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GENUINELY MULTI-DIMENSIONAL FINITE VOLUME SCHEMES FOR SYSTEMS OF CONSERVATION LAWS

RYZE MULTI-DIMENZIONÁLNÍ METODA KONEČNÝCH OBJEMŮ PRO SYSTÉMY ZÁKONŮ ZACHOVÁNÍ

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1 INTRODUCTION

Many physical problems can be modeled by systems of hyperbolic conservation laws. For example, time-dependent flow of an inviscid, compressible gas is described by the Euler equations. They are usually used for simulation in turbomachines, as well as in most external flows around a solid body, such as aircraft or spacecraft. Another example of system of conservation laws is provided by the shallow water equations. Many types of flows, not necessarily involving water, can be characterized as shallow water flows. They describe flows of fluids with a free surface under the influence of gravity, where any vertical scale is negligible in comparison to horizontal scales.

Usually systems of conservation laws are nonlinear and cannot be solved analytically. Therefore we are looking for numerical solutions to these systems of partial differential equations. Introduction to the numerical methods are given e.g. by LeVeque [7] or Feistauer [3]. Generally, we have three commonly used numerical techniques: finite difference, finite element and finite volume methods. There are the finite volume methods, which are mostly used for the approximation of conservation laws. The main reason is in their simplicity as well as automatic preservation of conservation principles. Most of the finite volume methods are based on the dimensional splitting and on an approximation of the one-dimensional simplified problem, the so-called Riemann problem. Considering just one-dimensional structures these methods lead correct resolutions. On the other hand, it was pointed out by many authors, see e.g. [1], [4], [7], [9], that this type of methods can produce large errors in the approximation of truly multi-dimensional structures such as circular or oblique shocks.

In the last decades we can find in literature several new truly multi-dimensional methods, e.g. the Method of transport [5], [14], the wave propagation algorithm [8], the method of Brio [1] and the Weighted Average Flux (WAF) method [17]. In the thesis we derive and study new multi-dimensional high-resolution finite volume evolution Galerkin (FVEG) methods for systems of nonlinear hyperbolic conservation laws. Characteristic Galerkin schemes were first considered by Morton [13] and Morton, Childs [2] for scalar problems. A generalization of characteristic Galerkin schemes to hyperbolic system in more than one space dimension were formulated by Ostkamp [15]. Multi-dimensional high-resolution FVEG methods for wave equation system were introduced in 2000 by Lukáčová, Morton and Warnecke, see [9]. Generalization to nonlinear system of the Euler equations is more deeply studied by Lukáčová, Saibertová and Warnecke [11]. In [10] is a new FVEG operator with the full stability (CFL = 1) derived. Application of the FVEG method to the linear system of Maxwell equations of electro-magnetic and implementation of several types of boundary conditions is studied in [19], [12].

The principal reason for developing computational methods which are truly multi-dimensional is that they are more intrinsically tied to the multi-dimensional physics of the flow. In multiply dimension there is, in general, no longer a finite number directions of propagation. The FVEG methods couple a finite volume formulation with approximate evolution Galerkin operators. They are constructed using the so-called bicharacteristics of the multi-dimensional hyperbolic system, such that all of the infinitely many directions of wave propagation are taken into account. In such a way approximate evolution operators for a hyperbolic system under consideration are derived. The first order schemes are obtained using the piecewise constant approximate functions. Second order resolution is obtained with a conservative piecewise bilinear recovery and the second order midpoint rule for the time integration. In both cases integrals along the Mach cone are evaluated exactly. Thus, all of the infinitely many directions of wave propagation are taken into account explicitly. The numerical experiments presented previously as well as in this thesis confirm higher accuracy and good multi-dimensional behaviour of the FVEG schemes.

The thesis is organized as follows. In the first sections we introduce general conservation laws governing motion of compressible fluids. Further, we derive several particularly interesting systems, e.g. the Euler equations of fluid dynamics, the wave equation of acoustic and the shallow water equations. In the Chapter 2 we present basic theoretical properties of hyperbolic conservation laws. We define the concept of a weak entropy solution and study the so-called Riemann problem for the shallow water equations.

Chapter 3 is devoted to the finite volume methods, particularly we present some truly multi-dimensional approaches: the wave propagation algorithm of LeVeque and the method of transport of Fey and Noelle.

The FVEG method is presented in Chapter 4. We rewrite the derivation of several approximate evolution operators. The main goal of thesis is the generalization of the FVEG methods to nonlinear hyperbolic conservation laws. We deal particularly with the Euler equations of gas dynamics as well as the shallow water equations and present the derivation of the EG operators and the linearization technique for the FVEG methods. A rigorous derivation of the approximate evolution operator EG5, using piecewise constant and piecewise bilinear data, for the Euler equations is a new result presented here for the first time.

In Chapter 5 we present more closely construction of the higher (second) order schemes. Several recovery techniques, conservative discontinuous or continuous bilinear recovery as well as recovery by means of incomplete biquadratic splines are studied. Further, we study the L^1-stability for a simplified problem of two-dimensional advection equation. The main point is to consider several suitable numerical quadratures for the approximation of cell integrals. We have shown that the trapezoidal rule leads to unconditionally unstable scheme, whereas Simpson's rule yields conditionally stable scheme. The CFL-stability condition is derived. In Section 5.3 the error analysis is presented. We study the global error as well as the linearization error in time. For linearized systems, such as the linearized Euler equations or the shallow water equations, as well as for the linear wave equation system with advection the truncation error is considered. We recall the results of our recent paper [11], where the first and the second order truncation error analysis is done for the finite volume EG1-EG3 schemes. Moreover, we present there a new result concerning the truncation error analysis of the finite volume EG5 scheme using piecewise constant as well as piecewise linear data.

Numerical experiments are presented in Chapter 6. Our aim was to demonstrate that the new finite volume evolution Galerkin schemes yield qualitatively correct resolutions, have good multi-dimensional behaviour as well as particularly high accuracy. All our numerical results confirm these conclusions. We have compared numerical solutions obtained by the FVEG schemes with the results of other commonly known schemes. The FVEG schemes yield qualitatively analogous results to the LeVeque scheme, however note a much higher accuracy of the FVEG schemes, which was reported for the linear problems in [10]. We have also shown that the FVEG schemes correctly resolve steady contact discontinuity (one-dimensional and two-dimensional static disc problem), which is not case of the MoT method, see [6].

In summary, the new finite volume evolution Galerkin schemes are promising truly multi-dimensional schemes having particularly high accuracy at least for linear problems and reasonable CFL stability conditions.

2 AIM OF WORK

The main goal of this thesis is the generalization of the finite volume evolution Galerkin methods, introduced in [9] for linear hyperbolic system of wave equation, to nonlinear hyperbolic conservation laws. First we concentrate our attention to the derivation of exact integral representations and approximate evolution operators for the Euler equations of gas dynamics, the shallow water equations and wave equation with advection. Next we show how to apply the approximate evolution operators in the finite volume framework in order to derive the finite volume evolution Galerkin schemes.

3 EULER EQUATIONS

In this section we consider the Euler equations. They can be written in a form of the system of nonlinear hyperbolic conservation laws that govern the dynamics of compressible materials, such as gases or liquids at high pressure, for which the effect of body forces, viscous stresses and heat flux can be neglected.

There is some freedom in choosing a set of variables to describe the flow under consideration. A possible choice is the so called primitive variable, namely the density $\rho(x, y, t)$, the pressure $p(x, y, t)$, the x-component of velocity $u(x, y, t)$ and the y-component of velocity $v(x, y, t)$. Another possibility is to take the conservative variables, which consist of the density ρ, the x-component of momentum ρu, the y-component of momentum ρv and the total energy E. Physically, these conserved quantities result naturally from the application of the fundamental laws of conservation of mass, of momentum and of energy.

Conservative formulation

The conservative formulation of the Euler equations, in differential form, in two space dimensions is

$$\underline{U}_t + \underline{F}(\underline{U})_x + \underline{G}(\underline{U})_y = 0, \tag{3.1}$$

where $\underline{U}$ is the vector of conserved variables, $\underline{F}(\underline{U})$ and $\underline{G}(\underline{U})$ are the fluxes. They are given as

$$\underline{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix}, \ \underline{F}(\underline{U}) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(E+p) \end{bmatrix}, \ \underline{G}(\underline{U}) = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(E+p) \end{bmatrix},$$

respectively. The equation of state for a polytropic gas yields

$$E = \rho \frac{u^2 + v^2}{2} + \frac{p}{\gamma - 1}$$

with $\gamma = \frac{c_p}{c_v}$ denoting the ratio of specific heats, $\gamma = 1.4$ for dry air.

Primitive variable formulation

If we rewrite the system (3.1) in the form of primitive or physical variables $W = (\rho, u, v, p)^T$. It has the form

$$\underline{W}_t + \underline{\underline{A}}_1(\underline{W})\underline{W}_x + \underline{\underline{A}}_2(\underline{W})\underline{W}_y = 0, \tag{3.2}$$

where

$$\underline{W} = \begin{bmatrix} \rho \\ u \\ v \\ p \end{bmatrix}, \ \underline{\underline{A}}_1(\underline{W}) = \begin{pmatrix} u & \rho & 0 & 0 \\ 0 & u & 0 & \frac{1}{\rho} \\ 0 & 0 & u & 0 \\ 0 & \gamma p & 0 & u \end{pmatrix}, \ \underline{\underline{A}}_2(\underline{W}) = \begin{pmatrix} v & 0 & \rho & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & v & \frac{1}{\rho} \\ 0 & 0 & \gamma p & v \end{pmatrix}.$$

For smooth solutions both formulations are equivalent. For solutions containing shock waves, however non-conservative formulation gives incorrect shock solution. In spite of this, non-conservative formulation has some advantages over its conservative counterpart, when analyzing the equations.

4 WAWE EQUATION

The second order wave equation in two space dimensions has the form

$$\frac{\partial^2 w}{\partial t^2} - c^2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = 0. \tag{4.1}$$

This can be rewritten as a first order system of conservation laws. By introducing new variables

$$\phi = w_t, \quad u = -c w_x, \quad v = -c w_y,$$

we obtain after the substitution into (4.1)

$$\underline{W}_t + \underline{\underline{A}}_1 \underline{W}_x + \underline{\underline{A}}_2 \underline{W}_y = 0, \tag{4.2}$$

where the vector $\underline{W} \in \mathbb{R}^3$ and the coefficient matrices $\underline{\underline{A}}_1, \underline{\underline{A}}_2 \in \mathbb{R}^{3\times 3}$ are defined by

$$\underline{W} = \begin{bmatrix} \phi \\ u \\ v \end{bmatrix}, \quad \underline{\underline{A}}_1 = \begin{pmatrix} 0 & c & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \underline{\underline{A}}_1 = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ c & 0 & 0 \end{pmatrix}.$$

Here $c \in \mathbb{R}$, $c = \text{const.}$ denotes the speed of sound.

Wave equation system is sometimes called the acoustic equation system since it describes propagation of acoustic waves in air. The wave equation system creates the key part of the Euler equations. If we linearize the system of the Euler equations (3.2) at some state $(\tilde{\rho}, \tilde{u}, \tilde{v}, \tilde{p})$ and we set $\tilde{\rho} = \frac{1}{\tilde{c}}$, we obtain by freezing the coefficient matrices $\underline{\underline{A}}_1, \underline{\underline{A}}_2$ the system

$$\underline{W}_t + \underline{\underline{A}}_1(\underline{\tilde{W}})\underline{W}_x + \underline{\underline{A}}_2(\underline{\tilde{W}})\underline{W}_y = 0, \tag{4.3}$$

where

$$\underline{W} = \begin{bmatrix} \rho \\ u \\ v \\ p \end{bmatrix}, \underline{\underline{A}}_1(\underline{\tilde{W}}) = \begin{pmatrix} \tilde{u} & \tilde{\rho} & 0 & 0 \\ 0 & \tilde{u} & 0 & \tilde{c} \\ 0 & 0 & \tilde{u} & 0 \\ 0 & \tilde{c} & 0 & \tilde{u} \end{pmatrix}, \underline{\underline{A}}_2(\underline{\tilde{W}}) = \begin{pmatrix} \tilde{v} & 0 & \tilde{\rho} & 0 \\ 0 & \tilde{v} & 0 & 0 \\ 0 & 0 & \tilde{v} & \tilde{c} \\ 0 & 0 & \tilde{c} & \tilde{v} \end{pmatrix}.$$

Now we remove the first row as well as the first column from the Jacobian matrices $\underline{\underline{A}}_1, \underline{\underline{A}}_2$ of system (4.3) and finally move the third equation for pressure to the first row. We obtain the so-called **wave equation system with advection**. Further, if the advection velocities are $\tilde{u} = \tilde{v} = 0$ and $\tilde{c} =$ const. we get the linear wave equation system (4.2). Thus by linearization the Euler equations at some state we can model propagation of disturbances of sound waves around the state $(\tilde{\rho}, \tilde{u}, \tilde{v}, \tilde{p})$.

5 SHALLOW WATER EQUATIONS

Numerical solution of the shallow water equation was one of the earliest applications of computers when they became available in the late 1940's. After then Charney, Fjörtoft and von Neumann produced the first weather forecast by simulating the two-dimensional shallow water equations that describe atmospheric flows. Today, weather forecasting is simulated using tree-dimensional models which incorporate a vast number of physical effects. Further examples of shallow water are rivers with their flood plains, currents along the coast or estuaries influenced by tides and wind, and flows in lakes generated by wind blows. In general, the shallow water equations describe a motion of incompressible fluids. The conditions that the water surface is free and the bottom surface is solid have to be implemented into the model. In conservative variables the shallow water equations read

$$\underline{U}_t + \underline{F}(\underline{U})_x + \underline{G}(\underline{U})_y = \underline{S}(\underline{U}), \tag{5.1}$$

with

$$\underline{U} = \begin{bmatrix} \phi \\ \phi u \\ \phi v \end{bmatrix}, \underline{F}(\underline{U}) = \begin{bmatrix} \phi u \\ \phi u^2 + \frac{g\phi^2}{2} \\ \phi uv \end{bmatrix}, \underline{G}(\underline{U}) = \begin{bmatrix} \phi v \\ \phi uv \\ \phi v^2 + \frac{g\phi^2}{2} \end{bmatrix}, \underline{S} = \begin{bmatrix} 0 \\ g\phi h_x \\ g\phi h_y \end{bmatrix},$$

where $\underline{U}$ is the vector of conservative variables, $\underline{F}(\underline{U})$ and $\underline{G}(\underline{U})$ are flux vectors and $\underline{S}(\underline{U})$ is a source term vector. Here ϕ denotes the depth of water and g is the acceleration of gravity, assumed constant.

We study only the homogeneous case. This strictly hyperbolic system can be written as

$$\underline{U}_t + \underline{F}(\underline{U})_x + \underline{G}(\underline{U})_y = 0. \tag{5.2}$$

Similarly to the Euler equations the shallow water equations can be rewritten into the primitive variable form

$$\underline{W}_t + \underline{\underline{A}}_1(\underline{W})\underline{W}_x + \underline{\underline{A}}_2(\underline{W})\underline{W}_y = 0, \tag{5.3}$$

where

$$\underline{W} = \begin{bmatrix} \phi \\ u \\ v \end{bmatrix}, \quad \underline{\underline{A}}_1(\underline{W}) = \begin{pmatrix} u & \phi & 0 \\ g & u & 0 \\ 0 & 0 & u \end{pmatrix}, \quad \underline{\underline{A}}_2(\underline{W}) = \begin{pmatrix} v & 0 & \phi \\ 0 & v & 0 \\ g & 0 & v \end{pmatrix}.$$

We can notice a common structure of the Jacobian matrices $\underline{\underline{A}}_1$, $\underline{\underline{A}}_2$ of the Euler equations (3.2) and the shallow water equations (5.3). Actually, they are sparse matrices, which diagonals are determined by the advective velocities (u, v). This fact will be used in what follows in order to derive approximate evolution operators.

6 FINITE VOLUME METHOD

Finite volume methods are numerical methods specially designed for the approximation of conservation laws. We restrict our derivation of the FVM onto two-dimensional case using a regular rectangular mesh. But schemes and results presented here can be extended to three-dimensional flow and to other types of grids, as well. For more details we refer the reader to the works [3], [7], [16] or [18].

We consider a general two-dimensional hyperbolic system

$$\underline{U}_t + \underline{F}(\underline{U})_x + \underline{G}(\underline{U})_y = 0 \quad \text{in } \Omega \times (0, \infty), \tag{6.1}$$

where $\underline{F} = (F_1, \dots, F_m)$, $\underline{G} = (G_1, \dots, G_m)$, $F_i, G_i \in C^1(\mathbb{R}^m)$. The computational domain Ω is divided into a finite number of control volumes Ω_{ij}, $\Omega_{ij} = [(i-\frac{1}{2})h_x,\ (i+\frac{1}{2})h_x] \times [(j-\frac{1}{2})h_y,\ (j+\frac{1}{2})h_y] = [x_{i-\frac{1}{2}},\ x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}},\ y_{j+\frac{1}{2}}]$ with a center $S_{ij} = (x_i h_x, y_j h_y)$, where $i, j \in \mathbb{Z}$, $h_x > 0$ and $h_y > 0$ are the mesh size parameters in the x-direction and the y-direction, respectively. Further, we choose a time step $\triangle t > 0$ and set $t_n = n\triangle t$, $n \in \mathbb{Z}^+$.

Denote by $V = \Omega_{ij} \times (t_n, t_{n+1})$ the space-time volume and integrate equation (6.1) over V. Applying Green's theorem on Ω_{ij} we obtain

$$0 = \int_V \partial_t \underline{U}(x,y,t)\mathrm{d}x\mathrm{d}y\mathrm{d}t + \int_V (\partial_x \underline{F}(\underline{U}) + \partial_y \underline{G}(\underline{U}))\mathrm{d}x\mathrm{d}y\mathrm{d}t =$$

$$= \int_{\Omega_{ij}} \underline{U}(x,y,t)\mathrm{d}x\mathrm{d}y \Big|_{t_n}^{t_{n+1}} + \int_{t_n}^{t_{n+1}} \int_{\partial\Omega_{ij}} (\underline{F}(\underline{U})n_x + \underline{G}(\underline{U})n_y)\mathrm{d}S\mathrm{d}t, \tag{6.2}$$

where (n_x, n_y) is the unit outer normal to $\partial\Omega_{ij}$. Now we replace the integrals in (6.2) by their suitable approximations

$$\int_{\Omega_{ij}} \underline{U}(x,y,t_n)\mathrm{d}x\mathrm{d}y \approx h_x h_y \underline{U}_{ij}^n, \tag{6.3}$$

$$\int_{\Omega_{ij}} \underline{U}(x,y,t)\mathrm{d}x\mathrm{d}y \Big|_{t_n}^{t_{n+1}} \approx h_x h_y (\underline{U}_{ij}^{n+1} - \underline{U}_{ij}^n), \tag{6.4}$$

$$\int_{t_n}^{t_{n+1}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} (\underline{F}(\underline{U})n_x + \underline{G}(\underline{U})n_y)\Big|_{x=x_{i+\frac{1}{2}}} \mathrm{d}S\mathrm{d}t \approx \triangle t h_y H_R(\underline{U}_{ij}^n, \underline{U}_{i+1j}^n), \tag{6.5}$$

where $\underline{U}_{ij}^n$ is a constant approximation of the exact solution $\underline{U}$ on the control volume Ω_{ij} at time t_n and the numerical flux $H_R(\underline{U}_{ij}^n, \underline{U}_{i+1j}^n)$ is an approximation to the exact flux function $\underline{F}(\underline{U})n_x + \underline{G}(\underline{U})n_y$ on $\partial\Omega_{ij} \cap \partial\Omega_{i+1j}$. Note that in classical dimensional splitting finite volume schemes the numerical flux $H_R(\underline{U}_{ij}^n, \underline{U}_{i+1j}^n)$ is derived using an approximation to an one-dimensional auxiliary problem in the direction of outer normal. Thus only the values $\underline{U}_{ij}^n$ and $\underline{U}_{i+1j}^n$ are used for the approximation. Analogously other numerical fluxes H_L, H_U, H_B on the left, upper and bottom cell interface are defined, respectively.

Substituting (6.3) - (6.5) into (6.2) we obtain the *finite volume* scheme

$$\begin{aligned} \underline{U}_{ij}^{n+1} = \underline{U}_{ij}^n &- \frac{\triangle t}{h_x}[H_R(\underline{U}_{ij}^n, \underline{U}_{i+1j}^n) - H_L(\underline{U}_{ij}^n, \underline{U}_{i-1j}^n)] \\ &- \frac{\triangle t}{h_y}[H_U(\underline{U}_{ij}^n, \underline{U}_{ij+1}^n) - H_B(\underline{U}_{ij}^n, \underline{U}_{ij-1}^n)]. \end{aligned} \tag{6.6}$$

Let $\underline{\underline{A}}_1(\underline{U})$ and $\underline{\underline{A}}_2(\underline{U})$ be the Jacobian matrices of the fluxes $\underline{F}(\underline{U})$ and $\underline{G}(\underline{U})$, respectively. Since the system (6.1) is hyperbolic there exists a diagonalizable matrix

$$\underline{\underline{P}}(\underline{U}, \underline{n}) = \underline{\underline{A}}_1(\underline{U})n_1 + \underline{\underline{A}}_2(\underline{U})n_2,$$

where $\underline{n} = (n_1, n_2)$ is unit vector, with real eigenvalues $\lambda_i, i = 1, \ldots, m$. This means that there exists a nonsingular matrix $\underline{\underline{T}} = \underline{\underline{T}}(\underline{U}, \underline{n})$ such that

$$\underline{\underline{T}}^{-1}\underline{\underline{PT}} = \underline{\underline{D}}(\underline{U}, \underline{n}) = \mathrm{diag}(\lambda_1, \ldots, \lambda_{\mathrm{m}}).$$

Let us denote by $\lambda^+ = \max(\lambda, 0)$ and $\lambda^- = \min(\lambda, 0)$. Set

$$\underline{\underline{D}}^{\pm} = \mathrm{diag}(\lambda_1^{\pm}, \ldots, \lambda_{\mathrm{m}}^{\pm}),$$

$$\underline{\underline{P}}^{\pm}(\underline{U}, \underline{n}) = \underline{\underline{P}}^{\pm} = \underline{\underline{T}}(\underline{U}, \underline{n})\underline{\underline{D}}^{\pm}(\underline{U}, \underline{n})\underline{\underline{T}}^{-1}(\underline{U}, \underline{n}).$$

Matrices $\underline{\underline{P}}^{\pm}$ are the so-called positive or negative parts of the matrix $\underline{\underline{P}}$, containing only positive or negative eigenvalues, respectively.

For example, in the finite volume method with Steger-Warming numerical flux H_R, H_L, H_U and H_B are define as follows

$$\begin{aligned}
H_R^{sw}(\underline{U}_{ij}, \underline{U}_{i+1j}) &= \underline{\underline{P}}^{+}\left(\underline{U}_{ij}, (1,0)\right)\underline{U}_{ij} + \underline{\underline{P}}^{-}\left(\underline{U}_{i+1j}, (1,0)\right)\underline{U}_{i+1j}, \\
H_L^{sw}(\underline{U}_{ij}, \underline{U}_{i-1j}) &= \underline{\underline{P}}^{+}\left(\underline{U}_{ij}, (-1,0)\right)\underline{U}_{ij} + \underline{\underline{P}}^{-}\left(\underline{U}_{i-1j}, (-1,0)\right)\underline{U}_{i-1j}, \\
H_U^{sw}(\underline{U}_{ij}, \underline{U}_{ij+1}) &= \underline{\underline{P}}^{+}\left(\underline{U}_{ij}, (0,1)\right)\underline{U}_{ij} + \underline{\underline{P}}^{-}\left(\underline{U}_{ij+1}, (0,1)\right)\underline{U}_{ij+1}, \\
H_B^{sw}(\underline{U}_{ij}, \underline{U}_{ij-1}) &= \underline{\underline{P}}^{+}\left(\underline{U}_{ij}, (0,-1)\right)\underline{U}_{ij} + \underline{\underline{P}}^{-}\left(\underline{U}_{ij-1}, (0,-1)\right)\underline{U}_{ij-1}.
\end{aligned}$$

This method belongs to the class of dimensional splitting schemes. Instead of solving a multi-dimensional problem the one-dimensional Riemann problems are solved in the direction of outer normals on cell interfaces. It was reported, see, e.g. [8], [9], that such dimensional splitting finite volume schemes can produce spurious oscillations in solutions, especially when shocks are propagating in directions that are oblique with respect to the orientation of mesh. In what follows we will describe the finite volume evolution scheme, which belongs to the class of genuinely multi-dimensional schemes. These methods are less dependent on mesh orientation and approximate fully multi-dimensional flows in a more accurate way.

7 FINITE VOLUME EVOLUTION GALERKIN SCHEME

In this section we will describe finite volume evolution Galerkin schemes for a general hyperbolic system in two space dimensions.

Let us consider a general hyperbolic conservation law in two space dimensions

$$\underline{U}_t + \sum_{k=1}^{2} (\underline{F}_k(\underline{U}))_{x_k} = 0, \quad \underline{x} = (x_1, x_2)^T \in \mathbb{R}^2, \tag{7.1}$$

where $\underline{F}_k = \underline{F}_k(\underline{U}), k = 1, 2$ represent given physical flux functions and the conservative variables are $\underline{U} = (u_1, \dots, u_m)^T \in \mathbb{R}^m$. Let us denote by $E(\tau) : X \to X$ the exact evolution operator associated with a time step τ acting on a suitable function space X for the system (7.1), i.e.

$$\underline{U}(\cdot, t + \tau) = E(\tau)\underline{U}(\cdot, t). \tag{7.2}$$

We limit our considerations to cases of constant time step Δt, i.e. $t_n = n\Delta t$. For further simplification let us use a uniform mesh consisting of squares of a uniform mesh size h. Generalization to nonuniform meshes can be done and it is only a question of implementation. We suppose that S_h^p is a finite element space consisting of piecewise polynomials of order $p \geq 0$ with respect to the given mesh. Let $\underline{U}^n$ be an approximation in the space S_h^p to the exact solution $\underline{U}(\cdot, t_n)$ at a time $t_n > 0$ and take $E_\tau : S_h^r \to X$ to be a suitable approximation to the exact evolution operator $E(\tau)$, $r \geq 0$. We denote by $R_h : S_h^p \to S_h^r$ a recovery operator, $r > p \geq 0$.

Definition 7.1 *Starting from some initial data $\underline{U}^0$ at time $t = 0$, the finite volume evolution Galerkin method (FVEG) is recursively defined by means of*

$$\underline{U}^{n+1} = \underline{U}^n - \frac{1}{h} \int_0^{\Delta t} \sum_{k=1}^{2} \delta_{x_k} \underline{F}_k(\underline{U}^{n+\tau/\Delta t})\, d\tau, \tag{7.3}$$

where the central difference $v(x + h/2) - v(x - h/2)$ is denoted by $\delta_x v(x)$ and $\delta_{x_k} \underline{F}_k(\underline{U}^{n+\tau/\Delta t})$ represents an approximation to the edge flux difference at intermediate time levels $t_n + \tau$, $\tau \in (0, \Delta t)$. The cell boundary flux $F_k\left(\underline{U}^{n+\tau/\Delta t}\right)$ is evolved using the approximate evolution operator E_τ to $t_n + \tau$ and averaged along the cell boundary, i.e. e.g. on vertical edges for $\underline{U}$ itself

$$\underline{F}_k(\underline{U}^{n+\tau/\Delta t}) = \frac{1}{h} \int_0^h \underline{F}_k(E_\tau R_h \underline{U}^n)\, dS_y. \tag{7.4}$$

This is the new key step in our FVEG methods. Analogous formula holds for horizontal edges.

For the computation of fluxes on cell interface solution of $\underline{U}^{n+\tau/\Delta t}$ has to be determined by means of an approximate evolution operator. If time integral from 0 to Δt is approximated by the midpoint rule and no recovery is used then the whole method is of first order. In this case the finite volume evolution Galerkin scheme (7.3), (7.4) gives

$$\underline{U}^{n+1} = \underline{U}^n - \frac{\Delta t}{h}\sum_{k=1}^{2} \delta_{x_k} \underline{F}_k(\underline{U}^*), \tag{7.5}$$

$$\underline{F}_k(\underline{U}^*) = \frac{1}{h}\int_0^h F_k(E_{\Delta t/2}\underline{U}^n)\mathrm{d}S. \tag{7.6}$$

However, the most important advantage of this formulation is that even a first order accurate approximation E_τ to the evolution operator $E(\tau)$ yields an overall second order update from $\underline{U}^n$ to $\underline{U}^{n+1}$. The second order scheme is obtained by a conservative discontinuous bilinear recovery R_h using the vertex values. The fluxes on cell interface are computed as

$$\underline{F}_k(\underline{U}^*) = \frac{1}{h}\int_0^h F_k(E_{\Delta t/2}R_h\underline{U}^n)\mathrm{d}S. \tag{7.7}$$

8 EXACT INTEGRAL REPRESENTATION AND APPROXIMATE EVOLUTION OPERATORS FOR THE EULER EQUATIONS

In [9] a general procedure for the derivation of integral equations for linear hyperbolic systems in d space dimensions have been described. In order to derive the integral equations for nonlinear hyperbolic systems a suitable linearization has to be done first. This is achieved by freezing the Jacobian matrices at a suitable state.

In this section we derive the integral equations for the linearized system of the Euler equations of gas dynamics and start with the system (3.2) in primitive variables. This is the simplest and most convenient form for studying the bicharacteristics of the system away from shocks and contact discontinuities.

To derive the integral equations we linearize system (3.2) by freezing the Jacobian matrices at a point $\tilde{P} = (\tilde{x}, \tilde{y}, \tilde{t})$. Denote by $\underline{\tilde{W}} = (\tilde{\rho}, \tilde{u}, \tilde{v}, \tilde{p})$ the local variables at the point $\tilde{P}$ and by $\tilde{c}$ the local speed of sound there, i.e. $\tilde{c} = \sqrt{\frac{\gamma\tilde{p}}{\tilde{\rho}}}$. Thus, the linearized system (3.2) has the form

$$\underline{W}_t + \underline{\underline{A}}_1(\underline{\tilde{W}})\underline{W}_x + \underline{\underline{A}}_2(\underline{\tilde{W}})\underline{W}_y = 0, \quad \underline{x} = (x, y)^T \in \mathbb{R}^2. \tag{8.1}$$

The eigenvalues of the matrix pencil $\underline{\underline{P}}(\underline{\tilde{W}}) = \underline{\underline{A}}_1(\underline{\tilde{W}})n_x + \underline{\underline{A}}_2(\underline{\tilde{W}})n_y$, where $\underline{n} = \underline{n}(\theta) = (n_x, n_y)^T = (\cos\theta, \sin\theta)^T \in \mathbb{R}^2$ are

$$\begin{aligned} \lambda_1 &= \tilde{u} \cos\theta + \tilde{v} \sin\theta - \tilde{c} \\ \lambda_2 &= \lambda_3 = \tilde{u} \cos\theta + \tilde{v} \sin\theta \\ \lambda_4 &= \tilde{u} \cos\theta + \tilde{v} \sin\theta + \tilde{c}, \end{aligned}$$

and the corresponding linearly independent right eigenvectors are

$$\underline{r}_1 = \begin{pmatrix} -\frac{\tilde{\rho}}{\tilde{c}} \\ \cos\theta \\ \sin\theta \\ -\tilde{\rho}\tilde{c} \end{pmatrix}, \ \underline{r}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \underline{r}_3 = \begin{pmatrix} 0 \\ \sin\theta \\ -\cos\theta \\ 0 \end{pmatrix}, \ \underline{r}_4 = \begin{pmatrix} \frac{\tilde{\rho}}{\tilde{c}} \\ \cos\theta \\ \sin\theta \\ \tilde{\rho}\tilde{c} \end{pmatrix}.$$

Let $\underline{\underline{R}}(\underline{\tilde{W}})$ be the matrix of the right eigenvectors. The inverse of $\underline{\underline{R}}(\underline{\tilde{W}})$ is

$$\underline{\underline{R}}^{-1}(\underline{\tilde{W}}) = \frac{1}{2} \begin{pmatrix} 0 & \cos\theta & \sin\theta & -1/(2\tilde{\rho}\tilde{c}) \\ 1 & 0 & 0 & -1/\tilde{c}^2 \\ 0 & \sin\theta & -\cos\theta & 0 \\ 0 & \cos\theta & \sin\theta & 1/(2\tilde{\rho}\tilde{c}) \end{pmatrix}.$$

Multiplying system (8.1) by $\underline{\underline{R}}^{-1}(\underline{\tilde{W}})$ from the left we obtain the characteristic system

$$\underline{V}_t + \underline{\underline{B}}_1(\underline{\tilde{W}})\underline{V}_x + \underline{\underline{B}}_2(\underline{\tilde{W}})\underline{V}_y = 0,$$

where

$$\begin{aligned} \underline{\underline{B}}_1(\underline{\tilde{W}}) &= \begin{pmatrix} \tilde{u} - \tilde{c}\cos\theta & 0 & -\frac{1}{2}\tilde{c}\sin\theta & 0 \\ 0 & \tilde{u} & 0 & 0 \\ -\tilde{c}\sin\theta & 0 & \tilde{u} & \tilde{c}\sin\theta \\ 0 & 0 & \frac{1}{2}\tilde{c}\sin\theta & \tilde{u} + \tilde{c}\cos\theta \end{pmatrix} \\ \underline{\underline{B}}_2(\underline{\tilde{W}}) &= \begin{pmatrix} \tilde{v} - \tilde{c}\sin\theta & 0 & \frac{1}{2}\tilde{c}\cos\theta & 0 \\ 0 & \tilde{v} & 0 & 0 \\ \tilde{c}\cos\theta & 0 & \tilde{v} & -\tilde{c}\cos\theta \\ 0 & 0 & -\frac{1}{2}\tilde{c}\cos\theta & \tilde{v} + \tilde{c}\sin\theta \end{pmatrix} \end{aligned}$$

and the characteristic variables $\underline{V}$ are

$$\underline{V} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \underline{\underline{R}}^{-1}(\underline{\tilde{W}})\underline{W} = \begin{pmatrix} \frac{1}{2}(-\frac{p}{\tilde{\rho}\tilde{c}} + u\cos\theta + v\sin\theta) \\ \rho - \frac{p}{\tilde{c}^2} \\ u\sin\theta - v\cos\theta \\ \frac{1}{2}(\frac{p}{\tilde{\rho}\tilde{c}} + u\cos\theta + v\sin\theta) \end{pmatrix}. \qquad (8.2)$$

The quasi-diagonalized system of the linearized Euler equations has the following form

$$
\begin{aligned}
\underline{V}_t \; + & \begin{pmatrix} \tilde{u}-\tilde{c}\cos\theta & 0 & 0 & 0 \\ 0 & \tilde{u} & 0 & 0 \\ 0 & 0 & \tilde{u} & 0 \\ 0 & 0 & 0 & \tilde{u}+\tilde{c}\cos\theta \end{pmatrix} \underline{V}_x \; + \\
+ & \begin{pmatrix} \tilde{v}-\tilde{c}\sin\theta & 0 & 0 & 0 \\ 0 & \tilde{v} & 0 & 0 \\ 0 & 0 & \tilde{v} & 0 \\ 0 & 0 & 0 & \tilde{v}+\tilde{c}\sin\theta \end{pmatrix} \underline{V}_y = \underline{S}
\end{aligned}
\tag{8.3}
$$

with

$$
\underline{S} = \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\tilde{c}(\sin\theta\frac{\partial v_3}{\partial x} - \cos\theta\frac{\partial v_3}{\partial y}) \\ 0 \\ \tilde{c}\sin\theta(\frac{\partial v_1}{\partial x} - \frac{\partial v_4}{\partial x}) - \tilde{c}\cos\theta(\frac{\partial v_1}{\partial y} - \frac{\partial v_4}{\partial y}) \\ \frac{1}{2}\tilde{c}(-\sin\theta\frac{\partial v_3}{\partial x} + \cos\theta\frac{\partial v_3}{\partial y}) \end{pmatrix}.
$$

This procedure was carried out in [9] for the wave equation system. Note that there the advection terms were not consider. These are present in the linearized Euler system. The system (8.3) would reduce to a diagonal system, i.e. $S = 0$, only in the special case when the Jacobian matrices $\underline{\underline{A}}_1$, $\underline{\underline{A}}_2$ commute, which is not the case for the two-dimensional Euler equations.

In what follows we will work with the concept of bicharacteristics. The ℓ-th bicharacteristic $\underline{x}_\ell$ corresponding to the ℓ-th equation of the system (8.1) is defined by

$$
\frac{\mathrm{d}\underline{x}_\ell}{\mathrm{d}t} = \underline{b}_{\ell\ell}(\underline{n}) := (b^1_{\ell\ell}, b^2_{\ell\ell})^T. \tag{8.4}
$$

The set of all bicharacteristics creates the so-called Mach cone, see Figure 1. We integrate the ℓ-th equation of the system (8.1) from the point $P = (x, y, t + \Delta t)$ down to the point $Q_\ell(\theta)$, where the bicharacteristic hits the plane through $P' = (x - \tilde{u}\Delta t, y - \tilde{v}\Delta t, t)$. Integrating the system (8.3) along the bicharacteristics from t up to $t + \triangle t$ gives the relations for the characteristic variables, which after the multiplication from the left by the matrix $\underline{\underline{R}}$ yield the exact integral representation

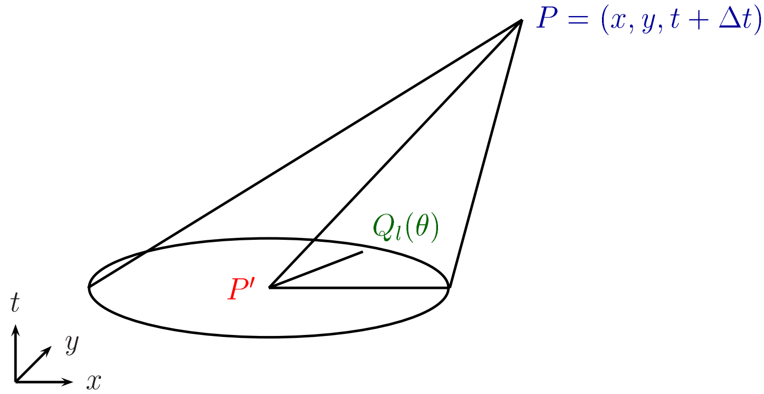

Figure 1: Bicharacteristics along the Mach cone through P and $Q_\ell(\theta)$.

$$
\underline{W}(P) = \frac{1}{2\pi}\int_0^{2\pi}\begin{pmatrix} -\frac{\tilde\rho}{\tilde c}v_1 + v_2 + \frac{\tilde\rho}{\tilde c}v_4 \\ v_1\cos\theta + v_3\sin\theta + v_4\cos\theta \\ v_1\sin\theta - v_3\cos\theta + v_4\sin\theta \\ -\tilde\rho\tilde c v_1 + \tilde\rho\tilde c v_4 \end{pmatrix} d\theta
$$

$$
+\frac{1}{2\pi}\int_0^{2\pi}\begin{pmatrix} -\frac{\tilde\rho}{\tilde c}S_1' + S_2' + \frac{\tilde\rho}{\tilde c}S_4' \\ S_1'\cos\theta + S_3'\sin\theta + S_4'\cos\theta \\ S_1'\sin\theta - S_3'\cos\theta + S_4'\sin\theta \\ -\tilde\rho\tilde c S_1' + \tilde\rho\tilde c S_4' \end{pmatrix} d\theta, \qquad (8.5)
$$

where $S_\ell' = \int_t^{t+\triangle t} S_\ell(\underline{x}_\ell(\tilde t,\theta),\tilde t,\theta)d\tilde t$ is an integral along the ℓ-th bicharacteristic. If we use the facts that $Q_1(\theta+\pi) = Q_4(\theta)$, all $v_\ell(x,y,t,\theta)$ and $Q_\ell(\theta)$, $\ell = 1,..,4$ are 2π-periodic and that $Q_2 = Q_3$ are independent on θ we obtain from (8.5) the following formulae for the exact solution $\underline{W}$ of the linearized system at the point $P = (\underline{x}, t + \Delta t)$. We put $Q := Q_1(\theta)$ and $P' := Q_2$.

$$
\begin{aligned}
\rho(P) &= \rho(P') - \frac{p(P')}{\tilde c^2} \\
&+ \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{p(Q)}{\tilde c^2} - \frac{\tilde\rho}{\tilde c}u(Q)\,\cos\theta - \frac{\tilde\rho}{\tilde c}v(Q)\,\sin\theta\right]\mathrm{d}\theta \\
&- \frac{\tilde\rho}{\tilde c}\frac{1}{2\pi}\int_0^{2\pi}\int_t^{t+\triangle t} S(\underline{x} - (\underline{\tilde u} - c\underline{n}(\theta))(t+\triangle t-\tilde t),\tilde t,\theta)\,\mathrm{d}\tilde t\,\mathrm{d}\theta,
\end{aligned} \qquad (8.6)
$$

$$
\begin{aligned}
u(P) &= \frac{1}{2\pi}\int_0^{2\pi}\left[-\frac{p(Q)}{\tilde{\rho}\tilde{c}}\ \cos\theta + u(Q)\ \cos^2\theta + v(Q)\ \sin\theta\ \cos\theta\right]\ \mathrm{d}\theta \\
&+ \frac{1}{2\pi}\int_0^{2\pi}\int_t^{t+\triangle t}\cos\theta\, S(\underline{x}-(\underline{\tilde{u}}-c\underline{n}(\theta))(t+\triangle t-\tilde{t}),\tilde{t},\theta)\,\mathrm{d}\tilde{t}\,\mathrm{d}\theta \\
&+ \frac{1}{2}u(P') - \frac{1}{2\tilde{\rho}}\int_t^{t+\triangle t} p_x(P'(\tilde{t}))\,\mathrm{d}\tilde{t}, \qquad (8.7) \\
v(P) &= \frac{1}{2\pi}\int_0^{2\pi}\left[-\frac{p(Q)}{\tilde{\rho}\tilde{c}}\ \sin\theta + u(Q)\ \cos\theta\ \sin\theta + v(Q)\ \sin^2\theta\right]\ \mathrm{d}\theta \\
&+ \frac{1}{2\pi}\int_0^{2\pi}\int_t^{t+\triangle t}\sin\theta S(\underline{x}-(\underline{\tilde{u}}-c\underline{n}(\theta))(t+\triangle t-\tilde{t}),\tilde{t},\theta)\,\mathrm{d}\tilde{t}\,\mathrm{d}\theta \\
&+ \frac{1}{2}v(P') - \frac{1}{2\tilde{\rho}}\int_t^{t+\triangle t} p_y(P'(\tilde{t}))\,\mathrm{d}\tilde{t}, \qquad (8.8) \\
p(P) &= \frac{1}{2\pi}\int_0^{2\pi}\left[p(Q) - \tilde{\rho}\tilde{c}u(Q)\ \cos\theta - \tilde{\rho}\tilde{c}v(Q)\ \sin\theta\right]\ \mathrm{d}\theta \\
&- \tilde{\rho}\tilde{c}\frac{1}{2\pi}\int_0^{2\pi}\int_t^{t+\triangle t} S(\underline{x}-(\underline{\tilde{u}}-c\underline{n}(\theta))(t+\triangle t-\tilde{t}),\tilde{t},\theta)\,\mathrm{d}\tilde{t}\,\mathrm{d}\theta, \\
& \qquad (8.9)
\end{aligned}
$$

where $P'(\tilde{t}) = (x-\tilde{u}\triangle t, y-\tilde{v}\triangle t,\tilde{t})$, $(\underline{x}-(\underline{\tilde{u}}-\tilde{c}\underline{n}(\theta))(t+\triangle t-\tilde{t})) =$ $= (x-(\tilde{u}-\tilde{c}\cos\theta)(t+\triangle t-\tilde{t}), y-(\tilde{v}-\tilde{c}\sin\theta)(t+\triangle t-\tilde{t}))$ and $S(\underline{x},t,\theta) :=$ $\tilde{c}[u_x(\underline{x},t,\theta)\sin^2\theta - (u_y(\underline{x},t,\theta)+v_x(\underline{x},t,\theta))\sin\theta\cos\theta + v_y(\underline{x},t,\theta)\cos^2\theta]$.

Note that we have derived the exact integral representation of the solution to the linearized Euler equations (8.1). This is a basis for our further numerical approximations. In derivation of individual approximate evolution operators the most attention need to be put on the approximation of the integrals of S over intermediate time levels and time integrals involving p_x and p_y. Here we present only resulting approximate evolution operators without detail derivations, which were done in Section 4.2 in the thesis.

Approximate evolution operator EG1

$$
\begin{aligned}
\rho(P) &= \frac{1}{2\pi}\int_0^{2\pi}\frac{p(Q)}{\tilde{c}^2} - 2\frac{\tilde{\rho}}{\tilde{c}}u(Q)\cos\theta - 2\frac{\tilde{\rho}}{\tilde{c}}v(Q)\sin\theta\mathrm{d}\theta \\
&+\rho(P') - \frac{p(P')}{\tilde{c}^2} + O(\Delta t^2)
\end{aligned}
$$

$$
\begin{aligned}
u(P) &= \frac{1}{\pi}\int_0^{2\pi} -\frac{p(Q)}{\tilde{\rho}\tilde{c}}\cos\theta + u(Q)(3\cos^2\theta - 1) + 3v(Q)\sin\theta\cos\theta \mathrm{d}\theta \\
&\quad + O(\Delta t^2), \\
v(P) &= \frac{1}{\pi}\int_0^{2\pi} -\frac{p(Q)}{\tilde{\rho}\tilde{c}}\sin\theta + 3u(Q)\sin\theta\cos\theta + v(Q)(3\sin^2\theta - 1)\mathrm{d}\theta \\
&\quad + O(\Delta t^2) \\
p(P) &= \frac{1}{2\pi}\int_0^{2\pi} p(Q) - 2\tilde{\rho}\tilde{c}u(Q)\cos\theta - 2\tilde{\rho}\tilde{c}v(Q)\sin\theta \mathrm{d}\theta + O(\Delta t^2)
\end{aligned}
$$

Approximate evolution operator EG2

$$
\begin{aligned}
\rho(P) &= \frac{1}{\pi}\int_0^{2\pi} \left[\frac{p(Q)}{\tilde{c}^2} - \frac{\tilde{\rho}}{\tilde{c}}u(Q)\ \cos\theta - \frac{\tilde{\rho}}{\tilde{c}}v(Q)\ \sin\theta\right] \mathrm{d}\theta \\
&\quad + \rho(P') - 2\frac{p(P')}{\tilde{c}^2} + O(\Delta t^3) \\
u(P) &= \frac{1}{\pi}\int_0^{\pi} \left[-\frac{p(Q)}{\tilde{\rho}\tilde{c}}\cos\theta + u(Q)(2\cos^2\theta - 1/2) + 2v(Q)\sin\theta\cos\theta\right] \mathrm{d}\theta \\
&\quad + O(\Delta t^3) \\
v(P) &= \frac{1}{\pi}\int_0^{2\pi} \left[-\frac{p(Q)}{\tilde{\rho}\tilde{c}}\sin\theta + 2u(Q)\cos\theta\sin\theta + v(Q)(2\sin^2\theta - 1/2)\right] \mathrm{d}\theta \\
&\quad + O(\Delta t^3) \\
p(P) &= -p(P') + \frac{1}{\pi}\int_0^{2\pi} [p(Q) - \tilde{\rho}\tilde{c}u(Q)\ \cos\theta - \tilde{\rho}\tilde{c}v(Q)\ \sin\theta]\, \mathrm{d}\theta + O(\Delta t^3)
\end{aligned}
$$

Approximate evolution operator EG3

$$
\begin{aligned}
\rho(P) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{p(Q)}{\tilde{c}^2} - 2\frac{\tilde{\rho}}{\tilde{c}}u(Q)\cos\theta - 2\frac{\tilde{\rho}}{\tilde{c}}v(Q)\sin\theta \mathrm{d}\theta \\
&\quad + \rho(P') - \frac{p(P')}{\tilde{c}^2} + O(\Delta t^2) \\
u(P) &= \frac{1}{2\pi}\int_0^{2\pi} -\frac{2}{\tilde{\rho}\tilde{c}}p(Q)\cos\theta + u(Q)(3\cos^2\theta - 1) + 3v(Q)\sin\theta\cos\theta \mathrm{d}\theta \\
&\quad + \frac{1}{2}u(P') + O(\Delta t^2) \\
v(P) &= \frac{1}{2\pi}\int_0^{2\pi} -\frac{2}{\tilde{\rho}\tilde{c}}p(Q)\sin\theta + 3u(Q)\sin\theta\cos\theta + v(Q)(3\sin^2\theta - 1)\mathrm{d}\theta \\
&\quad + \frac{1}{2}v(P') + O(\Delta t^2)
\end{aligned}
$$

$$p(P) = \frac{1}{2\pi}\int_0^{2\pi} p(Q) - 2\tilde{\rho}\tilde{c}u(Q)\cos\theta - 2\tilde{\rho}\tilde{c}v(Q)\sin\theta \mathrm{d}\theta + O(\Delta t^2)$$

Approximate evolution operator EG4

$$\begin{aligned}
\rho(P) &= \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{p(Q)}{\tilde{c}^2} - 2\frac{\tilde{\rho}}{\tilde{c}}u(Q)\ \cos\theta - 2\frac{\tilde{\rho}}{\tilde{c}}v(Q)\ \sin\theta\right]\ \mathrm{d}\theta \\
&\quad + \rho(P') - \frac{p(P')}{\tilde{c}^2} + O(\Delta t^2) \\
u(P) &= \frac{1}{2\pi}\int_0^{\pi}\left[-2\frac{p(Q)}{\tilde{\rho}\tilde{c}}\cos\theta + 2u(Q)\cos^2\theta + 2v(Q)\sin\theta\cos\theta\right]\ \mathrm{d}\theta + O(\Delta t^2) \\
v(P) &= \frac{1}{2\pi}\int_0^{2\pi}\left[-2\frac{p(Q)}{\tilde{\rho}\tilde{c}}\sin\theta + 2u(Q)\cos\theta\sin\theta + 2v(Q)\sin^2\theta\right]\ \mathrm{d}\theta + O(\Delta t^2) \\
p(P) &= \frac{1}{2\pi}\int_0^{2\pi}[p(Q) - 2\tilde{\rho}\tilde{c}u(Q)\ \cos\theta - 2\tilde{\rho}\tilde{c}v(Q)\ \sin\theta]\ \mathrm{d}\theta + O(\Delta t^2)
\end{aligned}$$

Approximate evolution operator EG5 for piecewise constant data

$$\begin{aligned}
\rho(P) &= \rho(P') - \frac{p(P')}{\tilde{c}^2} \\
&\quad + \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{p(Q)}{\tilde{c}^2} - \frac{\tilde{\rho}}{\tilde{c}}u(Q)\,\mathrm{sgn}(\cos\theta) - \frac{\tilde{\rho}}{\tilde{c}}v(Q)\,\mathrm{sgn}(\sin\theta)\right]\ \mathrm{d}\theta \\
u(P) &= \frac{1}{2\pi}\int_0^{2\pi}\left[-\frac{p(Q)}{\tilde{\rho}\tilde{c}}\,\mathrm{sgn}(\cos\theta) + v(Q)\ \sin\theta\ \cos\theta\right. \\
&\qquad\left. + u(Q)\left(\frac{1}{2} + \cos^2\theta\right)\right]\ \mathrm{d}\theta \\
v(P) &= \frac{1}{2\pi}\int_0^{2\pi}\left[-\frac{p(Q)}{\tilde{\rho}\tilde{c}}\,\mathrm{sgn}(\sin\theta) + u(Q)\ \cos\theta\ \sin\theta\right. \\
&\qquad\left. + v(Q)\left(\frac{1}{2} + \sin^2\theta\right)\right]\ \mathrm{d}\theta \\
p(P) &= \frac{1}{2\pi}\int_0^{2\pi}[p(Q) - \tilde{\rho}\tilde{c}u(Q)\,\mathrm{sgn}(\cos\theta) - \tilde{\rho}\tilde{c}v(Q)\,\mathrm{sgn}(\sin\theta)]\ \mathrm{d}\theta
\end{aligned}$$

Approximate evolution operator EG5 for piecewise linear data

$$\begin{aligned}
\rho(P) &= \rho(P') + \frac{1}{4}\int_0^{2\pi}\frac{1}{\tilde{c}^2}[p(Q) - p(P')]\ \mathrm{d}\theta \\
&\quad - \frac{1}{\pi}\int_0^{2\pi}\left[\frac{\tilde{\rho}}{\tilde{c}}u(Q)\ \cos\theta - \frac{\tilde{\rho}}{\tilde{c}}v(Q)\ \sin\theta\right]\ \mathrm{d}\theta + O(\Delta t^2)
\end{aligned}$$

$$
\begin{aligned}
u(P) &= u(P') + \frac{1}{\pi}\int_0^{2\pi} -\frac{p(Q)}{\tilde\rho\tilde c}\cos\theta\,\mathrm{d}\theta \\
&+\frac{1}{4}\int_0^{2\pi}\left[u(Q)\left(3\,\cos^2\theta-1\right)+3v(Q)\,\sin\theta\,\cos\theta-\frac{1}{2}u(P')\right]\mathrm{d}\theta \\
&+O(\Delta t^2) \\
v(P) &= v(P') + \frac{1}{\pi}\int_0^{2\pi} -\frac{p(Q)}{\tilde\rho\tilde c}\sin\theta\,\mathrm{d}\theta \\
&+\frac{1}{4}\int_0^{2\pi}\left[v(Q)\left(3\,\sin^2\theta-1\right)+3u(Q)\,\sin\theta\,\cos\theta-\frac{1}{2}v(P')\right]\mathrm{d}\theta \\
&+O(\Delta t^2) \\
p(P) &= p(P') + \frac{1}{4}\int_0^{2\pi}\left[p(Q)-p(P')\right]\mathrm{d}\theta \\
&-\frac{1}{\pi}\int_0^{2\pi}\tilde\rho\tilde c\left[u(Q)\,\cos\theta-v(Q)\,\sin\theta\right]\mathrm{d}\theta+O(\Delta t^2),
\end{aligned}
$$

where $Q=(x-\Delta t(\tilde u-\tilde c\cos\theta), y-\Delta t(\tilde v-\tilde c\sin\theta), t)$, $P'=(x-\Delta t\tilde u, y-\Delta t\tilde v, t)$, and $P=(x,y,t+\Delta t)$.

We want to point out that the discretization error of the evolution operator EG5 for piecewise constant data is $O(\Delta t^2)$ only if the CFL number equals 1, otherwise it is just $O(\Delta t)$. The accuracy of the whole finite volume evolution Galerkin scheme is imposed by the second, correction, step, which is the finite volume update.

In the thesis the rigorous derivation of the approximate evolution operators EG1-EG5 for the shallow water equations and wave equation with advection using the above design principles were done. We have decided not to present these rather lengthy calculations here, since it would enlarge the size of this paper substantially.

9 CONCLUSION

In thesis we studied numerically a complex problem of genuinely multi-dimensional structure of the hyperbolic systems of conservative laws. The main emphasis was put on derivation and study of new genuinely multi-dimensional numerical scheme based on the use of bicharacteristic, the so-called finite volume evolution Galerkin schemes. At the beginning of the thesis we deal with the simplified one-dimensional Riemann problem presenting a rich structure of its solution, such as rarefaction waves, shear waves and shocks. These can be seem in some our test cases in numerical experiments, cf. Chapter 6 of the thesis. In the thesis we have derived finite volume evolution Galerkin schemes for nonlinear systems of hyperbolic conservation laws in two space dimensions, namely for the Euler equations and for the shallow water equations. These methods couple a finite volume method with the approximate evolution Galerkin operator, which is construct using the bicharacteristics of multi-dimensional hyperbolic system such a way, that all infinitely many directions of wave propagation are taken into account. In Sections 4.2-4.4 of thesis we have derived precisely five evolution Galerkin operators EG1-EG5 for the Euler equations, the shallow water equations and for the wave equation with advection. The second order finite volume evolution Galerkin schemes are created by using the conservative piecewise bilinear recovery in space, appropriate approximation of the edge integrals and the midpoint rule approximation in time.

Further we have studied theoretically L^1-stability of the schemes. The L^1-stability was done for a simplified two-dimensional linear advection equation. The main point was to consider several suitable quadrature rules for the approximation of cell interface integrals. For the two-dimensional systems we have also analyzed the linearization error in time, global error in time and space under assumption of linearization and the truncation error for linearized hyperbolic systems. We have shown that the error of the finite volume evolution Galerkin scheme (7.5), (7.7) applied to the linearized Euler equations, shallow water equations and the wave equation system with advection is of second order. The error analysis concerning the second order scheme (7.5), (7.7) with the EG1-EG3 operators was already published in our paper [11]. The analysis for the operator EG4 is analogous. A new result presented in the thesis is the analysis of the truncation error for new operators using piecewise constant as well as piecewise linear data, cf. Section 5.3 of the thesis.

Many numerical experiments, presented at the end of thesis, illustrate good multi-dimensional behaviour as well as high global accuracy of our finite volume evolution Galerkin schemes.

There are still several open questions concerning the stability of the second order finite volume evolution Galerkin schemes. New finite volume EG5 operators increase the stability of the finite volume evolution Galerkin schemes considerably, however the desired limit of CFL=1 is not yet reached. Further we have studied in this thesis only homogeneous conservation laws, e.g. homogeneous shallow water equations. There is a possibility to extend our results in future to a non-homogeneous case, which is a particularly interesting question with respect to geophysical modeling.

10 ZÁVĚR

Předložená práce se zabývá matematickým modelováním proudění stlačitelných tekutin. Hlavní důraz byl kladen na odvození a studium nové ryze multi-dimenzionální numerické metody založené na metodě bicharakteristik. Tato metoda byla nazvána evoluční Galerkinova metoda konečných objemů. V úvodu se práce zabývá jedno-dimenzionálním Riemannovým problémem a popisuje jeho bohatou strukturu řešení, do které patří vlny zředění, kontakty nespojitosti a šoky. Tyto jednotlivé typy struktur se vyskytují i v řešení některých testovacích případů v kapitole numerických experimentů disertační práce. Dále je v práci odvozena evoluční Galerkinova metoda konečných objemů pro nelineární systémy hyperbolických zákonů zachování ve dvou dimenzích a to zejména pro Eulerovy rovnice a pro rovnice mělké vody. Evoluční Galerkinova metoda konečných objemů propojuje metodu konečných objemů s aproximativním evolučním Galerkinovým operátorem. Tento operátor je tvořen pomocí bicharakteristik multi-dimenzionálního hyperbolického systému tak, že pracuje s nekonečně mnoha směry proudění tekutiny. Podrobné odvození jednotlivých evolučních Galerkinových operátorů EG1-EG5 pro Eulerovy rovnice, rovnice mělké vody a vlnovou rovnici s advekcí je provedeno v oddílech 4.2-4.4. Metoda druhého řádu je vytvořena pomocí konzervativní po částech spojité bilineární rekonstrukce, vhodné numerické aproximace integrálu přes hranu konečného objemu a užitím obdélníkového pravidla pro aproximaci v čase.

Na případě dvou-dimenzionální lineární advektivní rovnice je v práci teoreticky studována L^1-stabilita evoluční Galerkinovy metody konečných objemů, kde uvažujeme několik vhodných numerických kvadratur pro aproximaci integrálu přes hranu konečného objemu. Analýza chyby linearizace v čase stejně tak jako globální chyby v čase a v prostoru za předpokladu linearizace a také chyby ze zanedbání (truncation error) pro linearizovaný hyperbolický sytém je vypracována pro systémy ve dvou dimenzích. Je zde ukázáno, že pokud evoluční Galerkinovu metodu konečných objemů (7.5), (7.7) aplikujeme na Eulerovy rovnice, rovnice mělké vody nebo vlnovou rovnici s advekcí, tak je chyba metody druhého řádu. Rozbor chyby metody (7.5), (7.7) druhého řádu s evolučními operátory EG1-EG3 byl již dříve publikován v našem článku [11]. Analýza chyby metody pro operátor EG4 je analogická. Novým výsledkem prezentovaným v disertační práci je analýza chyby pro nový evoluční operátor EG5 s po částech konstantními i s po částech lineárními daty, viz oddíl 5.3 disertační práce.

Dobré multi-dimenzionální chovaní naší metody i její vysoká přesnost jsou potvrzeny na mnoha numerických experimentech v kapitole 6 disertační práce.

Do budoucna zůstává stále otevřena otázka stability evoluční Galerkinovy metody konečných objemů druhého řádu. Nový evoluční operátor EG5 má sice vyšší stabilitu než ostatní evoluční operátory, ale magické hranice CFL=1 stále ještě nedosáhl.

Jelikož je práce zaměřena pouze na homogenní hyperbolické systémy, tj. homogenní rovnice mělké vody, je v budoucnu možnost rozšířit naše výsledky i na rovnice nehomogenní, což by mohla být zajímavá otázka vzhledem ke geofyzikálnímu modelování.

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