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Kvalitativní a numerická analýza nelineárních diferenciálních rovnic se zpožděním

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The Qualitative and Numerical Analysis of Nonlinear Delay Differential Equations

Zkrácená verze Ph.D. Thesis

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Obsah

1 INTRODUCTION

The Ph.D. thesis is focused on the investigation of asymptotic properties of solutions of some nonlinear differential equations with delayed argument and their discrete analogues (the corresponding difference equations). The content of this thesis is formed by three scientific papers which have been published (or submitted) in international mathematical journals.

1.1 RESEARCH MOTIVATION

Delay differential equations play an important role in the research field of various applied sciences such as control theory, electrical networks, population dynamics, environment science, biology and life science. Mathematical models employing delay differential equations turn out to be useful especially in the situation, where the description of investigated systems depends not only on the position of a system in the current time, but also in the past. In such cases the use of ordinary differential equations turns out to be insufficient. The presence of a delayed time argument in the investigated system may frequently influence properties of solutions. The survey of the theory related to delay differential equations can be found e.g. in books [3], [6], [17], [27] or [40].

It is known that the exact solution of delay differential equations can be found just in some special cases. There is no unified approach to solve the delayed differential equations, even in the linear case. The theory of ordinary differential equations gives various methods to obtain analytical solution (e.g. the variation of constants method, the separation of variables method and others). But these methods are inapplicable dealing with delay differential equations. Hence qualitative and numerical analysis of these equations gather great importance.

The importance of numerical solutions of delay differential equations has still been stated in the first chapter. Problems of numerical methods for (linear as well as nonlinear) delay differential equations have been investigated in many papers (see e.g. [7], [23], [24], [35], [37] or [43]). General reference is [6], where the overview of basic results from numerical analysis for differential equations with delayed argument can be found. In particular, numerical investigations of the θ-method for some linear delay differential equations are the subject of papers [5], [12], [19], [20], [32] and [44]. It can be stated that the analysis of the θ-method for nonlinear delay differential equations is just at its beginning.

For successful implementation of numerical methods it is often necessary to have general information about qualitative behaviour of solutions of the corresponding exact equation. In this sense the qualitative and numerical analysis of solutions of delay differential equations influence each other. As an example we can mention a simple initial value problem

$$x'(t) = -x(0.99\,t), \quad t \geq 0, \quad x(0) = 1.$$

Its solution (exact or numerical) takes within a long time interval almost zero values (e.g. $x(t) \approx 10^{-10}$ for $t \in (100; 200)$), consequently the numerical solution gives the identically zero solution after some critical instant due to the rounding errors. But it is contrary to qualitative behaviour of the exact solution, which is not stable (for more details see [35] and [31]). From this point of view, the simultaneous qualitative and numerical investigation of delay differential equations seems to be desirable.

1.2 Objectives of the thesis

The aim of the thesis is to investigate qualitative (especially asymptotic) properties of some nonlinear delay differential and difference equations.

The thesis deals with the behaviour of all solutions of the differential equation

$$x'(t) = a(t)x(t) + f(t, x(\tau(t))), \quad t \in I := [t_0, \infty), \tag{1.1}$$

where $x(t)$ represents a given state value, $\tau(t)$ is representing delayed argument and the function $f(t, x)$ fulfills the relation

$$|f(t, x)| \leq |b(t)||x|^r + |g(t)| \tag{1.2}$$

for suitable continuous functions $b(t)$, $g(t)$ and a positive real scalar r.

The first goal is to analyse asymptotic properties of this equation. The second aim of the thesis concerns asymptotic properties of solutions of the delay difference equation

$$\Delta y(n) = p(n)y(n) + \sum_{i=1}^{k} q_i(n)|y(\bar{\tau}_i(n))|^{r_i} \operatorname{sgn} y(\bar{\tau}_i(n)) + d(n), \; n \in \mathbb{N}(n_0), \tag{1.3}$$

where $0 < r_i \leq 1$ are real scalars, $p(n), q_i(n), d(n)$ are sequences of reals and $\bar{\tau}_i(n)$ are suitable sequences of integers. This difference equation has been obtained via the numerical discretization of the studied differential equation, where several delays instead of one delay have been considered. For this purpose, we utilized the Euler method as the probably simplest convergent numerical schema. Also other studied discrete equations correspond to selected numerical formulae, which can be used for approximate solutions of analysed equation.

Another task consists in comparisons of the results following from qualitative analysis of studied delay differential equations and corresponding difference equations. Due to these comparisons, we set up conditions on numerical parameters (the stepsize) preserving specific qualitative properties of underlying equations (stability solutions, asymptotic estimates, etc.).

1.3 Some preliminaries

In the thesis the problem of the asymptotic bounds of all solutions for the nonlinear delay differential equation (1.1) is studied, where $a : I \to \mathbb{R}$ is a continuous function, $f : I \times \mathbb{R} \to \mathbb{R}$ is a given continuous function and $\tau : I \to \mathbb{R}$ is a real continuous, increasing and unbounded function on I, which fulfills conditions $\tau(t) < t$ for all $t > t_0$ and $\tau(t_0) \leq t_0$. In particular, it also involves some special cases, e.g. $\tau(t) = t - \kappa$, $\kappa > 0$ (constant delay) or $\tau(t) = \lambda t$, $0 < \lambda < 1$, $t \geq 0$ (proportional delay).

Further we assume that the function $f(t, x)$ fulfils the relation (1.2) for a suitable real number $r > 0$, where $x \in \mathbb{R}$ and $b(t)$, $g(t)$ are continuous functions on I.

We discuss two cases of studied equations: sublinear delay differential equations ($0 < r \leq 1$, note that $r = 1$ corresponds to the linear case) and superlinear delay differential equations ($r > 1$). The example of a sublinear equation is

$$x'(t) = a(t)x(t) + b(t)|x(\tau(t))|^r \operatorname{sgn} x(\tau(t)) + g(t), \quad t \in I,\ 0 < r \leq 1. \quad (1.4)$$

Similarly, as an example of a superlinear equation we can mention the equation

$$x'(t) = a(t)x(t) + b(t)|x(\tau(t))|^r \operatorname{sgn} x(\tau(t)) + g(t), \quad t \in I,\ r > 1.$$

By a solution of (1.1) we understand a real valued function $x(t)$ which is continuous on $[\tau(t_0), \infty)$, continuously differentiable on I and satisfies (1.1) on I.

As far as the existence and uniqueness of solutions of (1.1) are concerned, assuming $\tau(t_0) < t_0$ we can apply the method of steps to show that there exists a unique solution of this equation coinciding with a given initial function on the initial interval $[\tau(t_0), t_0]$. But if $\tau(t_0) = t_0$ is valid, then the initial set degenerates to $\{t_0\}$ and instead of the initial function we prescribe the initial condition $x(t_0) = x_0$. To show the existence and uniqueness of the solution of the corresponding initial value problem, we can mention the following result issuing from Theorem 1 and Corollary 6 of [18].

Theorem 1.1. *Consider the equation (1.1) subject to the inequality (1.2). Then (1.1) has a solution on I for any initial value x_0. Furthermore, if $f(t, x)$ is Lipschitz continuous, then this solution is unique.*

The asymptotic properties of solutions of (1.1) depend on the sign of the coefficient $a(t)$. To describe the asymptotic behaviour of solutions of (1.1) with $a(t)$ negative, we introduce the following functional relations, namely the Abel functional equation

$$\psi(\tau(t)) = \psi(t) - 1, \quad t \in I, \quad (1.5)$$

the auxiliary nonlinear functional equation

$$|b(t)|\omega^r(\tau(t)) = |a(t)|\omega(t), \quad t \in I \quad (1.6)$$

and corresponding functional inequality

$$|b(t)|\omega^r(\tau(t)) \leq |a(t)|\omega(t) \quad t \in I. \quad (1.7)$$

The question of the existence and uniqueness of solutions of equations (1.5) and (1.6) can be found, e.g., in the monograph [29]. Here we recall the statement ensuring the existence of solutions of (1.5) which has some differential properties.

Proposition 1.2. *Let $\tau \in C^1(I)$, $\tau(t) < t$ and $\tau'(t) > 0$ for all $t \in I$. Then there exits a solution $\psi \in C^1([\tau(t_0), \infty))$ of (1.5) such that $\psi'(t) > 0$ for all $t \in I$.*

Remark 1.3. *Because of the assumption of Proposition 1.2 throughout the thesis we assume that $\tau(t) < t$ for all $t \in I$ (the case $\tau(t_0) = t_0$ does not enable to solve (1.5) on the whole I). However, we note that all the results presented in the thesis are valid (with some minor modifications) also for lags vanishing at t_0 because we are interested in the asymptotic behaviour of solutions as $t \to \infty$.*

Now we discuss properties of the nonlinear functional equation (1.6) which will be relevant in Section 2.2.

Proposition 1.4. *Consider the functional equation (1.6), where $a, b, \tau \in C^1(I)$, $a(t) < 0$, $b(t) \neq 0$, $\frac{|b(t)|}{|a(t)|}$ is nondecreasing on I, $\tau(t) < t$ for all $t \in I$, $\tau(t)$ is increasing on I and let $M > 0$ be arbitrarily large. Then there exists a positive and nondecreasing solution $\omega \in C^1(I)$ of (1.6) such that $\omega(t) > M$ for all $t \in [\tau(t_0), t_0]$.*

2 Main results

This chapter brings the overview of the main results presented in the Ph.D. thesis.

2.1 Asymptotic properties of solutions of sublinear delay differential equations

Sublinear differential equation is an equation of the form (1.1) satisfying condition (1.2) for a suitable $0 < r \leq 1$. The results presented in this section appeared in the paper [10].

First, we mention theorems which yield asymptotic estimates of solutions of (1.1), (1.2), where we distinguish the cases $a(t)$ positive and negative. Secondly, we formulate consequences of these estimates in some particular cases.

Theorem 2.1. *Consider the equation (1.1) subject to the condition (1.2) for a suitable real $0 < r \leq 1$, where $a, b, g, \tau \in C(I)$, $f \in C(I \times \mathbb{R})$, $\tau(t) < t$ for all $t \in I$, $\tau(t)$ is increasing and unbounded on I and let both the integrals*

$$\int_{t_0}^{\infty} |b(t)| \exp\{-(1-r)\int_{t_0}^{\tau(t)} a(u)\,\mathrm{d}u - \int_{\tau(t)}^{t} a(u)\,\mathrm{d}u\}\,\mathrm{d}t,$$

$$\int_{t_0}^{\infty} \exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}\,|g(t)|\,\mathrm{d}t$$

converge. Then for any solution $x(t)$ of (1.1) there exists a constant $L \in \mathbb{R}$ such that

$$\lim_{t\to\infty} x(t) \exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\} = L.$$

Remark 2.2. *The positivity of the coefficient $a(t)$ is not strictly required in the previous theorem. However, the convergence requirement put on both integrals would be too strict constraint in the opposite case.*

Now we presented the asymptotic properties of solutions of (1.1), (1.2) provided $a(t)$ is negative.

Theorem 2.3. *Consider the equation (1.1) subject to the condition (1.2) for a suitable real* $0 < r \leq 1$*, where* $a \in C(I)$ *is negative and nonincreasing on* I*,* $f \in C(I \times \mathbb{R})$*,* $b, g \in C(I)$*,* $\tau \in C^1(I)$*,* $\tau(t) < t$*,* $\tau'(t) > 0$ *for all* $t \in I$ *and* $\tau(t) \to \infty$ *as* $t \to \infty$*. Let* $\psi(t)$ *be a solution of (1.5) with the properties guaranteed by Proposition 1.2 such that* $\int_{t_0}^{\infty} \frac{\psi'(t)}{-a(t)}\,\mathrm{d}t < \infty$*. Further assume that there exists a positive function* $\omega \in C^2(I)$ *fulfilling the inequality (1.7) such that* $\omega' - \omega a > 0$ *on* I*,* $\omega'_*/(\omega' - \omega a)$ *is nonincreasing on* I *and*

$$\int_{t_0}^{\infty} \frac{\omega'_*(t)}{\omega'(t) - \omega(t)a(t)}\,\psi'(t)\,\mathrm{d}t < \infty,$$

where $\omega'_*(t) = (|\omega'(t)| - \omega'(t))/2$*,* $t \in I$*. If* $g(t) = O(\omega(t))$ *as* $t \to \infty$*, then*

$$x(t) = O(\omega(t)) \quad \text{as } t \to \infty$$

for any solution $x(t)$ *of (1.1).*

Remark 2.4. *The previous theorem essentially says that, under certain constraints, the solution* $x(t)$ *of the delay differential equation (1.1) can be estimated by a solution* $\omega(t)$ *of the functional nondifferential equation (1.7). However, finding such a solution is not a simple matter, in general.*

Remark 2.5. *It follows from the proof of Theorem 2.3 that considering (1.2) with* $g(t)$ *identically zero on* I *we can omit the assumptions that* $a(t)$ *is nonincreasing and* $\int_{t_0}^{\infty} \frac{\psi'(t)}{-a(t)}\,\mathrm{d}t < \infty$*. Similarly, if* $g(t)$ *is not identically zero on* I *and both the mentioned assumptions are replaced by* $a(t) \leq a < 0$ *for all* $t \in I$*, then using the same line of arguments as given in the proof of Theorem 2.3 we can modify the result of Theorem 2.3 as*

$$x(t) = O(\omega(t)\psi(t)) \text{ as } t \to \infty$$

for any solution $x(t)$ *of (1.1).*

Now we give some applications of Theorem 2.1 and Theorem 2.3. Particularly, we consider the equation (1.4) under various assumptions on $a(t)$, $b(t)$ and show that the previous assertions can yield effective asymptotic results.

Corollary 2.6. *Consider the sublinear delay differential equation*

$$x'(t) = a(t)x(t) + b(t)|x(\tau(t))|^r \operatorname{sgn} x(\tau(t)) + g(t), \quad t \in I,\ 0 < r < 1, \quad (2.1)$$

where $a, b, g \in C(I)$*,* $\tau \in C^1(I)$*,* $0 < |b(t)| \leq K|a(t)|$*,* $\tau(t) < t$*,* $\tau'(t) > 0$ *for all* $t \in I$ *and a suitable real* $K > 0$ *and* $\tau(t) \to \infty$ *as* $t \to \infty$*.*

(i) If $a(t)$ *is positive on* I *and* $\int_{t_0}^{\infty} \exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}|g(t)|\,\mathrm{d}t < \infty$*, then for any solution* $x(t)$ *of (2.1) there exists a constant* $L \in \mathbb{R}$ *such that*

$$\lim_{t\to\infty} x(t) \exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\} = L.$$

(ii) *If $a(t)$ is negative and $g(t)$ is identically zero on I, then any solution $x(t)$ of (2.1) is bounded on I.*

(iii) *If $a(t)$ is negative and nonincreasing on I, $g(t)$ is bounded on I and $\int_{t_0}^{\infty} \frac{\psi'(t)}{-a(t)}\,\mathrm{d}t$ converges, where $\psi(t)$ is a solution of the Abel equation (1.5) with the properties guaranteed by Proposition 1.2, then any solution $x(t)$ of (2.1) is bounded on I.*

Further we consider only the cases when $a(t)$ is negative and $g(t)$ is identically zero on I in (1.4). The extension of the next results also to $a(t)$ positive and $g(t)$ nonzero can be easily done by use of Theorem 2.1 and Theorem 2.3.

Corollary 2.7. *Consider the equation without forcing term*

$$x'(t) = a(t)x(t) + b(t)|x(\tau(t))|^r \operatorname{sgn} x(\tau(t)), \quad t \in I,\ 0 < r < 1, \tag{2.2}$$

where $I = [t_0, \infty)$ with $t_0 > 0$, $a, b \in C(I)$, $\tau \in C^1(I)$, $a(t) < 0$, $b(t) \neq 0$, $\tau(t) < t$, $\tau(t_0) > 0$, $\tau'(t) > 0$ for all $t \in I$, $\tau(t) \to \infty$ as $t \to \infty$ and assume that $0 < |b(t)| \leq K|a(t)|t^{\alpha}(\tau(t))^{-r\alpha}$ for suitable $K, \alpha \in \mathbb{R}$, $K > 0$ and all $t \in I$.

(i) *If $\alpha \geq 0$, then*

$$x(t) = O(t^{\alpha}) \quad \text{as } t \to \infty \tag{2.3}$$

for any solution $x(t)$ of (2.2).

(ii) *If $\alpha < 0$, $a(t) < \frac{\alpha}{t}$ for all $t \in I$, $a(t)t$ is nonincreasing on I and*

$$\int_{t_0}^{\infty} \frac{\psi'(t)}{\alpha - a(t)t}\,\mathrm{d}t < \infty,$$

where $\psi(t)$ is a solution of (1.5) with the properties guaranteed by Proposition 1.2, then (2.3) holds for any solution $x(t)$ of (2.2).

2.2 Asymptotic properties of solutions of superlinear delay differential equations

We derive the asymptotic properties of the solution of the superlinear differential equation (1.1) satisfying condition (1.2) for a suitable $r > 1$. The results stated in this section were presented in the paper [15].

Comparing with the sublinear case we impose two restrictions. We assume that the function $g(t)$, appearing in (1.2), is identically zero, and, furthermore, $a(t)$ is negative.

Theorem 2.8. *Let $x(t)$ be a solution of (1.1) and (1.2) holds for a suitable real $r > 1$, where $g(t) \equiv 0$, $a(t)$, $b(t)$ are continuously differentiable functions on I such that $a(t)$ is negative, $b(t)$ is nonzero on I. Further, let $\tau \in C^1(I)$, $\tau(t) < t$ and $\tau'(t) > 0$ for all $t \in I$ and let $\psi(t)$ be a solution of (1.5) with the properties guaranteed by Proposition 1.2. Finally assume that $\omega \in C^1(I)$ is a positive and nondecreasing function satisfying (1.6) and let $M_0 = \sup\left\{\frac{|x(t)|}{\omega(t)}, t \in [\tau(t_0), t_0]\right\}$.*

(i) *If $M_0 \leq 1$, then $|x(t)| \leq \omega(t)$ for all $t \geq t_0$.*

(ii) *If* $M_0 > 1$*, then* $|x(t)| \le \omega(t)\, M_0^{r^{\psi(t)+1-\psi(t_0)}}$ *for all* $t \ge t_0$*.*

Remark 2.9. *It follows from the proof of Theorem 2.8 that the functional equation (1.6) can be replaced by the functional inequality (1.7) and the assertion of Theorem 2.8 remains valid. Moreover, in this case it is not necessary to require the differentiability of* $a(t)$*,* $b(t)$ *and we can omit the monotonic assumption on* $\frac{|b(t)|}{|a(t)|}$*.*

Remark 2.10. *The conclusions of Theorem 2.8 can be modified in the following way. By Proposition 1.4, we are able to make the function* $\omega(t)$ *arbitrarily large on* $[\tau(t_0), t_0]$*. This implies, among others, that we can choose a solution* $\omega(t)$ *of (1.6) such that* $M_0 \le 1$*. However, it does not mean that this procedure automatically yields the "better" estimate than the original result of Theorem 2.8 (corresponding to the case* $M_0 > 1$*) yields. More details concerning this question will be discussed in the next text.*

Corollary 2.11. *Consider the superlinear delay differential equation*

$$x'(t) = a(t)x(t) + b(t)|x(\tau(t))|^r \operatorname{sgn} x(\tau(t)), \quad t \in I,\ r > 1, \tag{2.4}$$

where $a, b \in C(I)$*,* $\tau \in C^1(I)$*,* $a(t) < 0$*,* $0 < |b(t)| \le K|a(t)|$*,* $\tau(t) < t$ *and* $\tau'(t) > 0$ *for all* $t \in I$ *and a suitable real* $K > 0$*. Let* $x(t)$ *be a solution of the equation (2.4) and let* $X_0 = \sup\{|x(t)|, t \in [\tau(t_0), t_0]\}$*.*

(i) *If* $X_0 \le K^{\frac{1}{1-r}}$*, then* $|x(t)| \le K^{\frac{1}{1-r}}$ *for all* $t \ge t_0$*.*

(ii) *If* $X_0 > K^{\frac{1}{1-r}}$*, then* $|x(t)| \le K^{\frac{1}{1-r}} \left(X_0 K^{\frac{-1}{1-r}}\right)^{r^{\psi(t)+1-\psi(t_0)}}$ *for all* $t \ge t_0$*, where* $\psi(t)$ *is a solution of (1.5) with the properties guaranteed by Proposition 1.2.*

Example 2.12. Consider the superlinear delay differential equation with a proportional delay, i.e. the equation

$$x'(t) = ax(t) + b|x(\lambda t)|^r \operatorname{sgn} x(\lambda t), \quad t \ge t_0 > 0\,, \tag{2.5}$$

where $r > 1$, $0 < \lambda < 1$, $a < 0$, $b \ne 0$ are real constants.

In the case of the proportional delay, the function $\psi(t) = \frac{\log t}{\log \lambda^{-1}}$ is a solution of the Abel equation (1.5) and fulfills assumptions described in Proposition 1.2. If $K = |\frac{b}{a}|$, then, by Corollary 2.11,

$$|x(t)| \le \left|\frac{b}{a}\right|^{\frac{1}{1-r}} \left[X_0 \left|\frac{b}{a}\right|^{\frac{-1}{1-r}}\right]^{r^{\frac{\log t - \log t_0}{\log \lambda^{-1}}+1}}, \quad t \in I$$

for any solution $x(t)$ of (2.5). Moreover, if $X_0 \le \left|\frac{b}{a}\right|^{\frac{1}{1-r}}$, then $|x(t)| \le \left|\frac{b}{a}\right|^{\frac{1}{1-r}}$ for all $t \in I$.

Remark 2.13. *Consider the equation (2.4) under assumptions of Corollary 2.11. Substituting into (1.7), we can easily verify that the system*

$$\omega(t) = K^{\frac{1}{1-r}} \exp\{\alpha r^{\psi(t)}\}, \quad t \in I,\ r > 1, \tag{2.6}$$

where α is a real parameter and $\psi(t)$ is a solution of the Abel equation (1.5), forms the one-parameters family of solutions of (1.7). Hence (2.6) satisfies the auxiliary functional inequality (1.7). Moreover, choosing α large enough we can fulfill the required inequality $\omega(t) > M$ for all $t \in [\tau(t_0), t_0]$ and M being arbitrarily large (see also Proposition 1.4). In particular, if

$$\alpha = r^{1-\psi(t_0)} \log\left(X_0 K^{-\frac{1}{1-r}}\right), \qquad X_0 = \sup\{|x(t)|, t \in [\tau(t_0), t_0]\},$$

then $M_0 = \sup\left\{\frac{|x(t)|}{\omega(t)}, t \in [\tau(t_0), t_0]\right\} \le 1$ and, by Theorem 2.8, the estimate

$$|x(t)| \le \omega(t) = K^{\frac{1}{1-r}} \left(X_0 K^{\frac{-1}{1-r}}\right)^{r^{\psi(t)+1-\psi(t_0)}} \qquad t \in I$$

holds for any solution $x(t)$ of (2.4). It may be interesting to note, that this estimate coincides with the result obtained in Corollary 2.11 (ii).

Remark 2.14. *The term $\psi(t) - \psi(t_0)$ appearing in the previous asymptotic estimates will be further studied. It holds that if $\psi(t)$ is a solution of the Abel equation (1.5), then $\psi(t) + \alpha, \alpha \in \mathbb{R}$ is also a solution of (1.5). This implies that without the loss of validity we can choose the solution $\psi(t)$ of (1.5) with the property $\psi(t_0) = 0$ and then omit $\psi(t_0)$ in all formulae involving the term $\psi(t) - \psi(t_0)$.*

2.3 ASYMPTOTIC ESTIMATES OF SOLUTIONS OF LINEAR OR SUBLINEAR DIFFERENCE EQUATIONS

The results presented in this section will appear in the paper [11]. We discuss some asymptotic properties of the delay difference equation (1.3), where $n_0 \in \mathbb{Z}$, $n_0 \ge 0$, $\mathbb{N}(n_0) = \{n_0, n_0 + 1, n_0 + 2, \dots\}$, $0 < r_i \le 1$ are real scalars, $p(n)$, $q_i(n)$, $d(n)$ are sequences of reals and $\bar{\tau}_i(n)$ are nondecreasing unbounded sequences of integers such that $\bar{\tau}_i(n) < n$ for all $n \in \mathbb{N}(n_0)$ ($i = 1, \dots, k$). The equation (1.3) is a discrete analogue of the delay differential equation

$$x'(t) = a(t)x(t) + \sum_{i=1}^{k} b_i(t)|x(\tau(t))|^{r_i} \operatorname{sgn} x(\tau(t)) + g(t), \qquad t \ge t_0.$$

This differential equation with $k = 1$ has been discussed in Section 2.1. Considering the discrete case, we consider $k \in \mathbb{N}$ to generalize some known results of the qualitative theory of difference equations. Note that the extension of the results of Section 2.1 for the case of several delays is only a technical matter.

Let $n_{-1} = \min\{\bar{\tau}_i(n_0) : i = 1, \dots, k\}$. By a solution of (1.3) we mean a sequence $y(n)$ of real numbers which is defined for $n \geq n_{-1}$ and satisfies (1.3) for $n \geq n_0$, $n \in \mathbb{Z}$. It is easy to see that for any given $n_0 \in \mathbb{N}(0)$ and initial conditions $y(n) = y_0(n)$, $n_{-1} \leq n \leq n_0$, the equation (1.3) has a unique solution satisfying these initial conditions.

Consequently, we formulate an upper bound for solutions $y(n)$ of (1.3). Before doing this, we introduce some necessary notations and auxiliary relations. Put

$$\begin{aligned}\sigma_{-1} &= n_{-1},\\ \sigma_0 &= n_0,\\ \sigma_{m+1} &= \max\{n \in \mathbb{N}(n_0) :\ \bar{\tau}_i(n) \leq \sigma_m \text{ for all } i = 1, \dots, k\}, \quad m = 0, 1, 2, \dots\end{aligned}$$

and consider two difference inequalities

$$\bar{\psi}(\sigma_{m+1}) \geq \bar{\psi}(\sigma_m) + 1, \qquad m = 0, 1, 2, \dots \tag{2.7}$$

and

$$\sum_{i=1}^{k} |q_i(n)| (\bar{\omega}(\bar{\tau}_i(n)))^{r_i} \leq (1 - |1 + p(n)|)\bar{\omega}(n), \qquad n \in \mathbb{N}(n_0)\,. \tag{2.8}$$

Note that previous inequalities correspond to the auxiliary relations imposed in Section 1.3. More precisely, the relation (2.7) is an analogue of the Abel equation (1.5) and the inequality (2.8) is consistent with the auxiliary functional relation (1.7). In addition, sequences $\bar{\psi}(n)$, $\bar{\omega}(n)$ are discrete analogues of functions $\psi(t)$, $\omega(t)$, respectively.

Further, for $m = 0, 1, 2, \dots$ we denote

$$u(m) = \min\{\frac{\Delta\bar{\omega}(\nu)}{1 - |1 + p(\nu)|} : \sigma_m \leq \nu \leq \sigma_{m+1}\} \tag{2.9}$$

$$v(m) = \max\{\frac{|d(\nu)|}{(1 - |1 + p(\nu)|)\bar{\omega}(\nu)} : \sigma_m \leq \nu \leq \sigma_{m+1}\}\,. \tag{2.10}$$

Theorem 2.15. *Consider the equation (1.3), where $|1 + p(n)| < 1$ for all $n \in \mathbb{N}(n_0)$. Further, let $\bar{\omega}(n)$ be a positive monotonous sequence satisfying (2.8), let $\bar{\psi}(n)$ be a positive increasing sequence satisfying (2.7) and let $u(m)$, $v(m)$ be given by (2.9) and (2.10), respectively.*

(i) If $\bar{\omega}(n)$ is nondecreasing, then for any solution $y(n)$ of (1.3) there exists a constant $L > 0$ such that

$$|y(n)| \leq \Big(L + \sum_{i=0}^{\lfloor \bar{\psi}(n) \rfloor} v(i)\Big)\bar{\omega}(n) \tag{2.11}$$

for all $n \in \mathbb{N}(n_0)$. (The symbol $\lfloor\ \rfloor$ means an integer part.)

(ii) *If $\bar{\omega}(n)$ is decreasing, then for any solution $y(n)$ of (1.3) there exists a constant $L > 0$ such that*

$$|y(n)| \leq \Big(L + \sum_{i=0}^{\lfloor \bar{\psi}(n) \rfloor} v(i)\Big)\bar{\omega}(n) \prod_{s=0}^{\lfloor \bar{\psi}(n) \rfloor} \Big(1 - \frac{u(s)}{\bar{\omega}(\sigma_{s+1})}\Big) \tag{2.12}$$

for all $n \in \mathbb{N}(n_0)$.

Remark 2.16. *By Theorem 2.15, any solution $y(n)$ of the delay difference equation (1.3) with a forcing term $d(n)$, can be estimated in terms of solutions of difference inequalities (2.7) and (2.8). Moreover, if $d(n)$ is identically zero, then $v(i)$ is also identically zero and both the estimates (2.11) and (2.12) are significantly simplified.*

Further we apply our general asymptotic result to some important particular cases to demonstrate, how it can be turned into effective asymptotic criterions.

Corollary 2.17. *Consider (1.3), where $d(n) \equiv 0$ and let $r = \max\{r_1, \dots, r_k\}$. Then any solution $y(n)$ of (1.3) is bounded if either*

$$r = 1, \quad |1 + p(n)| < 1 \quad \text{and} \quad |1 + p(n)| + \sum_{i=1}^{k} |q_i(n)| \leq 1, \quad n \in \mathbb{N}(n_0)$$

or

$$0 < r < 1 \qquad \text{and} \qquad 0 < \frac{\sum_{i=1}^{k} |q_i(n)|}{1 - |1 + p(n)|} < K, \quad n \in \mathbb{N}(n_0),$$

where K is a suitable scalar.

As another consequence, we discuss the sublinear difference equation

$$\Delta y(n) = p(n)y(n) + q(n)|y(\lfloor \lambda n \rfloor)|^r \operatorname{sgn} y(\lfloor \lambda n \rfloor), \qquad n \in \mathbb{N}(n_0)\,, \tag{2.13}$$

where $0 < \lambda, r < 1$, originating from the numerical discretization of the sublinear pantograph equation. We present conditions under which all its solutions tend to zero and derive also the rate of this convergency.

Corollary 2.18. *Consider (2.13), where $|1 + p(n)| \leq \bar{p} < 1$ for all $n \in \mathbb{N}(n_0)$ and $q(n) = O(n^{\alpha(1-r)})$ as $n \to \infty$ for a real scalar α. Then*

$$y(n) = O(n^{\alpha}) \qquad \text{as } n \to \infty$$

for any solution $y(n)$ of (2.13).

2.4 Asymptotic estimates for the Euler discretization of (1.4)

Applications of qualitative results derived in Section 2.3 to the Euler discretization are presented in this section.

First we mention the derivation of the Euler formula of the equation (1.4). We set the discretization equidistant grid $t_n := t_0 + nh$, where $n \in \mathbb{N}$ and $h > 0$ is the stepsize. We denote $\bar{\tau}(n) := \left\lfloor \frac{\tau(t_n)-t_0}{h} \right\rfloor$, where symbol $\lfloor\ \rfloor$ means an integer part. Then the explicit Euler discretization of (1.4) yields

$$\Delta y(n) = p(n)y(n) + q(n)|y(\bar{\tau}(n))|^r \operatorname{sgn} y(\bar{\tau}(n)) + d(n), \quad n \in \mathbb{N}(0)\,, \tag{2.14}$$

where $0 < r \leq 1$,

$$p(n) := ha(t_n), \qquad q(n) := hb(t_n), \qquad d(n) := hg(t_n). \tag{2.15}$$

The replacement $y(n) \approx x(t_n)$ is provided as usually. The replacement $y(\bar{\tau}(n)) \approx x(\tau(t_n))$ is performed by the piecewise constant interpolation with the left grid point.

We can easily check that the assumptions imposed on $\tau(t)$ in Section 1.3 ensure that properties assumed in Section 2.3 are valid. Then (2.14) is a particular case of the difference equation (1.3) considered in the previous section.

Our aim is to show that asymptotic bounds of solutions valid in the continuous case holds (under some restrictions) also in the corresponding discrete case. Doing this, we formulate an upper bound for solutions $y(n)$ of (2.14), which corresponds to the results mentioned in Theorem 2.3.

Now we state some notes on the initial conditions and necessary auxiliary relations. Similarly to Section 2.3, we set $n_{-1} := \bar{\tau}(0)$. The equation (2.14) has a unique solution satisfying initial conditions

$$y(n) = y_0(n), \qquad n_{-1} \leq n \leq 0,\ n \in \mathbb{Z}.$$

These initial conditions originates from the prescribed initial functions defined on the initial interval $[\tau(t_0), t_0]$. This is obvious that if $\tau(t_0) = t_0$, then $n_{-1} = 0$ and the initial condition is $y(0) := x(t_0)$ $(= x_0)$.

Furthermore we put $\sigma_{-1} = n_{-1}$, $\sigma_0 = 0$, $\sigma_{m+1} = \max\{n \in \mathbb{N}(0) :\ \bar{\tau}(n) \leq \sigma_m\}$, $m = 0, 1, 2, \dots$. We consider the following auxiliary difference inequality

$$|q(n)|\bar{\omega}(\bar{\tau}(n))^r \leq (1 - |1 + p(n)|)\bar{\omega}(n), \qquad n \in \mathbb{N}(0)\,. \tag{2.16}$$

This relation is a simplification of difference inequality (2.8). In addition, note that here we have to use relation (2.7) mentioned in the previous section, too. The sequences $u(m)$, $v(m)$ given by (2.9) and (2.10), respectively, for $m = 0, 1, 2, \dots$ are also used.

Theorem 2.19. *Let $p(n), q(n), d(n)$ be given by (2.15) and let $|1 + p(n)| < 1$ for all $n \in \mathbb{N}(0)$. Further, let $u(m)$, $v(m)$ be given by (2.9) and (2.10), respectively. Let $\bar{\omega}(n)$ be a positive monotonous sequence satisfying (2.16) and $\bar{\psi}(n)$ a positive and increasing sequence satisfying (2.7). Finally let $y(n)$ be a solution of (2.14).*

(i) *If $\bar{\omega}(n)$ is nondecreasing, then there exists a constant $L > 0$ such that*

$$|y(n)| \le \Big(L + \sum_{i=0}^{\lfloor \bar{\psi}(n) \rfloor} v(i)\Big)\bar{\omega}(n) \qquad \text{for all } n \in \mathbb{N}(0).$$

(ii) *If $\bar{\omega}(n)$ is decreasing, then there exists a constant $L > 0$ such that*

$$|y(n)| \le \Big(L + \sum_{i=0}^{\lfloor \bar{\psi}(n) \rfloor} v(i)\Big)\bar{\omega}(n) \prod_{s=0}^{\lfloor \bar{\psi}(n) \rfloor} \Big(1 - \frac{u(s)}{\bar{\omega}(\sigma_{s+1})}\Big) \qquad \text{for all } n \in \mathbb{N}(0).$$

Corollary 2.20. *Consider the equation (2.14), where $d(n) \equiv 0$, $p(n), q(n)$ be given by (2.15) and $|1 + p(n)| < 1$ for all $n \in \mathbb{N}(0)$.*

(i) *Let $r = 1$ and*

$$|1 + p(n)| + |q(n)| \le 1, \qquad n \in \mathbb{N}(0).$$

Then any solution $y(n)$ of (2.14) is bounded.

(ii) *Let $0 < r < 1$. Assume that there exists an arbitrary $K \ge 0$ such that*

$$0 < \frac{|q(n)|}{1 - |1 + p(n)|} \le K, \quad n \in \mathbb{N}(0). \tag{2.17}$$

Then any solution $y(n)$ of (2.14) is bounded.

Remark 2.21. *Note that condition (2.17) is a discrete analogue of the condition $0 < |b(t)| \le K|a(t)|$ from Corollary 2.6 formulated for the exact delay differential equation.*

Example 2.22. Now we consider the sublinear delay differential equation with constant coefficients

$$x'(t) = ax(t) + b|x(\tau(t))|^r \operatorname{sgn} x(\tau(t)) + g(t), \quad t \ge t_0, \quad 0 < r < 1\,. \tag{2.18}$$

Assume that $a < 0$ and $g(t)$ is bounded on I. Then, by Theorem 2.3 with respect to Remark 2.5,

$$x(t) = O(\psi(t)) \qquad \text{as } t \to \infty \tag{2.19}$$

for any solution $x(t)$ of (2.18), where $\psi(t)$ is a solution of the Abel equation (1.5).

In particular, if $\tau(t) = \lambda t$, $0 < \lambda < 1$ (the proportional delay), then the Abel equation (1.5) admits the solution $\psi(t) = \frac{\log t}{\log \lambda^{-1}}$ and from (2.19) we get

$$x(t) = O(\log t) \qquad \text{as } t \to \infty. \tag{2.20}$$

The Euler discretization of (2.18) is

$$\Delta y(n) = hay(n) + hb|y(\bar{\tau}(n))|^r \operatorname{sgn} y(\bar{\tau}(n)) + d(n), \qquad n \in \mathbb{N}(0), \tag{2.21}$$

where $0 < r < 1$, $d(n) = hg(t_n)$ and $y(n)$ is an approximation of $x(t)$ at $t = t_n$.

We set the stepsize h such that $|1 + ha| < 1$. It is clear that the sequence $d(n)$ is bounded. Then from (2.10) the sequence $v(m)$ is bounded too. From (2.9) the sequence $u(m)$ is zero. Then Theorem 2.19 implies that

$$y(n) = O(\bar{\psi}(n)) \qquad \text{as } n \to \infty \tag{2.22}$$

for any solution $y(n)$ of (2.21), where the sequence $\bar{\psi}(n)$ satisfies (2.7).

If we consider the proportional delay, then $\bar{\tau}(n) = \lfloor \lambda n \rfloor$, $0 < \lambda < 1$ and $\sigma_m = \lfloor \frac{\sigma_{m-1}}{\lambda} \rfloor$. We can choose $\bar{\psi}(n) = \frac{\log(n - \lambda/(1-\lambda))}{\log \lambda^{-1}}$. This sequence satisfies (2.7) and the estimate (2.22) becomes

$$y(n) = O(\log n) \qquad \text{as } n \to \infty\,. \tag{2.23}$$

Comparing (2.20), (2.23), we can observe the resemblance of the asymptotics of solutions of (2.18) and (2.21) provided that $|1 + ha| < 1$.

2.5 Asymptotic estimates for the θ-method discretization of (1.4)

The popular discretization of (1.4) is the θ-method involving e.g. Euler methods and trapezoidal rule as particular cases. Similarly as in the previous section we set $t_n := t_0 + nh$, $n \in \mathbb{N}$, $h > 0$ is the stepsize and we denote $\bar{\tau}(n) := \left\lfloor \frac{\tau(t_n) - t_0}{h} \right\rfloor$. In our case, we consider the θ-method in the form

$$\begin{aligned} \Delta y(n) = h\big((1-\theta)a(t_n)y(n) + \theta a(t_{n+1})y(n+1) + (1-\theta)b(t_n)|y^h(\bar{\tau}(n))|^r \\ \times \operatorname{sgn} y^h(\bar{\tau}(n)) + \theta b(t_{n+1})|y^h(\bar{\tau}(n+1))|^r \operatorname{sgn} y^h(\bar{\tau}(n+1)) \\ + (1-\theta)g(t_n) + \theta g(t_{n+1})\big), \end{aligned}$$

where $0 \le \theta \le 1$, $0 < r \le 1$ and the values $y^h(\bar{\tau}(n))$, $y^h(\bar{\tau}(n+1))$ are given by the linear interpolation utilizing the left and right neighbours of $\bar{\tau}(n)$ and $\bar{\tau}(n+1)$, respectively. Namely

$$y^h(\bar{\tau}(n)) := (1 - s_n)y(\bar{\tau}(n)) + s_n y(\bar{\tau}(n) + 1),$$

where $s_n := \frac{\tau(t_n) - t_0}{h} - \left\lfloor \frac{\tau(t_n) - t_0}{h} \right\rfloor$. The interpolation value of point $\tau(t_{n+1})$ is performed analogously.

Let $1 - \theta h a(t_{n+1}) \neq 0$. Then the previous equation can be also rewritten as the difference equation

$$\begin{aligned} \Delta y(n) = p(n)y(n) + q(n)\big|\mu(n)y(\bar{\tau}(n)) + \eta(n)y(\bar{\tau}(n) + 1)\big|^r \\ \times \operatorname{sgn}\big(\mu(n)y(\bar{\tau}(n)) + \eta(n)y(\bar{\tau}(n) + 1)\big) \\ + \hat{q}(n)\big|\hat{\mu}(n)y(\bar{\tau}(n+1)) + \hat{\eta}(n)y(\bar{\tau}(n+1) + 1)\big|^r \\ \times \operatorname{sgn}\big(\hat{\mu}(n)y(\bar{\tau}(n+1)) + \hat{\eta}(n)y(\bar{\tau}(n+1) + 1)\big) + d(n), \end{aligned} \tag{2.24}$$

$n \in \mathbb{N}(0)$, where $0 < r \le 1$,

$$p(n) := \frac{(1-\theta)ha(t_n) + \theta ha(t_{n+1})}{1 - \theta ha(t_{n+1})}, \qquad q(n) := \frac{hb(t_n)}{1-\theta ha(t_{n+1})},$$
$$d(n) := \frac{(1-\theta)hg(t_n) + \theta hg(t_{n+1})}{1 - \theta ha(t_{n+1})}, \qquad \hat{q}(n) := \frac{hb(t_{n+1})}{1-\theta ha(t_{n+1})} \tag{2.25}$$

and

$$\eta(n) := (1-\theta)^{\frac{1}{r}}\Big(\frac{\tau(t_n)-t_0}{h} - \Big\lfloor \frac{\tau(t_n)-t_0}{h} \Big\rfloor\Big),$$
$$\hat{\eta}(n) := \theta^{\frac{1}{r}}\Big(\frac{\tau(t_{n+1})-t_0}{h} - \Big\lfloor \frac{\tau(t_{n+1})-t_0}{h} \Big\rfloor\Big), \tag{2.26}$$
$$\mu(n) := (1-\theta)^{\frac{1}{r}} - \eta(n), \qquad \hat{\mu}(n) := \theta^{\frac{1}{r}} - \hat{\eta}(n)$$

In this section, we derive conditions which imply that the solution sequence of the θ-method discretization of (1.4) has asymptotic behaviour analogous to the behaviour of the exact solution. These conditions depend on coefficients $a(t)$, $b(t)$, the stepsize h and the parameter θ.

As in the previous section, we introduce a sequence σ_m and an auxiliary relation. Put $\sigma_{-1} = n_{-1} = \bar{\tau}(0)$, $\sigma_0 = 0$, $\sigma_{m+1} = \max\{n \in \mathbb{N}(0) : \bar{\tau}(n+1) + 1 \le \sigma_m\}$, $m = 0, 1, 2, \ldots$ and consider a difference inequality

$$|q(n)| \cdot \big|\mu(n)\bar{\omega}(\bar{\tau}(n)) + \eta(n)\bar{\omega}(\bar{\tau}(n)+1)\big|^r + |\hat{q}(n)| \cdot \big|\hat{\mu}(n)\bar{\omega}(\bar{\tau}(n+1)) + \hat{\eta}(n)\bar{\omega}(\bar{\tau}(n+1)+1)\big|^r \le (1 - |1+p(n)|)\bar{\omega}(n) \tag{2.27}$$

for all $n \in \mathbb{N}(0)$. This relation corresponds to auxiliary inequality (2.8). The following theorem is a direct consequence of Theorem 2.15.

Theorem 2.23. *Let $\eta(n)$, $\mu(n)$, $\hat{\eta}(n)$, $\hat{\mu}(n)$ be given by (2.26), $p(n)$, $q(n)$, $\hat{q}(n)$, $d(n)$ begiven by (2.25) such that $|1 + p(n)| < 1$ for all $n \in \mathbb{N}(0)$. Further, let $u(m)$, $v(m)$ be given by (2.9) and (2.10), respectively. Let $\bar{\omega}(n)$ be a positive monotonous sequence satisfying (2.27) and $\bar{\psi}(n)$ a positive and increasing sequence satisfying (2.7). Finally let $y(n)$ be a solution of (2.24).*

(i) If $\bar{\omega}(n)$ is nondecreasing, then there exists a constant $L > 0$ such that

$$|y(n)| \le \Big(L + \sum_{i=0}^{\lfloor \bar{\psi}(n) \rfloor} v(i)\Big)\bar{\omega}(n) \qquad \text{for all } n \in \mathbb{N}(0).$$

(ii) If $\bar{\omega}(n)$ is decreasing, then there exists a constant $L > 0$ such that

$$|y(n)| \le \Big(L + \sum_{i=0}^{\lfloor \bar{\psi}(n) \rfloor} v(i)\Big)\bar{\omega}(n) \prod_{q=0}^{\lfloor \bar{\psi}(n) \rfloor} \Big(1 - \frac{u(q)}{\bar{\omega}(\sigma_{q+1})}\Big) \qquad \text{for all } n \in \mathbb{N}(0).$$

Using the notation $\bar{q}(n) := \max(|q(n)|, |\hat{q}(n)|)$ we get the following corollary.

Corollary 2.24. *Consider the equation (2.24), where $d(n) \equiv 0$, $p(n)$, $q(n)$, $\hat{q}(n)$ begiven by (2.25) such that $|1 + p(n)| < 1$ for all $n \in \mathbb{N}(0)$ and $\eta(n)$, $\mu(n)$, $\hat{\eta}(n)$, $\hat{\mu}(n)$ be given by (2.26).*

(i) *Let $r = 1$ and*

$$0 < |1 + p(n)| + \bar{q}(n) \le 1, \quad n \in \mathbb{N}(0).$$

Then any solution $y(n)$ of (2.24) is bounded.

(ii) *Let $0 < r < 1$. Assume that there exists an arbitrary $K \ge 0$ such that*

$$0 < \frac{\bar{q}(n)}{1 - |1 + p(n)|} \le K, \quad n \in \mathbb{N}(0).$$

Then any solution $y(n)$ of (2.24) is bounded.

Example 2.25. We consider the sublinear differential equation

$$x'(t) = ax(t) + b|x(\tau(t))|^r \operatorname{sgn} x(\tau(t)), \qquad t \ge t_0,\ 0 < r < 1, \tag{2.28}$$

where $a < 0$, $b \ne 0$ are real constants. By Corollary 2.6 (ii), any solution $x(t)$ of (2.28) is bounded.

Now we describe the asymptotic estimate of solutions of the corresponding θ-method discretization (2.24), where $p(n) \equiv \frac{ha}{1-\theta ha}$, $q(n) = \hat{q}(n) \equiv \frac{hb}{1-\theta ha}$ and $\eta(n)$, $\mu(n)$, $\hat{\eta}(n)$, $\hat{\mu}(n)$ are given by (2.26). By Corollary 2.24 (with $K = \frac{h|b|}{|1-\theta ha|-|1+(1-\theta)ha|}$), the solution $y(n)$ is bounded if $|1 + \frac{ha}{1-\theta ha}| < 1$ and b is arbitrary.

2.6 Stability analysis of the θ-method for the sublinear equation

The aim of this section is to analyse the stability of the numerical method originating from the θ-method discretization of (1.4). This analysis substantially utilizes qualitative properties of studied differential equations and their discretizations (from related papers we refer to [25], [26], [36] and [38]).

We consider the test equation

$$x'(t) = ax(t) + b|x(\tau(t))|^r \operatorname{sgn} x(\tau(t)), \quad t \ge t_0,\ 0 < r < 1, \tag{2.29}$$

where $a, b \in \mathbb{R}$, $a, b \ne 0$, and its θ-method discretization

$$\begin{aligned}\Delta y(n) = py(n) + q\Big(&\big|\mu(n)y(\bar{\tau}(n)) + \eta(n)y(\bar{\tau}(n)+1)\big|^r \operatorname{sgn}\big(\mu(n)y(\bar{\tau}(n)) \\ &+ \eta(n)y(\bar{\tau}(n)+1)\big) + \big|\hat{\mu}(n)y(\bar{\tau}(n+1)) + \hat{\eta}(n)y(\bar{\tau}(n+1)+1)\big|^r \\ &\times \operatorname{sgn}\big(\hat{\mu}(n)y(\bar{\tau}(n+1)) + \hat{\eta}(n)y(\bar{\tau}(n+1)+1)\big)\Big),\end{aligned} \tag{2.30}$$

$n \in \mathbb{N}(0)$, where $0 < r < 1$,

$$p = \frac{ha}{1 - \theta ha}, \quad q = \frac{hb}{1 - \theta ha}, \tag{2.31}$$

$\eta(n)$, $\mu(n)$, $\hat{\eta}(n)$, $\hat{\mu}(n)$ are given by (2.26) and $h > 0$ is the stepsize. We do not consider the pure delayed case ($a = 0$) in this section.

An important theoretical question on these numerical approximations is a problem whether the numerical and exact solutions have the related asymptotic behaviour on the unbounded domain. More precisely, if all solutions of a given differential equation have certain asymptotic properties, then we investigate if the solutions of corresponding discretization have the same properties (regardless of the stepsize h and the delayed argument $\tau(t)$). We pay a special attention to boundedness property.

Corollary 2.6 (ii) implies

Theorem 2.26. *Let $x(t)$ be a solution of (2.29), where $a < 0$ and $b \neq 0$. Then $x(t)$ is bounded as $t \to \infty$.*

The following property is taken from the standard notions of stability of numerical methods for linear equations.

Definition 2.27. *The numerical method (2.30) is called* stable *if any application of the method to the equation (2.29), where $a < 0$, generates a numerical solution $y(n)$ that is bounded for any $h > 0$.*

Procedures performed in Example 2.25 can be summarized as follows.

Theorem 2.28. *Let $y(n)$ be a solution of the θ-method discretization (2.30) with $\eta(n)$, $\mu(n)$, $\hat{\eta}(n)$, $\hat{\mu}(n)$ given by (2.26), p, q given by (2.31), where $a, b \neq 0$ and*

$$0 < |1 - \theta ha| - |1 + (1 - \theta)ha|. \tag{2.32}$$

Then $y(n)$ is bounded.

The condition (2.32) implies the following statement.

Theorem 2.29. *Let $a < 0$, $b \neq 0$. The θ-method discretization (2.30) is stable if and only if*

$$\frac{1}{2} \leq \theta \leq 1.$$

Example 2.30. Consider the initial value problem for the delay differential equation with constant coefficients

$$x'(t) = -4x(t) + \sqrt{|x(t/2)|}\ \operatorname{sgn} x(t/2), \qquad t \geq 0, \quad x(0) = 1. \tag{2.33}$$

By Theorem 2.26, the solution $x(t)$ of (2.33) is bounded.

Consider the θ-method discretization obtained from (2.30). We discuss the bouneness of the corresponding discretization with respect to changing $h > 0$ and $0 \leq \theta \leq 1$. By Theorem 2.28, the solution $y(n)$ of (2.30) is bounded if the condition

$$0 < 1 + 4\theta h - |1 - 4h(1 - \theta)| \tag{2.34}$$

$h\backslash nh$	50	150	500	1000
0.01	0.06419	0.06306	0.06267	0.06258
0.1	0.06482	0.06327	0.06273	0.06261
0.5	0.06787	0.06414	0.06301	0.06275
1	0.06992	0.06513	0.06321	0.06285
5	0.10704	0.07189	0.06484	0.06366
10	0.21264	0.08504	0.06626	0.06435
50	1	0.06832	0.11672	0.08539

Table 2.1: The solution $x(nh)$ for $a = -4$ and $\theta = 0.8$

holds.

First we consider $1/2 \leq \theta \leq 1$, e.g. $\theta = 0.8$. In this case, by Theorem 2.29, the solution $y(n)$ is bounded for all $h > 0$ and the method (2.30) is stable. The situation is illustrated by Table 2.1.

Further we assume $0 \leq \theta < 1/2$, e.g. $\theta = 0.3$. It follows from (2.34) that the solution $y(n)$ is bounded provided

$$h < \frac{1}{2(1-2\theta)} = 1.25.$$

Table 2.2 demonstrate the strictness of this stepsize condition.

$h\backslash nh$	50	150	500	1000
0.01	-0.05831	0.06234	0.06244	-0.06247
0.5	-0.06773	-0.06388	-0.06203	0.06226
1	0.09807	-0.06138	-0.05605	-0.05931
1.24	0.9753	0.4786	0.09373	-0.02712
1.2499	1.3398	1.3434	1.3666	1.1991
1.255	-0.8951	-1.1907	2.7191	11.9956
2	-779.004	-2.857 E9	2.689 E32	1.776 E65

Table 2.2: The solution $x(nh)$ for $\theta = 0.3$

3 Conclusion

In the Ph.D. thesis, there are presented the results concerning with the asymptotic behaviour of the nonlinear delay differential equation

$$x'(t) = a(t)x(t) + f(t, x(\tau(t))), \qquad t \in [t_0, \infty),$$

where the right-hand side fulfills the relation

$$|f(t,x)| \leq |b(t)||x|^r + |g(t)|, \qquad t \in [t_0, \infty), \quad r > 0.$$

We derived two different types of asymptotics, which depends on the sign of the function $a(t)$ provided that the studied equation is of the sublinear type ($0 < r < 1$). Asymptotic estimates of the superlinear equation ($r > 1$) were provided only for negative values of $a(t)$. Obtained results were demonstrated by several corollaries and illustrating examples.

Since the searching for an analytical solution of studied nonlinear equations turned out to be impossible, we need to discuss the numerical solution. The appropriate numerical formulae are constructed as difference equations. Consequently, the second part of the thesis is already concerned with problems of sublinear difference equations (with one or more delays). Considering these equations, main qualitative properties (especially asymptotic) were derived. Using these results we discussed the stability property of the θ-method discretization. It was shown, that for $\frac{1}{2} \leq \theta \leq 1$ this method is stable. Further, in several examples we compared asymptotic estimates of both exact and numerical solutions.

There are several directions, where the results obtained in this work can be further developed. It can be useful to focus on improvement of some asymptotic estimates. Some numerical experiments indicate that some of these estimates can be improved. Another development may consist in considering the corresponding differential equations of neutral type. Finally, obtained results for differential and difference equations can be unified and generalized in the frame of the time scale theory.

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ABSTRACT

The Ph.D. thesis formulates the asymptotic estimates of solutions of the so-called sublinear and superlinear differential equations with a delayed argument. These estimates are given in terms of auxiliary functional equations and inequalities. Further the thesis discusses the qualitative properties of some delay difference equations originating from discretizations of studied differential equations. We also deal with the resemblances between asymptotic behaviour of solutions of given equations in the continuous and discrete form, considering general as well as particular cases. We discuss stability properties of the θ-method discretizations, too. Several examples illustrating the obtained results are included in the thesis.