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Estimation of Solutions of Differential Systems with Delayed Argument of Neutral Type

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ESTIMATION OF SOLUTIONS OF DIFFERENTIAL SYSTEMS WITH DELAYED ARGUMENT OF NEUTRAL TYPE

ODHADY ŘEŠENÍ DIFERENCIÁLNÍCH SYSTÉMŮ SE ZPOŽDĚNÝM ARGUMENTEM NEUTRÁLNÍHO TYPU

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1 Introduction

Dynamical processes are those whose state depends on the prehistory and are described by differential equations with deviating argument, i.e. by such equations in which the unknown function of one scalar argument (time) and its derivatives are at different values of the argument.

Such equations are widely used in mathematical modeling of processes in control theory, economics, population dynamics, and medicine. Processes taking place in these systems depend both on the current time, and on the prehistory.

One of the main characteristics of system dynamics is the stability of the process. We will investigate the stability of given problems. Emphasis is placed on the second (direct) method of Lyapunov (see [25]).

Mathematical models described by the functional-differential equations more adequately describe the bulk of dynamic objects. In real objects, almost always, there are elements that cause delayed effects.

Natural and technical causes of the delay may be transportation delays, delays in transmission of information received by the delay in decision-making, etc. There may be other factors. These mostly include natural delay in the simulation of economic objects, objects in the environment, medicine, population dynamics, etc. In the chemical-technological processes, the delay is caused by the fact that the reactions required for the passage of time determined by the properties of the reactants, [1, 2, 4, 7, 9]. Dynamics of vehicles on the water is very different from the dynamics on the ground. Features can be taken into account by introducing a delay. There are other physical and technical interpretations.

2 Preliminaries

There exists the following simple classification of equations with constant delay [7]. Let us consider a scalar differential equation with one constant delay
$\dot{x}(t) =$

$$f\left(t, x(t), x'(t), x''(t), ..., x^{(n)}(t), x(t-\tau), x'(t-\tau), x''(t-\tau), ..., x^{(m)}(t-\tau)\right).$$

1. Let $n > m$. Then the equation is called differential equation with delay.
2. Let $n = m$. Then the equation is called differential equation of neutral type.
3. Let $n < m$. Then the equation is called differential equation with an advancing argument.

For systems of ordinary differential equations of the first order without delay the basic initial problem (Cauchy problem) is as follows. We need to find the solution $x(t)$ of the system $\dot{x}(t) = f(t, x(t))$, satisfying the initial condition $x(t_0) = x_0$.

For the differential equation with one constant delay, i.e. $\dot{x}(t) = f(t, x(t), x(t-\tau))$, $\tau > 0$, $t > t_0$, Cauchy problem consists in finding of a solution $x(t)$, satisfying the initial condition $x(t) = \varphi(t)$, $t_0 - \tau \leqslant t \leqslant t_0$, where $\varphi(t)$ is an arbitrary continuous function, also called the initial function. Set $E_{t_0} = \{t : t_0 - \tau \leqslant t \leqslant t_0\}$ is called initial set. The "condition of gluing" $\varphi(t_0) = x(t_0 + 0)$ is a natural condition.

Consider an initial problem

$$\dot{x}(t) = f(t, x(t), x(t-\tau)), \quad x(t) = \varphi(t), \quad t_0 - \tau \leqslant t \leqslant t_0. \tag{1}$$

Theorem 2.1 *(Existence and uniqueness of solutions of the Cauchy problem) Let in the parallelepiped* $D = \{(t, y, z) : |t - t_0| \leqslant a, |y - \varphi(t_0)| \leqslant b, |z - \varphi(t_0 - \tau)| \leqslant b\}$ *be defined function* $f(t, y, z)$ *satisfying the properties*

1. *Function* $f(t, y, z)$ *is continuous for all variables in* D.
2. *Function* $f(t, y, z)$ *satisfies the Lipschitz condition for variables* y, z *with constant* L, *i.e.* $|f(t, y_1, z) - f(t, y_2, z)| \leqslant L|y_1 - y_2|$, $|f(t, y, z_1) - f(t, y, z_2)| \leqslant L|z_1 - z_2|$.

Then for $t_0 \leqslant t \leqslant t_0 + h$, *where* $h = \min\left\{a, \frac{b}{N}, \frac{1}{2L}\right\}$, $N = \max\{|f(t, y, z)|, (t, y, z) \in D\}$ *there is only one solution of the differential equation* (1) *satisfying* $x(t) = \varphi(t)$, $t_0 - \tau \leqslant t \leqslant t_0$.

3 Current State

One can hardly name a branch of natural science or technology in which problems of stability do not attract the attention of scholars, engineers, or experts who investigate natural phenomena or operate designed machines or systems. If for a process or a phenomenon, for example, atom oscillations or a supernova explosion, a mathematical model is constructed in the form of a system of differential equations, the investigation of the latter is possible by a direct (numerical as a rule) integration of the equations or by its analysis by qualitative methods.

Dynamics of systems is a branch of science that studies actual equilibriums and motions of natural or artificial real objects. However, it is known that hardly every state of a really functioning system as observed in practice corresponds to a mathematically strict solution of either equilibrium or differential motion equations. It has been found out that only those equilibriums and motions of real systems are evident which possess certain "resistance" to outer perturbations. The equilibrium states of this kind are referred to as stable, while the other ones are called unstable.

The notion of stability is intuitivelly clear, but difficult to formulate and only Lyapunov (see [25]) attempted to formulate it.

Direct Lyapunov method based on scalar auxiliary function proved to be a powerful technique of qualitative analysis of the real world phenomena.

The most frequently used method for investigating the stability of functional-differential systems is the method of Lyapunov-Krasovskii functionals [20, 21]. Usually, it uses positive definite functionals of a special quadratic form and the integral (over the interval of delay [19]) of a quadratic form.

Literature on the stability and estimation of solutions of neutral differential equations is enormous. Tracing previous investigations on this topic, we emphasize that a Lyapunov function $v(x) = x^T Hx$ has been used to investigate the stability in [9] (see [16] as well).

The stability of linear neutral systems, yet with different delays h_1 and h_2, is studied in [14].

In [15, 17], functionals depending on derivatives are also suggested for investigating the asymptotic stability of neutral nonlinear systems. The investigation of nonlinear neutral delayed systems with two time-dependent bounded delays in [22] to determine the global asymptotic and exponential stability uses special functionals as well.

Delay independent criteria of stability for some classes of delay neutral systems are developed in [13]. The stability of systems with time dependent delays is investigated in [29]. For recent results on the stability of neutral equations, see [22]–[24], [27] and the references therein. Papers [23, 27] deal with delay independent criteria of the asymptotical stability.

3.1 Studying the stability of solutions. Second Lyapunov method.

The problem of the stability of movement is a problem of technical origin. The problems of the stability of the balance of bodies or of mechanical systems were the first ones that were solved in kinetics. Once equilibrium has been determined, the question of its stability arises. Only stable states are of practical importance. These states are characterized by the fact that when such a system deflects from its equilibrium, it is automatically returned.

Stability has different meanings for different researchers. We draw the reader's attention to the following concepts of stability:

1. The notion of stability movement system which is associated with the behavior of its solutions (of the bundle of solutions). Generally accepted definition of stability in Lyapunov sense as well as asymptotic stability, stability with regard to some of the variables, and all the combinations and modifications of these definitions belongs to this class.

2. Definition of stability that is characterized by the behavior of one individual trajectory of the motion. Such is the concept of stability according to Poisson, Lagrange, and others.

3. The stability of the system as a whole, i.e. its “rigidity” with regard to the perturbations, or stability in all the trajectories of the system. The concepts such as “stiff dynamical systems”, workable, $\Omega-$stability, “robust” belong to this class.

In the classical theory of the stability of movement in the first sense (as described above), A. M. Lyapunov developed two approaches.

The first (analytic) method of Lyapunov consists in expressing the solutions of the systems in terms of power series with the given initial deviations and with the theory of eigenvalues of solutions of a system of linear approximations. This approach was used also for problems in the theory of oscillations and it has wide application in mechanics, physics, and technology.

The second (direct) Lyapunov method is based on introducing a special auxiliary function (or a functional) and on obtaining the results about the stability on the basis of the behavior of this function (or a functional) along trajectories of solutions of this system.

In the following, we will use the first concept of stability as listed above. We will consider the system of ordinary differential equations

$$\dot{y} = F\left(y, t\right), \quad y \in R^n, \quad t \geqslant t_0. \tag{2}$$

Vector function $F\left(y, t\right)$ is such that conditions of existence and uniqueness of solutions of Cauchy problem hold for $t_0 \leqslant t < +\infty$, $y \in R^n$. Let us denote by $y = \varphi\left(y_0, t_0, t\right)$ the solution of the system passing in $t = t_0$ through the point $y_0 \in R^n$, and by $y = y\left(y_0^*, t_0, t\right)$ another (perturbed) solution, which for $t = t_0$ goes through the point $y_0^* \in R^n$.

In the following we define norm $|\cdot|$ of the vector $y = \left(y_1, y_2, ..., y_n\right)^T$, as $|y| = \sqrt{\sum\limits_{i=1}^{n} y_i^2}$.

Definition 3.1 *Solution $y = \varphi\left(y_0, t_0, t\right)$ is called stable in Lyapunov sense if for arbitrary $\varepsilon > 0$ and t_0 there exists $\delta\left(\varepsilon, t_0\right) > 0$ such that for every other solution $y = y\left(y_0^*, t_0, t\right)$ of the system holds the following: if only $\left|y_0^* - y_0\right| < \delta\left(\varepsilon, t_0\right)$, then*

$$\left|y\left(y_0^*, t_0, t\right) - \varphi\left(y_0, t_0, t\right)\right| < \varepsilon,$$

for $t > t_0$.

Definition 3.2 *Let Definition* 3.1 *be valid and let*

$$\lim_{t \to +\infty} \left|y\left(y_0^*, t_0, t\right) - \varphi\left(y_0, t_0, t\right)\right| = 0.$$

Then the solution $y = \varphi\left(y_0, t_0, t\right)$ is called asymptotically stable.

Usually, we substitute for $y = \varphi\left(y_0, t_0, t\right) + x$ in the source system and we get the system $\dot{x} = f\left(x, t\right)$ of equations for the perturbed movements. Then we study not the stability of the solution $y = \varphi\left(y_0, t_0, t\right)$ of the source system, but the stability of the zero solution $x\left(t\right) \equiv 0$ of the system of perturbed equations.

For definitions of the basic theorems on stability, see [25], [26].

Let $\mathcal{D} \subset R^n$ be a domain containing the origin of coordinates. $R^+ = \{t \in R, T > 0\}$.

Theorem 3.3 *(The first Lyapunov's theorem on stability). Let in* $\mathcal{D} \times R^{+}$ *a continuously differentiable function* $V(x,t)$*, satisfying the following conditions be given:*

1. *Function* $V(x,t)$ *is positive definite, i.e. there exist a continuous function* $w(x)$*,* $w(0) = 0$*, such that for* $(x,t) \in D \times R^{+} \setminus \{(0,t_0)\}$*,* $V(x,t) \geqslant w(x) > 0$*,* $V(0,t) \equiv 0$*.*

2. *The total derivative of a function* $V(x,t)$ *along trajectories of the system* (2)

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x,t) = \frac{\partial V(x,t)}{\partial t} + \langle gradV_x(x,t), f(x,t)\rangle \leqslant 0,$$

   i.e., is a non-positive function.

Then the zero solution of the system is stable in Lyapunov sense.

Theorem 3.4 *(Second Lyapunov's theorem on asymptotical stability). Let in* $\mathcal{D} \times R^{+}$ *a continuously differentiable function* $V(x,t)$*, satisfying the following conditions be given:*

1. *Function* $V(x,t)$ *is positive definite, i.e. there exist a continuous function* $w(x)$*,* $w(0) = 0$*, such that for* $(x,t) \in D \times R^{+} \setminus \{(0,t_0)\}$*,* $V(x,t) \geqslant w(x) > 0$*,* $V(0,t) \equiv 0$*.*

2. *For function* $V(x,t)$*, the upper limit is infinitesimal, i.e. there exists a continuous function* $W(x)$*,* $W(0) = 0$*, such that* $V(x,t) \leqslant W(x)$*.*

3. *The total derivative of a function* $V(x,t)$ *along trajectories of the system* (2)

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x,t) = \frac{\partial V(x,t)}{\partial t} + \langle gradV(x,t), f(x,t)\rangle$$

   is negative definite.

Then the zero solution is asymptotically stable.

3.2 Stability of solutions of equations with delay

Formal transfer of direct Lyapunov method onto a system with delay does not pose difficulties. Analogous theorems to the ones above can be formulated as follows:

- If the differential-functional equation of perturbed movement is such that we can find a positive definite function $V(x,t)$, the total derivative of it along the trajectories of the system is a functional which is always negative or identically equals to zero, then the non perturbed movement is stable on Lyapunov.

- If the differential-functional equation of perturbed movement is such that we may find a positive definite function $V(x,t)$ with the infinitesimal upper limit and the total derivative of $V(x,t)$ along the trajectories of the system is negative definite, then the non-perturbed movement is asymptotically stable.

The problems of the applicability of the above-stated claims are essential for even the simplest scalar equations.

It was suggested that the studying solution of problem of stability of functional differential equations can be developed in two directions.

1) **Method of functionals of Lyapunov-Krasovskii.** [20, p157]. Is is advisable to take as an element of the trajectory $x(x_0(\theta_0), t_0, t)$ not the vector $\{x(x_0(\theta_0), t_0, t)\}$, itself but a vector-interval of this trajectory $\{x(x_0(\theta_0), t_0, t+\theta)\}$, $-h \leqslant \theta \leqslant 0$.

Theorem 3.5 [20, p.172] *If a differential equation with delay is such that it is possible to find a functional* $V(x(\theta), t)$ *which is positive definite, has infinitesimal upper limit, and such that the value*

$$\lim_{\Delta t \to +0} \sup \frac{V(x(x_0(\theta), t_0, t + \Delta t + \theta), t + \Delta t) - V(x(x_0(\theta_0), t_0, t + \theta), t)}{\Delta t}$$

is negative definite along trajectories of the equation, then the solution $x = 0$ *is asymptotically stable.*

2) **Method of Lyapunov functions with Razumikhin condition** .

Theorem 3.6 [28, p.36]. *If the differential-functional equation of perturbed movement is such that we may find a positive definite function* $V(x, t)$ *such that the function*

$$R(x, t) = \sup \left\{ \dot{V}(x_{t,h}, t) \,|V(x(\theta), \theta) \leqslant V(x(t), t), t - h \leqslant \theta \leqslant 0, x(t) = x \right\}$$

is always negative or identically equals to zero, then the non perturbed movement is stable according to Lyapunov.

Theorem 3.7 [28, p.40]. *If the differential-functional equation of perturbed movement is such that we may find a positive definite function with infinitesimal upper limit* $V(x, t)$ *such that*

$$R(x, t) = \sup \left\{ \dot{V}(x_{t,h}, t) \,|V(x(\theta), \theta) \leqslant V(x(t), t), t - h \leqslant \theta \leqslant 0, x(t) = x \right\}$$

is negative definite on the domain $t \geqslant T \geqslant t_0 + h$*, then the non-perturbed movement is asymptotically stable.*

4 Estimation of solutions

Dynamical systems with inaccurately defined parameters have been studied for quite a long time. A.M. Lyapunov studied in great detail the stability of solutions of differential equations with perturbation of initial data [25]. Further was introduced the notion of "stability, with permanent perturbations" [26]. Next, mathematicians began to study dynamic systems subjected to stochastic perturbations [8], [3]. Theory of stability of stochastic differential equations began to develop as a separate scientific direction (see , e.g. I.A. Dzhalladova [5], [6]).

At the end of the last century, mathematicians began to consider systems of differential equations whose parameters change in predefined intervals. A sub-discipline of the so-called interval's stability emerged. The results of V.L. Kharitonov, known as "big and small" Kharitonov's theorem [12, 13] are rather important. Unfortunately, the results for linear stationary differential equations have not been as effective in other classes of dynamical systems.

In this work we consider dynamical systems described by linear differential-difference equations of neutral type [10, 11, 14]. The study uses the second Lyapunov method, which is essentially a "robust", in the sense that if the total derivative of the Lyapunov function (or functional Lyapunov-Krasovskii) is negative definite, then a whole class of systems with allowed perturbations is stable [17]-[19].

The main findings presented in this part were published in the papers [30]-[33].

4.1 Equation of neutral type. The estimation of dynamics of solutions

We will consider differential-difference equations of neutral type with constant coefficients

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t)-dx(t-\tau)\right]=f\left(x\left(t\right),x\left(t-\tau\right)\right), \tag{3}$$

$t\geqslant 0$, $\tau>0$, $x(t)\in R^1$. We assume that the initial conditions are of the form $x(t)=\varphi(t)$, $x'\left(t\right)=\varphi'\left(t\right)$, $-\tau\leqslant t\leqslant 0$, where $\varphi\left(t\right)-$ is a continuously differentiable initial function. The solution of equation (3) is in parts continuously differentiable function $x\left(t\right)$, which might have jumps in derivation at points $t=k\tau$, $k=0,1,2,...$, and satisfies initial conditions $x(t)=\varphi(t)$, $x'\left(t\right)=\varphi'\left(t\right)$, $-\tau\leqslant t\leqslant 0$.

Here and later we will use the following vector and matrix norms

$$\|x(t)\|:=\sqrt{\sum_{i=1}^{n}x_i^2(t)},\quad \|x(t)\|_\tau=\max_{-\tau\leqslant s\leqslant 0}\left\{|x(s+t)|\right\},$$

$$\|x(t)\|_{\tau,\beta}=\left\{\int\limits_{t-\tau}^{t}\mathrm{e}^{-\beta(t-s)}x^2\left(s\right)\mathrm{d}s\right\}^{\frac{1}{2}},\quad \|\dot{x}(t)\|_{\tau,\beta}=\left\{\int\limits_{t-\tau}^{t}\mathrm{e}^{-\beta(t-s)}\dot{x}^2\left(s\right)\mathrm{d}s\right\}^{\frac{1}{2}},$$

where β is a parameter.

One of the commonly used methods of obtaining estimations of perturbation solutions of functional-differential equations is Lyapunov-Krasovskii functionals method [20]. The most commonly used functional is

$$V\left[x(t)\right]=\left[x(t)-dx(t-\tau)\right]^2+\int\limits_{t-\tau}^{t}gx^2(s)\mathrm{d}s.$$

The constant $g>0$ is selected on the basis of conditions for negative definiteness of the functional along the trajectory of solutions to the equation. However, this functional will only allow us to arrive to the assertion on asymptotic stability of the solution [19, 20]. At the same time, the estimate of the solution is a significant part of the task. Therefore we will use Lyapunov-Krasovskii functional of the quadratic form both from the current coordinates and from its derivatives [30]-[32], [34]-[37]. The study of unstable systems will use the exponential multiplier.

Definition 4.1 *A zero solution of the equation of neutral type is called exponentially stable in the metric C^0 if there exist constants $N_i>0$, $i=1,2$ and $\gamma>0$ such that for any solution $x(t)$ of equation for $t>0$ the following inequality holds*

$$|x\left(t\right)|\leqslant\left[N_1\left\|x\left(0\right)\right\|_\tau+N_2\left\|\dot{x}\left(0\right)\right\|_\tau\right]\exp\left\{-\frac{1}{2}\gamma t\right\},\quad t>0.$$

Since the derivative $\dot{x}(t)$ of the solution at zero can have a jump of the first kind, the derivative at zero means a right-hand derivative.

Definition 4.2 *A zero solution of the equation of neutral type is called exponentially stable in the metric C^1 if it is stable in metric C^0 and there exist constants $R_i > 0$, $i = 1, 2$ and $\varsigma > 0$ such that for any solution $x(t)$ of equation for $t > 0$, the following inequality holds*

$$|\dot{x}(t)| \leqslant [R_1 \|x(0)\|_\tau + R_2 \|\dot{x}(0)\|_\tau] \exp\left\{-\frac{1}{2}\varsigma t\right\}, \quad t > 0.$$

4.2 Estimations of convergence of solutions of scalar equations

Consider a linear scalar equation

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - dx(t-\tau)] = -ax(t) + bx(t-\tau), \quad t \geqslant 0, \tag{4}$$

defined on interval $0 \leqslant t \leqslant m\tau$, where $m > 1$ is an integer, and $a, b \in R$. For the study we will use the functional of the type

$$V_0[x(t), t] = x^2(t) + \int\limits_{t-\tau}^{t} \mathrm{e}^{-\beta(t-s)} \left\{g_1 x^2(s) + g_2 \dot{x}^2(s)\right\} \mathrm{d}s, \; g_1 > 0, \; g_2 > 0, \; \beta > 0. \tag{5}$$

We denote

$$S[\beta, g_1, g_2] = \begin{bmatrix} 2a - g_1 - a^2 g_2 & -b(1 - ag_2) & -d(1 - ag_2) \\ -b(1 - ag_2) & \mathrm{e}^{-\beta\tau} g_1 - b^2 g_2 & -bdg_2 \\ -d(1 - ag_2) & -bdg_2 & \left(\mathrm{e}^{-\beta\tau} - d^2\right) g_2 \end{bmatrix}.$$

Theorem 4.3 *Let parameters a, b, d of equation (4) and constants $g_1 > 0$, $g_2 > 0$, $\beta > 0$ be such that the matrix $S[\beta, g_1, g_2]$ is positive definite. Then the zero solution of equation (4) is exponentially stable in the metric C^1.*

At the same time, the following estimates of the convergence hold for the solution $x(t)$, $(m-1)\tau \leq t \leq m\tau$:

$$|x(t)| \leqslant [(1 + \tau\sqrt{g_1}) \|x(0)\|_\tau + \tau\sqrt{g_2} \|\dot{x}(0)\|_\tau] \mathrm{e}^{-\frac{1}{2}\gamma t},$$

$$|\dot{x}(t)| \leqslant \left[\left(\frac{|b|}{|d|} + M(1 + \tau\sqrt{g_1})\right) \|x(0)\|_\tau + (1 + M\tau\sqrt{g_2}) \|\dot{x}(0)\|_\tau\right] \mathrm{e}^{-\frac{1}{2}\varsigma t},$$

where $M = \dfrac{|ad - b|}{1 - |d|\,\mathrm{e}^{\frac{1}{2}\gamma\tau}} \mathrm{e}^{\frac{1}{2}\gamma\tau} \left[1 - \left(|d|\,\mathrm{e}^{\frac{1}{2}\gamma\tau}\right)^{m-1}\right]$,

$$\gamma = \min\left\{\lambda_{\min}(S[\beta, g_1, g_2]), \beta\right\}, \quad \varsigma = \min\left\{\frac{2}{\tau} \ln \frac{1}{|d|}, \gamma\right\}.$$

We will get the estimation of the behavior of solutions to the equation (4) without any assumptions about its stability. To obtain these estimates, we will use non-autonomous Lyapunov-Krasovskii functional of the following type

$$V[x(t), t] = \mathrm{e}^{\gamma t}\left\{x^2(t) + \int\limits_{t-\tau}^{t} \mathrm{e}^{-\beta(t-s)} \left[g_1 x^2(s) + g_2 \dot{x}^2(s)\right] \mathrm{d}s\right\}.$$

This functional includes an exponential multiplier, which allows us to assess the estimate of the "divergence" of the solution even in the case of instability. Denote

$$S\left[\beta, g_1, g_2, \gamma\right] = \left[\begin{array}{lll} 2a - g_1 - a^2 g_2 - \gamma & -b\left(1 - a g_2\right) & -d\left(1 - a g_2\right) \\ -b\left(1 - a g_2\right) & \mathrm{e}^{-\beta\tau} g_1 - b^2 g_2 & -b d g_2 \\ -d\left(1 - a g_2\right) & -b d g_2 & \left(\mathrm{e}^{-\beta\tau} - d^2\right) g_2 \end{array}\right].$$

Theorem 4.4 *Let there exist constants $\beta > 0$, $g_1 > 0$, $g_2 > 0$, γ such that the matrix $S\left[\beta, g_1, g_2, \gamma\right]$ be positive definite. Then for any solution $x(t)$, $(m-1)\tau \leqslant t \leqslant m\tau$, the following estimates hold*

$$|x(t)| \leqslant \left[\left(1 + \tau\sqrt{g_1}\right)\|x(0)\|_\tau + \tau\sqrt{g_2}\,\|\dot{x}(0)\|_{\tau,\beta}\right]\mathrm{e}^{-\frac{1}{2}(\xi+\gamma)t},$$

$$|\dot{x}(t)| \leqslant \left[\left(\frac{|b|}{|d|} + M\left(1 + \tau\sqrt{g_1}\right)\right)\|x(0)\|_\tau + \left(1 + M\tau\sqrt{g_2}\right)\|\dot{x}(0)\|_\tau\right]\mathrm{e}^{-\frac{1}{2}\varsigma t},$$

$$M = \frac{|ad - b|}{1 - |d|\mathrm{e}^{\frac{1}{2}\gamma\tau}}\mathrm{e}^{\frac{1}{2}\gamma\tau\left[1 - \left(|d|\mathrm{e}^{\frac{1}{2}\gamma\tau}\right)^{m-1}\right]},$$

$$\xi = \min\left\{\lambda_{\min}\left(S\left[\beta, g_1, g_2\right]\right), (\beta - \gamma)\right\}, \quad \varsigma = \min\left\{\frac{2}{\tau}\ln\frac{1}{|d|}, \gamma\right\}.$$

4.3 Systems of neutral type

We will consider linear systems of neutral differential equations with constant coefficients and a constant delay

$$\dot{x}(t) = D\dot{x}(t-\tau) + Ax(t) + Bx(t-\tau) \tag{6}$$

where $t \geq 0$ is an independent variable, $\tau > 0$ is a constant delay, A, B and D are $n \times n$ constant matrices, and $x\colon [-\tau, \infty) \to \mathbb{R}^n$ is a column vector-solution. The sign " $\cdot$ " denotes the left-hand derivative. Let $\varphi\colon [-\tau, 0] \to \mathbb{R}^n$ be a continuously differentiable vector-function. The solution $x = x(t)$ of problem (6), (7) on $[-\tau, \infty)$ where

$$x(t) = \varphi(t),\ \dot{x}(t) = \dot{\varphi}(t),\ t \in [-\tau, 0] \tag{7}$$

is defined in the classical sense (we refer, e.g. to [14]) as a function continuous on $[-\tau, \infty)$, continuously differentiable on $[-\tau, \infty)$ except for points τp, $p = 0, 1, \dots$, and satisfying the equation (6) everywhere on $[0, \infty)$ except for points τp, $p = 0, 1, \dots$.

Let $\mathcal{F}$ be a rectangular matrix. We will use the matrix norm $\|\mathcal{F}\| := \sqrt{\lambda_{\max}(\mathcal{F}^T\mathcal{F})}$ where the symbol $\lambda_{\max}(\mathcal{F}^T\mathcal{F})$ denotes the maximal eigenvalue of the corresponding square symmetric positive semi-definite matrix $\mathcal{F}^T\mathcal{F}$.

The most frequently used method for investigating the stability of functional-differential systems is the method of Lyapunov-Krasovskii functionals [20, 21].

We will use Lyapunov-Krasovskii quadratic type functionals of the dependent coordinates and their derivatives

$$V_0[x(t), t] = x^T(t)Hx(t) + \int\limits_{t-\tau}^{t} \mathrm{e}^{-\beta(t-s)}\left[x^T(s)\,G_1 x(s) + \dot{x}^T(s) G_2 \dot{x}(s)\right]\mathrm{d}s \tag{8}$$

and $V[x(t),t] = \mathrm{e}^{pt} V_0[x(t),t]$, i.e.,

$$V[x(t),t] = \mathrm{e}^{pt}\left[x^T(t)Hx(t) + \int_{t-\tau}^{t} \mathrm{e}^{-\beta(t-s)}\left[x^T(s)G_1x(s) + \dot{x}^2(s)G_2\dot{x}^2(s)\right]\mathrm{d}s\right] \tag{9}$$

where x is a solution of (6), β and p are real parameters, the $n \times x$ matrices H, G_1 and G_2 are positive definite, and $t > 0$.

To the best of our knowledge, the general functionals (8), (9) have not yet been applied as suggested to the study of stability and estimates of solutions of (6).

Definition 4.5 *The zero solution of the system of equations of neutral type* (6) *is called exponentially stable in the metric C^0 if there exist constants $N_i > 0$, $i = 1, 2$ and $\mu > 0$ such that, for an arbitrary solution $x = x(t)$ of* (6), *the inequality*

$$\|x(t)\| \le \left[N_1 \|x(0)\|_\tau + N_2 \|\dot{x}(0)\|_\tau\right] \mathrm{e}^{-\mu t}$$

holds for $t > 0$.

Definition 4.6 *The zero solution of the system of equations of neutral type* (6) *is called exponentially stable in the metric C^1 if it is stable in the metric C^0 and, moreover, there exist constants $R_i > 0$, $i = 1, 2$ and $\nu > 0$ such that, for an arbitrary solution $x = x(t)$ of* (6), *the inequality*

$$\|\dot{x}(t)\| \le \left[R_1 \|x(0)\|_\tau + R_2 \|\dot{x}(0)\|_\tau\right] \mathrm{e}^{-\nu t}$$

holds for $t > 0$.

We will give estimates of solutions of the linear system (6) on the interval $(0, \infty)$ using the functional (8).

We will use an auxiliary $3n \times 3n$-dimensional matrix

$$S = S(\beta, G_1, G_2, H) := \begin{pmatrix} -A^TH - HA - G_1 - A^TG_2A & -HB - A^TG_2B & -HD - A^TG_2D \\ -B^TH - B^TG_2A & \mathrm{e}^{-\beta\tau}G_1 - B^TG_2B & -B^TG_2D \\ -D^TH - D^TG_2A & -D^TG_2B & \mathrm{e}^{-\beta\tau}G_2 - D^TG_2D \end{pmatrix}$$

depending on the parameter β and the matrices G_1, G_2, H. Next we will use the numbers

$$\varphi(H) := \frac{\lambda_{\max}(H)}{\lambda_{\min}(H)}, \quad \varphi_1(G_1, H) := \frac{\lambda_{\max}(G_1)}{\lambda_{\min}(H)}, \quad \varphi_2(G_2, H) := \frac{\lambda_{\max}(G_2)}{\lambda_{\min}(H)}.$$

Theorem 4.7 *Let there exist a parameter $\beta > 0$ and positive definite matrices G_1, G_2, H such that matrix S is also positive definite. Then the zero solution of system* (6) *is exponentially stable in the metric C^0. Moreover, for the solution $x = x(t)$ of* (6), (7) *the inequality*

$$\|x(t)\| \le \left[\sqrt{\varphi(H)}\|x(0)\| + \sqrt{\tau\varphi_1(G_1, H)}\,\|x(0)\|_\tau + \sqrt{\tau\varphi_2(G_2, H)}\,\|\dot{x}(0)\|_\tau\right] \mathrm{e}^{-\gamma t/2}$$

holds on $(0, \infty)$ where $\gamma \le \gamma_0 := \min\left(\beta, \dfrac{\lambda_{\min}(S)}{\lambda_{\max}(H)}\right)$.

Theorem 4.8 *Let the matrix D be nonsingular and $\|D\| < 1$. Let the assumptions of Theorem 4.7 with $\gamma < (2/\tau)\ln(1/\|D\|)$ and $\gamma \le \gamma_0$ be true. Then the zero solution of system (6) is exponentially stable in the metric C^1. Moreover, for a solution $x = x(t)$ of (6), (7), the inequality*

$$\|\dot{x}(t)\| \le \left[\left(\frac{\|B\|}{\|D\|} + M\left(\sqrt{\varphi(H)} + \sqrt{\tau\varphi_1(G_1, H)}\right)\right)\|x(0)\|_\tau \right.$$

$$\left. + \left(1 + M\sqrt{\tau\varphi_2(G_2, H)}\right)\|\dot{x}(0)\|_\tau\right] \mathrm{e}^{-\gamma t/2}$$

where

$$M = \|A\| + \|DA + B\|\mathrm{e}^{\gamma\tau/2}(1 - \|D\|\mathrm{e}^{\gamma\tau/2})^{-1} \tag{10}$$

holds on $(0, \infty)$.

Now we will estimate the norms of solutions of (6) and the norms of their derivatives in the case of the assumptions of Theorem 4.7 or Theorem 4.8 being not necessarily satisfied. It means that the estimates derived will cover the case of instability as well. For obtaining such type of results we will use a functional of Lyapunov-Krasovskii in the form (9). This functional includes an exponential factor, which makes it possible, in the case of instability, to get an estimate of the "divergence" of solutions. Functional (9) is a generalization of (8) because the choice $p = 0$ gives $V[x(t), t] = V_0[x(t), t]$.

We define an auxiliary $3n \times 3n$ matrix

$$S^* = S^*(\beta, G_1, G_2, H, p)$$

$$:= \begin{pmatrix} -A^TH - HA - G_1 - A^TG_2A - pH & -HB - A^TG_2B & -HD - A^TG_2D \\ -B^TH - B^TG_2A & \mathrm{e}^{-\beta\tau}G_1 - B^TG_2B & -B^TG_2D \\ -D^TH - D^TG_2A & -D^TG_2B & \mathrm{e}^{-\beta\tau}G_2 - D^TG_2D \end{pmatrix}$$

depending on the parameters p, β and the matrices G_1, G_2 and H. The parameter p plays a significant role for the positive definiteness of the matrix S^*. Particularly, a proper choice of $p \ll 0$ can cause the positivity of S^*.

Theorem 4.9 A) *Let p be a fixed real number, β a positive constant and G_1, G_2, H positive definite matrices such that the matrix S^* is also positive definite. Then a solution $x = x(t)$ of problem (6), (7) satisfies on $(0, \infty)$ the inequality*

$$\|x(t)\| \le \left[\sqrt{\varphi(H)}\,\|x(0)\| + \sqrt{\tau\varphi_1(G_1, H)}\,\|x(0)\|_\tau + \sqrt{\tau\varphi_2(G_2, H)}\,\|\dot{x}(0)\|_\tau\right]\mathrm{e}^{-\gamma t/2}$$

where $\gamma \le \gamma^* := \min\left(\beta, p + \dfrac{\lambda_{\min}(S^*)}{\lambda_{\max}(H)}\right)$.

B) *Let the matrix D be nonsingular and $\|D\| < 1$. Let all the assumptions of part A) with $\gamma < (2/\tau)\ln(1/\|D\|)$ and $\gamma \le \gamma^*$ be true. Then the derivative of the solution $x = x(t)$ of problem (6), (7) satisfies on $(0, \infty)$ the inequality*

$$\|\dot{x}(t)\| \le \left[\left(\frac{\|B\|}{\|D\|} + M\left(\sqrt{\varphi(H)} + \sqrt{\tau\varphi_1(G_1, H)}\right)\right)\|x(0)\|_\tau + \right.$$

$$\left. + \left(1 + M\sqrt{\tau\varphi_2(G_2, H)}\right)\|\dot{x}(0)\|_\tau\right]\mathrm{e}^{-\gamma t/2}$$

where M is defined by (10).

Example 4.10 *We will investigate system* (6) *where* $n = 2$, $\tau = 1$,

$$D = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, \qquad A = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0.6213 \\ 0.6213 & 0 \end{pmatrix},$$

i.e., the system

$$\dot{x}_1(t) = 0.1\dot{x}_1(t-1) - 3x_1(t) - 2x_2(t) + 0.6213x_2(t-1), \tag{11}$$

$$\dot{x}_2(t) = 0.1\dot{x}_2(t-1) + 1x_1(t) \qquad\qquad + 0.6213x_1(t-1), \tag{12}$$

with initial conditions (7)*. Set* $\beta = 0.1$ *and*

$$G_1 = \begin{pmatrix} 0.5 & 0.1 \\ 0.1 & 0.1 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, \qquad H = \begin{pmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{pmatrix}.$$

For the eigenvalues of matrices G_1, G_2 *and* H, *we get* $\lambda_{\min}(G_1) \doteq 0.0764$, $\lambda_{\max}(G_1) \doteq 0.5236$, $\lambda_{\min}(G_2) = \lambda_{\max}(G_2) = 0.1$ $\lambda_{\min}(H) = 0.2$, $\lambda_{\max}(H) = 1$. *The matrix* $S = S(\beta, G_1, G_2, H)$ *takes the form*

$$S \doteq \begin{pmatrix} 1.3000 & 1.1000 & -0.3106 & -0.1864 & -0.0300 & -0.0500 \\ 1.1000 & 1.1000 & -0.3728 & -0.1243 & -0.0200 & -0.0600 \\ -0.3106 & -0.3728 & 0.4138 & 0.0905 & 0 & -0.0062 \\ -0.1864 & -0.1243 & 0.0905 & 0.0519 & -0.0062 & 0 \\ -0.0300 & -0.0200 & 0 & -0.0062 & 0.0895 & 0 \\ -0.0500 & -0.0600 & -0.0062 & 0 & 0 & 0.0895 \end{pmatrix}$$

and $\lambda_{\min}(S) \doteq 0.00001559$. *Because all eigenvalues are positive, matrix* S *is positive definite. Since all conditions of Theorem 4.7 are satisfied, the zero solution of system* (11), (12) *is asymptotically stable in the metric* C^0. *Further we have*

$$\varphi(H) = \frac{1}{0.2} = 5, \;\; \varphi_1(G_1, H) \doteq \frac{0.5236}{0.2} \doteq 2.618, \;\; \varphi_2(G_2, H) = \frac{0.1}{0.2} = 0.5,$$

$$\gamma_0 \doteq \min(0.1,\, 0.00001559) = 0.00001559,$$

$$\|A\| \doteq 3.7025, \;\; \|B\| \doteq 0.6213, \;\; \|D\| = 0.1, \;\; \|DA + B\| \doteq 0.8028, \;\; M \doteq 4.5945.$$

Since $\gamma_0 < (2/\tau)\ln(1/\|D\|) = 2\ln 10 \doteq 4.6052$, *all conditions of Theorem 4.8 are satisfied and, consequently, the zero solution of* (11), (12) *is asymptotically stable in the metric* C^1. *Finally, from* (4.7) *and* (**??**) *follows that the inequalities*

$$\begin{aligned} \|x(t)\| &\le \left[\sqrt{5}\|x(0)\| + \sqrt{2.618}\,\|x(0)\|_1 + \sqrt{0.5}\;\|\dot{x}(0)\|_1\right] \mathrm{e}^{-0.00001559t/2} \\ &\doteq [2.2361\|x(0)\| + 1.6180\,\|x(0)\|_1 + 0.7071\;\|\dot{x}(0)\|_1]\,\mathrm{e}^{-0.00001559t/2}, \\ \|\dot{x}(t)\| &\le \left[\left(6.213 + 4.5945\left(\sqrt{5} + \sqrt{2.618}\right)\right)\;\|x(0)\|_1 \right. \\ &\quad \left. + \left(1 + 4.5945\sqrt{0.5}\right)\|\dot{x}(0)\|_1\right] \mathrm{e}^{-0.00001559t/2} \\ &\doteq [23.9206\,\|x(0)\|_1 + 4.2488\,\|\dot{x}(0)\|_1]\,\mathrm{e}^{-0.00001559t/2} \end{aligned}$$

hold on $(0, \infty)$.

4.4 Equations with special type of nonlinearity

We will consider nonlinear differential-differential equations of neutral type

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x\left(t\right)-dx\left(t-\tau\right)\right]=ax\left(t\right)+bx\left(t-\tau\right)+f\left(x\left(t\right)\right),\quad t\geqslant 0, \tag{13}$$

where $f\left(x\right)-$ continuous function satisfying the Lipschitz condition and the so-called "sector condition"

$$\left[kx-f\left(x\right)\right]\sigma>0,\quad k>0. \tag{14}$$

As a solution we will understand a piecewise continuously differentiable function $x\left(t\right)$, satisfying identically equation (13) and initial conditions $x\left(t\right)=\varphi\left(t\right)$, $x'\left(t\right)=\psi\left(t\right)$, where $\varphi\left(t\right)$, $\psi\left(t\right)$ are arbitrary continuous functions defined on interval $-\tau\leqslant t\leqslant 0$.

Definition 4.11 *A zero solution to the equation* (13) *is called exponentially stable in the metric C^0, if there exist constant $N_i>0$, $i=1,2$ and $\gamma>0$ such that for arbitrary solution $x(t)$ of equation* (13) *for $t>0$ the following inequality holds*

$$\left|x\left(t\right)\right|\leqslant\left[N_1\left\|x\left(0\right)\right\|_\tau+N_2\left\|\dot{x}\left(0\right)\right\|_\tau\right]\exp\left\{-\frac{1}{2}\gamma t\right\},\quad t>0.$$

Definition 4.12 *A zero solution to the equation* (13) *is called exponentially stable in the metric C^1, if it is stable in metric C^0 and there exist constants $R_i>0$, $i=1,2$ and $\eta>0$ such that for arbitrary solution $x(t)$ of equation* (13) *for $t>0$, the following inequality holds*

$$\left|\dot{x}\left(t\right)\right|\leqslant\left[R_1\left\|x(0)\right\|_\tau+R_2\left\|\dot{x}\left(0\right)\right\|_\tau\right]\exp\left\{-\frac{1}{2}\eta t\right\},\quad t>0.$$

To obtain conditions of stability, we will be using the functional of Lyapunov-Krasovskii of the quadratic type which will depend both on the current coordinates and the derivatives

$$V\left[x(t)\right]=x^2(t)+\int\limits_{t-\tau}^{t}\mathrm{e}^{-\varsigma(t-s)}\left\{g_1x^2\left(s\right)+g_2\left(\dot{x}\left(s\right)\right)^2\right\}\mathrm{d}s\quad+\beta\int\limits_{0}^{x(t)}f\left(s\right)\mathrm{d}s,$$

$$\beta>0,\quad g_1>0,\quad g_2>0. \tag{15}$$

We denote

$$S_1\left[\beta,\varsigma,\nu,g_1,g_2\right]=\begin{bmatrix} s_{11}^1 & s_{12}^1 & s_{13}^1 & s_{14}^1\\ s_{12}^1 & s_{22}^1 & 0 & s_{24}^1\\ s_{13}^1 & 0 & s_{33}^1 & s_{34}^1\\ s_{14}^1 & s_{24}^1 & s_{34}^1 & s_{44}^1\end{bmatrix},$$

$$\begin{gathered} s_{11}^1=-2a-g_1-a^2g_2,\quad s_{12}^1=-b-abg_2,\quad s_{13}^1=-d-\left(a+b\right)d,\\ s_{14}^1=-1-abg_2-\frac{1}{2}\left(\beta a+\nu\right),s_{22}^1=\mathrm{e}^{-\varsigma\tau}g_1-b^2g_2,s_{24}^1=-b\left(g_2+\frac{1}{2}\beta\right),\\ s_{33}^1=\mathrm{e}^{-\varsigma\tau}g_2-d^2g_2,\quad s_{34}^1=-d\left(g_2+\frac{1}{2}\beta\right),\quad s_{44}^1=-g_2-\beta+\frac{1}{k}\nu, \end{gathered} \tag{16}$$

where $\nu>0$ is a positive constant.

Theorem 4.13 *Let $|d| < 1$, and there exist constants $g_1 > 0$, g_2, $\beta > 0$, $\varsigma > 0$, $\nu > 0$, such that the matrix $S_1[\beta, \varsigma, \nu, g_1, g_2]$ be positive definite. Then the zero solution of equation* (13) *is asymptotically stable in the metric C^1.*

For arbitrary solution $x(t)$, $t > 0$ the following estimates of convergence hold:

$$|x(t)| \leqslant \left[\sqrt{1 + \frac{1}{2}\beta k}\, |x(0)| + \sqrt{g_1}\, \|x(0)\|_{\tau,\varsigma} + \sqrt{g_2\, \|\dot{x}(0)\|_{\tau,\varsigma}}\right] \mathrm{e}^{-\frac{1}{2}\gamma t},$$

$$|\dot{x}(t)| \leqslant \left\{ M\sqrt{1 + \frac{1}{2}\beta k}\, |x(0)| + \left[M\tau\sqrt{g_1} + \frac{|b|}{|d|}\right] \|x(0)\|_{\tau} \right. \tag{17}$$

$$\left. + \left(1 + M\tau\sqrt{g_2}\right) \|\dot{x}(0)\|_{\tau} \right\} \mathrm{e}^{-\frac{1}{2}\gamma t},$$

$$M = [|a| + |a|\, k] \frac{[|ad + b| + |d|\, k]}{|d| \left[1 - |d|\, e^{\frac{1}{2}\gamma\tau}\right]}, \qquad \gamma < \min\left\{\varsigma, \frac{2}{\tau}\ln\frac{1}{|d|}, \frac{\lambda_{\min}\left(S_1[\beta, \varsigma, \nu, g_1, g_2]\right)}{1 + \frac{1}{2}\beta k}\right\}.$$

As a rule, in the practical problems we do not known precisely the parameters of the model. We will study the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - dx(t - \tau)\right] = (a + \Delta a)\, x(t) + (b + \Delta b)\, x(t - \tau) + f(x(t)), \tag{18}$$

where parameters Δa and Δb take their values from intervals

$$|\Delta a| \leqslant \alpha, \quad |\Delta b| \leqslant \beta, \quad \alpha, \beta \in R^{+}. \tag{19}$$

Nonlinear function $f(\sigma)$, as in the previous part, satisfies condition (14).

Definition 4.14 *A zero solution of equation* (13) *is called intervals stable if it is exponentially stable for an arbitrary function $f(x)$, which satisfies the "sector condition"* (15), *and perturbations Δa and Δb satisfy the condition* (19).

Theorem 4.15 *Let $|d| < 1$ and let there exist $g_1 > 0$, $g_2 > 0$, $\beta > 0$, $\varsigma > 0$, $\nu > 0$ such that the matrix $S_1[\beta, \varsigma, \nu, g_1, g_2]$ is positive definite and the following inequalities hold:*

$$|\Delta a| \leqslant \frac{1}{R_2}\left[\sqrt{[1 + |ag_2|]^2 + (1 - \xi^2)(1 - \eta^2)\, \lambda_{\min}\left(S_1[\beta, \varsigma, \nu, g_1, g_2]\right) R_2}\right.$$

$$\left. - [1 + |ag_2|]\right],$$

$$|\Delta b| \leqslant \min\left\{\sqrt{\frac{1 - \xi^2}{2} \frac{\lambda_{\min}\left(S_1[\beta, \varsigma, \nu, g_1, g_2]\right)}{|d|}}\eta,\right.$$

$$\left. \frac{1}{R_1}\left[\sqrt{|g_2 b|^2 + (1 - \xi^2)\, \lambda_{\min}\left(S_1[\beta, \varsigma, \nu, g_1, g_2]\right) R_1} - |g_2 b|\right]\right\},$$

where

$$R_1 = \frac{|1 + g_2 b| + 2\xi^2 \left|+\frac{1}{2}\beta\right|^2}{\xi^2 \lambda_{\min}\left(S_1[\beta, \varsigma, \nu, g, g_2]\right)} + |g_2|\left(1 + \frac{1}{\alpha^2}\right),$$

$$R_2 = \frac{|g_2 b| + 2\xi^2 \left(|d|^2 + \left| g_2 b + \frac{1}{2}\beta \right|^2 \right)}{\xi^2 \lambda_{\min} \left(S_1 \left[\beta, \varsigma, \nu, g_1, g_2 \right] \right)} + |g_2| + \alpha^2,$$

and $0 < \xi < 1$, $0 < \eta < 1$, α *are arbitrary constants.*

Then the zero solution of (18) *is absolutely interval stable in the metric* C^1. *For arbitrary solution* $x\,(t)$, $t > 0$ *the following estimates of convergence hold:*

$$|x\,(t)| \leqslant \left[\sqrt{1 + \frac{1}{2}\beta k}\, |x\,(0)| + \sqrt{g_1}\, \|x\,(0)\|^2_{\tau,\varsigma} + \sqrt{g_2}\, \|\dot{x}\,(0)\|^2_{\tau,\varsigma} \right] \mathrm{e}^{-\frac{1}{2}\gamma t},$$

$$|\dot{x}\,(t)| \leqslant \left\{ M \sqrt{1 + \frac{1}{2}\beta k}\, |x\,(0)| + \left[M\tau\sqrt{g_1} + \frac{|b + \Delta b|}{|d|} \right] \|x\,(0)\|_\tau \right.$$

$$\left. + \left(1 + M\tau\sqrt{g_2}\right) \|\dot{x}\,(0)\|_\tau \right\} \mathrm{e}^{-\frac{1}{2}\gamma t},$$

$$M = \left[|a + \Delta a| + |d|\, k \right] + \frac{\left[|d\,(a + \Delta a) + (b + \Delta b)| + |d|\, k \right]}{|d| \left[1 - |d|\, \mathrm{e}^{\frac{1}{2}\gamma\tau} \right]},$$

$$\gamma < \min \left\{ \gamma, \frac{2}{\tau} \ln \frac{1}{|d|}, \frac{\theta\,[\cdot]}{1 + \frac{1}{2}\beta k} \right\},$$

$$\theta\,[\cdot] \leqslant \left(1 - \xi^2\right) \left(1 - \eta^2\right) \lambda_{\min} \left(S_1 \left[\beta, \varsigma, \nu, g_1, g_2 \right] \right) - 2\left[1 + |a g_2| \right] |\Delta a|$$

$$- \left[|g_2| + \frac{|g_2 b|^2 + 2\xi^2 \left(|d|^2 + \left| g_2 + \frac{1}{2}\beta \right|^2 \right)}{\xi^2 \lambda_{\min} \left(S_1 \left[\beta, \varsigma, \nu, g_1, g_2 \right] \right)} + \alpha^2 \right] |\Delta a|^2.$$

4.5 Estimates of convergence of solutions of systems of nonlinear equations of neutral type

We will consider systems of nonlinear differential-difference equations with delayed argument of neutral type

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[x\,(t) - Dx\,(t - \tau) \right] = Ax\,(t) + Bx\,(t - \tau) + bf\,(\sigma\,(t)), \tag{20}$$

$$\sigma\,(t) = c^T x\,(t), \quad t \geqslant 0.$$

Here, $x\,(t) \in R^n$, A, B, D are square matrices with constant coefficients, $b, c \in R^n$, $\tau > 0$, $f\,(\sigma)$ is continuous function satisfying the Lipschitz condition, and $f\,(0) = 0$. As a solution of the system we will understand a piecewise continuous differentiable function $x\,(t)$, that would identically satisfy the system (20) and initial conditions $x\,(t) = \varphi\,(t)$, $x'\,(t) = \varphi'\,(t)$, where $\varphi\,(t)$ is an arbitrary continuous function, defined on $-\tau \leqslant t \leqslant 0$.

Definition 4.16 *A zero solution of the system* (20) *of neutral type is called exponentially stable in the metric* C^0, *if there exist constants* $N_i > 0$, $i = 1, 2$ *and* $\gamma > 0$ *such that for an arbitrary solution* $x(t)$ *the following inequality holds*

$$|x\,(t)| \leqslant \left[N_1 \|x\,(0)\|_\tau + N_2 \|\dot{x}\,(0)\|_\tau \right] \exp\left\{ -\frac{1}{2}\gamma t \right\}, \quad t > 0.$$

Definition 4.17 *A zero solution of the system* (20) *of neutral type is called exponentially stable in the metric C^1, if it is stable in metric C^0 and if there exist constants $R_i > 0$, $i = 1, 2$ and $\eta > 0$ such that for arbitrary solution $x(t)$ for $t > 0$ the following inequality holds*

$$|\dot{x}(t)| \leqslant [R_1 \|x(0)\|_\tau + R_2 \|\dot{x}(0)\|_\tau] \exp\left\{-\frac{1}{2}\eta t\right\}, \quad t > 0.$$

Definition 4.18 *System* (20) *is called absolutely stable, if its zero solution is exponentially stable for arbitrary function $f(\sigma)$, satisfying the "sector condition"*

$$[k\sigma - f(\sigma)]\,\sigma > 0, \quad k > 0. \tag{21}$$

In this part we will use Lyapunov-Krasovskii functional of the quadratic type from the current coordinates as well as their derivatives

$$V[x(t)] = x^T(t)\,Hx(t) + \int\limits_{t-\tau}^{t} \mathrm{e}^{-\varsigma(t-s)} \left\{x^T(s)\,G_1 x(s) + \dot{x}^T(s)\,G_2 \dot{x}(s)\right\} \mathrm{d}s$$

$$+\beta \int\limits_{0}^{\sigma(t)} f(\sigma)\,\mathrm{d}\sigma, \;\; \sigma(t) = c^T x(t), \quad \beta > 0$$

with positive definite matrices H, G_1, G_2.

Denote

$$S_1[\beta, \varsigma, \nu, H, G_1, G_2] = \begin{bmatrix} S_{11}^1 & S_{12}^1 & S_{13}^1 & S_{14}^1 \\ (S_{12}^1)^T & S_{22}^1 & \Theta & S_{24}^1 \\ (S_{13}^1)^T & \Theta & S_{33}^1 & S_{34}^1 \\ (S_{14}^1)^T & (S_{24}^1)^T & (S_{34}^1)^T & S_{44}^1 \end{bmatrix},$$

$$S_{11}^1 = -A^T H - HA - G_1 - A^T G_2 A, \quad S_{12}^1 = -HB - A^T G_2 B,$$

$$S_{13}^1 = -HD - (A+B)^T D, \quad S_{14}^1 = -Hb - A^T G_2 b - \frac{1}{2}\left(\beta A^T + \nu I\right) c,$$

$$S_{22}^1 = \mathrm{e}^{-\varsigma\tau} G_1 - B^T G_2 B, \quad S_{24}^1 = -B^T \left(G_2 b + \frac{1}{2}\beta c\right), \quad S_{33}^1 = \mathrm{e}^{-\varsigma\tau} G_2 - D^T G_2 D,$$

$$S_{34}^1 = -D^T \left(G_2 b + \frac{1}{2}\beta c\right), \quad S_{44}^1 = -b^T G_2 b - \beta c^T b + \frac{1}{k}\nu,$$

$\nu > 0$, Θ is $n \times n$ zero matrix,

$$\varphi_{11}(H) = \frac{\lambda_{\max}(H) + \frac{1}{2}\beta k\,|c|^2}{\lambda_{\min}(H)}, \quad \varphi_{12}(H, G_1) = \frac{\lambda_{\max}(G_1)}{\lambda_{\min}(H)}, \quad \varphi_{13}(H, G_2) = \frac{\lambda_{\max}(G_2)}{\lambda_{\min}(H)}.$$

Theorem 4.19 *Let $|D| < 1$ and there exist positive definite matrices H, G_1, G_2 and parameters $\beta > 0$, $\varsigma > 0$, $\nu > 0$, such that the matrix $S_1[\beta, \varsigma, \nu, H, G_1, G_2]$ is positive definite. Then the zero solution of system (20) is absolutely stable in the metric C^1.*

For arbitrary solution $x(t)$, $t > 0$ the following estimates of convergence hold:

$$|x(t)| \leqslant \left[\sqrt{\varphi_{11}(H)}\,|x(0)| + \sqrt{\varphi_{12}(H, G_1)}\,\|x(0)\|_{\tau,\varsigma}^2\right. \tag{22}$$

$$+\sqrt{\varphi_{13}\left(H,G_2\right)}\left\|\dot{x}\left(0\right)\right\|_{\tau,\varsigma}^2\Big]\,\mathrm{e}^{-\frac{1}{2}\gamma t},$$

$$|\dot{x}\left(t\right)|\leqslant\left\{M\sqrt{\varphi_{11}(H)}\left|x\left(0\right)\right|+\left[M\tau\sqrt{\varphi_{12}\left(H,G_1\right)}+\frac{|B|}{|D|}\right]\left\|x\left(0\right)\right\|_{\tau}\right.$$

$$\left.+\left(1+M\tau\sqrt{\varphi_{13}\left(H,G_2\right)}\right)\left\|\dot{x}\left(0\right)\right\|_{\tau}\right\}\mathrm{e}^{-\frac{1}{2}\gamma t},$$

$$M=\left[|A|+|D|\,|b|\,k\,|c|\right]+\frac{\left[|DA+B|+|D|\,|b|\,k\,|c|\right]}{|D|\left[1-|D|\,\mathrm{e}^{\frac{1}{2}\gamma\tau}\right]},$$

$$\gamma<\min\left\{\varsigma,\frac{2}{\tau}\ln\frac{1}{|D|},\frac{\lambda_{\min}\left(S_1\left[\beta,\varsigma,\nu,H,G_1,G_2\right]\right)}{\lambda_{\max}\left(H\right)+\frac{1}{2}\beta k\left|c\right|^2}\right\},$$

Typically, the parameters of such systems are unknown. They take their values from some of the predefined intervals. We will consider the system of direct control, which is described by a system of differential-difference equations of neutral type with coefficients given in the interval form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x\left(t\right)-Dx\left(t-\tau\right)\right]=\left(A+\Delta A\right)x(t)+\left(B+\Delta B\right)x(t-\tau)+bf\left(\sigma(t)\right), \tag{23}$$

$$\sigma(t)=c^Tx(t),\quad x\left(t\right)\in R^n,\quad t\geqslant 0.$$

Here, matrices ΔA and ΔB may have their values in certain defined fixed intervals

$$\Delta A=\left\{\Delta a_{ij}\right\},\ \Delta B=\left\{\Delta b_{ij}\right\},\ \left|\Delta a_{ij}\right|\leqslant\alpha_{ij},\ \left|b_{ij}\right|\leqslant\beta_{ij},\ i,j=1,2,\ldots,n. \tag{24}$$

Systems of this type are called intervals systems. Non-linear function $f\left(\sigma\right)$, as in the previous part, satisfies the condition (21).

Denote

$$\left\|\Delta A\right\|=\max_{\Delta a_{ij}}\left\{\left|\Delta A\right|\right\},\quad\left\|\Delta B\right\|=\max_{\Delta a_{ij}}\left\{\left|\Delta B\right|\right\}.$$

Definition 4.20 *System* (20) *is called absolutely interval stable, if its zero solution is exponentially stable for an arbitrary function $f\left(\sigma\right)$ which satisfies "the sector condition"* (21) *and for arbitrary matrices A, B satisfying the conditions* (24).

We get conditions for absolute stability of interval system (20), similar to the ones for the systems without interval perturbations.

Denote

$$S_2\left[\beta,H,G_2\right]=\left[\begin{array}{llll}S_{11}^2 & S_{12}^2 & S_{13}^2 & S_{14}^2\\ \left(S_{12}^2\right)^T & S_{22}^2 & \Theta & S_{24}^2\\ \left(S_{13}^2\right)^T & \Theta & \Theta & \Theta\\ \left(S_{14}^2\right)^T & \left(S_{24}^2\right)^T & \Theta & \Theta\end{array}\right],$$

$$S_{11}^2=-\Delta A^TH-H\Delta A--A^TG_2\Delta A-\Delta A^TG_2A-\Delta A^TG_2\Delta A,$$

$$S_{12}^2=-H\Delta B-A^TG_2\Delta B-\Delta A^TG_2A-\Delta A^TG_2\Delta B,$$

$$S_{13}^2=-\left(\Delta A+\Delta B\right)^TD,\quad S_{14}^1=-\Delta A^TG_2b-\frac{1}{2}\beta\Delta A^Tc,$$

$$S_{22}^2=-B^TG_2\Delta B-\Delta B^TG_2B-\Delta B^TG_2\Delta B,\quad S_{24}^2=-\Delta B^T\left(G_2b+\frac{1}{2}\beta c\right).$$

We get the following statement.

Theorem 4.21 *Let $|D| < 1$ and there exist positive definite matrices H, G_1, G_2 and parameters $\beta > 0$, $\varsigma > 0$, $\nu > 0$ such that the matrix $S_1[\beta,\varsigma,\nu,H,G_1,G_2]$ is positive definite and the following inequalities hold:*

$$\|\Delta A\| \leqslant \frac{1}{R_2}\left[\sqrt{[|H| + |A^T G_2|]^2 + (1-\xi^2)(1-\eta^2)\lambda_{\min}(S_1[\beta,\varsigma,\nu,H,G_1,G_2])R_2} - \left[|H| + \left|A^T G_2\right|\right]\right],$$

$$\|\Delta B\| \leqslant \min\left\{\sqrt{\frac{1-\xi^2}{2}}\frac{\lambda_{\min}(S_1[\beta,\varsigma,\nu,H,G_1,G_2])}{|D|}\eta,\right.$$

$$\left.\frac{1}{R_1}\left[\sqrt{|G_2B|^2 + (1-\xi^2)\lambda_{\min}(S_1[\beta,\varsigma,\nu,H,G_1,G_2])R_1} - |G_2B|\right]\right\},$$

$$R_1 = \frac{|H + G_2B| + 2\xi^2\left|G_2b + \frac{1}{2}\beta c\right|^2}{\xi^2\lambda_{\min}(S_1[\beta,\varsigma,\nu,H,G_1,G_2])} + |G_2|\left(1 + \frac{1}{\alpha^2}\right),$$

$$R_2 = \frac{|G_2B| + 2\xi^2\left(|D|^2 + \left|G_2b + \frac{1}{2}\beta c\right|^2\right)}{\xi^2\lambda_{\min}(S_1[\beta,\varsigma,\nu,H,G_1,G_2])} + |G_2| + \alpha^2,$$

where $0 < \xi < 1$, $0 < \eta < 1$, α are arbitrary constants.

Then system (20) is absolutely interval stable in the metric C^1.

For arbitrary solution $x(t)$, $t > 0$ the following estimates of convergence hold:

$$|x(t)| \leqslant \left[\sqrt{\varphi_{11}(H)}\,|x(0)| + \sqrt{\varphi_{12}(H,G_1)}\,\|x(0)\|^2_{\tau,\varsigma} + \sqrt{\varphi_{13}(H,G_2)}\,\|\dot{x}(0)\|^2_{\tau,\varsigma}\right]\mathrm{e}^{-\frac{1}{2}\gamma t},$$

$$|\dot{x}(t)| \leqslant \left\{M\sqrt{\varphi_{11}(H)}\,|x(0)| + \left[M\tau\sqrt{\varphi_{12}(H,G_1)} + \frac{|B+\Delta B|}{|D|}\right]\|x(0)\|_\tau + \left(1 + M\tau\sqrt{\varphi_{13}(H,G_2)}\right)\|\dot{x}(0)\|_\tau\right\}\mathrm{e}^{-\frac{1}{2}\gamma t},$$

$$M = [|A+\Delta A| + |D|\,|b|\,k\,|c|] + \frac{[|D(A+\Delta A) + (B+\Delta B)| + |D|\,|b|\,k\,|c|]}{|D|\left[1 - |D|\,\mathrm{e}^{\frac{1}{2}\gamma\tau}\right]},$$

$$\gamma < \min\left\{\gamma, \frac{2}{\tau}\ln\frac{1}{|D|}, \frac{\theta[\cdot]}{\lambda_{\max}(H) + \frac{1}{2}\beta k|c|^2}\right\},$$

$$\theta[\cdot] \leqslant \left(1-\xi^2\right)\left(1-\eta^2\right)\lambda_{\min}(S_1[\beta,\varsigma,\nu,H,G_1,G_2]) - 2\left[|H| + \left|A^T G_2\right|\right]|\Delta A| - \left[|G_2| + \frac{|G_2B|^2 + 2\xi^2\left(|D|^2 + \left|G_2b + \frac{1}{2}\beta c\right|^2\right)}{\xi^2\lambda_{\min}(S_1[\beta,\varsigma,\nu,H,G_1,G_2])} + \alpha^2\right]|\Delta A|^2.$$

5 Conclusions

In Part 4.3 we derived statements on the exponential stability of system (6) as well as on estimates of the norms of its solutions and their derivatives in the case of exponential stability and in the case of exponential stability being not guaranteed. To obtain these results, special Lyapunov functionals in the form (8) and (9) were utilized as well as a method of constructing a reduced neutral system with the same solution on the intervals indicated as for the initial neutral system (6). The flexibility and power of this method was demonstrated using examples and comparisons with other results in this field. Considering further possibilities along these lines, we conclude that, to generalize the results presented to systems with bounded variable delay $\tau = \tau(t)$, a generalization of some auxiliary results is needed. This can cause substantial difficulties in obtaining results which are easily presentable. An alternative would be to generalize only the part of the results related to the exponential stability in the metric C^0 and the related estimates of the norms of solutions in the case of exponential stability and in the case of the exponential stability being not guaranteed (omitting the case of exponential stability in the metric C^1 and estimates of the norm of a derivative of solution). Such an approach will probably permit a generalization to variable matrices ($A = A(t)$, $B = B(t)$, $D = D(t)$) and to a variable delay ($\tau = \tau(t)$) or to two different variable delays. Nevertheless, it seems that the results obtained will be very cumbersome and hardly applicable in practice.

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Abstract

The Ph.D. thesis discusses the solutions to the differential equation and to systems of differential equations. The main attention is paid to study of asymptotical properties of equations with delay and systems of equations with delay.
In the first chapter are given physical and technical examples described by differential equations with delay and their systems. The classification of equations with delay is given and basic notions of theory of stability are formulated (mainly with the emphasis on the Lyapunov second method).
In the second chapter estimates of solutions of equations of neutral type are studied.
The third chapter deals with systems of differential equations of neutral type. Asymptotic estimates for solutions and their derivatives are proved. At the end of the chapter examples and comparisons of our results and of other authors are given. The calculation were performed with the MATLAB software.
Last, the fourth chapter deals with asymptotical properties of systems having a special type of non-linearities, so called "sector nonlinearities". Properties and estimations of solutions and derivatives are derived. The basic tools used in the dissertation are the Lyapunov second method and functionals of Lyapunov-Krasovskii type.

Abstrakt

Tato disertační práce pojednává o vlastnostech řešení diferenciálních rovnic a systémů diferenciálních rovnic. Hlavní pozornost je věnována asymptotickým vlastnostem rovnic se zpožděním a systémů rovnic se zpožděním.
V první kapitole jsou uvedeny fyzikální a technické příklady popsané pomocí diferencíálních rovnic a jejich systémů se zpožděním. Je uvedena klasifikace rovnic se zpožděním a jsou zformulovány základní pojmy stability s důrazem na druhou metodu Ljapunova.
Ve druhé kapitole jsou studovány odhady řešení rovnic neutrálního typu.
Třetí kapitola se zabývá systémy diferenciálních rovnic neutrálního typu. Jsou odvozeny asymptotické odhady pro řešení i pro derivace řešení. V závěru kapitoly jsou uvedeny příklady a srovnání výsledků s pracemi jiných autorů. Výpočty byly prováděny pomocí programu MATLAB.
Poslední, čtvrtá kapitola, se zabývá asymptotickými vlastnostmi systémů se speciálním typem nelinearity, tzv. sektorové nelinearity. Jsou odvozeny vlastnosti řešení a derivace řešení.
Zkladní metodou pro důkazy je v celé práci druhá Ljapunovova metoda a použití funkcionálů Ljapunova-Krasovského.