω reproduces the fundamental vector fields and ω is an absolute parallelism. The homogeneous model for Cartan geometries of type (G, P) is the canonical bundle $p : G \to G/P$ endowed with the left Maurer–Cartan form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$.

Definition 4.1. [71] Let $\mathfrak{g}$ be a $|k|$–graded semisimple Lie algebra. A *parabolic geometry* is a Cartan geometry of type (G, P), where G is a semisimple Lie group and $P \subset G$ is the subgroup of all elements of G whose adjoint action preservers the filtration associated to a $|k|$–grading of the Lie algebra $\mathfrak{g}$ of G

One can find the theory of parabolic geometries in the book "Parabolic geometries I, Background and general theory" [71] by Andreas Čap and Jan Slovák published in 2009 in AMS Publishing House. There are also many articles, for example [69, 74, 20, 72, 70, 65, 73, 66, 67]. The general idea of Cartan geometries is to model the individual tangent spaces by the Lie algebra $\mathfrak{g}/\mathfrak{p}$, i.e. the tangent space to the homogeneous model in the origin inclusive its algebraic structure. In the special case of parabolic geometries, this amounts to special understanding of the corresponding $|k|$–gradings of semisimple Lie algebras. Note that a parabolic geometry, after selecting the reductive part $\mathfrak{g}_0$ of $\mathfrak{p}$, is canonically a split Cartan geometry, since we always have the subalgebras $\mathfrak{g}_-$, which is complementary to the subalgebra $\mathfrak{p} \subset \mathfrak{g}$. This complement is however very far from being $\mathfrak{p}$–invariant.

Geometric structures of our interest are the examples of so–called $|1|$–graded parabolic geometries, i.e. geometries determined by $|1|$–gradation. Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a $|1|$–graded semisimple Lie algebra, let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $P \subset G$ be a parabolic subgroup for the given grading with Levi subgroup $G_0 \subset P$.

4.1 Torsions of Weyl connections

In the general theory, the classical prolongation procedure for G–structures starts with finding a minimal available torsion for a connection belonging to the structure on the given manifold M. Unlike the projective and conformal Riemannian structures where torsion free connections always exist, the torsion has to be allowed for the almost quaternionic structures in general in dimensions bigger than four, for the almost product and almost complex projective structures in general in dimensions bigger than two.

The standard normalization comes from the general theory of parabolic geometries and we shall not need this in the sequel. The details may be found for example in [70], [20], for another and more classical point of view see [60]. The only essential point for us is that all connections compatible with the given geometry sharing the unique normalized torsion are parametrized by smooth one–forms on M. In analogy to the conformal Riemannian geometry we call them Weyl connections for the given geometry on M.

In more detail, parabolic geometry $(p : \mathcal{G} \to M, \omega)$ is called *normal* if its curvature κ satisfies $\partial^* \kappa = 0$, where ∂^* is and adjoint for cohomology operator ∂ and in this case the *harmonic curvature* κ_H is defined to be the image of κ in the space of sections of the bundle $\mathcal{G}_0 \times_{G_0} H^n(\mathfrak{g}_-, \mathfrak{g})$. The harmonic curvature of normal parabolic geometries is much simpler object and the most important fact is that there is an algorithm to compute the harmonic curvature [75, 71]. For normal parabolic geometries vanishing of κ_H implies vanishing of κ and finally in the case $|1|$–graded parabolic geometries $(\kappa_H)_1$ coincides with the torsion of Weyl connections.

The almost quaternionic (or almost complex projective, or an almost product projective) structures on a manifold M admitting a linear connection compatible with the structure and without torsion are called quaternionic (or complex projective, or product projective) geometries. In the case, the Weyl connections on M are just the connections without torsion and compatible with the G–structure. In general, the Weyl connection are all compatible connections sharing the unique normalized torsion.

Finally note, that $|1|$–graded parabolic geometries are normally completely given by certain classical G–structures on the underlying manifolds, as we will see for an almost quaternionic. In other two mentioned cases however the semisimple Lie algebra belongs to the series of exceptions and only the choice of an appropriate class of connections defines the Cartan geometry completely [70, 71]. Moreover in these two cases the Lie algebra $\mathfrak{g}$ is a real form of semisimple (not simple) Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}' \oplus \mathfrak{g}''$. Note that in case of semisimple $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}''$ parabolic geometries, the appropriate cohomology is computed by the Künneth formula from the classical Kostant's formulae.

4.2 Projective geometries

Very well known example of $|1|$–graded parabolic geometry is *projective geometry.* Classically, two connections are called projectively equivalent if they share the geodesics as unparametrized curves. Let us remind that in both smooth and holomorphic settings this means

$$\tilde{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi, \tag{4.1}$$

for smooth or holomorphic one form Υ. A smooth manifold M equipped with a projective class of connections (4.1) is called *projective* [4, 5]. In parabolic geometry language, the projective geometry is the split real form of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$, i.e. $\mathfrak{g} = \mathfrak{sl}(2n, \mathbb{R})$. In the Satake diagram, we obtain the appropriate parabolic subalgebra $\mathfrak{p}$ by setting $|1|$-gradation by crossing the first root, i.e.

$$\times\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\cdots-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\circ.$$

In matrix form, we can view elements of $\mathfrak{g}$ as block matrices

$$\begin{pmatrix} -\mathrm{tr}(A) & Z \\ X & A \end{pmatrix},$$

with $X \in \mathbb{R}^n$, $Z \in \mathbb{R}^{n*}$ and $A \in \mathfrak{gl}(n, \mathbb{R})$, such that

$$\begin{pmatrix} 0 & 0 \\ Z & 0 \end{pmatrix} \in \mathfrak{g}_{-1}, \begin{pmatrix} -\mathrm{tr}(A) & 0 \\ 0 & A \end{pmatrix} \in \mathfrak{g}_0, \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_1.$$

The group $SL(n, \mathbb{R})$ consists of all invertible linear endomorphisms of $\mathbb{R}^n$ with determinant equal of one. We define $G = PSL(n+1, \mathbb{R})$ as the quotient of $SL(n+1, \mathbb{R})$ by its center $\{\pm \mathrm{id}\}$. Then subgroup P by definition is the stabilizer of the line generated by the first vector in the standard basis and $G/P \cong \mathbb{PR}^n$ and the subgroup G_0 is equal to $GL(n, \mathbb{R})$.

In order to understand the latter formulae, we introduce the so called adjoint tractors. They are sections of the vector bundle (called usually adjoint tractor bundle) over M

$$\mathcal{A} = \mathcal{G}_0 \times_{G_0} \mathfrak{g}.$$

The transformation formulae for the Weyl connections for $|1|$–graded geometries are generally given by the Lie bracket in the algebra in question.

$$\hat{\nabla}_X Y = \nabla_X Y + [[X, \Upsilon], Y] \tag{4.2}$$

where we use the frame forms $X, Y : \mathcal{G} \to \mathfrak{g}_{-1}$ of vector fields, and similarly for $\Upsilon : \mathcal{G} \to \mathfrak{g}_1$. Consequently, $[\Upsilon, X]$ is a frame form of an affinor valued in $\mathfrak{g}_0$ and the bracket with Y expresses the action of such affinor on the vector field. According to the general theory, this transformation rule works for all covariant derivatives ∇ of Weyl connections. If we rewrite the rule for the Weyl connection transformation, we obtain exactly (4.1) and if we realize that G_0 contains all invertible matrices $n \times n$ and thus the reduction of the principal frame bundle is avoided, we find that parabolic geometry corresponds exactly to the choice of a projective connection class.

We will see that for an almost complex projective geometry, the rule (4.2) coincides with (4.3) and for an almost product projective geometry, the rule (4.2) coincides with (4.4).

4.3 Examples

In the last part we show the impact of the theory on three important examples which are parabolic and Cliffordian projective simultaneously. In these cases we show that the class of Weyl connections corresponds to the class of $[\nabla]_{\mathcal{O}}$–projective connections and that the appropriate group G_0 coincides with the group of morphisms preserving A. As the parabolic geometry class of Weyl connections is given, both geometries are identical. Furthermore, we know that the structure morphisms are exactly the generalized projective transformations. We will do it in five steps:

1.	Define an A–structure, where $A = \mathcal{O}$ is a Clifford algebra
2.	Define an Cliffordian projective geometry $(M, \mathcal{O}, [\nabla]_{\mathcal{O}})$
3.	Define parabolic geometry, such that $\mathcal{G}_0$ is a geometry from (2)
4.	Proof the coincidence of class $[\nabla]_{\mathcal{O}}$ and Weyl connections
5.	Discus geometric interpretation of torsion
6.	Formulate results on projective transformations

ad 4. The key is to compute the difference tensor of Weyl connections $[[X, \Upsilon], Y]$. The computations are quite technical and can be found for particular geometries in [25, 31] and [32].

4.3.1 Almost quaternionic structures

Almost hypercomplex structures [82, 2, 20, 14] are smooth manifolds equipped with a smooth linear hypercomplex structure on each tangent space, but they do not possess any invariant connection. An almost quaternionic structure is an A–structure on smooth manifold M, such that the fibre A_x is isomorphic to hypercomplex numbers (quaternions) for any $x \in M$.

1) Let M be a smooth manifold of dimension $4n$. An almost hypercomplex structure on M is a triple (I, J, K) of smooth affinors in $\Gamma(T^*M \otimes TM)$ satisfying $I^2 = J^2 = K^2 = -id_{TM}$, $K = IJ = -JI$. An *almost quaternionic structure* is an A–structure, where A is locally generated by the identity E and and a hypercomplex structure.

2) These G–structures are of finite type. For an almost quaternionic structure there is a class of $\mathcal{D}$–connections parametrized also by all smooth one–forms Υ, but with the transformation rule

$$\begin{aligned}\tilde{\nabla}_\xi \eta = \nabla_\xi \eta &- \Upsilon(I\xi)I\eta + \Upsilon(\xi)\eta - \Upsilon(I\eta)I\xi + \Upsilon(\eta)\xi \\ &- \Upsilon(J\xi)J\eta + \Upsilon(\xi)\eta - \Upsilon(J\eta)J\xi + \Upsilon(\eta)\xi \\ &- \Upsilon(K\xi)K\eta + \Upsilon(\xi)\eta - \Upsilon(K\eta)K\xi + \Upsilon(\eta)\xi\end{aligned}$$

3) The structure group of an almost quaternionic structure is $G_0 = GL(n, \mathbb{H}) \times_{\mathbb{Z}_2} Sp(1)$, where $Sp(1)$ are unit quaternions in $GL(n, \mathbb{H})$. Let us reformulate $G_0 \subset P$ as the subgroup of $SL(n+1, \mathbb{H})$:

$$G_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix} : A \in GL(n, \mathbb{H}), \mathrm{Re}(a\, \mathrm{det} A) = 1 \right\},$$

$$P = \left\{ \begin{pmatrix} a & Z \\ 0 & A \end{pmatrix} : A \in GL(n, \mathbb{H}), \mathrm{Re}(a\, \mathrm{det} A) = 1, Z \in (H^n)^* \right\}.$$

Now, since P is an parabolic subgroup the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{H})$ carries G_0–invariant grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix} : A \in \mathfrak{gl}(n, \mathbb{H}), a \in \mathbb{H}, \mathrm{Re}(a + \mathrm{Tr} A) = 0 \right\},$$

$$\mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} : Z \in (H^n)^* \right\}, \quad \mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} : X \in (H^n) \right\}.$$

In other words, an almost quaternionic geometry corresponds to the real form $\mathfrak{sl}(n+1, \mathbb{H})$ of the complex algebra $\mathfrak{gl}(2n+2, \mathbb{C})$ and we obtain the appropriate parabolic subalgebra $\mathfrak{p}$ by setting $|1|$-gradation by crossing the second (firs noncompact) root, i.e. corresponding to the Satake diagram:

$$\bullet\!-\!\!-\times-\!\!-\bullet-\!\!-\cdots-\!\!-\bullet-\!\!-\circ-\!\!-\bullet$$

5) The torsion has to be allowed for the almost quaternionic structures in general in dimensions bigger than four. Essential point for us is that all connections compatible with the given geometry sharing the unique normalized torsion are parameterized by smooth one–forms on M. In analogy to the conformal Riemannian geometry we call them Weyl connections for the given almost quaternionic geometry on M.

6.) Let $f : M \to M'$ be a diffeomorphism between two almost quaternionic manifolds of dimension at least eight. Then f is a morphisms of the geometries if and only if it preserves the class of unparametrized geodesics of all Weyl connections on M and M'.

4.3.2 Almost complex projective structures

An almost complex structure [79, 6, 54, 44] is a smooth manifold equipped with a smooth linear complex structure on each tangent space.

1) Let M be a smooth manifold of dimension $2n$. An *almost complex structure* on M is a smooth trace–free affinor J in $\Gamma(T^\star M \otimes TM)$ satisfying $J^2 = -id_{TM}$. We can equivalently define an almost complex structure (M, J) as a reduction of the linear frame bundle P^1M to the structure group preserving this affinor J, i.e. for our choice of affinor J the group is

$$\left\{ \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{pmatrix} \middle| A_{i,j} = \begin{pmatrix} a_{i,j} & -b_{i,j} \\ b_{i,j} & a_{i,j} \end{pmatrix}, det(A) \neq 0 \right\} \cong GL(n, \mathbb{C}).$$

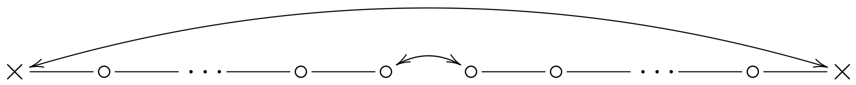
2) These G–structures are of infinite type but each choice of ∇ compatible with the affinor J, i.e. $\nabla J = 0$, defines a geometry of finite type (with morphisms given by the affine maps)[40]

$$\tilde{\nabla}_\xi \eta = \nabla_\xi \eta - \Upsilon(J\xi)J\eta + \Upsilon(\xi)\eta - \Upsilon(J\eta)J\xi + \Upsilon(\eta)\xi. \tag{4.3}$$

3) The homogeneous model (Klein geometry) is a homogeneous space $M \cong G/P$ together with a transitive action on M by a Lie group G, which acts as the symmetry group of the geometry [71]. In our example, the Lie group G is $SL(n+1,\mathbb{C})$ and P is the usual parabolic subgroup of all matrices of the form

$$\left\{ \begin{pmatrix} c & W \\ 0 & C \end{pmatrix}, \text{ where } C \in GL(n,\mathbb{C}),\ c \in GL(1,\mathbb{C}),\ c \cdot \det(C) = 1 \right\}$$

and C has positive real determinant. The Lie algebra of P is a parabolic subalgebra of the real form $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{C})$ of the complex algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(n+1,\mathbb{C}) \oplus \mathfrak{sl}(n+1,\mathbb{C})$ corresponding to the Satake diagram:



The Maurer–Cartan form on G provides the homogeneous model by the structure of $|1|$-graded parabolic geometries of type (G,P). In the matrix form, we can illustrate the grading from our example as:

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

where

$$\mathfrak{g}_{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ A_{2,1} & 0 & \cdots & 0 \\ \vdots & & & \\ A_{m,1} & 0 & \cdots & 0 \end{pmatrix}, \mathfrak{g}_0 = \begin{pmatrix} A_{1,1} & 0 & \cdots & 0 \\ 0 & A_{2,2} & \cdots & A_{2,m} \\ 0 & \vdots & \cdots & \vdots \\ 0 & A_{m,2} & \cdots & A_{m,m} \end{pmatrix},$$

$$\mathfrak{g}_1 = \begin{pmatrix} 0 & A_{1,2} & \cdots & A_{1,m} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

5) In the our case, however the semisimple Lie algebra belongs to the the series of exceptions and only the choice of an appropriate class of connections defines the Cartan geometry completely [70, 71]. For normal structures the torsion has to be of type $(0,2)$, but it is well known that for a linear connection which preserves an almost complex structure the $(0,2)$–component of the torsion is a non–zero multiple of the Nijenhuis tensor.

6) Let $f : M \to M'$ be a diffeomorphism between two almost complex projective manifolds of dimension at least four. Then f is a homomorphism ($f^*J = J$) or an anti-homomorphism ($f^*J = -J$) of the almost complex projective structures if and only if it preserves the class of unparametrized geodesics of all Weyl connections on M and M'.

4.3.3 Almost product projective structures

An almost product structure [79, 6, 54] is a smooth manifold equipped with a smooth linear product structure on each tangent space.

1) Let M be a smooth manifold of dimension $2m$. *An almost product structure* on M is a smooth affinor J in $\Gamma(T^\star M \otimes TM)$ satisfying $J^2 = \mathrm{id}_{TM}$. For better understanding, we describe an almost product structure at each tangent space in a fixed basis, i.e. with the help of real matrices:

$$J := \begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & -\mathbb{I}_m \end{pmatrix}.$$

The eigenvalues of J have to be ± 1 and $T_xM = T^L_xM \oplus T^R_xM$, where the subspaces are of the form

$$T^LM := J_+ = \left\{ \begin{pmatrix} c \\ 0 \end{pmatrix} \mid c \in \mathbb{R}^m \right\}, \; T^RM := J_- = \left\{ \begin{pmatrix} 0 \\ c \end{pmatrix} \mid c \in \mathbb{R}^m \right\}.$$

Thus, we can equivalently define an almost product structure J on M as a reduction of the linear frame bundle P^1M to the appropriate structure group, i.e. as a G–structure with the structure group L of all automorphisms preserving the affinor J:

$$L := \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} |A, B \in GL(m, \mathbb{R}) \right\} \cong GL(m, \mathbb{R}) \times GL(m, \mathbb{R}).$$

2) These G–structures are of infinite type, however each choice of a linear connection ∇ compatible with the affinor J, determines a finite type geometry similar to products of projective structures, which we shall study below. Instead, we shall consider a class of connections parametrized also by all smooth one–forms Υ, but with the transformation rule

$$\begin{aligned} \hat{\nabla}_{\xi^L+\xi^R}(\eta^L + \eta^R) = \nabla_{\xi^L+\xi^R}(\eta^L + \eta^R) + \Upsilon^L(\xi^L)\eta^L + \Upsilon^L(\eta^L)\xi^L \\ + \Upsilon^R(\xi^R)\eta^R + \Upsilon^R(\eta^R)\xi^R, \end{aligned} \tag{4.4}$$

where the indices at the forms and fields indicate the components in the subbundles T^LM and T^RM, respectively. Clearly such a transformed connection will make J parallel again.

3) Let us consider the homogeneous space $M = G/P$ given as the product of two projective spaces $G_L/P_L \times G_R/P_R$, i.e. $G_L = G_R = SL(n+1, \mathbb{R})$ while $P = P_L \times P_R$, where $P_R = P_L$ is the usual parabolic subgroup corresponding to the block upper triangular matrices of the block sizes $(1, n)$. Clearly, any product connection on M built of the linear connections ∇^L and ∇^R from the two projective classes on the product components provides a homogeneous example of an almost product projective structure. At the same time, the Maurer–Cartan form on $G = G_L \times G_R$ provides the homogeneous model of the $|1|$-graded parabolic geometries of type (G, P). The Lie algebra of $P \times P$ is a parabolic subalgebra of the real form $\mathfrak{sl}(n+1, \mathbb{R}) \oplus \mathfrak{sl}(n+1, \mathbb{R})$ of the complex algebra $\mathfrak{sl}(n+1, \mathbb{C}) \oplus \mathfrak{sl}(n+1, \mathbb{C})$ corresponding to the Satake diagram:

$$\times\!-\!\circ\!-\!\cdots\!-\!\circ \qquad \circ\!-\!\cdots\!-\!\circ\!-\!\times\, .$$

In the matrix form, we can illustrate the grading from our example as:

$$\mathfrak{g} = \begin{pmatrix} \mathfrak{g}^L_0 & \mathfrak{g}^L_1 & 0 & 0 \\ \mathfrak{g}^L_{-1} & \mathfrak{g}^L_0 & 0 & 0 \\ 0 & 0 & \mathfrak{g}^R_0 & \mathfrak{g}^R_1 \\ 0 & 0 & \mathfrak{g}^R_{-1} & \mathfrak{g}^R_0 \end{pmatrix}$$

5) In our case, the appropriate cohomology is computed by the Künneth formula from the classical Kostant's formulae and the computation [75] provides all six irreducible components of the curvature, only two of which are of torsion type. Of course, the integrability obstructions of the bundles T^LM and T^RM are just those two. Therefore, a *normal almost product projective structure* $(M, J, [\nabla])$ has this minimal torsion. I.e. if $(M, J, [\nabla])$ is a normal almost projective product structure on a smooth manifold M then the Weyl connections coincide exactly with the distinguish class $[\nabla]$.

6) Let $f : M \to M'$ be a diffeomorphism between two almost product projective manifolds of dimension at least four. Then f is a morphism of the almost product structures if and only if it preserves the class of unparameterized geodesics of all Weyl connections on M and M'.

5 Special geometric structures in engineering

In this chapter, we show a few common applications of Clifford algebras in engineering. In the 21' century, Clifford algebra engineering application are widely developed area. Our applications are mostly based on Euclidean Clifford algebras and conformal Clifford algebras. Euclidean Clifford algebras represent rigid body motions of Euclidean space. Conformal Clifford algebras cover projective transformations in addition. For more information one can see the books [59, 21, 62]. In the last section we briefly show the impact of the jet theory in to material sciences, for more details see [12, 10, 34].

5.1 Clifford Algebra for the Euclidean Group

To work with Euclidean geometry, we have to extend the basic definition of Clifford algebra from section 3.2. The Clifford algebra $\mathcal{C}l(s,t,r)$ is a free unitary anti commutative algebra generated by elements $f_j,\ j=1,\dots,s;\ e_i, i=1,\dots t$, and $g_k, k=1,\dots r$ such that $f_j^2=E$, $e_i^2=-E$ and $g_k^2=0$. In particular, the Clifford algebras $\mathcal{C}l(s,t,0)$ are denoted as $\mathcal{C}l(s,t)$.

On any Clifford algebra there is a conjungation *, we define $c_i^*=-c_i$ for Clifford algebra generators and $(c_ic_j)^*=c_j^*c_i^*$ for any pair of Clifford algebra elements. Finally, we can split any Clifford algebra into even and odd degree subspaces

$$\mathcal{C}l(s,t,r)=\mathcal{C}l^+(s,t,r)\oplus\mathcal{C}l^-(s,t,r)$$

and define

$$\mathrm{Spin}(n)=\{g\in\mathcal{C}l^+(0,n,0)|\ gg^*=1, gxg^*\in\mathbb{V},\ \text{for all } x\in\mathbb{V}\}$$

as a Spinor group of rotations, where $\mathbb{V}=\langle e_1,,\dots,e_n\rangle$ is a vector space isomorphically equivalent to $\mathbb{R}^n$.

Example 5.1. In the Clifford algebra $\mathcal{C}l(0,3,0)$ we have four even elements

$$1,\ e_1e_2,\ e_1e_3,\ e_2e_3\in\mathcal{C}l^+(0,3,0).$$

One can see that $(e_1e_2)^2=e_1e_2e_1e_2=-1$, $(e_1e_3)^2=-1$ and $(e_2e_3)^2=-1$ and we can identify $\mathcal{C}l^+(0,3,0)$ with quaternion numbers $\mathbb{H}$. The conjugation of element

$$c=\alpha_0+\alpha_1e_1e_2+\alpha_2e_1e_3+\alpha_3e_2e_3\in\mathcal{C}l^+(0,3,0),\ \alpha_i\in\mathbb{R}$$

is

$$c^*=\alpha_0-\alpha_1e_1e_2-\alpha_2e_1e_3-\alpha_3e_2e_3\in\mathcal{C}l^+(0,3,0),$$

and our conjugation is natural quaternionic conjugation

$$(\alpha_0+\alpha_1i+\alpha_2j+\alpha_3k)^*=\alpha_0-\alpha_1i-\alpha_2j-\alpha_3k.$$

Finally, we can identify Spin(3) with unitary quaternions which are usually used to rotate objects in $\mathbb{R}^3$.

In general dimension, one can represent the elements of $\mathbb{R}^n$ as the set

$$\{1+xe\mid x=x_1e_1+\cdots x_ne_n\}\subset\mathcal{C}l(0,n,1)$$

where $e^2=0$, $e_i^2=-1$, and define the group of rotors

$$\mathrm{Rot}(n)=\{g+\frac{1}{2}tge\mid\ g\in\mathrm{Spin}(n),\ t=t_1e_1+\cdots+t_ne_n\}\subset\mathcal{C}l(0,n,1),$$

then the action of group $\mathrm{Rot}(n)$ on the $\mathbb{R}^n$ is given by conjugation

$$(g+\frac{1}{2}tge)(1+xe)(g+\frac{1}{2}tge)^*=1+(gxg^*+t)e.$$

In fact, the group $\mathrm{Rot}(n)\cong\mathrm{Spin}(n)\ltimes\mathbb{R}^n$ double covers the group of proper rigid motions $SE(n)$. For $\mathbb{R}^3$ we can identify Spin(3) with $\mathbb{H}$ (see Example 5.1) and then $\mathrm{Rot}(n)\cong\mathbb{H}\ltimes\mathbb{R}^n$.

5.1.1 Clifford algebra of Points, Lines and Planes in $\mathbb{R}^3$

The vector equation of the plane is given by $n \cdot r = d$, where r is any point on the plane, n is a unit normal vector and $d \in \mathbb{R}$ is a perpendicular distance from the origin. In Euclidean Clifford algebra $\mathcal{C}l(0,3,1) = \langle e_1, e_2, e_3, e \rangle$, plane can by represented by the elements of the form

$$\pi = n_x e_1 + n_y e_2 + n_z e_3 + de$$

and the points $[x, y, z]$ are represented by the elements of the form

$$p = e_1 e_2 e_3 + x e_2 e_3 e + y e_3 e_1 e + z e_1 e_2 e.$$

Lines in $\mathbb{R}^3$ can by determined by a pair of vectors:

1. unit direction vector v,
2. momentum vector $u = r \times v$, where r is the position of any point on the line.

In Euclidean Clifford algebra $\mathcal{C}l(0,3,1)$ we will represent a line by the element of the form

$$l = (v_x e_2 e_3 + v_y e_3 e_1 + v_z e_1 e_2) + (u_x e_1 e + u_y e_2 e + u_z e_3 e).$$

On the Clifford algebra there are algebraic operations of incidence, meets and joints. For example, by direct computation of exterior product we have

$$\pi \wedge p := \frac{1}{2}(\pi p - p\pi) = (x n_x + y n_y + z n_z - d) e_1 e_2 e_3 e$$

and this expression of exterior product vanishes if the point lies on the line. It is not difficult to see that the same property holds for any combination of points, lines and planes. The intersection of pair of linear subspaces can be found by taking their exterior product and then dividing by a constant.

5.1.2 Robot Kinematics

Let us show the description of T^3 robot by means of a Clifford algebra. The acronym T^3 was intended to stand for "The Tomorrow Tool", the large industrial 6–R robot manufactured by Cincinnati Milacron [62]. The following Clifford algebra elements $\ell_1, \dots, \ell_6 \in \mathcal{C}l(0,3,1)$ are joint axes of our robot in initial position.

$$\begin{aligned}
\ell_1 &= e_1 e_2, \\
\ell_2 &= e_2 e_3, \\
\ell_3 &= e_2 e_3 + l_1 e_2 e, \\
\ell_4 &= e_2 e_3 + (l_1 + l_2) e_2 e, \\
\ell_5 &= e_3 e_1 - (l_1 + l_2 + l_3) e_1 e, \\
\ell_6 &= e_1 e_2
\end{aligned}$$

where the numbers l_1, l_2 and l_3 are the constant design parameters of the robot. The second, third and fourth joint axes of this robot are parallel. One can work with the system in the following way: to compute $\ell_1 \ell_1^* = e_1 e_2 (e_1 e_2)^* = e_1 e_2 (-e_2)(-e_1) = 1$, it is easy to see that $\ell_1 \in \mathrm{Spin}(n)$ and one can define the element $r_1 = \ell_1 + \frac{1}{2} t \ell_1 e$, where $t = t_1 e_1 + \cdots t_n e_n$. Now, the conjugation by element ℓ_1 represents rotation around the axis ℓ_1 and conjugation by element r_1 represents screw motion around the axis ℓ_1.

In the book [62] on can see strength of our approach to verify assumption of very well known Piper's theorem which showed that any 6–R robot that has three consecutive joint axes meeting at a point (or three consecutive joints axes parallel) has solvable inverse kinematics.

5.2 Conformal Clifford Algebra

In optics, computer vision or neurogeometry we mostly work with projective geometry and usually use stereographic projection. In Clifford algebra language, we extend the space $\mathcal{C}l(0,3,0)$ by adding two base vectors, such that the extended Clifford algebra is $\mathcal{C}l(1,4,0)$, i.e. it becomes mixed signature and is defined by the basis $\{e_1, e_2, e_3, e, \bar{e}\}$, where e and $\bar{e}$ are defined so that

$$e^2 = 1, \ \ \bar{e}^2 = -1, \ \ e \cdot \bar{e} = 0, \ \ e \cdot e_i = \bar{e} \cdot e_i = 0, \ \ i \in \{1, 2, 3\}.$$

Now, if we define so called null vectors as

$$n = e + \bar{e}$$
$$\bar{n} = e - \bar{e} \ (n^2 = \bar{n}^2 = 0)$$

then the transformation

$$F(x) = \frac{1}{2}(x^2 n + 2x - \bar{n}) \equiv X, \ \ (\text{where } F^{-1} = (X \wedge n) \cdot e, \ X \neq n)$$

maps $\mathbb{R}^3$ to the null vectors, i.e. $(F(x))^2 = 0$, $x \in \langle e_1, e_2, e_3 \rangle$. The origin is mapped approximately (multiple by scalar) to $\bar{n}$ and we define n as the image of infinity, i.e.

$$F(0) \sim \bar{n}, \ \ F(\infty) \sim n.$$

One can see, that rotations and other natural transformations work on null space as a conjungation:

1. Rotations:

$$RF(x)R^* = F(RxR^*)$$

2. Translations:

$$T_a = \exp(\frac{na}{2}) = 1 + \frac{na}{2}, \ \ F(x + a) = T_a F(x) T_a^*$$

3. Dilations:

$$D_\alpha = \exp(\frac{\alpha e \bar{e}}{2}), \ \ F(e^{-\alpha} x) \sim D_\alpha F(x) D_\alpha^*$$

4. Inversion $x \mapsto \frac{x^2}{x}$ may be represented as

$$F(x) \mapsto eF(x)e,$$

Representation of geometric objects in conformal Clifford algebra is then the following:

generalized circles	lines	$X_i \wedge X_j \wedge n$
	circles	$X_i \wedge X_j \wedge X_k$
generalized spheres	planes	$X_i \wedge X_j \wedge X_k \wedge n$
	spheres	$X_i \wedge X_j \wedge X_k \wedge X_l$

For example, if

$$M : F(x_1) \wedge F(x_2) \wedge F(x_3) \wedge F(x_4)$$

is a generalized sphere which passes through the points $x_1, \ldots x_4$, then

$$RM\tilde{R} : RF(x_1)\tilde{R} \wedge RF(x_2)\tilde{R} \wedge RF(x_3)\tilde{R} \wedge RF(x_4)\tilde{R}$$

is sphere after an effect of rotor R and

$$RM\tilde{R} : F(Rx_1\tilde{R}) \wedge F(Rx_2\tilde{R}) \wedge F(Rx_3\tilde{R}) \wedge F(Rx_4\tilde{R})$$

is generalized sphere passing through the points $RF(x_1)\tilde{R}, \ldots, RF(x_4)\tilde{R}$.

5.2.1 Saccadic eye movement

There are several types of human eye movement but this note is concerned with *sacadic eye movement* which is used to fix targets in the scene. In the brain sciences, the very well Listing's law asserts further that there is a unique gaze direction p, called the primary direction, such that, for any g, $S(g)$ is obtained by a parallel transport along a geodesic from p. Let p be a reference gaze direction, a saccade from p to a new direction g can be described by saccade spinor S satisfying

$$g = SpS^{-1}$$

Listing's law states that the actual orientation of the eye, as p is fixated is consistent with rotation of the primary line of the sight z around a perpendicular axis w. It follows that, although the axis of rotation depends on the target direction, it must lie in Listing's plane, which is itself perpendicular to the reference direction. Listing's plane, which is determined by experiment, is approximately parallel to the face.

In the visual analysis, we use the stereographic projection to describe the retinal projection of the scene. This procedure is defined as follows. The eyeball is represented by the unit sphere S and the stereographic center of projection $s_0 = (0, 0, -1)$ is fixed in the back of the eyeball. A projection plane $\mathcal{T}$ is fixed in space, perpendicular to the z axis. Note that, in coordinates, the axis is defined as Doner's low asserts that gaze attitude $S = S(g)$ for any gaze direction is unique and independent of the path by which the eye arrived at g. Accordingly, Listing's law is expressed by the formula

$$S(g) = 1 + gp = 1 + g \cdot p + I(g \times p),$$

where $I := e_1e_2e_3e$ is so called *pseudoscalar*. A saccade from p in a new direction g together with stereographic projection can be described by a spinor G, such that

$$g = GF(p)G^{-1},$$

where $G = eS$ by formula

$$\begin{aligned} G(g) = e + egp = e + e(g \cdot p) + eI(g \times p) = \\ = e + (g \cdot p)e + eI(g \times p) = \\ = e + (g \cdot p)e - e_1e_2e_2e(g \times p). \end{aligned}$$

5.3 Material science

In continuum mechanics, a *material body* $\mathcal{B}$ is defined as a three–dimensional differentiable manifold that can be covered with a single coordinate chart. We recall that a configuration of material body $\mathcal{B}$ is an embedding

$$\kappa : \mathcal{B} \to \mathbb{E}^3$$

and the single coordinate chart $\kappa_0 : \mathcal{B} \to \mathbb{R}^3$ is usually identified as a reference configuration. Once a reference configuration has been fixed, one can associate with any given configuration κ the deformation $\chi := \kappa \circ \kappa^{-1}$ and the non–singular Jacobian matrix $F = \frac{\partial \chi}{\partial x}$ evaluated in any point called *deformation gradient*. The constitutive equation of material body $\mathcal{B}$ is a mathematical expression of thermomechanical behaviour. In the case of hyperelasticity, the constitutive equation is of the form $\psi = \psi(F, \kappa)$.

We define a *Cosserat body* as the principal frame bundle $\mathcal{P}^1\mathcal{B}$ of an ordinary material body $\mathcal{B}$. The physical meaning is that the underling body represents the *macromedium* and each fiber $\mathcal{P}^1_x\mathcal{B}$ represents *microparticle* or grain at $x \in \mathcal{B}$ [11, 9].

In fact, the use of differential geometry in material science is based on 1-jet calculus. This technique is described for example in [12, 10]. A material body endowed with a constitutive equation induces naturally a linear connection, and several important physical properties of the material are described by means of its geodesics. The cited books handle one constitutive equation and thus one appropriate linear connection. In case that a material is endowed with more than one constitutive equation, that is, by more than one connection, the topic of higher order connections appears. Note that the topic of higher order connections is widely studied; see, for example, [42]. Such approach is not established so far in the material science, and the paper [34] thus formulates introductory principles and problems of the theory of materials endowed with more than one constitutive equation.

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Abstract

This thesis deals with Clifford algebras and geometric structures over them. Motivated by the fact that the choice of the projective structure corresponds to the choice of the class of connections sharing geodesics and the choice of "complex" projective structure corresponds to the choice of connections sharing the J–planar curves, we define the projective A–structures as A–structures together with the choice of connection class sharing A–planar curves. If A is a

Clifford algebra, we obtain a class of so–called Cliffordian projective structures. While almost quaternionic structure is the parabolic geometry, an almost complex and almost product are not parabolic and become parabolic geometries just after the choice of a special class of connections. As our main result, we have shown that the morphisms of these projective structures are exactly the morphisms preserving the class of geodesics of Weyl connections, i.e. the generalized projective transformations with respect to the class of Weyl connections.