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OPTIMUM SUBBAND CODING OF CYLOSTATIONARY SIGNALS WITH MAXIMALLY DECIMATED FILTER BANKS

OPTIMÁLNÍ SUBPÁSMOVÉ KÓDOVÁNÍ CYKLOSTACIONÁRNÍCH SIGNÁLŮ S POUŽITÍM MDFB

SHORT VERSION OF PH.D. THESIS

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Zisk kódování, decimace, interpolace, banka filtrů, polyfázová reprezentace, matice spektrálních hustot výkonu, systém s vícerychlostním vzorkováním, subpásmové kódování, zhušťovací filtr, simulace, úplná rekonstrukce, majorizace, kvantizační šum, unitární transformace, adaptivní modulace.

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ABSTRACT

Předkládaná disertační práce si klade za cíl vybudování ucelené teorie subpásmového kódování cyklostacionárních signálů s nulovou střední hodnotou a s N-periodickou statistikou druhého řádu. Uvažovaný subpásmový kodér využívá ortonormální uniformní banku filtru s M subpásmovými kanály, která z hlediska systémové analýzy reprezentuje lineární, časově periodicky proměnný objekt. Vlastní teorie zahrnuje pravidla pro optimální přidělování bitů jednotlivým subpásmovým A/D převodníkům a zejména obecné zásady návrhu banky filtrů za účelem maximalizace kódového zisku. Kritériem pro vyhodnocení optimality analyzovaného návrhu je průměrný výkon kvantizačního šumu ve všech kanálech kodéru.

Jádro disertační práce spočívá v definování matematického formalizmu použitého pro popis subpásmového kodéru cyklostacionárních signálů, dále v odvození a důkazu základních požadavků kladených na matici spektrálních hustot subpásmových signálů, jež zajistí maximalizaci kódového zisku a tím zvýšení efektivity kódování.

Z požadavků kladených na matici spektrálních hustot subpásmových signálů, je proces optimálního „stlačování" energie (*optimum compaction process*) identifikován jakožto jedno z teoreticky možných řešení návrhu banky filtrů, vedoucí k výborným výsledkům při kódování cyklostacionárních signálů.

Jednoduchý simulační model, vytvořený za pomoci programového balíku Matlab, využívá výkonově symetrickou QM (*quadrature mirror*) banku filtrů za účelem dekorelace a uspořádání prvků matice spektrálních hustot subpásmových signálů, jež zajistí maximální kódový zisk. Na rozdíl od rigorózního teoretického přístupu preferovaného v prvních čtyřech článcích práce, jsou výsledky simulace především určeny k podpoře kvantitativní představy o možné efektivitě subpásmového kódování použitého pro konkrétní typ vstupního signálu. Ve zvoleném případě má tento signál biperiodickou statistiku druhého řádu a představuje jednu z nejjednodušších možných realizací.

První článek práce obsahuje základní definice cyklostacionarity, subpásmového kódování a polyfázové reprezentace banky filtrů, čímž buduje aparát nezbytný pro orientaci v následujících kapitolách. Článek druhý stručně představuje vybrané publikované výsledky v oboru kódování stacionárních signálů, na které tato práce navazuje. Hlavní výsledek třetího článku práce dokazuje nutné a postačující podmínky kladené na matici spektrálních hustot, za účelem dosažení maximálního kódového zisku. Tyto podmínky jsou pak v následující sekci vztaženy k procesu optimálního „stlačování" energie. Poslední, pátý článek prezentuje některé z výsledků počítačové simulace dvoukanálového subpásmového kodéru, získané z programového balíku Matlab.

1. INTRODUCTION

1.1 Subband Coder Topology

This project aims to study fundamental principles of design and optimization of maximally decimated uniform subband coders for Wide Sense Cyclostationary (WSCS) signals. Signal *x(k)* will be considered WSCS with period *N,* provided that obeys

$$E[x(k)] = m_x(k) = m_x(k+N) \quad \text{and} \quad E[x(k)x(l)] = R_x(k,l) = R_x(k+N, l+N), \qquad (1.1)$$

for all *k, l,* where *E[.]* is the expectation operator. If not noted otherwise, this work will stay focused on zero mean WSCS signals, without loss of generality.

Such sort of signal class captures a variety of man-made signals encountered in communication, telemetry, radar and sonar systems. These include amplitude, frequency and phase modulated waveforms, periodic keying of amplitude, phase and frequency in digital modulation systems and in addition also some naturally generated signals, including time series data obtained in meteorology, climatology, atmospheric science, oceanography, hydrology and astronomy [9].

There has been a considerable activity in the field of subband coding in recent years, resulting in successful applications in speech coding and image compression. Since results on optimal subband coding are mainly confined to WSS (Wide Sense Stationary) signals, the necessity of extending these results to subband coding of WSCS signals seems to emerge naturally, to cope with whole bunch of processes, which can be better represented under the umbrella of cyclostationarity. This work seeks to define the limits of what is achievable and tries to develop some framework of filter design that permits realization of these performance limits, under specified conditions. Goals achieved by the theory of subband coding of WSS signals, is often used as a starting base for treating WSCS cases.

The results described are mainly focused on filter bank structures, uniform rather then non-uniform ones. To explain the basic idea of proposed work, M-channel uniform maximally decimated filter bank is focused, as depicted in Fig. 1. Here, *x(k)* is the WSCS signal to be coded, $H_i(k,z^{-1})$ and $F_i(k,z^{-1})$ are known as analysis and synthesis filters respectively, Q_i are $b_i(k)$ bit quantizers converting the signal continuous in amplitude into its digital representation, the blocks to the right of the analysis filters are M-fold decimators that discard all but every M-th sample, and the blocks to the left of synthesis filters are M-fold interpolators that raise the sampling rate by a factor M, by inserting (M-1) zero samples between two consecutive samples of $w_i(k)$.

In particular, when *x(k)* is WSS, the analysis and synthesis filters are chosen to be Linear Time Invariant (LTI). However, in case of *x(k)* being WSCS, it seem to be more appropriate to consider Linear Periodically Time Varying (LPTV) analysis and synthesis filter bank. The presence of time sample *k* in $H_i(k,z^{-1})$ and $F_i(k,z^{-1})$ anticipates this fact. One can think about the arrangement to the left of the quantizers (known as Analysis Bank) as being located at the transmitter side, likewise the arrangement to the right of the quantizers (known as Synthesis bank) being located at the receiver side. Quantizers themselves can belong to communication channel. The goal is to take the signals $w_i(k)$ and synthesize a signal $\hat{x}(k)$ that is as close to $x(k)$ as possible.

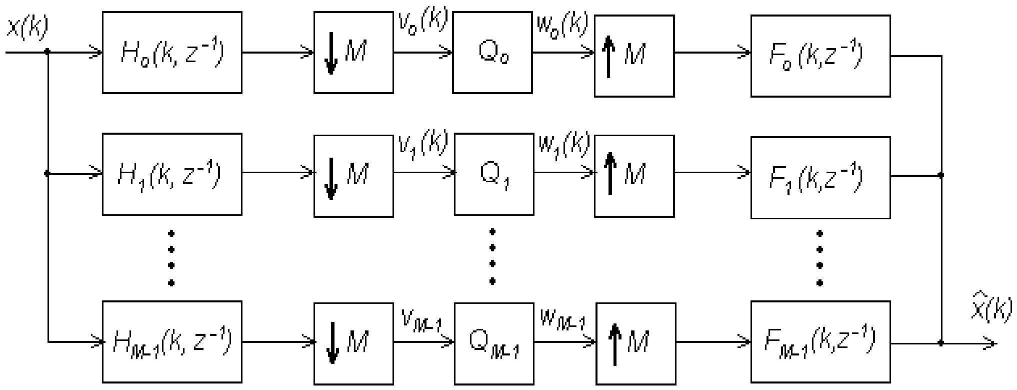

Figure 1: A Maximally Decimated Uniform Filter Bank

For the purpose of following derivations, quatizers Q_i are assumed to be sources of additive, uncorrelated, zero mean quantizing noise $q_i(k)$, the variance of which is

$$\sigma_{qi}^2(k) = c2^{-2b_i(k)}\sigma_{vi}^2(k). \tag{1.2}$$

Here $\sigma_{vi}^2(k)$ is the variance of i-th subband signal and c is a constant determined by the signal distribution and probability of overflow. Assumption on decorrelation of quantizer noise holds for sufficiently large number of allocated bits $b_i(k)$ and expression (1.2) results from standard quatizer noise analysis.

Different probabilities of overflow, as well as different distributions of random variable *x* yield different values of constant *c,* while the expression (1.2) still holds.

1.2 Coding Gain

As mentioned above, the point of optimum subband coder design is in minimization of the distortion $\hat{x}(k) - x(k)$, while keeping the average transmission bit rate *b* constant for each *k*,

$$b = \frac{1}{M}\sum_{i=0}^{M-1} b_i(k) \quad . \tag{1.3}$$

Point of comparison is the distortion caused by the straightforward transmission of *x(k)* through a b-bit quantizer (standard PCM channel), resulting $\tilde{x}(k)$. Hence the *Coding Gain* of the arrangement in Fig. 1 is defined as

$$G_{SBC} = \frac{E([\tilde{x}(k) - x(k)]^2)}{E([\hat{x}(k) - x(k)]^2)} \quad . \tag{1.4}$$

Obviously, minimizing the output distortion variance in Fig.1 is equivalent to maximizing the coding gain according to (1.4).

The systems involving WSCS signals will assume LPTV analysis and synthesis filters to be present. In explaining the ideas of subband coding, we will suppose that coding signals that are WSCS with period N, will be processed by analysis and synthesis filters that are also N-periodic. Thus, their impulse response $h(k,m)$ obeys for all *k, m,*

$$h(k+N,m+N)=h(k,m) \quad . \tag{1.5}$$

However, in general a periodicity of the analysis and synthesis filters is not necessarily restricted to that of input signal statistics. If both the analysis and synthesis filters, and the WSCS signal *x(k)* have the same period N, the variance of $\hat{x}(k)$ as well as subband signal variances are themselves N-periodic, i.e.

$$\sigma_{vi}{}^2(k+N)=\sigma_{vi}{}^2(k) \quad , \quad \sigma_{\hat{x}}{}^2(k+N)=\sigma_{\hat{x}}{}^2(k) \quad . \tag{1.6}$$

It is assumed, that $\{\sigma_x^2(0),\sigma_x^2(1),.....,\sigma_x^2(N-1)\ \}$ were extracted from input signal *x(k)* and they are available. Extending the above ideas on the quantizers Q_i, it seems to be natural to allocate $b_i(k)$ bits at the *k*-th instant within the period of cyclostationarity N, thus making $b_i(k)$ N-periodic as well. Further, at all time instants, the average bit rate *b* remains fixed by (1.3). Hence, **Periodically Dynamic Bit Allocation** has been defined.

Unlike static bit allocation expressed by (1.7), periodically dynamic bit allocation shall better reflect the WSCS nature of coded signals and hopefully provide better results as far as coding gain is concerned. Yet compared to static bit allocation it is obviously more difficult to implement.

$$b=\frac{1}{M}\sum_{i=0}^{M-1}b_i \tag{1.7}$$

Suppose *x(k)* is WSCS with period N, and both analysis and synthesis filters are N-periodic. Recalling (1.6), subband variances $\sigma_{vi}^2(k)$ are also N-periodic. It is thus natural to take the average distortion over the period N to be the appropriate measure for optimization. Under orthonormality condition {to be explained later}, the coding gain (1.4) becomes

$$G_{SBC}=\frac{c2^{-2b}\sigma_x^2}{\frac{c}{MN}\sum_{k=0}^{N-1}\sum_{i=0}^{M-1}2^{-2b_i(k)}\sigma_{vi}^2(k)} \quad , \tag{1.8}$$

where numerator represents distortion of a direct b-bit PCM coder and denominator represents averaged distortion in the output of the subband coder.

Applying AM-GM inequality on (1.8) and substituting periodically dynamic bit allocation (1.3) for *b*, the value of the expression in denominator is limited from the bottom by

$$\frac{c}{MN}\sum_{k=0}^{N-1}2^{-2b}\left(\prod_{i=0}^{M-1}\sigma_{vi}^2(k)\right)^{1/M} \quad . \tag{1.9}$$

Hence, the largest possible value of the coding gain can be expressed as

$$G_{SBC} = \frac{MN\sigma_x^2}{\sum_{k=0}^{N-1}\left(\prod_{i=0}^{M-1}\sigma_{vi}^2(k)\right)^{1/M}} \quad , \tag{1.10a}$$

being a function of the statistics of the input signal *x(k),* number of branches in the subband coder *M*, as well as function of the statistics of the subband signals $v_i(k)$. The maximization of coding gain is achieved under the *optimum bit allocation scheme.* Using conditions from AM-GM inequality, the *optimum bit allocation scheme* is such that for each *k* within a period of cyclostationarity:

$$2^{-2b_i(k)}\sigma_{vi}^2(k) = 2^{-2b_{i+1}(k)}\sigma_{v(i+1)}^2(k) \quad . \tag{1.10b}$$

Under a pertinent optimum bit allocation strategies, maximization of the coding gain can be reduced to minimization of the denominator of (1.10a), denoted as

$$J_{CG} = \sum_{k=0}^{N-1}\left(\prod_{i=0}^{M-1}\sigma_{vi}^2(k)\right)^{1/M} \quad . \tag{1.11}$$

Expression (1.11) is the function to be minimized to achieve optimum performance of the subband coder. Especially for *M>2* it is not a trivial task, and considerable part of this dissertation is devoted to the solution of this problem.

1.3 Polyphase Representation

To achieve considerable simplification of theoretical results as well as computationally efficient implementations of decimation and interpolation filters, filter banks and other topological structures, so called *polyphase decomposition* has been introduced into the analysis concept. The basic idea behind polyphase representation dwells in rearranging the expressions for analysis filters $H_i(k,z^{-1})$ from Fig.1 into a form

$$H_i(k,z^{-1}) = E_{i,0}(k,z^M) + E_{i,1}(k,z^M)z^{-1} + ... + E_{i,M-1}(k,z^M)z^{-M+1} = \sum_{l=0}^{M-1} z^{-l}E_{i,l}(k,z^M) \tag{1.12}$$

where for fixed *k* and *i*, polyphase components $E_{i,l}(k,z^M)$ constitute row elements in a *polyphase matrix.* Using original impuls response $h_i(k,l)$ of $H_i(k,z^{-1})$, one can write

$$E_{i,l}(k,z) = \sum_{n=-\infty}^{\infty} e_{i,l}(k,n)z^{-n} \quad , \tag{1.13a}$$

with

$$e_{i,l}(k,n) \equiv h_i(k,Mn+l) \quad , \quad 0 \le l \le M-1 \quad . \tag{1.13b}$$

Expression (1.12) defines type-1 polyphase representation of the analysis filter $H_i(k,z^{-1})$. Likewise, type-2 polyphase representation for the synthesis filters $F_i(k,z^{-1})$ is defined as

$$F_i(k,z^{-1}) = \sum_{l=0}^{M-1} z^{-(M-1-l)}R_{i,l}(k,z^M) \quad , \tag{1.14}$$

where type-2 polyphase components $R_{i,l}(k,z)$ are permutations of $E_{i,l}(k,z)$, i.e. $R_{i,l}(k,z) = E_{i,M-1-l}(k,z)$.

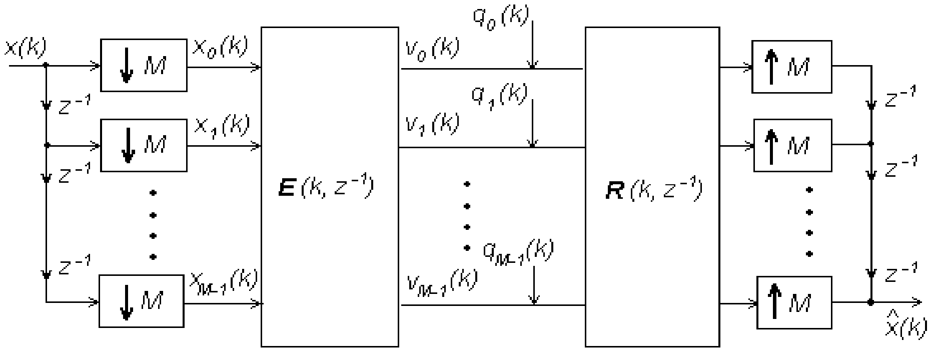

Figure 2: Maximally Decimated Filter Bank Represented by Polyphase Decomposition

Once the polyphase decomposition has been applied, the maximally decimated uniform filter bank can be redrawn according to Fig.2. Here, the quantizers are represented by additive, uncorrelated, zero mean quantizing noise $q_i(k)$, as already described in section 1.1, and both decimators and interpolators have been moved outward, by using so called *Nobel Identities*, [10]. A filter bank is said to be Biorthogonal if

$$R(k,z^{-1}) = E^{-1}(k,z^{-1}), \tag{1.15}$$

i.e. in the absence of quantizers, $\hat{x}(k) = x(k)$. This is a strong version of so called *Perfect reconstruction* (PR) property. A filter bank is called Orthonormal, if in addition to (1.15), $E(k,z^{-1})$ is all pass, i.e. the input energy is equal to its output energy. In LTI case, this requiments constitutes a need for matrices $E(e^{-j\omega})$ and $R(e^{-j\omega})$ to be unitary at each frequency:

$$E^T(e^{j\omega})E(e^{-j\omega}) = R^T(e^{j\omega})R(e^{-j\omega}) = \bar{I}\ . \tag{1.16}$$

This ensures that the energy contained in all $v_i(k)$, equals to the energy contained in *x(k)*. In other words both the analysis and synthesis filter banks are lossless. It is common to say, that orthonormality implies both the analysis and synthesis banks to be *Power Complementary (PC).*

1.4 Blocking and Problem Formulation

Since the representation by polyphase decomposion depicted in Fig. 2 still retains its N periodic nature, one can find convenient to work with higher dimensional LTI operators, using the blocking approach, as described below. Observing Fig. 3 and having in mind that decimators, interpolators and delay chains lay outside depicted structure (just for the sake of simplicity), matrices $\widetilde{E}(z^{-1})$, $\widetilde{R}(z^{-1})$ represent blocked polyphase decomposition of the analysis bank and synthesis bank, respectively.

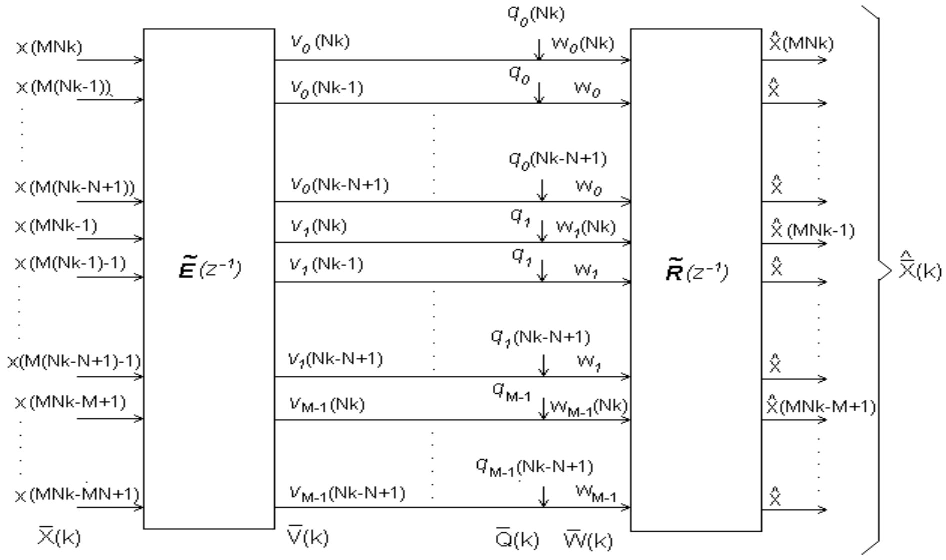

Figure 3: Blocked Polyphase Representation of the Filter Banks

Observing Fig.2 for *i=0,1, ... M-1,* denote

$$x_i(k) = x(Mk - i) \quad , \tag{1.17}$$

and define the N-fold blocked version of $x_i(k)$ as a column vector

$$\widetilde{x}_i(k) = [x(MNk - i), x(M(Nk-1) - i), ..., x(M(Nk - N + 1) - i)]^T \quad . \tag{1.18}$$

The above definition is no doubt consistent with the requirement that $\widetilde{x}_i(k)$ contains each *i-th* sample extracted from *x(k)* within a period of M samples. Calling

$$\widetilde{x}(k) = \left[\widetilde{x}_0{}^T(k), \widetilde{x}_1{}^T(k), ..., \widetilde{x}_{M-1}^T(k)\right]^T \tag{1.19}$$

and referring to Fig.4 with $H(k, z^{-1})$ N-periodic, assume the impulse response of $H(k, z^{-1})$ is *h(k,l),* obeying (1.5).

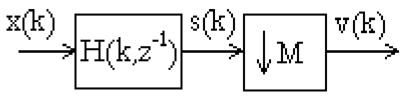

Figure 4: Cascade of the Analysis Filter and Decimator

Let $\widetilde{h}_{m,n}(k,l)$ be the $N \times N$ impulse response matrix of a system relating $\widetilde{x}_n(k)$ and $\widetilde{s}_m(l)$. Clearly, for any $0 \le p, q \le N-1$, the *pq*-element of this matrix obeys:

$$\left[\widetilde{h}_{mn}(k,l)\right]_{pq} = h(M(Nk - p) - m, M(Nl - q) - n) = \left[\widetilde{h}_{mn}(k - l)\right]_{pq} \quad . \tag{1.20}$$

Formula (1.20) now represents LTI systems for which we can define Z-transform

$$\widetilde{H}_{mn}(z^{-1}) = \sum_{k=-\infty}^{\infty} \widetilde{h}_{mn}(k) z^{-k} \ . \tag{1.21}$$

Turning our attention to Fig.1, define similarly for each N-periodic filter $H_i(k, z^{-1})$, *i=0,1, ... M-1,* the $N \times N$ matrix transfer functions $\widetilde{H}_{i0}(z^{-1})$, $\widetilde{H}_{i1}(z^{-1})$, … $\widetilde{H}_{i(M-1)}(z^{-1})$ that respectively relate the N-fold blocked 1-st, 2-nd, …, M-th samples within a period of M samples, to the N-fold blocked M-th samples of the output of $H_i(k, z^{-1})$.

Indeed, the $MN \times MN$ blocked LTI operator $\widetilde{E}(z^{-1})$ relates the $MN \times 1$ blocked vector $\widetilde{x}(k)$ to the $MN \times 1$ blocked vector $\widetilde{v}(k)$. In particular this LTI operator can be expressed using respective submatrices as

$$\widetilde{E}(z^{-1}) = \begin{bmatrix} \widetilde{H}_{00}(z^{-1}) & . & . & \widetilde{H}_{0(M-1)}(z^{-1}) \\ . & & . & . \\ . & & . & . \\ \widetilde{H}_{(M-1)0}(z^{-1}) & . & . & \widetilde{H}_{(M-1)(M-1)}(z^{-1}) \end{bmatrix} \ . \tag{1.22}$$

When *x(k)* is WSCS with period N, both $\widetilde{v}(k)$ and $\widetilde{x}(k)$ are WSS with PSD matrices $\overline{S}_{\widetilde{v}}(\omega)$ and $\overline{S}_{\widetilde{x}}(\omega)$, respectively. It should be noted, that from the periodic autocorrelations $E\left[x(k)x^*(l)\right]$ all the elements of PSD matrix can be calculated, hence they are supposed to be available.

The *Perfect Reconstruction* condition, which already has been introduced by (1.15) is equivalent to the requirement of

$$\widetilde{R}(z^{-1}) = \widetilde{E}^{-1}(z^{-1}) \ . \tag{1.23}$$

Orthonormality further imposes the requirement, that for all square summable *x(k),*

$$\sum_{k=-\infty}^{\infty} \widetilde{x}^T(k)\widetilde{x}(k) = \sum_{k=-\infty}^{\infty} \widetilde{v}^T(k)\widetilde{v}(k) \ . \tag{1.24}$$

Thus $\widetilde{E}(z^{-1})$ must be all-pass, and because of (1.23), one can arrive to (1.16). A distortion measure, defined as

$$\hat{q}(k) = \hat{x}(k) - x(k) \quad , \tag{1.25}$$

is also WSCS with period *MN.* It seems to be natural to propose the average variance of $\hat{q}(k)$ to be minimized for achieving optimality

$$E[\sigma_{\hat{q}}^2] = \frac{1}{MN} \sum_{k=0}^{MN-1} \sigma_{\hat{q}}^2(k) \ . \tag{1.26}$$

Because of (1.16) it follows from orthonormality and from PR property, that

$$\sum_{k=-\infty}^{\infty} \widetilde{v}^T(k)\widetilde{v}(k) = \sum_{k=-\infty}^{\infty} \widetilde{w}^T(k)\widetilde{w}(k) = \sum_{k=-\infty}^{\infty} \left|\hat{x}(k)\right|^2 \ . \tag{1.27}$$

Thus from (1.25) and from the fact that $q_i(k)$ and $v_i(k)$ are zero mean, mutually independent and WSCS signals with period N, it follows readily that

$$\sum_{k=0}^{MN-1} \sigma_{\hat{q}}^2(k) = \sum_{k=0}^{N-1} \sum_{i=0}^{M-1} \sigma_{qi}^2(k) \ . \tag{1.28}$$

Last expression justifies the choice of the denominator in the coding gain definition (1.8).

After a struggling effort to provide sufficient background for description of orthonormal uniform maximally decimated filter banks and coders, the optimization problem can be mathematically precisely formulated.

The optimization problem: *Consider the* $MN \times MN$ *system* $\widetilde{E}(z^{-1})$ *with WSS input vector* $\widetilde{x}(k)$ *with positive definite Hermitian-symetric PSD matrix* $S_{\widetilde{x}}(\omega)$. *Suppose* $\widetilde{v}(k)$ *is the output of* $\widetilde{E}(z^{-1})$. *Find* $\widetilde{E}(z^{-1})$ *such that (1.11) is minimized subject to (1.16).*

It just remains to note, that calculation of $\widetilde{E}(z^{-1})$ inherently provides $\widetilde{R}(z^{-1})$, by using (1.23). Finally $H_i(k,z^{-1})$ and $F_i(k,z^{-1})$ are obtained by unblocking $\widetilde{E}(z^{-1})$ and $\widetilde{R}(z^{-1})$, respectively.

2. STATE OF ART

2.1 Subband Coding of WSS Signals

Available results in the existing literature on optimal subband coders are limited to the case of WSS *x(k),* with LTI analysis and synthesis filters and orthonormal filter bank, i.e. *Transform coders* where $E(k,z^{-1})$ is a constant unitary matrix. Quite often, the asymptotic solution for M aproaching infinity is discussed, but the work which influenced most this research [4], treats the problem for finite M, assuming *x(k)* being WSS, zero mean, with known second order statistics. Under these conditions, Vaydianathan provides complete solution of obtaining the uniform optimal M-channel orthonormal filter bank that minimizes the coding gain.

Therefore, before discussing WSCS case, let observe some basic results emerged from WSS theory, to get an idea behind optimal bit allocation. Consider a two channel filter bank in Fig.5, at this time with LTI filters and WSS *x(k).* The analysis filters split the signal into two subbands. If one subband has greater energy, then assigning more bits to this signal in preference to the other subband, will enhance fidelity of compression. Hence to exploit fully the enhanced fidelity, one has to assign bits in proportion to variance of signals, while choosing one subband to have largest possible energy, subject to the orthonormality constraint. Such a process defining principle of selecting the analysis filters is named *Energy Compaction.*

Solution for M-channel case provided in [4] accords with the idea presented above. Following notation similar to that in the introductory part of this work, quantizing noise model for quantizer Q_i, which has been allocated b_i bits, assumes its output to obey

$$w_i = v_i + q_i \quad , \tag{2.1}$$

where q_i is zero mean, white, independent from v_i and has a variance

$$\sigma_{qi}^2 = c2^{-2b_i}\sigma_{vi}^2 \quad . \tag{2.2}$$

Here σ_{vi}^2 is the variance of the subband signal v_i and *c* again is a constant determined by the signal distribution as well as probability of overflow. These assumptions hold well at high bitrates and provide good insight into the mechanism that guides subband coding. Under orthonormality constrain, the total distortion variance at the output of the filter bank is

$$\frac{1}{M}\sum_{i=0}^{M-1}\sigma_{qi}^2 \quad . \tag{2.3}$$

Invoking already mentioned AM-GM inequality, stating that arithmetic mean of a set of numbers is bounded from below by their geometric mean, one can write,

$$\frac{1}{M}\sum_{i=0}^{M-1}\sigma_{qi}^2 \geq c2^{-2b}\left(\prod_{i=0}^{m-1}\sigma_{vi}^2\right)^{1/M} \quad ,$$

with equality if and only if for all i , j,

$$2^{-2b_i}\sigma_{vi}^2 = 2^{-2b_j}\sigma_{vj}^2 \quad . \tag{2.4}$$

The later expression constitutes an optimum bit allocation strategy. Thus for a given set of subband signals, the best achievable coding gain under the orthonormality condition,

$$G_{SBC} = \frac{\sigma_x^2}{\left(\prod_{i=1}^{M-1}\sigma_{vi}^2\right)^{1/M}} \quad . \tag{2.5}$$

Consequently, subject to the orthonormality condition, the filter design reduces to the minimization of

$$J_{SBC} = \prod_{i=0}^{M-1}\sigma_{vi}^2 \quad . \tag{2.6}$$

As shown in [4], necessary and sufficient conditions under which (2.6) is minimized are following. At the first it is necessary that the subband signals are mutually decorrelated, i.e. for all $i \neq j$, and k, l :

$$E[v_i(k)v_j(l)] = 0 \quad . \tag{2.7}$$

At the second, if one expresses

$$\bar{S}_v(\omega) = diag\{\lambda_0(\omega),\cdots\cdots,\lambda_{M-1}(\omega)\} \quad , \tag{2.8}$$

then optimality is ensured if the $\lambda_i(\omega)$ obey a consistent magnitude ordering at each frequency, i.e. for all ω :

$$\lambda_i(\omega) \geq \lambda_{i+1}(\omega) \quad . \tag{2.9}$$

Call the vector of inputs to $E(z^{-1})$ in Fig.2 (index k has no relevance for WSS case) $\bar{x}$, and its PSD matrix $\bar{S}_{\bar{x}}(\omega)$. Then the subband spectra:

$$\bar{S}_v(\omega) = E(e^{-j\omega})\bar{S}_{\bar{x}}(\omega)E^T(e^{j\omega}) \quad . \tag{2.10}$$

Snce (1.16) holds, $\lambda_i(\omega)$ are simply eigenvalues of the matrix $\bar{S}_{\bar{x}}(\omega)$. This constitutes an energy compaction condition, which together with subband decorrelation ensures that the first subband has largest energy, the second subband has the second largest and so on.

To achieve optimality for general finite M, it is shown in [4] that $H_0(z^{-1})$ is just the *optimum compaction filter* for *x(k),* i.e. it is Nyquist-M filter, which obeys

$$\sum_{k=0}^{M-1}\left|H_0(e^{(j\omega-2\pi k/M)})\right|^2 = M \, , \tag{2.11}$$

for all ω, and subject to (2.11) it maximizes the output variance of $v_i(k)$'s .To summarize the results for WSS case, two requirements have to be fulfilled:

1. *The subband signals have to be decorrelated* .
2. *The subband signal PSD's have to obey a consistent ordering property.*

Presented solution leads to analysis and synthesis filters that are ideal. In practice, finite dimension filters, usually standard FIR approximations are used. Taking another approach, one can try to maximize the coding gain for *a priori* fixed FIR filter order.

2.2 Two Channel Filter Bank for WSCS Signals

Available studies on maximally decimated uniform filter banks applied on WSCS signals are focused on *M=2* case so far (see Fig. 5). Work [2] presents conditions for optimal design of a maximally decimated uniform two-channel filter bank, employing polyphase decomposition and blocking as described in subsections 1.3, 1.4, respectively. It is probably not surprising, that equivalent blocked LTI system calls for similar requirements as the filter bank intended for processing WSS signals.

Indeed, all the subband signals at all time instants *k*, have to be decorrelated, i.e. the blocked PSD matrix $\bar{S}_{\tilde{v}}(\omega)$ has to be diagonal at almost all ω. This leads to diagonal elements of $\bar{S}_{\tilde{v}}(\omega)$ to be the eigenvalues of the blocked PSD matrix $\bar{S}_{\tilde{x}}(\omega)$ of the input signal *x(k)* and (2.8) still holds for blocked structure. However, the ordering of the diagonal elements of $\bar{S}_{\tilde{v}}(\omega)$ is not that straightforward as in (2.9).

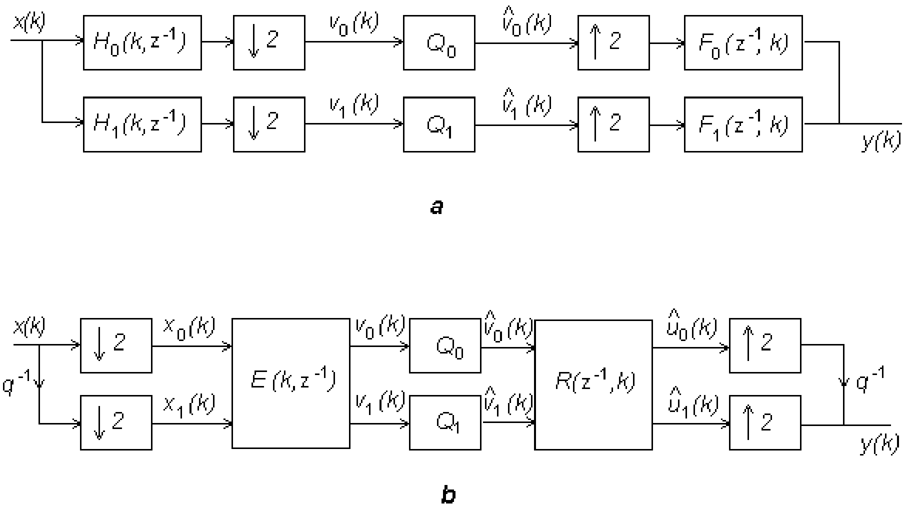

Figure 5: Two-channel Filter Bank for WSCS Signals

For two-channel case the coding gain (1.10a) takes the form of

$$G_{SBC} = \frac{2N\sigma_x^2}{\sum_{k=0}^{N-1}\sqrt{\sigma_{v1}^2(k)\sigma_{v2}^2(k)}} \quad , \tag{2.12}$$

hence the expression in denominator to be minimized subject to orthonormality condition (1.16) can be written as

$$J_{CG} = \sum_{k=0}^{N-1} \sqrt{\sigma_{v1}^2(k)\sigma_{v1}^2(k)} \quad . \tag{2.13}$$

Unlike WSS case, where (2.6) represents a product of subband variances, in (2.13) one can observe a sum of square-roots of products of subband variances. This fact substantially changes requirements on ordering of diagonal elements of $\overline{S}_{\tilde{v}}(\omega)$, thus changing the level of mathematical difficulty. However, for two-channel filter bank, there is a unique way leading to general algorithm for finding an optimal solution to the optimization problem. Here is presented without proof.

Properties of so called majorized sequences of numbers, allowed Schwartz in [2] to determine requirements on pairing diagonal elements of $\overline{S}_{\tilde{v}}(\omega)$ in such a way, that signal with largest energy (variance) is paired in the product with the signal having the smallest energy, second largest with the second smallest etc. Mathematically formally, we can state:

Whenever for some $0 \le i \le N-1$ and $0 \le l \le 2N-1$

$$\left[\overline{S}_{\tilde{v}}(\omega)\right]_{ii} = \lambda_l(\omega) \qquad \text{, one must have} \tag{2.14}$$

$$\left[\overline{S}_{\tilde{v}}(\omega)\right]_{2N-i-1,2N-i-1} = \lambda_{2N-l-1}(\omega) \quad , \tag{2.15}$$

where $\lambda_l(\omega)$ represent eigenvalues of $\overline{S}_{\tilde{x}}(\omega)$, ordered according to (2.9).

The existence of a unique ordering of the eigenvalues of PSD matrix, i.e. of the subband variances, leads Schwartz to the conclusion on optimal solution for unrestricted filter order of the analysis and synthesis filters, which similarly to that of Vaidyanathan [4], identifies energy compaction as a condition to achieve minimum distortion of the subband coder.

How fortunate the existence of unique ordering for solution of a two-channel filter bank is, can be underlined by the fact that for *M=3* (nor for higher values of M), there is no general way, independent of particular values of the numbers in the sequence *A*, that would allow the sum of products $\sqrt{a_{i_n} a_{l_n} a_{m_n}}$ (for *i, l, m* being disjoint partitions of the set $\{1,2,...,3N\}$) to be minimized, defining criteria similar to (2.14) and (2.15).

3. OPTIMUM SOLUTION

3.1 Mathematical Concept

In following definitions, lemmas and theorems, author's effort is to prove that optimal performance of the LPTV Multirate Subband Coder, i.e. maximized Coding Gain as defined in (1.4), (1.8), is achieved if subband signals $v_i(k)$ are mutually decorrelated and diagonal elements of the PSD matrix of the vector $\overline{V}(k)$ obeys a specific ordering for almost all ω.

Definition 3.1: Consider a sequence $A = \{a_i\}_{i=1}^{MN}$, such that $a_1 \ge a_2 \ge ..... \ge a_{MN} \ge 0$, where $MN = M \times N$. Define a set of integers $S = \{1,2,3,.....,MN\}$, its partitioning $\{S_1, S_2,..., S_N\}$, such that $card[S_i] = M$, $\forall i \in \{1,2,..., N\}$.

Define a *Characteristic Sum*

$$J(A,S_1,S_2,\ldots,S_N)=\sum_{i=1}^{N}\left(\prod_{l\in S_i}a_l\right)^{1/M} \quad , \tag{3.1}$$

and a *Minimum Sum*

$$J^*(A)=\min_{S_1,S_2,\ldots,S_N} J(A,S_1,S_2,\ldots,S_N) \ , \tag{3.2}$$

where the sequence-argument

$$\{S^*(A)\}=\{S_1{}^*(A),S_2{}^*(A),\ldots,S_N{}^*(A)\} \tag{3.3}$$

represents *The Best Ordering* of a sequence *A*, that may not be unique.

Lemma 3.1: Consider sequence *A*, and *J*, *J**, *{S*(A)}* as in (3.1) through (3.3). For $k\in S_i{}^*(A)$, $l\in S_j{}^*(A)$, where $k,l\in\{1,2,\ldots,MN\}: k\neq l$, $i,j\in\{1,2,\ldots,N\}: i\neq j$. Let introduce:

$$\bar{a}_{i,k}=\prod_{q\in S_i^*(A),q\neq k}a_q \ , \tag{3.4}$$

and

$$\bar{a}_{j,l}=\prod_{q\in S_j^*(A),q\neq l}a_q \ . \tag{3.5}$$

If $a_k>a_l$, then $\bar{a}_{i,k}\leq\bar{a}_{j,l}$. Proof is available in complete version of the dissertation.

Definition 3.2 *Majorization:* Let's take sequence $A=\{a_i\}_{i=1}^{MN}$ and $\Lambda=\{\lambda_i\}_{i=1}^{MN}$, with elements labeled such that $a_1\geq a_2\geq\ldots\ldots\geq a_N\geq 0$, $\lambda_1\geq\lambda_2\geq\ldots\ldots\geq\lambda_N\geq 0$. Suppose that *A* and Λ obey relation:

$$\sum_{i=1}^{k}a_i\leq\sum_{i=1}^{k}\lambda_i \quad , \tag{3.6}$$

for all $1\leq k\leq N$, with equality at *k=N*. Then we say that Λ majorizes *A*, or *A* is majorized by Λ, noting $A\prec\Lambda$. There are several important consequences implied by majorization.

Definition 3.3 *Schur-concave function:* A real-valued function $\Phi(x_1,x_2,\ldots,x_n)$, defined on a subset $\Im\subset\Re^n$ is said to be strictly Schur-concave on this subset, whenever for $X=\{x_i\}_{i=1}^n$ and $Y=\{y_i\}_{i=1}^n$, $X\prec Y$ implies

$$\Phi(X)\geq\Phi(Y) \quad , \tag{3.7}$$

with equality if and only if *X=Y* in terms of equal elements.

Theorem 3.1: Let $\Phi(x_1,x_2,\ldots,x_n)$ be a real-valued function, where $x_1\geq x_2\geq\ldots\geq x_n\geq 0$, defined on a subset $\Im\subset\Re^n$ and twice differentiable on its interior. Also $\Phi(x_1,x_2,\ldots,x_n)$ is symmetric on $\Im$. Denote [2] :

$$\varphi_{(k)}(X)=\frac{\partial\Phi(X)}{\partial x_k} \qquad \text{and}$$

$$\varphi_{(k,l)}(X)=\frac{\partial^2\Phi(X)}{\partial x_k\partial x_l} \ .$$

Then $\Phi(X)$ is strictly Schur-concave on $\Im$ if:

1. $\varphi_{(k)}(X)$ is increasing in *k*
2. $\varphi_{(k)}(X)=\varphi_{(k+1)}(X)$ implies: $\varphi_{(k,k)}(X)-\varphi_{(k,k+1)}(X)-\varphi_{(k+1,k)}(X)+\varphi_{(k+1,k+1)}(X)<0$

Proof of this theorem is available in [3].

Lemma 3.2: The *Minimum Sum* as presented in (3.2) is strictly Schur-concave function and hence for all *A* and Λ, $A \prec \Lambda$ implies

$$J^*(A) \geq J^*(\Lambda) \quad , \tag{3.8}$$

with equality if and only if $A = \Lambda$ in terms of equal elements. Proof of Lemma 3.2 uses rules outlined in Theorem 3.1 and is available in the complete version of this dissertation.

Fact 3.1: Consider a $MN \times MN$ positive semidefinite, Hermitian symmetric matrix $\overline{S}$, sequence of its diagonal elements $A = \{a_i\}_{i=1}^{MN}$, such that $a_1 \geq a_2 \geq .....\geq a_{MN} \geq 0$. Consider sequence of eigenvalues $\Lambda = \{\lambda_i\}_{i=1}^{MN}$ of this positive semidefinite, Hermitian symmetric matrix, such that $\lambda_1 \geq \lambda_2 \geq .....\geq \lambda_N \geq 0$. Then

$$A \prec \Lambda \quad . \tag{3.9}$$

Further, the sequence of the diagonal elements equals to the sequence of eigenvalues, if and only if $\overline{S}$ is a diagonal matrix.

3.2. Existence of the Optimum Solution

The optimization problem, as introduced in section 1.4 will be dealt with here-in-after and author's effort is to prove that such solution exists and it is unique. The assumed all-pass nature of the analysis and synthesis filter bank allows us to take an advantage of the results gained above. Perfect reconstruction calls for (1.23) while orthonormality implies (1.16). Obviously, expression (1.23) is just an LTI version of (1.15), now applied on blocked polyphase matrices.

Taking $S_{\overline{X}}(\omega)$ the PSD matrix of $\overline{X}(k)$, which is positive definite Hermitian symmetric, we can write

$$S_{\overline{X}} = \widetilde{U}(e^{-j\omega})\Lambda(\omega)\left[\widetilde{U}(e^{-j\omega})\right]^H \quad , \tag{3.10}$$

where $\widetilde{U}(e^{-j\omega})$ is a unitary matrix at all ω, and

$$\Lambda(\omega) = diag\{\lambda_0(\omega), \lambda_1(\omega), ..., \lambda_{MN}(\omega)\} \tag{3.11}$$

obeys a *Canonic ordering* at all ω:

$$\lambda_i(\omega) \geq \lambda_{i+1}(\omega) > 0 \, . \tag{3.12}$$

Also, it follows from Fig. 3 and appendix C of [10], that PSD matrix $S_V(\omega)$ of $\overline{V}(k)$ obeys

$$S_{\overline{V}} = \widetilde{E}(e^{-j\omega})S_{\overline{X}}(\omega)\left[\widetilde{E}(e^{-j\omega})\right]^H \quad . \tag{3.13}$$

Let's return to the denominator of the Coding Gain (1.11) that concerns the subband variances $\sigma_{vi}{}^2(j)$ for $i \in \{0,..,M-1\}$, $j \in \{0,...,N-1\}$. Let's introduce sequence of subband variances

$$\Sigma_{\overline{V}} = \{\sigma_{v0}^2(0), \sigma_{v0}^2(-1), ..., \sigma_{v0}^2(-N+1), \sigma_{v1}^2(0), ..., \sigma_{v(M-1)}^2(0), ..., \sigma_{v(M-1)}^2(-N+1)\} \quad , \tag{3.14}$$

which can be also written as

$$\Sigma_{\overline{V}} = \{\sigma_{vi}(j)\}_{ij} = \left\{ \frac{1}{2\pi}\left(\int_0^{2\pi} S_{\overline{V}}(\omega)d\omega \right)_{kk} \middle| 0 \leq k \leq MN-1 \right\} . \tag{3.15}$$

We shall call the sequence Σ *achievable,* if under orthonormality condition, (3.11) through (3.13) and (3.15) hold.

Lemma 3.3: Consider all the quantities defined in the formulation of the "Optimization Problem" and an achievable Σ as in (3.14). Then having $\lambda_i(\omega)$ as in (3.10) through (3.12), and under orthonormality condition (1.16),

$$\Sigma \prec \left\{ \frac{1}{2\pi} \left(\int_0^{2\pi} \lambda_k(\omega) d\omega \right)_{kk} \middle| 0 \le k \le MN-1 \right\}. \tag{3.16}$$

Proof: Suppose that the elements of $S_{\overline{V}}(\omega)$ are not ordered in a consistent way at all ω, e.g. around some frequency ω_1, $\left(S_{\overline{V}}(\omega_1)\right)_{ii} > \left(S_{\overline{V}}(\omega_1)\right)_{ll}$, but around another frequency ω_2 $\left(S_{\overline{V}}(\omega_2)\right)_{ii} < \left(S_{\overline{V}}(\omega_2)\right)_{ll}$, see Fig. 6-A. Then, at each ω there exists a permutation matrix $P(\omega)$ such that for all ω,

$$S^*{}_{\overline{V}} = P(\omega) S_{\overline{X}}(\omega) P^T(\omega) \tag{3.17}$$

obeys

$$\left(S^*{}_{\overline{V}}(\omega)\right)_{ii} \ge \left(S_{\overline{V}}(\omega)\right)_{ll} \quad . \tag{3.18}$$

See Fig. 6-B, demonstrating action of $P(\omega)$ on neighboring elements of $S_{\overline{V}}(\omega)$, to achieve (3.18).

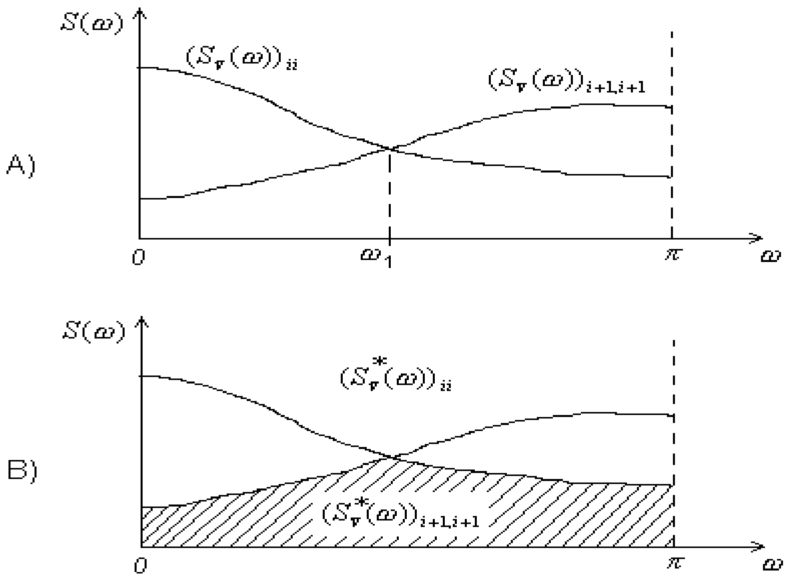

Figure 6: Example of Power Spectra of Two Neighbor Elements of PSD Matrix $S_{\overline{V}}(\omega)$ before (A) and after (B) the Action of $P(\omega)$

Define

$$\Sigma^*_{\overline{V}} = \left\{ \frac{1}{2\pi} \left(\int_0^{2\pi} S^*{}_{\overline{V}}(\omega) d\omega \right)_{kk} \middle| 0 \le k \le MN-1 \right\} \quad . \tag{3.19}$$

Clearly, for all $0 \le l \le MN-1$ and all ω:

$$\sum_{k=0}^{l}\left(S_{\overline{V}}(\omega)\right)_{kk} \le \sum_{k=0}^{l}\left(S^{*}{}_{\overline{V}}(\omega)\right)_{kk} \quad , \tag{3.20}$$

with equality at $l = MN-1$. Hence,

$$\sum_{k=0}^{l}\int_{0}^{2\pi}\left(S_{\overline{V}}(\omega)\right)_{kk} d\omega \le \sum_{k=0}^{l}\int_{0}^{2\pi}\left(S^{*}_{\overline{V}}(\omega)\right)_{kk} d\omega \,. \tag{3.21}$$

From (3.19), and (3.15) then

$$\sum\nolimits_{\overline{V}} \prec \sum\nolimits^{*}_{\overline{V}} \quad . \tag{3.22}$$

Unless $P(\omega)$ in (3.21) is frequency invariant almost everywhere (word *almost* refers to possible countable set of values) on $[0,2\pi)$, i.e. unless for almost all ω

$$P(\omega) = P \quad , \tag{3.23}$$

at least one inequality in (3.21) is strict. Hence, for (3.23)

$$\sum\nolimits_{\overline{V}} = \sum\nolimits^{*}_{\overline{V}} \,. \tag{3.24}$$

Also, $P(\omega)$ is a unitary permutation matrix. Thus, from orthonormality (1.16), (3.10) through (3.13), (3.15) and (3.17), $\lambda_i(\omega)$ are the eigenvalues of $S_{\overline{X}}(\omega)$. Then from the Fact 3.1, for all ω and $0 \le l \le MN-1$,

$$\sum_{k=0}^{l}\left(S^{*}{}_{\overline{V}}(\omega)\right)_{kk} \le \sum_{k=0}^{l}\lambda_k(\omega) \quad , \tag{3.25}$$

with equality at $l = MN-1$. Result (3.16) follows directly from (3.25) which is the definition of majorization as in Definition 3.2. After proving the first result, suppose that equality in (3.16) holds. It implies that (3.24) holds as well. Thus the permutation matrix $P(\omega)$ obeys (3.23) for almost every ω. To complete the proof one shall show that for almost every ω

$$S_{\overline{V}}{}^{*}(\omega) = \Lambda(\omega) \quad . \tag{3.26}$$

Suppose there does exist an interval within $[0,2\pi)$, in which (3.26) fails. Since $\lambda_i(\omega)$ are the eigenvalues of $S_{\overline{V}}(\omega)$ and (3.14) as well as (3.18) apply, from Fact 3.1, on such interval for some $0 \le l \le MN-1$

$$\sum_{k=0}^{l}\left(S^{*}{}_{\overline{V}}(\omega)\right)_{kk} < \sum_{k=0}^{l}\lambda_k(\omega) \quad .$$

Thus, since (3.25) holds at other frequencies, for the above particular $0 \le l \le MN-1$

$$\sum_{k=0}^{l}\int_{0}^{2\pi}\left(S^{*}{}_{\overline{V}}(\omega)\right)_{kk} d\omega < \sum_{k=0}^{l}\int_{0}^{2\pi}\lambda_k(\omega)d\omega \quad .$$

Definition of majorization establishes a contradiction.

Theorem 3.2: The optimum performance of the multirate subband coder with uniform, maximally decimated filter-bank for WSCS signals, i.e. the maximum coding gain, is attained if and only if both of the following hold:

1. The subband signals $v_i(k)$ are totally decorrelated for all k, i.e. the blocked PSD matrix $S_V(\omega)$ is diagonal.
2. The diagonal elements of $S_V(\omega)$ obey a specific magnitude ordering at each ω, although possibly through a different frequency invariant permutation, then that indicated by the canonic ordering.

Proof: Requirement on both diagonalization and magnitude ordering follows directly from Lemmas 3.2, 3.3 and from Fact 3.1, since diagonal PSD matrix $\Lambda(\omega)$ as defined in (3.10) through (3.12), which was obtained by unitary transform (filtering by all-pass filter bank), indeed contains eigenvalues of $S_X(\omega)$ on the principal diagonal and the set of eigenvalues is unique.

Applying (3.8), (3.9), and (3.16), which can be written in a shorthand notation as $\Sigma \prec \int \Lambda$, we get

$$J^*(\Sigma) \geq J^*(\int \Lambda) \quad , \tag{3.27}$$

where minimum sums are used in accordance with Definition 3.1. Minimum sum plays an important part in minimizing the denominator of (1.10a).

Using (3.22), expression (3.27) can be extended to

$$J^*(\Sigma) \geq J^*(\Sigma^*) \geq J^*(\int \Lambda) \quad , \tag{3.28}$$

in which the first inequality refers to the requirement of consistent magnitude ordering at each ω, while the second refers to the necessity diagonalization of the PSD matrix $S_V(\omega)$. Hence, letting $\widetilde{E}(e^{-j\omega}) = [\widetilde{U}(e^{-j\omega})]^H$ is one possible way to fulfill requirements of the Theorem 3.2, getting $S_{\overline{V}}(\omega) = \Lambda(\omega)$.

3.3 Remarks on Minimum Sum

While Theorem 3.2 constitutes necessary and sufficient condition for maximizing coding gain, it assumes implicitly that sequence of matrix elements of $S_{\overline{V}}(\omega)$ follows *the best ordering* as defined in Definition 3.1, in order to permute the elements in a way to minimize (1.11), (3.1). However, no answer is provided on how to reach *the best ordering*.

Recall result (2.6), presented in Chapter 2 for WSS signals. There is no need for similar ordering in a simple product of subband variances. In WSCS case however, one is not concerned with simple products of diagonal elements of the normalized integral of $S_{\overline{V}}(\omega)$, but rather with a sum of M-roots of M-products. For WSCS signals, and for particular case when M=2 (two-channel filter bank), (2.14) and (2.15) quite fortunately specify a unique ordering of subband variance. For general M and N however, there is no equivalent to (2.14) and (2.15), hence *the best ordering* (3.3) has to be found numerically, based on the exact knowledge of $S_{\overline{X}}(\omega)$ upon which the diagonalization and consistent magnitude ordering has been applied for almost all ω, as stated in Theorem 3.2.

Assume that after diagonalization and consistent magnitude ordering for all ω, as required by Theorem 3.2, $\widetilde{E}(z^{-1})$ has the form of (1.22), transforming $S_{\overline{V}}(\omega)$ into diagonal matrix (3.11) with canonically ordered diagonal elements like in (3.12). Once the best ordering (3.3) has been found, the blocked polyphase matrix $\widetilde{E}(z^{-1})$ shall adopt form

$$\widetilde{E}_{J*}(z^{-1}) = \begin{bmatrix} \widetilde{H}_{00}(z^{-1}) & . & . & . & \widetilde{H}_{0(M-1)}(z^{-1}) \\ \Gamma_1\widetilde{H}_{10}(z^{-1}) & . & & & \Gamma_1\widetilde{H}_{1(M-1)}(z^{-1}) \\ . & & . & & . \\ . & & . & & . \\ \Gamma_{M-1}\widetilde{H}_{(M-1)0}(z^{-1}) & . & . & . & \Gamma_{M-1}\widetilde{H}_{(M-1)(M-1)}(z^{-1}) \end{bmatrix} , \qquad (3.29)$$

with $\widetilde{H}_{mn}(z^{-1})$ as in (1.21).

Here Γ_i for $1 \le i \le M-1$ are frequency invariant (constant) $N \times N$ permutation matrices, i.e. matrices having just one unity element on each row and in each column, with other elements equal to zero. Position of the unity element on each row is governed by *the best ordering* (3.3) and is determined by particular values of subband variances obtained from normalized integral of diagonal elements of $S_{\overline{V}}(\omega)$. As already mentioned above, no general algorithm is available.

Summarizing the above derivation: applying *the best ordering* on the blocked polyphase matrix $\widetilde{E}(z^{-1})$, constructed according to Theorem 3.2, via permutation matrices Γ_i, will allow the subband coder for WSCS signals to reach the coding gain

$$G_{SBC} = \frac{MN\sigma_x^2}{J*(\int \Lambda)} , \qquad (3.30)$$

where $J*(\int \Lambda)$ denoting the minimum sum as in (3.2), with argument

$$J*(\int \Lambda) \equiv J*(\left\{ \frac{1}{2\pi} \left(\int_0^{2\pi} \lambda_k(\omega) d\omega \right)_{kk} \middle| 0 \le k \le MN-1 \right\}) . \qquad (3.31)$$

Obviously, expressions (3.30), (3.31) represent themselves a general solution to *the optimization problem* stated in the later part of Section 1.

4. OPTIMUM ENERGY COMPACTION

The main intention of this section is to show formally, that the analysis filters yielded by the optimum solution which itself calls for a canonical ordering of the subband variances, as described in the previous section, are in fact optimum compaction filters for the WSCS input signal.

4.1 Nyquist-M filter

Definition and a list of fundamental properties of Nyquist-M filters (or M-th band filters) used for WSS signals are presented in [10]. Consider a polyphase decomposition of $H(z^{-1})$ (see Fig.1, with omitted index *k*). Suppose the 0-th polyphase component $E_0(z^{-1})$ is a constant *c*, making

$$H(z^{-1}) = c + z^{-1}E_1(z^M) + ... + z^{-(M-1)}E_{M-1}(z^M) . \qquad (4.1)$$

Then the output

$$Y(z^{-1}) = X(z^M)H(z^{-1}) = cX(z^M) + \sum_{l=1}^{M-1} z^{-l}E_l(z^M)X(z^M) . \qquad (4.2)$$

Expression (4.2) implies that $y(Mn) = cx(n)$ in the time domain. In practical applications one can scale the filter such that *c=1*. Thus, even though the interpolation filter inserts new samples, the existing samples in the input sequence *x(n)* are communicated to the output without distortion. An impulse response having the above property satisfies condition

$$h(Mn) = \begin{cases} c, \ldots n = 0 \\ 0, \ldots otherwise \end{cases} \quad . \tag{4.3}$$

In other words, *h(n)* has periodic zero-crossings separated by M samples, with exception of *h(0)=c*. After convolving *x(n)* with the impulse response *h(n),* nonzero samples of *x(n)* are unaffected, except for a scaling factor *c*. In frequency domain (Z-domain), the Nyquist-M property is manifested as well. If $H(z^{-1})$ satisfies (4.1), it can be shown that

$$\sum_{k=0}^{M-1} H(z^{-1}W_M^k) = Mc \quad , \tag{4.4}$$

where core of the DFT $W_M = e^{-j2\pi/M}$. The frequency response of $H(z^{-1}W_M^k)$ is the shifted version $H(e^{-j(\omega+2\pi k/M)})$ of $H(e^{-j\omega})$. Finally, it can be concluded that all the M uniformly shifted versions of $H(e^{-j\omega})$ add up to an allpass filter.

4.2 Nyquist-M Process for WSCS Signals

To deal with optimum energy compaction solution for the subband coder treated in this dissertation, one must first define N-periodic optimum compaction of an N-periodic WSCS process. Definition of optimum compaction of WSS signals described in [4], involves an LTI filter $H(z^{-1})$ for which $H(z^{-1})H^*(z)$ is Nyquist-M. For an LPTV system $H(k, z^{-1})$ let's define the *adjoined filter* $H^a(k, z^{-1})$, whose implulse response $h^a(k,l)$ relates to the impulse response $h(k,l)$ of $H(k, z^{-1})$ by

$$h^a(k,l) = h^*(l,k) \quad . \tag{4.5}$$

It can be observed that the adjoined of an LTI system with transfer function $H(z^{-1})$, has the transfer function $H^*(z)$. Thus an analogy of a system with transfer function $H(z^{-1})H^*(z)$ in the LTV (Linear Time-Varying) case is the LTV system $H(k, z^{-1})H^a(k, z^{-1})$. Following definition constitutes LTV Nyquist-M filter.

Definition 4.1: ***LTV Nyquist-M Filter*** Consider the arrangement as in Fig. 4 with $H(k, z^{-1})$ LTV filter with impulse response *h(k,l)*. Then $H(k, z^{-1})$ is Nyquist-M if for all integers *n* and *m*, following equality holds

$$h(Mn, Mm) = c\delta(n-m) \quad , \tag{4.6}$$

where δ denotes Kronecker delta.

Recall from Chapter 1, that $\widetilde{H}_{00}(z^{-1})$, $\widetilde{H}_{01}(z^{-1})$,…, $\widetilde{H}_{0(M-1)}(z^{-1})$ respectively represent an LTI systems relating the blocked 0-th, 1-st,…,(M-1)-th samples within a period of M samples, to the blocked M-th samples of *s(k),* when $H(k, z^{-1})$ is WSCS with period N. Thus, the comparable result for WSCS case would be as follows: for N-periodic $H(k, z^{-1})$, the product $H(k, z^{-1})H^a(k, z^{-1})$ is Nyquist-M if and only if for all ω,

$$\left[\widetilde{H}_{00}(e^{-j\omega}),\widetilde{H}_{01}(e^{-j\omega}),\ldots,\widetilde{H}_{0(M-1)}(e^{-j\omega})\right]\begin{bmatrix}[\widetilde{H}_{00}(e^{-j\omega})]^H\\ [\widetilde{H}_{01}(e^{-j\omega})]^H\\ \vdots\\ [\widetilde{H}_{0(M-1)}(e^{-j\omega})]^H\end{bmatrix}=\bar{I} \quad . \tag{4.7}$$

$\bar{I}$ is $N\times N$ identity matrix.

In general, the values of $h(k,l)$ playing a part in (4.7) are only those that correspond to the M-th values of k, within a period of M samples.

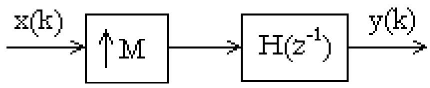

Figure 7: An Interpolator Cascaded with Interpolation Filter

The second reason for preferring definition (4.6) goes beyond the application of Nyquist-M filters in this dissertation. As already indicated in section 4.1 for WSS case, a key advantage of LTI Nyquist-M filters dwells in their usefulness for M-fold interpolation process. Then, an LTI $H(z^{-1})$ is Nyquist-M if and only if for all n and $x(n)$, $y(Mn)=x(n)$. Obviosly, only the M-th samples of the input and output of $H(z^{-1})$ in Fig.7 are pertinent to this requirement.

4.3 Optimum Energy Compaction with Nyquist-M Filter

In the following section, reader's attention shall be turned to the Nyquist-M property of the blocked submatrices $\widetilde{H}_{00}(z^{-1})$, $\widetilde{H}_{01}(z^{-1})$,…, $\widetilde{H}_{0(M-1)}(z^{-1})$ respectively, as they were defined in (1.20) through (1.22).

Theorem 4.1 Consider an N-periodic analysis filter $H(k,z^{-1})$, its adjoined filter $H^a(k,z^{-1})$ and blocked sub-matrices $\widetilde{H}_{mn}(z^{-1})$, defined in (1.20) and (1.21) for $m,n=0,1,\ldots,M-1$. Then $H(k,z^{-1})H^a(k,z^{-1})$ is Nyquist-M if and only if (4.7) holds.

Proof: Consider a structure in Fig.8 and define for $i=0,1,\ldots,M-1$,

$$r_i(k)=r(Mk-i) \quad . \tag{4.8}$$

Define the N-fold blocked version of $r_i(k)$ as

$$\widetilde{r}_i(k)=\left[r(MNk-i),r(M(Nk-1)-i),\ldots,r(M(Nk-N+1)-i)\right]^T \tag{4.9}$$

and

$$\widetilde{r}(k)=\left[\widetilde{r}_0^{\,T}(k),\widetilde{r}_1^{\,T}(k),\ldots,\widetilde{r}_{M-1}^{\,T}(k)\right]^T \quad . \tag{4.10}$$

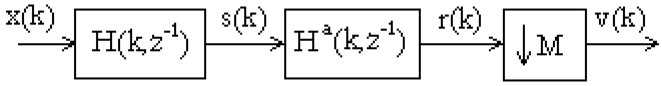

Figure 8: Concatenation of the Analysis Filter, its Adjoined Counterpart and the M-fold Decimator

Then the LTI system relating $\tilde{s}(k)$ to $\tilde{r}(k)$, has transfer function

$$\tilde{E}^a(z^{-1}) = \begin{bmatrix} \tilde{H}^H{}_{00}(z) & . & . & \tilde{H}^H{}_{(M-1)0}(z) \\ . & & . & . \\ . & & . & . \\ \tilde{H}^H{}_{0(M-1)}(z) & . & . & \tilde{H}^H{}_{(M-1)(M-1)}(z) \end{bmatrix} . \quad (4.11)$$

Thus, the system relating $\tilde{x}_0(k)$ to $\tilde{r}_0(k)$ has transfer function

$$\left[\tilde{E}^a(z^{-1}).\tilde{E}(z^{-1})\right]_{00} = \tilde{H}_{00}(z^{-1})\tilde{H}^H_{00}(z) + \tilde{H}_{10}(z^{-1})\tilde{H}^H_{10}(z) + .. + \tilde{H}_{(M-1)0}(z^{-1})\tilde{H}^H_{(M-1)0}(z) .$$

The result then establishes analogy to the LTI case mentioned earlier.

Definition 4.2: ***Optimum Compaction Process*** Consider Fig.4, with *x(k)* WSCS with period N, and $H(k,z^{-1})$ LPTV with period N. Then, $H(k,z^{-1})$ is an optimum compaction filter for *x(k),* if subject to being Nyquist-M and for some index set $\{k_0, k_1, ...k_i ..., k_{N-1}\} = \{0,1,...N-1\}$, it simultaneously maximizes the partial sum of variances of *v(k)* : $\sum_{i=0}^{l} \sigma_v^2(k_i)$, for all $0 \le l \le N-1$.

It can be observed, that the above definition targets to accommodate the fact that *v(k)* is WSCS with period N. Consequently, N variance values in total have to be considered. One can call the optimum compaction filter a *canonical filter*, if $k_i = i$. It shall be noted however, that even canonical filter is not unique. This is consistent with properties of compaction filters for WSS processes [4]. The main result of this section follows.

Theorem 4.2 Recall Fig.1 with $H_i(k,z^{-1})$ N-periodic and *x(k)* WSCS with period N. Then the $H_0(k,z^{-1})$ provided by the solution to the Theorem 3.2 is an N-periodic canonical optimum compaction filter for *x(k).*

Proof: Consider any N-periodic canonical optimum compaction filter $H(k,z^{-1})$ of signal *x(k).* Consider the $N \times MN$ matrix

$$\tilde{H}_0(e^{-j\omega}) = \left[\tilde{H}_{00}(e^{-j\omega}), \tilde{H}_{01}(e^{-j\omega}), ..., \tilde{H}_{0(M-1)}(e^{-j\omega})\right] \quad , \quad (4.12)$$

with $\tilde{H}_{mn}(z^{-1})$ for *m,n=0,1,...,M-1*, defined in (1.20) and (1.21). It can be observed that by Definition 4.2 , $H(k,z^{-1})$ is Nyquist-M filter. Hence, referring to Theorem 4.1, condition (4.7) holds. For $H(k,z^{-1})$ being a canonical optimum compaction filter, the sum of partial variances of subband signals

$$\sum_{i=0}^{l}\int_{0}^{2\pi}\left[\widetilde{H}_0(e^{-j\omega})S_{\overline{X}}(\omega)\widetilde{H}_0^H(e^{-j\omega})\right]_{ii} d\omega \quad , \tag{4.13}$$

has to be maximized for all $0 \le l \le N-1$. The expression inside brackets represents a unitary transform of positive semi-definite Hermitian symmetric matrix $S_{\bar{x}}(\omega)$. Due to the Fact 3.1, Definition 3.2 and under (4.7),

$$\sum_{i=0}^{l}\left[\widetilde{H}_0(e^{-j\omega})S_{\overline{X}}(\omega)\widetilde{H}_0^H(e^{-j\omega})\right]_{ii} \tag{4.14}$$

is maximized for all $0 \le l \le N-1$, if and only if

$$\left[\widetilde{H}_0(e^{-j\omega})S_{\overline{X}}(\omega)\widetilde{H}_0^H(e^{-j\omega})\right]_{ii} = \lambda_i(\omega) \quad . \tag{4.15}$$

When arguing in a similar way as in the proof of Lemma 3.3, the implication is such that partial sums in (4.13) are simultaneously maximized if and only if

$$\widetilde{H}_0(e^{-j\omega})S_{\overline{X}}(\omega)\widetilde{H}_0^H(e^{-j\omega}) = diag\{\lambda_0(\omega), \lambda_1(\omega), ..., \lambda_{N-1}(\omega)\} \, , \tag{4.16}$$

where $\lambda_n(\omega) \ge \lambda_{n+1}(\omega)$ for almost all ω. This is condition from Theorem 3.2, met by Nyquist-M $H_0(k, z^{-1})$. Hence the result.

Through the above derivation, we have approached the expression for blocked polyphase representation (1.22). Indeed, $\widetilde{H}_0(e^{-j\omega})$ stands for the first row of $\widetilde{E}(e^{-j\omega})$. Other rows can be obtained in a straightforward way via interchanging of subscript in $H_i(k, z^{-1})$. However, recalling remarks in section 3, it is (3.29) that represents optimum solution with respect to maximized coding gain. The optimum compaction filter according to Theorem 4.2 is truly optimum solution for M=2 only, since for this particular case, the permutation matrix Γ is uniquely fixed by (2.14) and (2.15).

5. FILTER BANK MODEL FOR WSCS SIGNALS

In this chapter, the author's aim is to show that application of a subband coder with an LPTV maximally decimated filter bank, leads to the coding gain greater then unity. It shall be underlined from the very beginning, that the aim to show is quite different from the aim to prove. While previous chapters 3 and 4 treated the problem in a consistent way, in order to provide most general solution, following paragraphs will focus on a very particular example of a two-channel LPTV maximally decimated filter bank with two types of synthesized WSCS input signals.

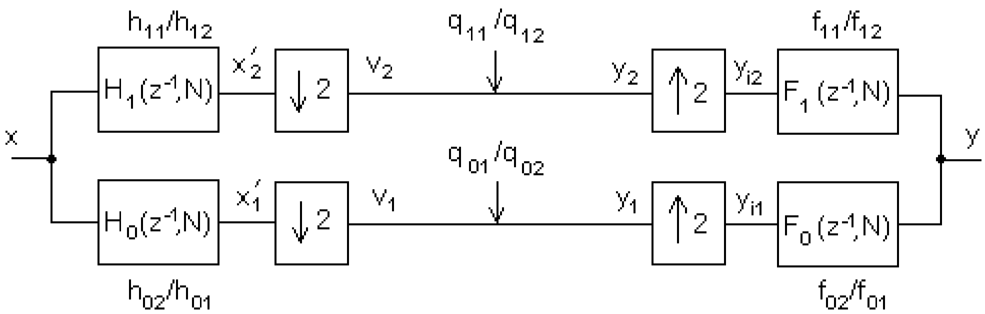

Figure 9: Two-channel Maximally Decimated Filter Bank Used for Simulation

A topology of the two-channel filter bank has already been introduced in subsection 2.2, here it is depicted once more in Fig. 9. In the above structure, LPTV analysis FIR filters $H_i(z^{-1},N)$ and synthesis filters $F_i(z^{-1},N)$, represented by their respective impulse responses h_{i1}, h_{i2} and f_{i1}, f_{i2}, are pertinent to be used for WSCS signal with N=2. In other words, a WSCS input waveform x, satisfying (1.1) for N=2, is processed by the filters alternating between impulse responses h_{i1} and h_{i2}, as well as between f_{i1} and f_{i2}.

While action of decimators and interpolators has already been described in this work, quantities q_{i1}, q_{i2} substitute for a quantizing noise in the Matlab model. The average power of the quantizing noise depends on the statistics of the particular subband signal v_i and number of bits allocated to the A/DC, as governed by (1.2). Due to the *Periodically Dynamic Bit Allocation* introduced in chapter 1.2, the simulated source of quantizing noise also alternates between values q_{i1} and q_{i2}.

Since it is not a trivial task to obtain a real cyclostationary signal with period N=2 to serve as an input for the subbband coder, two artificial signals were synthesized instead, only one of them strictly adheres to the condition specified by (1.1). The first input signal

$$x_1(2i-1) = 1.\sin\left(\frac{2\pi}{8}i\right) \quad , \tag{5.1a}$$

$$x_1(2i) = 0{,}5.\sin\left(\frac{6\pi}{8}i\right) \quad , \tag{5.1b}$$

for which $E[x(k)] = m_x(k) = m_x(k+2)$ and $E[x(k)x(l)] = R_x(k,l) = R_x(k+2,l+2)$. The second input signal

$$x_2(i) = 1.\sin\left(\frac{\pi}{8}i\right) + 0{,}25.\sin\left(\frac{7\pi}{8}i\right) \text{ for } 0 \le i(\mathrm{mod}\,2N) \le N \quad , \tag{5.2a}$$

$$x_2(i) = 0{,}75.\sin\left(\frac{\pi}{8}i\right) + 0{,}5.\sin\left(\frac{7\pi}{8}i\right) \text{ for } N \le i(\mathrm{mod}\,2N) \le 2N \, , \tag{5.2b}$$

for which $m_x(k) = m_x(k+1) = \ldots = m_x(k+N-1) = m_x(k+2N) = \ldots = m_x(k+3N-1)$, $R_x(k,l) = R_x(k+1,l+1) = \ldots = R_x(k+N-1,l+N-1) = \ldots = R_x(k+3N-1,l+3N-1)$, where N is sufficiently large number.

Both the above-presented signals are no doubt purely deterministic waveforms, hence being stationary in nature. But for the purpose of subsequent simulations, interlaced sinusoidal waveforms will serve as a trivial representation of WSCS signal, each sinusoid representing a WSS element within a period of cyclostationarity. One has to note, that signal represented by (5.1) has some limitations as for the flexibility of shaping its spectral components for testing purposes. In fact, it represents a sum of interpolated sinusoids (interpolation coefficient I=2), which results in a symmetry of its spectrum around relative angular frequency $\pi/2$. It shows up, that the symmetry prevents the tester to explore some of the consequences implied by Theorem 3.2.

More flexibility, as far as spectral shaping is concerned, is provided by signal represented by (5.2), in which interlaced samples are substituted by interlaced blocks of samples of length N. Such violation of (1.1) significantly affects possible interpretation of the results acquired on such a signal and must be treated carefully.

5.1 Quadrature Mirror Filter-bank

The core element in the test structure used to demonstrate subband coder functionalities is a power symmetric Quadrature Mirror Filter-bank (QMF), the basic topology of which has already been introduced in Fig. 5-A, B, (with omitted index k, since WSS structure is concerned in this subsection). Power symmetric QMF itself represents a special class of alias free digital systems with low complexity and reasonably low amplitude distortion. The distortion function of alias-free QMF can be written as

$$T(z) = \frac{1}{2}[H_0(z)H_1(-z) - H_1(z)H_0(-z)] \quad . \tag{5.3}$$

Invoking a *power symmetry* condition for $H_0(z)$, which calls for

$$\widetilde{H}_0(z)H_0(z) + H_0(-z)\widetilde{H}_0(-z) = 1 \quad , \tag{5.4}$$

where $\widetilde{H}_0(z)$ stands for $H^*(1/z^*)$, and putting $H_1(z) = H_0(-z)$ we get the allpass property as required by (1.16). On the unit circle

$$\left|H_0(e^{j\omega})\right|^2 + \left|H_1(e^{j\omega})\right|^2 = 1 \quad . \tag{5.5}$$

The product of $\widetilde{H}_0(z)H_0(z)$ is a *half-band filter,* satisfying $\widetilde{H}_0(z)H_0(z)\Big|_{\downarrow 2} = 0.5$.

As a matter of fact, the assumption of $H_1(z) = H_0(-z)$ is excessively strong for a perfect reconstruction system to be designed. Assuming now that $H_0(z)$ is power symmetric, compare (5.4) with (5.3). It can be seen immediately, that if the filter $H_1(z)$ is designed such that

$$H_1(z) = -z^{-N}\widetilde{H}_0(-z) \quad , \tag{5.6}$$

for some odd N, then (5.3) is reduced into $T(z) = 0.5z^{-N}$, i.e. to the perfect reconstruction system. To achieve a realizable system, filters $H_0(z)$, $H_1(z)$ have to be FIR (to avoid instability of their paraconjugate counterparts). The synthesis filters are given by [4]

$$F_0(z) = z^{-N}\widetilde{H}_0(z) \quad \text{and} \quad F_1(z) = z^{-N}\widetilde{H}_1(z) \quad . \tag{5.7}$$

All the above formulae for analysis filter $H_1(z)$ and synthesis filters can be respectively rewritten in the time domain as

$$h_1(k) = (-1)^k h_0^*(N-k) \quad , \tag{5.8a}$$

$$f_0(k) = h_0^*(N-k) \quad , \tag{5.8b}$$

$$f_1(k) = (-1)^k h_0(N-k) \quad . \tag{5.8c}$$

Once filter $H_0(z)$ is causal, one can see that all other filters are causal, as long as $N \geq$ *order of* $H_0(z)$. Summary of the results of this subsection, pertinent for the purpose of this dissertation, then dwells in the following statement:

Let's have a power symmetric and real coefficient FIR low-pass analysis filter

$$H_0(z) = \sum_{k=0}^{N} h_0(k)z^{-k} \ .$$

Then the design of the high-pass analysis filter $H_1(z)$, *as well as design of both the synthesis filters of a perfect reconstruction QMF subband coder is governed by (5.8).*

All the QMF results presented above, are related to a WSS signal *x(k)*. To accommodate for WSCS signals, the QMF has to be incorporated into more complex LPTV structure, in which time varying filter coefficients will provide processing power in accordance with results in sections 3 and 4. Such structure, still simple enough to remain easy to use for the simulation purposes, will be introduced in subsection 5.3.

5.2 Design of the Power Symmetric Filters

A creative part of the design procedure is mostly limited to the construction of a real coefficient low-pass and power symmetric filter $H_0(z)$. Four filters were tested by the author of this dissertation, the basic parameters of them are reviewed in Table 1.

Design number	Min. stopband attenuation	Transition bandwidth
1	30 dB	0.1π
2	30 dB	0.3π
3	20 dB	0.1π
4	20 dB	0.3π

Table 1: Basic Specifications of Analysis Filters Used for Simulation

Resulting magnitude responses of both the low-pass and high-pass analysis filters are shown in Fig. 10, for design #1 ($A_{HOS} = 30dB$, transition bandwidth 0.1π).

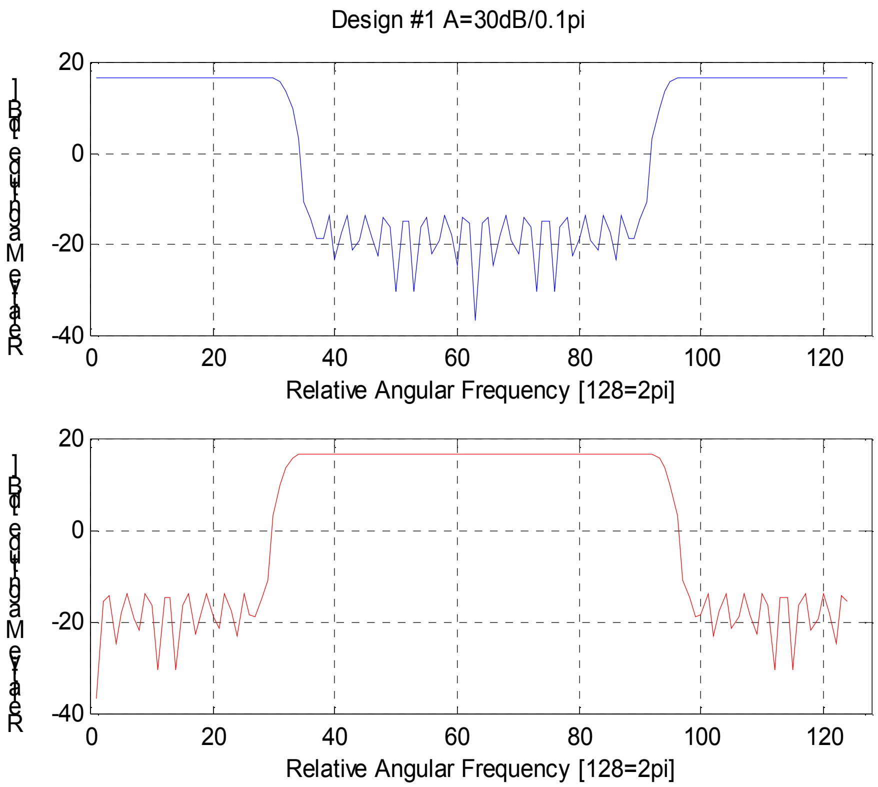

Figure 10: Amplitude Responses of the Low-pass and High-pass Analysis Filters Calculated for Design # 1

The amplitude distortion function $|T(e^{j\omega})|$ as defined by (5.3), is depicted in last figure of this subsection. For the sake of lucidity, the frequency-domain responses in Fig.11 were normalized, by dividing each sample of sequences $h_0(k)$, $h_1(k)$, $f_0(k)$, $f_1(k)$ by the square root of the mean value of $|T(e^{j\omega})|$ calculated for a particular design. This operation will finally avoid undesired gain of the QMF structure, which might distort the simulation results.

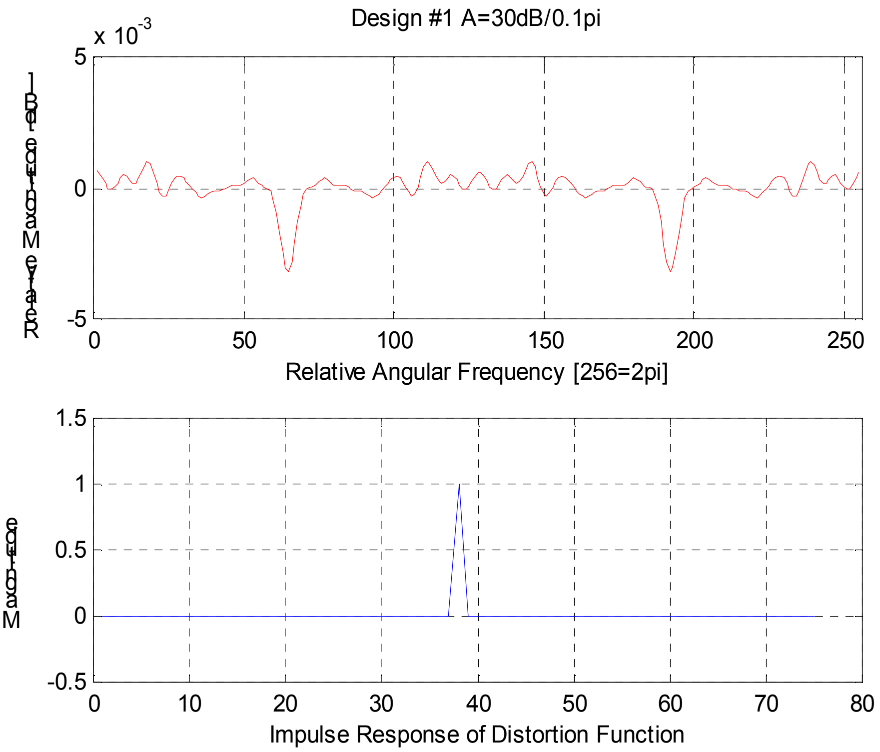

Figure 11: Distortion Function of the QMF Bank Calculated for Design # 1

Looking at the peak values of amplitude ripple of the distortion function $|T(e^{j\omega})|$, one can conclude, that the designed power symmetry QMF is indeed very close to the perfect reconstruction system. The narrow peak of the time-domain impulse response evidently supports such a conclusion (position of the impulse goes on account of processing delay).

5.3 Description of the Simulation Model

A functioning of the simulation model targets fulfillment of requirements specified in the main result of section 3, Theorem 3.2. In other words, based on a particular type of input signal, the simulation model shall manipulate its spectral components in such a way, to achieve decorrelation of the subband signals and appropriate ordering of their power spectra.

Basic topology of the simulation model for WSCS-like signal is depicted in Fig. 12. The outward QMF, which consists of the filters $H_0(z^{-1})$, $H_1(z^{-1})$, $F_0(z^{-1})$, $F_1(z^{-1})$ and works with input signal x and output signal y, cuts the total frequency band by half, allocating low-frequency and high-frequency components to the subband signals x_1' and

x_2', respectively. Indeed, using two non-overlapping filters in the analysis bank brings decorrelation to the subband signals. Since for the chosen testing inputs, as defined by (5.1a) through (5.2b), the energy of the train of sinusoidal waveforms is well concentrated far from $\pi/2$, the decorrelation is almost perfect.

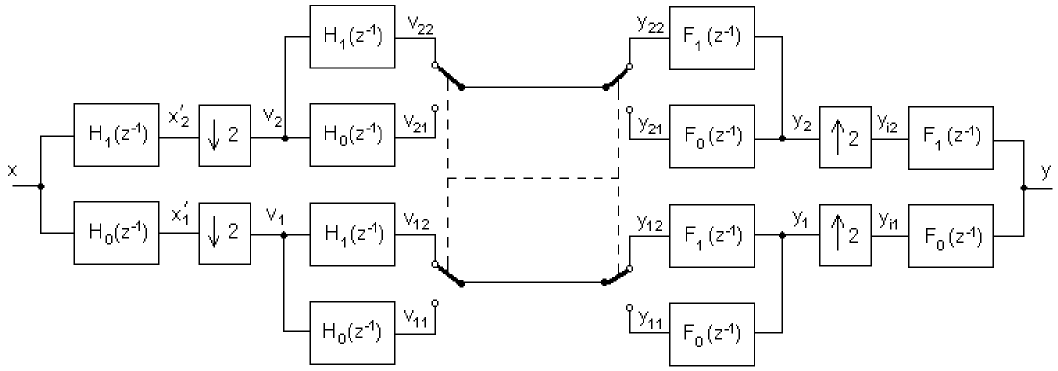

Figure 12: Topology of the Simulation Model with Tree-structured QMF

The purpose of the imbedded QMF structure, having its outputs connected to switches, dwells in the decorrelation of the frequency components that originate from samples (or possibly blocks of samples) arriving to the input of the WSCS subband coder in different time instants, within a period o cyclostationarity.

Looking at (5.1), one can notice, that the input signal x_1' has only two quasi-discrete frequency lines in the frequency band $(0,\pi/2)$, each being originated by signal coming from different time instant within the period of cyclostationarity, and they are symmetrical around $\pi/4$. The same holds for the frequency band $(\pi/2,\pi)$. A similar fact can be observed on (5.2), if one considers blocks of N samples, instead of individual samples.

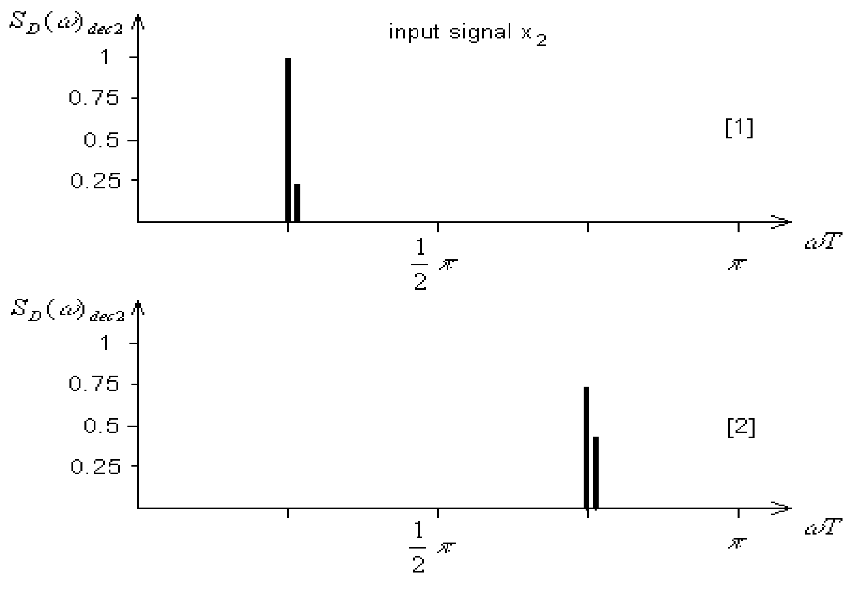

Figure 13: Ordering of the Subband Signals for Input Signal x2

By using synchronous switching of the time-multiplexed components of the input signal, the simulation structure effectively creates four non-overlapping filters of equal bandwidth, that separate frequency components of the input signal, hence decorrelating them in the sense required by Theorem 3.2, condition 1.

Frequency spectra of the decimated subband signals, calculated for the input signal (5.2) are depicted in Fig. 13, using relative frequency scale ωT. The upper figure [1] shows situation during the first half of the period of cyclostationarity, as defined by (5.2a), the lower one [2] shows situation during the second half. One can notice that the image of the high frequency component of the original input signal falls to the same relative frequency at which the low frequency component is located. This fact holds for both half-periods of the period of cyclostationarity. Each of the two components is present in different channel of the subband coder, since they were separated by the outward QMF.

5.4 Quantizing Noise Model Description

To evaluate the coding gain reached by the application of subband coding on a particular input signal, the appropriate quantizing noise model has to be implemented into Matlab code. The values of coding gain, as defined by (1.10a) and calculated for a plethora of input signals, can finally prove the robustness of the presented theory, or make her fall into oblivion.

The quantizing noise has been generally defined by (1.2). In the simulation model, a random noise with uniform probability density function is generated by *rand()* function from Matlab tool-kit. Values of the noise are distributed within amplitude interval $\langle -0.5;0.5 \rangle$, the variance is adjusted via suitable multiplicative constant, depending on the magnitude of the particular subband signal. Such noise model is believed to represent well a real A/D converter.

Values of the quantizing noise are calculated using (1.2) and listed in Table 2 for test input signal x_2 (constant c was neglected).

Input signal x2				
Subband	Amplitude	Power	Allocated bits	Quantizing Noise
lower-low	1	1	9	3,81E-06
higher-low	0,75	0,5625	8,3	5,66E-06
lower-high	0,5	0,25	7,7	5,78E-06
higher-high	0,25	0,0625	7	3,81E-06

Table 2: Variance and Quantizing Noise Values for Test Input Signal x2

5.5 Analysis of the Results

Results gained from Matlab simulation model, that carries out processing of WSCS signal by a subband coder with maximally decimated filter banks, are presented here-in-after within text and in graphical charts. While numbers allow the reader to compare most testifying quantity associated with a subband coder, *the coding gain*, graphical outputs show changes of the input signal, as far as perfect reconstruction is concerned. Topology and functionality of the simulation model has been thoroughly described in previous subsections.

Coding gain acquired for test input signal x_2 *reached 10.40 dB for design #1, 10.32 dB for design #2, 10.30 dB for design #3 and 10.19 dB for design #4*. See all four power-symmetric filter designs specified by parameters in Table 1. For all four designs examined, the coding gain exceeds 10 dB and varies a little with changing the filter order. This insensitivity to change of the filter order shall be contributed to the narrowband character of the spectra of test input signal, which doesn't contain much energy in the vicinity of subband borders. Three simulations were performed for each design and averages were calculated. Value of **10,3 dB** of the coding gain represents almost 2-bit savings in comparison with standard PCM coders. Reader shall kindly recall from Section 1 that each bit contributes by 6 dB to the signal to quantizing noise ratio. Respective time waveforms and spectra of both the input signal x_2 and output o the synthesis filter bank are depicted in Fig. 15, for filter design 1. See again Table 1 for design specifications.

Noticeable discontinuities in the output waveform originate in the transients caused by switching between signal blocks of the length N. They are more profound for higher filter orders as the delay chain of FIR filter increases. Fundamental explanation of the discontinuities dwells in violation of (1.1) by the test input signal x_2, as already emphasized in the beginning of section 5. Indeed the test input signal x_2 serves for evaluating the coding gain for suitable arrangement of input spectra and doesn't fall within the class of signals to be processed by the multi-rate subband coder for WSCS signals. On the other hand, in frequency domain, the distortion of spectra is far from being obvious. Despite of the above-mentioned constrictions, the coding gain obtained from simulation results for x_2 induce some promising expectations.

Results calculated for test input signal x_1*, which strictly obeys definition (1.1) of the WSCS signal are 6.33 dB for design #1, 6.06 dB for design #2, 6.10 dB for design #3 and 5.98 dB for design #4.* Unlike in the previous case, the coning gain reaches the values around **6 dB**, representing just one bit savings with respect to standard PCM coder.

Due to the symmetry of spectra with respect to $\pi/2$ coming from the nature of test input signal x_1 (sum of two interpolated sinusoidal trains), the coding cannot benefit from pairing of the subband signals as defined by (2.14), (2.15). In fact both lines in each pair formed within the period of cyclostationarity N=2 have the same energy. Hence the coding gain of 6 dB shall be fully attributed to the decorrelation of subband signals. This result allows the reader to build a specific feeling for judging individually the importance of the rules specified in Theorem 3.2.

Time waveforms depicted in Fig. 14, for filter design #1 indicate perfect reconstruction without discontinuities. Simulated quantizing noise in the output is clearly visible due to the multiplication constant *1000,* which was incorporated to the Matlab code to minimize round-off errors while calculating with small numbers. In reality, the quantizing noise would not be visible. On the other hand, the output spectrum shows small deformations dependent on the order of the filters used in the filter-bank.

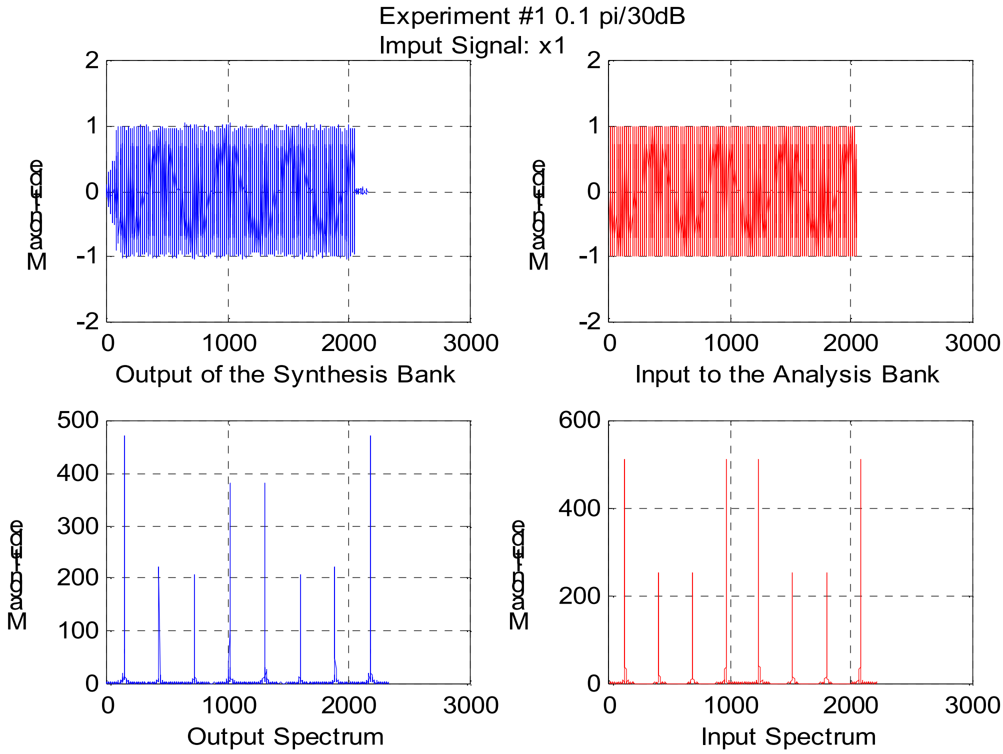

Figure 14: Output and Input Waveforms plus Spectra of the Test Signal x1 for Filter-bank Design #1

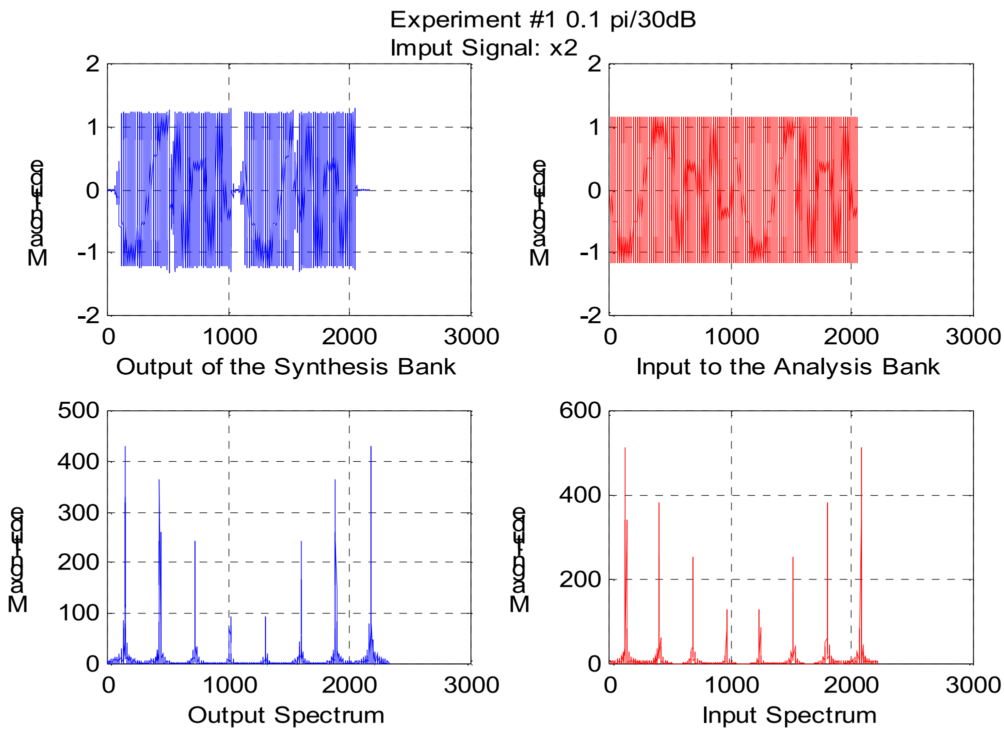

Figure 15: Output and Input Waveforms plus Spectra of the Test Signal x2 for Filter-bank Design #1

6. CONCLUSION

This dissertation aimed to develop a consistent theory of subband coding of zero mean wide-sense cyclostationary signals, with N-periodic statistic. An M-channel, orthonormal, uniform, maximally decimated filter bank, employing linear periodically time-varying filters, has been used to form the subband coder.

Using an average variance to evaluate the output distortion, two fundamental conditions were determined for the optimal subband coding of WSCS signals with period of cycloctationarity N. These conditions first require the blocked subband signals to be totally decorrelated. Secondly, the subband power spectrum densities have to obey specific ordering, in order to achieve maximized coding gain, although possibly through a different and frequency invariant permutation, than that indicated by the canonic ordering. The “best ordering” permutation itself has to be found numerically, for each type of input signal statistics. Only for two-channel subband coder, general and unique “best ordering” permutation can be found. These conditions form the main result of section 3.

In section 4, the optimum solution was tied to optimum compaction process. A N-periodic Nyquist-M filter has been defined and studied to establish analogy to well described WSS subband coder solutions. Through the Nyquist-M filter, a canonical N-periodic optimum compaction process for WSCS signal with period N, has been identified as a possible solution fulfilling conditions specified in the previous section.

Few simulations were carried out in Matlab to support expectations emerging from theoretical results of this work, using two input test signals processed by 2-channel subband coder for 2-periodic WSCS signals. One can state based on the results presented, the subband coding of WSCS signals brings measurable outcome. Despite of the artificial nature of simple test input signals, the coding gain expectations fall within interval from *6 dB* to *12 dB*, effectively saving one or two bits in comparison with standard PCM coder. Simulations were performed using one of the simplest input signal, with period of cyclostationarity N=2, processed by 2-channel filter bank. For more complex input signals with higher values of N, the benefit of subband coding can possibly be even higher.

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