Current density in a group of long parallel conductors

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Published in: *Phys. Scr.* **94** (2019), 12, article ID 125504 (11pp), September 11, 2019.

Abstract

An original method is proposed for the calculation of current density in a finite number of long parallel solid conductors of arbitrary cross section. Some pairs of the conductors under examination can be connected to a voltage source; they are active. The other conductors are passive. The currents, voltages and magnetic fields are assumed to be quasi-stationary and the displacement current is neglected. The permeability of the conductor material is constant and equals the vacuum permeability. No method has so far been published that would allow an exact calculation of current density in all the cases that satisfy the above assumptions. The application of the method is demonstrated by solving an example with two active conductors, one of which is of circular and the other of rectangular cross section, while a third passive conductor is of rectangular cross section.

Keywords: applied classical electromagnetism; magnetic induction; induced currents; numerical simulation

1 Introduction

Maxwell [1], Art. 689 and 690, proposed a method for the calculation of current density in a solitary long cylindrical conductor supplied with variable current. The author of the present paper proposed [2–4] a method for the calculation of current density in a pair of conductors connected to a voltage source on the assumption that the conductors have similar cross sections. In the present paper, an original method is proposed for the calculation of current density in a finite number of long parallel solid conductors of arbitrary cross section.

The method proposed is a generalization of the method for calculating current density in a conductor that is not connected to a source and lies in an external time-varying magnetic field. This method was published in [5]. The currents, voltages and magnetic fields are assumed to be quasi-stationary and the displacement current is neglected. A quasi-stationary, or slowly varying quantity means that at any instant its magnitude is the same on a straight line parallel to the conductors considered. It actually means that an infinitely high propagation velocity of electromagnetic field is assumed.

The Cartesian coordinate system xyz is chosen such that the axis z is parallel to the conductors examined. The conductor cross section does not depend on z. The permeability of the conductor material is constant and equals the vacuum permeability μ_0. The resistivity of the conductors does not depend on z, it is a function of

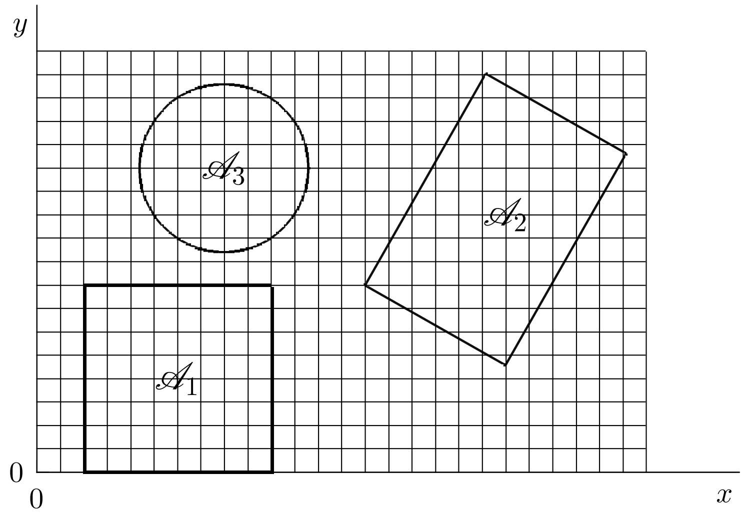

Figure 1: Cross section of conductors $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$ and net of rectangles.

x and y. In view of each of the conductors being determined by its cross section in the plane xy, the conductors will be designated by the same symbols as their cross sections. Generally, the conductors considered are $\mathscr{A}_\ell$, where $\ell = 1, 2, \ldots, c, c > 0$. Each cross section $\mathscr{A}_\ell$ is in fact a closed connected set with continuous boundary [6] and thus it can be approximated with arbitrary precision by the cross section $\mathrm{a}(\mathscr{A}_\ell)$, which is a sum of disjoint rectangles [5]. These rectangles can form part of the net $\mathscr{R}$ of identical rectangles formed with the aid of the parallels

$$x = k_x\,\Delta x, \quad y = k_y\,\Delta y,$$

where k_x, k_y are integers and Δx and Δy the chosen lengths of the rectangle sides. The cross section $\mathrm{a}(\mathscr{A}_\ell)$ is the aggregate of rectangles in the net $\mathscr{R}$

$$\mathrm{a}(\mathscr{A}_\ell) = \bigcup_{j=1}^{N_\ell} A_{\ell j}, \quad \ell = 1, 2, \ldots, c. \tag{1}$$

Fig. 1 illustrates the rectangular cross sections $\mathscr{A}_1, \mathscr{A}_2$ and circular cross section $\mathscr{A}_3$ together with a net of rectangles. There are several possibilities of forming the aggregate $\mathrm{a}(\mathscr{A}_\ell)$ in a given net $\mathscr{R}$. If $A_{\ell j} \in \mathscr{R}$, then $A_{\ell j} \subset \mathrm{a}(\mathscr{A}_\ell)$ if either $A_{\ell j} \subset \mathscr{A}_\ell$ or $A_{\ell j} \cap \mathscr{A}_\ell \neq \emptyset$ or $(X_{\ell j}, Y_{\ell j}) \in \mathscr{A}_\ell$, where $(X_{\ell j}, Y_{\ell j})$ is the centre of the rectangle $A_{\ell j}$. The possibilities given do not exhaust all the possibilities. What is important is that it holds

$$\mathrm{a}(\mathscr{A}_\ell) \to \mathscr{A}_\ell \text{ for } d \to 0,$$

where d is the length of the diagonal of rectangle $A_{\ell j}$. The net $\mathscr{R}$ need not be the same for all the conductors $\mathscr{A}_\ell$. Each conductor $\mathscr{A}_\ell$ can have its own net $\mathscr{R}_\ell$ and its own coordinate system. For example, in the case of rectangular cross section $\mathscr{A}_\ell$ the coordinate system and the sides $\Delta x, \Delta y$ can be chosen such that the sides of the net rectangles are parallel to the axes of coordinates and that it holds

$$\mathrm{a}(\mathscr{A}_\ell) = \mathscr{A}_\ell,$$

as shown in Fig. 2. The sets A_i are assumed to be rectangles but they can also have a different shape; in [4], for example, they are sectors of circular ring.

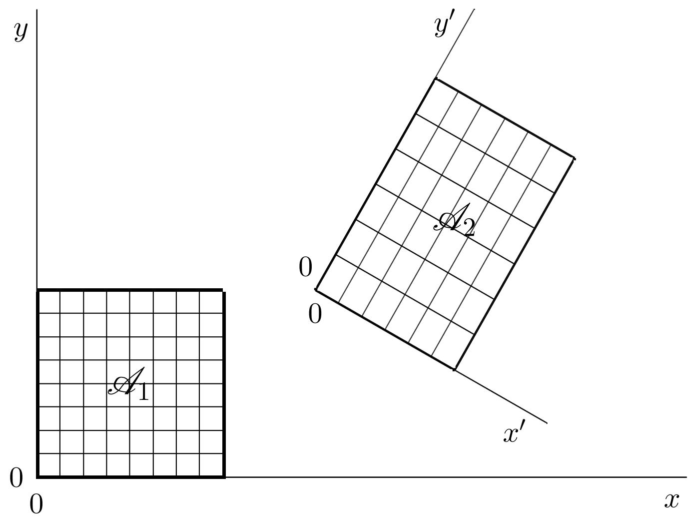

Figure 2: Example of rectangular cross sections and their approximations by aggregates in different nets of rectangles.

The resistivity of conductors ϱ is given, it depends on x and y, $\varrho = \varrho(x, y)$. If the point (x, y) does not lie in any of the aggregates $\mathrm{a}(\mathscr{A}_\ell), \ell = 1, 2, \ldots, c$, then $\varrho(x, y) = 0$.

2 Active, passive and partial conductors

Some of the conductors considered can be connected to the voltage source. They are called active conductors. Conductors connected to one source form a loop, with one conductor connected to each terminal. This assumption is without loss of generality. If a group of conductors are connected to one terminal of the source, this group will in the following be taken for a single conductor. The cross section of such a conductor is not a connected set [6] but several connected sets.

In addition to active conductors there can also be passive conductors among the conductors considered, i.e. conductors that are not connected to a source. Theoretically, one conductor could be connected to the ideal current source [7]. There is in fact no ideal current source. A replacement of a real current source via a parallel connection of ideal current source and parallel conductance means adding a further conductor to the conductors considered. The added conductor and the original conductor would form a loop of active conductors, in which the current source can be replaced by an equivalent voltage source. Thus it is not necessary to separately examine such a case. Nevertheless, the proposed method for calculating current density also covers the case when the conductor is connected to the ideal current source. Admittedly, the case is unrealizable but it is the generalization of a historically significant method for calculating current density in a conductor of circular cross section [1, 8, 9], Art. 689, 690. An erroneous interpretation of the result obtained by this method has unfavourable consequences even today [3, 10–12]. Conductors connected to the ideal current source will in the following be included among the passive conductors.

A pair of passive conductors can form a loop. This pair is regarded as a pair of active conductors connected to a source of zero voltage.

Using the index ℓ, the conductors are numbered such that conductors with the index $\ell = 1, 2, \ldots, c2$, where $c2$ is even, are active conductors while the remaining conductors, with the index $\ell = c2+1, c2+2, \ldots, c$, are passive conductors. Each pair of active conductors $\mathscr{A}_\ell, \mathscr{A}_{\ell+1}$, where $\ell < c2$ is an odd number, is connected to the voltage source such that all the odd-index conductors are connected to a source terminal of the same polarity.

Induced in an arbitrary closed curve C according to Faradays law of electromagnetic induction is the voltage

$$U_C = \frac{\mathrm{d}\Phi}{\mathrm{d}t}, \tag{2}$$

where Φ is a linked flux to the curve C, It is the flux of the vector

$$\boldsymbol{B} = \boldsymbol{B}^{\mathrm{ex}} + \boldsymbol{B}^{\mathrm{co}}$$

through a continuous surface S_C bounded by the curve C. Φ does not depend on the shape of the surface S_C. If the whole curve C lies in an electrically conductive medium, then U_C will produce a conductive electric current, i.e. induced current. $\boldsymbol{B}^{\mathrm{co}}$ is the magnetic field produced by the conductors examined, $\boldsymbol{B}^{\mathrm{ex}}$ is the external field, which is not affected by the currents induced in the conductors considered and is perpendicular to them. The non-zero component of the density of the induced current is only the z-component,

$$\boldsymbol{J} = \langle 0, 0, J(x, y, t) \rangle,$$

because the conductors are parallel to the axis z and the field $\boldsymbol{B}$ is perpendicular to the conductor, as assumed.

The net of rectangles $\mathscr{R}_\ell, \ell = 1, 2 \ldots, c$ is assumed to be assigned to the conductor $\mathscr{A}_\ell$. All the rectangles in $\mathscr{R}_\ell$ are of the same dimensions and equally large area Δ_ℓ. For each $\mathscr{A}_\ell$ there exists in the net $\mathscr{R}_\ell$ its approximation $\mathrm{a}(\mathscr{A}_\ell)$. The rectangle $A_{\ell j}$ can be regarded as the cross section of a partial conductor, which is designated by the same symbol $A_{\ell j}$. Partial conductors can be distinguished by one index i instead of two indices ℓ and j. The partial conductors $A_i, i = N_{\ell-1}+1, N_{\ell-1}+2, \ldots, N_\ell$, where $N_0 = 0, N_c = N$, form one conductor $\mathrm{a}(\mathscr{A}_\ell)$ and, by (1), they approximate the conductor $\mathscr{A}_\ell$.

Arbitrary two partial conductors $A_i, A_j, i \neq j$ can form a partial loop if

$$[A_i \subset \mathrm{a}(\mathscr{A}_\ell)] \wedge [A_j \subset \mathrm{a}(\mathscr{A}_\ell)], \quad 1 \leq \ell \leq c,$$

or

$$[A_i \subset \mathrm{a}(\mathscr{A}_\ell)] \wedge [A_j \subset \mathrm{a}(\mathscr{A}_{\ell+1})], \quad \ell \text{ is odd}, \ \ell < c2,$$

or

$$[A_i \subset \mathrm{a}(\mathscr{A}_\ell)] \wedge [A_j \subset \mathrm{a}(\mathscr{A}_{\ell-1})], \quad \ell \text{ is even}, \ \ell \leq c2.$$

Voltage is induced in partial conductors and current flows through them.

3 Calculation of current density in conductors

The conductors under consideration are infinitely long. The voltage of voltage sources, which produces a drop in the potential along the conductors, would have to be infinitely large for the potential drop to be non-zero in the final segment of active conductors. The magnetic field along the conductors does not change and its

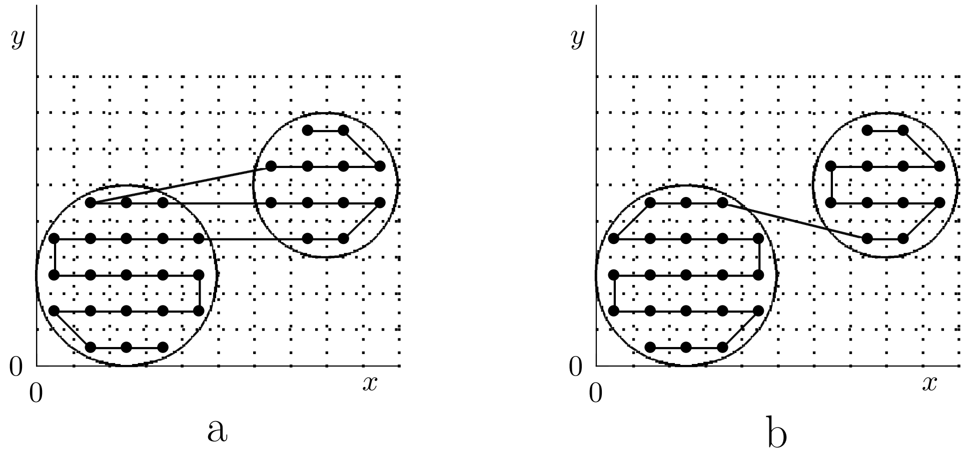

Figure 3: Example of two path graphs in the loop of two cylindrical conductors. Dotted line segments represent the net of rectangles, the centres of partial conductor cross sections, i.e. graph vertices are designated by $\bullet$, the line segments connecting the graph vertices are its edges.

flux through the loops is infinitely large too. For these reasons, only a part of the space and conductors between the planes $z = z_1$ and $z = z_2, z_2 > z_1$ is examined in the following. This approach does not indicate a change in the original objective, i.e. determination of current density in conductors, as proved in [5].

In the calculation of current density it is assumed that current density in the cross section $A_i, i = 1, 2, \ldots, N$, of a partial conductor is constant and has the value J_i. The value of resistivity in a partial conductor, too, is assumed to be constant and have the value $\varrho_i = \varrho(X_i, Y_i)$, where (X_i, Y_i) is the centre of the rectangle A_i.

The partial conductors $A_i, i = N_{\ell-1} + 1, N_{\ell-1} + 2, \ldots, N_\ell$, which form a passive aggregate $\mathrm{a}(\mathscr{A}_\ell)$, must be numbered such that the graph with $N_\ell - N_{\ell-1}$ vertices (X_i, Y_i) and $N_\ell - N_{\ell-1} - 1$ edges is a path graph [5,13]. Similarly, as a generalization of the results in [5], the conductors are numbered in each pair of aggregates $\mathrm{a}(\mathscr{A}_\ell)$, $\mathrm{a}(\mathscr{A}_{\ell+1})$, $\ell = 1, 3, \ldots, c2 - 1$, connected to the same source. The graph with $N_{\ell+1} - N_{\ell-1}$ vertices (X_i, Y_i) and $N_{\ell+1} - N_{\ell-1} - 1$ edges must be a path graph [13]. Fig. 3 gives an example of the path graph in the cross section of a loop of two cylindrical conductors. For the given graph vertices (X_i, Y_i) there usually exist several path graphs; it is important that there always exists at least on path graph [13]. Creating a path graph is usually connected with renumbering the vertices (X_i, Y_i). The indices of vertices and thus also partial conductors must follow one after another, the same as vertices in the path graph.

The cross section of each conductor in the plane $z = \text{const}$ is an equipotential surface. Between the cross sections of the active conductors $\mathrm{a}(\mathscr{A}_\ell)$ and $\mathrm{a}(\mathscr{A}_{\ell+1}), \ell = 1, 3, \ldots, c2-1$, (forming a loop) there is potential difference $V_\ell(z, t)$, provided the two cross sections lie in the same plane $z = \text{const}$. The active conductors are numbered such that the conductors $\mathrm{a}(\mathscr{A}_\ell), \ell = 1, 3, \ldots, c2 - 1$, are connected to the + terminal of the respective voltage source. In the loop of active conductors $\mathrm{a}(\mathscr{A}_\ell), \mathrm{a}(\mathscr{A}_{\ell+1})$ are p_ℓ partial conductors, designated by an increasing sequence of indices

$$i_1, i_2, \ldots i_{p_\ell}, \text{ where } A_{i_1} \subset \mathrm{a}(\mathscr{A}_\ell),\, p_\ell > 0,\, i_{p_\ell} < N_{\ell+1}, \tag{3}$$

such that either

$$(A_{i_k} \subset \mathrm{a}(\mathscr{A}_\ell)) \wedge (A_{i_k+1} \subset \mathrm{a}(\mathscr{A}_{\ell+1})), \tag{4}$$

or

$$(A_{i_k} \subset \mathrm{a}(\mathscr{A}_{\ell+1})) \wedge (A_{i_k+1} \subset \mathrm{a}(\mathscr{A}_\ell)). \tag{5}$$

The segment of each partial loop between the planes $z = z_1$ and $z = z_2$ can be replaced by a lumped-elements circuit. Applying the loop currents method to independent partial loops will yield equations for determining current density in conductors. According to [5] the path graph determines the independent partial loops. The ith partial loop is formed by the partial conductors A_i and A_{i+1}. It is necessary to distinguish partial loops formed from partial conductors which lie

1. inside the same passive conductor $\mathrm{a}(\mathscr{A}_\ell), \ell = c2+1, c2+2, \ldots, c$,

2. in the same loop of active conductors $\mathrm{a}(\mathscr{A}_\ell), \mathrm{a}(\mathscr{A}_{\ell+1}), \ell = 1, 3, \ldots, c2-1$.

As for 1: The equivalent circuit of the ith partial loop with the respective loop currents is given in [5] in Fig. 3. $N - N_{c2}$ partial conductors are divided into $c - c2$ passive aggregates $\mathrm{a}(\mathscr{A}_\ell), \ell = c2+1, c2+2, \ldots, c$. In the aggregate $\mathrm{a}(\mathscr{A}_\ell)$ are $N_\ell - N_{\ell-1} - 1$ independent partial loops. In all the passive aggregates there are $N - N_{c2} + c2 - c$ independent partial loops, with the following equation holding for them

$$(z_2 - z_1)\left[\varrho_i J_i(t) - \varrho_{i+1} J_{i+1}(t)\right] + \frac{\mathrm{d}}{\mathrm{d}t}\Phi_i(t) = 0, \tag{6}$$

where $\mathrm{d}\Phi_i(t)/\mathrm{d}t$ is the voltage induced in the segment of the ith partial loop.

As for 2: Equation (6) holds for each ith partial loop

$$\begin{aligned} i \;\; = \;\; & N_{\ell-1}+1, N_{\ell-1}+2, \ldots, i_1 - 1, \\ & i_1+1, i_1+2, \ldots, i_2-1, \\ & i_2+1, i_1+2, \ldots, i_{p_\ell}-1, \\ & i_{p_\ell}+1, i_{p_\ell}+2, \ldots, N_{\ell+1}-1, \end{aligned}$$

which makes a total of $N_{\ell+1} - N_{\ell-1} - 1 - p_\ell$ equations. The pair of partial conductors A_{i_k}, A_{i_k+1} for $k = 1, 2, \ldots, p_\ell$ forms independent loops. The equivalent circuit of these loops can be seen in Fig. 4 for the case of (4). The equivalent circuit for the case of (5) differs from the circuit in Fig. 4 only in that the arrows for the voltages $V_\ell(z_1, t)$ and $V_\ell(z_2, t)$ are of opposite directions. The equation holding for the circuit in Fig. 4 is

$$(z_2 - z_1)\left[\varrho_i J_i(t) - \varrho_{i+1} J_{i+1}(t)\right] + \frac{\mathrm{d}}{\mathrm{d}t}\Phi_i(t) = \bar{\delta}\left[V_\ell(z_1,t) - V_\ell(z_2,t)\right], \quad i = i_k, \tag{7}$$

where $\bar{\delta} = 1$. In the case of (5) it holds $\bar{\delta} = -1$. For one pair of active conductors it is possible to write p_ℓ equations (7), which is the number of the terms of sequence (3). The total number of independent equations (6) and (7) which belongs to all the pairs of active conductors considered is $N_{c2} - c2/2$.

The systems of equations (6) and (7) can be written as a system of $N - c + c2/2$ equations

$$(z_2 - z_1)\left[\varrho_i J_i(t) - \varrho_{i+1} J_{i+1}(t)\right] + \frac{\mathrm{d}}{\mathrm{d}t}\Phi_i(t) = \delta_i\left[V_\ell(z_1,t) - V_\ell(z_2,t)\right], \tag{8}$$

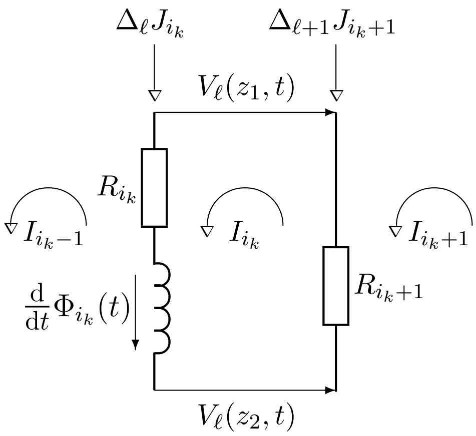

Figure 4: The equivalent circuit of the i_kth partial loop segment between the planes $z = z_1$ and $z = z_2$ in the case of (4).

where

$$i \neq N_2, N_4, \ldots, N_{c2}; N_{c2+1}, N_{c2+2}, \ldots, N_c \ (= N), \tag{9}$$

$$\delta_i = \begin{cases} 1 & \text{if } i \text{ is a term of (3) and relation (4) holds} \\ -1 & \text{if } i \text{ is a term of (3) and relation (5) holds} \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Equation (8) thus does not hold for each index i from the set $\{1, 2, \ldots, N\}$ but only for the index $i \in \mathscr{I}$, where $\mathscr{I} \subset \{1, 2, \ldots, N\}$. The flux $\Phi_i(t)$ is a flux linked to C_i of the vector $\boldsymbol{B}$. The closed curve C_i is composed of four line segments ls_1, ls_2, ls_3 and ls_4, which are sets of points (x, y, z) and are determined by the endpoints

ls_1: (X_i, Y_i, z_1) and (X_i, Y_i, z_2),

ls_2: (X_i, Y_i, z_2) and (X_{i+1}, Y_{i+1}, z_2),

ls_3: (X_{i+1}, Y_{i+1}, z_2) and (X_{i+1}, Y_{i+1}, z_1),

ls_4: (X_{i+1}, Y_{i+1}, z_1) and (X_i, Y_i, z_1).

The flux $\Phi_i = \Phi_i(t)$ is the sum of fluxes $\Phi_i^{\mathrm{co}}(t)$ and $\Phi_i^{\mathrm{ex}}(t)$. The flux Φ_i^{co} is the sum of the fluxes produced by the partial conductors

$$\Phi_i^{\mathrm{co}}(t) = \frac{\mu_0}{4\pi} \sum_{k=1}^{N} J_k(t)(z_2 - z_1)\phi_{ik}, \tag{11}$$

where $J_k(t)(z_2 - z_1)\mu_0\, \phi_{ik}/(4\pi)$ is the contribution of the kth partial conductor. For the flux Φ_i^{ex} of the vector $\boldsymbol{B}^{\mathrm{ex}}(x, y, t)$ linked to C_i it holds

$$\Phi_i^{\mathrm{ex}}(t) = (z_2 - z_1)\, \phi_i^{\mathrm{ex}} \mathscr{T}(t). \tag{12}$$

The quantity ϕ_i^{ex} does not depend on t. $\mathscr{T}(t)$ is the given function of t. The notation $\forall k$ will be used instead of writing $k = 1, 2, \ldots, N$. The coefficients $\phi_{ik}, i \in \mathscr{I}, \forall k$, do not depend on t and their calculation is given in the Appendix of [5], where are also the formulae for the calculation of the coefficients ϕ_i^{ex} in (12) if the field $\boldsymbol{B}^{\mathrm{ex}}$

is homogeneous or is produced by a current filament parallel to the axis z. After specifying the flux Φ_i and after rearranging, equation (8) has the form

$$\varrho_i\, J_i(t) - \varrho_{i+1}\, J_{i+1}(t) + \frac{\mu_0}{4\pi}\sum_{k=1}^{N}\phi_{ik}\,\frac{\mathrm{d}}{\mathrm{d}t}J_k(t) = \delta_i\, U_\ell(t) - \phi_i^{\mathrm{ex}}\,\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{T}(t), \quad i\in\mathscr{I}, \tag{13}$$

where $U_\ell(t) = [V_\ell(z_1,t) - V_\ell(z_2,t)]/(z_2 - z_1)$. Equation (13), complemented with relations (9) and (10), is a system of $N - c + c2/2$ ordinary differential first-order equations with unknown N current densities $J_1(t), J_2(t), \ldots, J_N(t)$. To determine all the current densities, it is necessary to add $c - c2/2$ equations. These equations are the conditions that the total current through each pair of active conductors is zero

$$\sum_{k=N_{\ell-1}+1}^{N_\ell}\Delta_\ell J_k + \sum_{k=N_\ell+1}^{N_{\ell+1}}\Delta_{\ell+1} J_k = 0, \quad \ell = 1,3,\ldots,c2-1, \tag{14}$$

and the total current is also zero in each passive conductor

$$\sum_{k=N_{\ell-1}+1}^{N_\ell} J_k = 0, \quad \ell = c2+1, c2+2,\ldots,c. \tag{15}$$

Equation (15) expresses that the total current is equal to zero after division by the area Δ_ℓ of the net rectangle. If a ℓth conductor is connected to the ideal current source, equation (15) is replaced by the equation

$$\sum_{k=N_{\ell-1}+1}^{N_\ell}\Delta_\ell J_k = I_{0\ell},$$

where $I_{0\ell}$ is the inner current of the current source.

In the calculation of the magnetic field and its fluxes it is necessary to take into consideration the potentially different coordinate systems in which the cross sections are specified (Fig.2). The calculation of the magnetic field and its flux produced by a conductor of rectangular cross section is the simplest in a coordinate system whose axes x and y are parallel to the cross section sides (see the Appendix of [5]). When calculating the flux through an area it is therefore necessary to transform the area into the coordinates in which the conductor cross section is defined.

Consistent with the path graph (see Fig. 3) is a system of independent loops and independent equations (13) but the calculated current density does not depend on the choice of the path graph.

4 Steady sinusoidal state

The proposed method for calculating the current density in a long parallel conductors is described by the system of equations (13), (14) and (15). This method allows solving many cases in both steady and transient states. Demonstrating all potential cases in one paper is not possible so that only the simplest case is chosen, namely the steady sinusoidal state for $\Phi_i^{\mathrm{ex}} = 0$.

If each voltage $U_\ell(t), i\in\mathscr{I}$ in equation (13) is either equal to zero or sinusoidal

$$U_\ell(t) = \hat{U}_\ell \sin(\omega t + \alpha_\ell),$$

then after a sufficiently long time after the connection of all voltages the current density can be regarded as steady sinusoidal with angular frequency ω

$$J_i(t) = \hat{J}_i \sin(\omega t + \varepsilon_i), \quad \forall i.$$

In steady state, the fluxes

$$\Phi_i^{\mathrm{co}}(t) = \hat{\Phi}_i^{\mathrm{co}} \sin(\omega t + \varphi_i), \quad i \in \mathscr{I}$$

will also be steady. It follows from the above that the system of equations (13), (14) and (15) can be solved in the complex domain if the complex voltage and current density

$$\delta_i \, \underline{U}_\ell \exp(\mathrm{j}\omega t), \; i \in \mathscr{I}; \quad \underline{J}_i \exp(\mathrm{j}\omega t), \; \forall i,$$

are taken into consideration, where the underlined symbols denote the phasors

$$\underline{U}_\ell = \hat{U}_\ell \exp(\mathrm{j}\alpha_\ell), \; \ell = 1, 3, \ldots, c2-1; \quad \underline{J}_i = \hat{J}_i \exp(\mathrm{j}\varepsilon_i), \quad \forall i.$$

Substituting the complex quantities into equation (13), (14) and (15) and rewriting will yield a system of N linear algebraic equations in the complex domain

$$\varrho_i \underline{J}_i - \varrho_{i+1} \underline{J}_{i+1} + \frac{\mathrm{j}\,\mu_0\,\omega}{4\pi} \sum_{k=1}^{N} \phi_{ik}\, \underline{J}_k = \delta_i \underline{U}_\ell, \quad i \in \mathscr{I}, \tag{16}$$

$$\sum_{k=N_{\ell-1}+1}^{N_\ell} \Delta_\ell\, \underline{J}_k + \sum_{k=N_\ell+1}^{N_{\ell+1}} \Delta_{\ell+1}\, \underline{J}_k = 0, \quad \ell = 1, 3, \ldots, c2-1, \tag{17}$$

$$\sum_{k=N_{\ell-1}+1}^{N_\ell} \underline{J}_k = 0, \quad \ell = c2+1, c2+2, \ldots, c. \tag{18}$$

Solving the system of equations (16), (17) and (18) gives the current density phasors $\underline{J}_i, \forall i$.

5 Example

Prior to calculating current density by the proposed method (equations (8) and (13)) it is necessary to know the values of the resistivity of the conductors under consideration in dependence on x and y. Assumed in the subsequent example is constant resistivity over the cross section of conductors; a more complex resistivity pattern was assumed in [5]. The resistivity values have been taken over from [14], the resistivity of Cu conforms to annealed Cu. The resistivity of Cu depends substantially on the production method of conductors and on their deformation [15–17]. The present paper deals with infinitely long conductors and this means that relevant quantities cannot depend on the coordinate z. To apply the effect of the structure, defects, dislocations, etc. to resistivity, it would be necessary to know their effect over the cross section of the conductors (in dependence on x and y) and, on top of that, this effect could in no way depend on z.

Three conductors $\mathscr{A}_1$, $\mathscr{A}_2$ and $\mathscr{A}_3$ are examined at a temperature of 25°C, the dimensions of their cross sections are evident from Fig. 5. The conductors $\mathscr{A}_1$ and $\mathscr{A}_2$ form an active loop, the conductor $\mathscr{A}_3$ is passive. The $\mathscr{A}_1$ and $\mathscr{A}_2$ conductors are of copper, $\varrho_1 = 1.712 \times 10^{-8}$ Ω·m [14], the $\mathscr{A}_3$ conductor is of aluminium, $\varrho_2 = 2.709 \times 10^{-8}$ Ω·m [14].

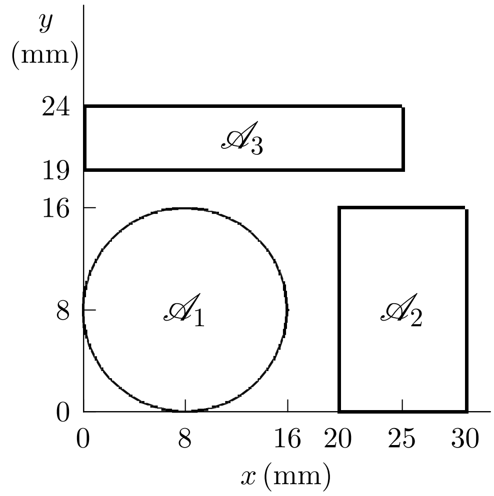

Figure 5: Cross sections of the conductors $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$ in the Example.

Four variants are considered. In all the variants the position of the conductor $\mathscr{A}_1$ remains unchanged. In variant 1 the position of the conductors is the same as in Fig.5. Variants 2, 3, and 4 differ from variant 1 by the position of the conductors $\mathscr{A}_2$ and $\mathscr{A}_3$. Two cases are examined in each variant. Considered in the first case are only the conductors $\mathscr{A}_1$ and $\mathscr{A}_2$, i.e. $c = 2$, in the second all the three conductors, i.e. $c = 3$.

The cross sections of the partial conductors form in the cross section of the ℓth, $\ell = 1, 2, 3$, conductor n_ℓ layers parallel with the axis x, with each layer containing m_ℓ cross sections of the partial conductors. The calculated current density $J_i, \forall i$, is constant in the cross section of the ith partial conductor and is thus piecewise constant, which is an approximation of the real state. The calculation result is the more precise, the larger the value of N. In the Figures given below, the discontinuous values of the amplitude $\hat{J}_i$ and the initial phase ε_i of the current density are approximated by the continuous curves $\hat{J}(x, y)$ and $\varepsilon(x, y)$, respectively. For a constant y the curves begin at the point $x = X_1$ and terminate at the point $x = X_{m_\ell}$. With increasing N these points get closer to the conductor surface.

The voltage on the active loop (see equation (16)) was chosen to be $\underline{U}_1 = 1\ \mathrm{V{\cdot}m^{-1}}\angle 0°$. The values of the current density in the Figures are divided by the value $J_{\mathrm{dc}} = 3.253 \times 10^7\ \mathrm{A{\cdot}m^{-2}}$, which is the magnitude of current density in the conductor $\mathscr{A}_2$ when the frequency f of the voltage source is zero, $f = \omega/(2\pi)$. Besides current density, the parameter values were also determined. The parameters are understood to be: the amplitude $\hat{I}$ of the current and its initial phase β in the conductor $\mathscr{A}_1$, the real and the imaginary component $\Re(\underline{Z})$ and $\Im(\underline{Z})$, respectively, the impedance $\underline{Z}$ of a segment of the active conductors $\mathscr{A}_1$ and $\mathscr{A}_2$. The current through the conductor $\mathscr{A}_2$ is of the same amplitude as the current through the conductor $\mathscr{A}_1$, its initial phase is equal to $180° + \beta$. The parameter values are given in Table 1.

The square net $\mathscr{R}, \Delta x = \Delta y$, was chosen in all the variants, with $\Delta x = 0.2$ mm and $\Delta x = 0.1$ mm in variants 1, 2 and 3, 4, respectively.

Variant 1
The voltage frequency $f = 50$ Hz. Fig. 6 gives the dependence of the normed amplitude of current density on x for a constant y in variant 1. The dependence

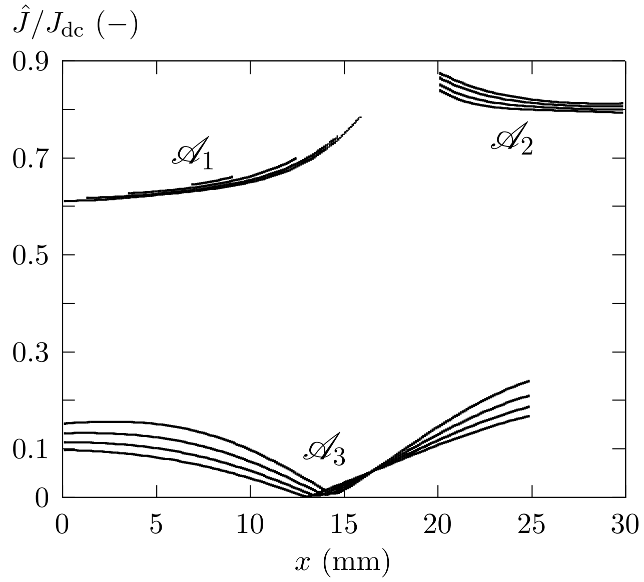

Figure 6: Variant 1: Dependence of normed amplitude of current density on x in the cross sections $\mathscr{A}_1$, $\mathscr{A}_2$ for $y = 0.1, 1.5, 3.9, 7.9$ mm and in the cross section $\mathscr{A}_3$ for $y = 19.1, 20.7, 22.3, 23.9$ mm.

Table 1: Dependence of active-loop parameters on the variant and on the presence of passive conductor ($c = 3$).

varianta	c	f (kHz)	$\hat{I}$ (A)	β (°)	$\Re(\underline{Z})$	$\Im(\underline{Z})$
1	2	0.05	4210	−33.93	1.971×10^{-4}	1.326×10^{-4}
2	2	0.05	4485	−26.81	1.990×10^{-4}	1.006×10^{-4}
3	2	1	387.3	−79.00	4.926×10^{-4}	2.534×10^{-3}
4	2	100	12.61	−84.29	7.883×10^{-3}	7.888×10^{-2}
1	3	0.05	4180	−33.48	1.995×10^{-4}	1.320×10^{-4}
2	3	0.05	4432	−26.15	2.026×10^{-4}	9.946×10^{-5}
3	3	1	395.6	−71.70	7.936×10^{-4}	2.400×10^{-3}
4	3	100	12.76	−84.30	7.784×10^{-3}	7.799×10^{-2}

of $\hat{J}$ and ε on $y, 0 < y < 16$ mm for a constant x is decreasing in the conductor $\mathscr{A}_2$ and $\mathscr{A}_3$ for $x = 20$ mm and $x = 0$, respectively. The cross sections $\mathscr{A}_1$ and $\mathscr{A}_2$ are symmetrical with the straight line $y = 8$ mm and therefore symmetrical with respect to this straight line is also the current density for the case of $c = 2$. The presence of the conductor $\mathscr{A}_3$ disturbs the symmetry. The symmetry disturbance is low with respect to the extent of the values on the vertical axis in Figs 6 and 7. Fig. 8 shows the dependence of $\hat{J}/J_{\rm dc}$ on x for two pairs of symmetrical values y, y' in the conductor $\mathscr{A}_2$.

Variant 2
The mutual distances between the conductors are zero and zero thickness insulation is assumed between the conductors. The cross sections of the conductor $\mathscr{A}_2$ and the conductor $\mathscr{A}_3$ are $[16, 26] \times [0, 16]$ mm^2 and $[0, 25] \times [16, 21]$ mm^2, respectively. The voltage source frequency is the same as in variant 1, $f = 50$ Hz. The parameters are given in Table 1. It can be seen from a comparison of variants 1 and 2 that the distance between the conductors affects the parameters.

Varianta 3

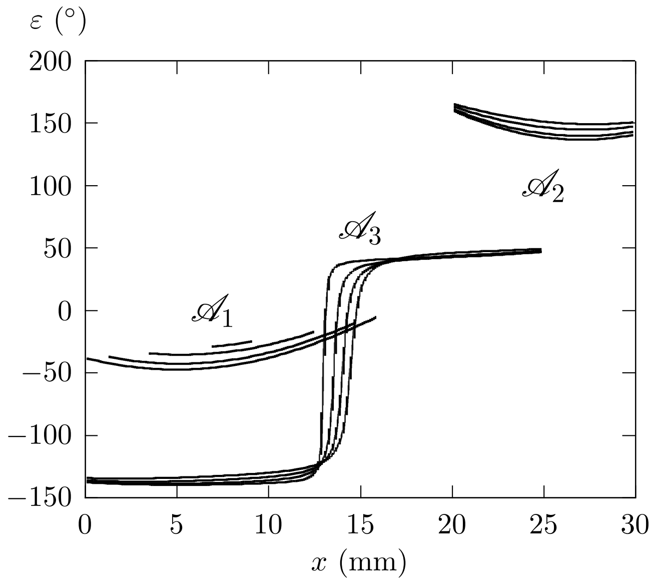

Figure 7: Variant 1: Dependence of the initial phase of current density on x for a constant y in the cross sections $\mathscr{A}_1$, $\mathscr{A}_2$ and $\mathscr{A}_3$; the values y are the same as in Fig. 6.

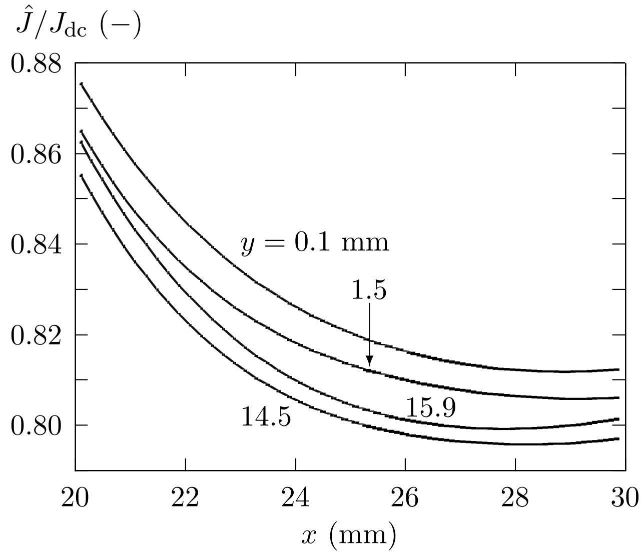

Figure 8: Variant 1: Dependence of the normed amplitude of current density on x for two pairs $(0.1, 15.9)$ mm and $(1.5, 14.5)$ mm of symmetrical values of y with respect to the straight line $y = 8$ mm in the cross section $\mathscr{A}_2$.

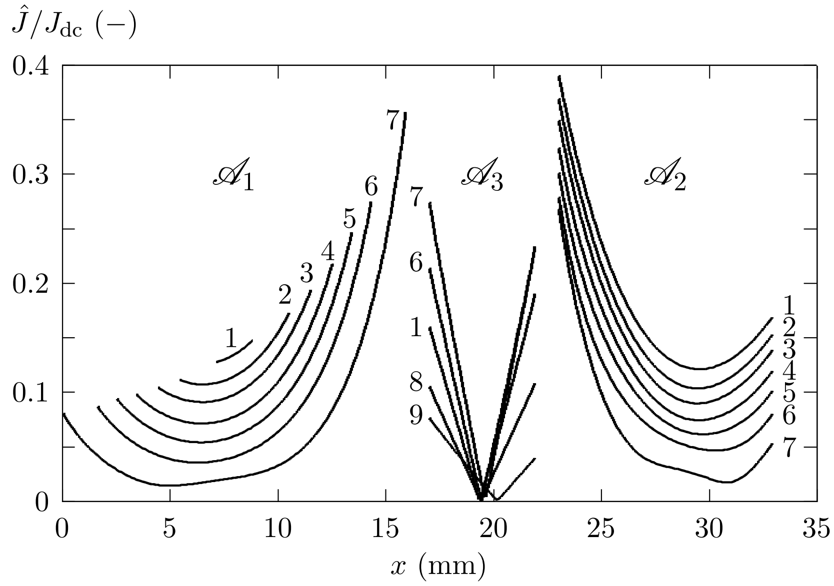

Figure 9: Variant 3: Dependence of the normed amplitude of current density on x for a constant y (expressed in mm) 1: 0.05, 2: 0.45, 3: 0.85, 4: 1.45, 5: 2.15, 6: 3.25, 7: 7.95, 8: 19.25, 9: 24.95.

The cross sections of the conductors $\mathscr{A}_2$ and $\mathscr{A}_3$ are $[23, 33] \times [0, 16]$ mm^2 and $[17, 22] \times [0, 25]$ mm^2, respectively. The voltage source frequency is $f = 1$ kHz. The parameters are given in Table 1 for $c = 2$ and $c = 3$. Figs 9 and 10 illustrate the dependence of the normed phasor of current density on x for a constant y. With a view to the symmetry of the cross sections $\mathscr{A}_1$ and $\mathscr{A}_2$, the values $y < 8$ mm are chosen in these cross sections in Figs 9 and 10. Fig. 11 gives the dependence of the instantaneous value of normed current density at time t, when in the conductor $\mathscr{A}_1$ is the maximum current $I(t) = \hat{I}\sin(\omega t + \beta)$. The current $I(t)$ is at its maximum if $\omega t + \beta = (k + 0.5)\pi$, where k is a non-negative integer.

Varianta 4
The cross sections of the conductors $\mathscr{A}_2$ and $\mathscr{A}_3$ are $[17, 27] \times [0, 16]$ mm^2 and $[-6, -1] \times [0, 25]$ mm^2, respectively. The voltage source frequency is $f = 100$ kHz. The parameters are given in Table 1 for $c = 2$ and $c = 3$. Fig. 12 gives the dependence of the normed amplitude $\hat{J}/J_{dc}$ on x for several values of y in the lower half of the cross section of the conductors $\mathscr{A}_1$ and $\mathscr{A}_2$. The values in the upper half of the cross sections are not given because with the large extent of the values $\hat{J}/J_{dc}$ in Fig. 12 they would merge with the values in the lower half of the cross sections. The large extent of the current density values is also the cause of the values $\hat{J}/J_{dc}$ lower than ca 10^{-11} being affected by rounding errors in the calculation, as evident in Fig. 12. The effect of rounding errors can be reduced by calculating with greater than double precision, which is common in PCs and was also used in the calculation of the results presented in the present paper. The simplest, of course, is to regard the values of $\hat{J}/J_{dc}$ that are less than 10^{-11} as zero because such low values of current density cannot affect the conductor parameters. From a comparison of the parameters in Table 1 for $c = 2$ and $c = 3$ it results that the presence of the conductor $\mathscr{A}_3$ affects the parameters of the conductors $\mathscr{A}_1$ and $\mathscr{A}_2$ only a little. The normed amplitude of current density in the conductor $\mathscr{A}_3$ is illustrated in Fig. 13.

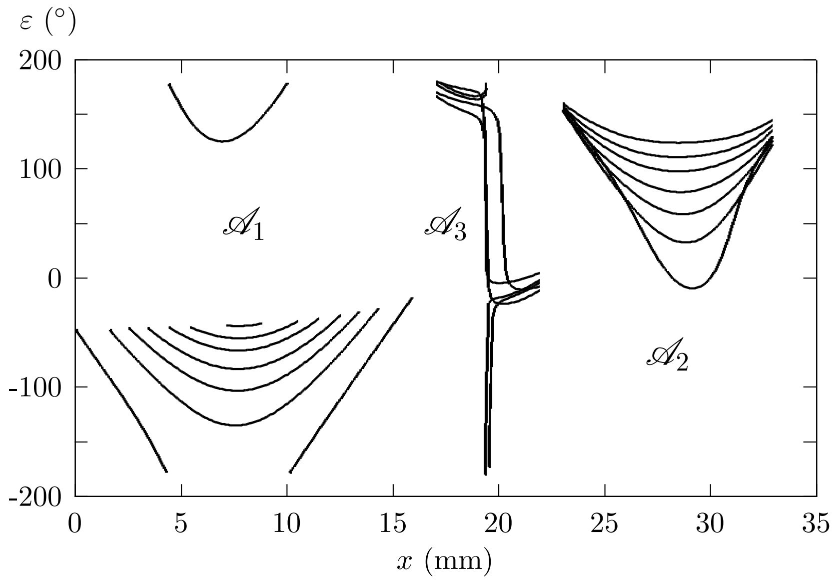

Figure 10: Variant 3: Dependence of the initial phase of current density on x for a constant y; the values y are the same as in Fig. 9 (1: 0.05, 2: 0.45, 3: 0.85, 4: 1.45, 5: 2.15, 6: 3.25, 7: 7.95, 8: 19.25, 9: 24.95).

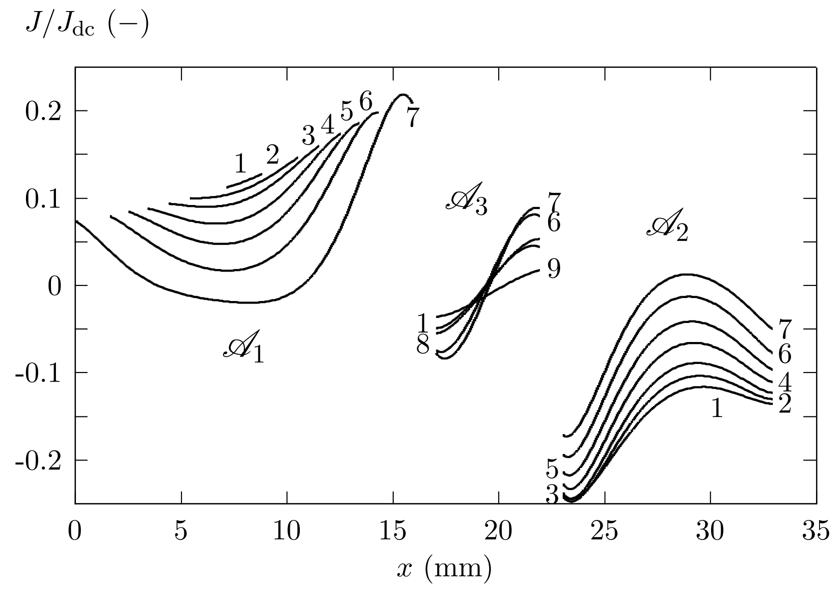

Figure 11: Variant 3: Dependence of the normed current density on x for a constant y at time $t = 0.44917/f$, when current in the conductor $\mathscr{A}_1$ is at its maximum; the values y are the same as in Fig. 9 (1: 0.05, 2: 0.45, 3: 0.85, 4: 1.45, 5: 2.15, 6: 3.25, 7: 7.95, 8: 19.25, 9: 24.95).

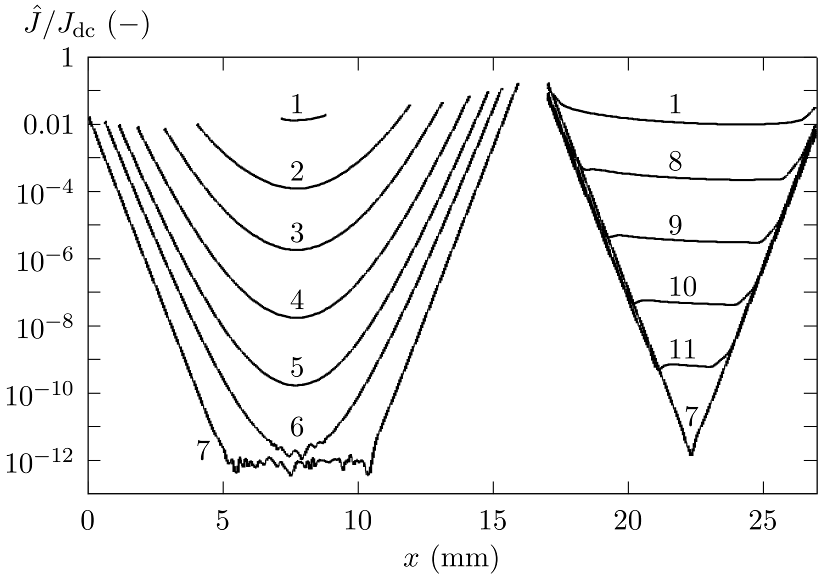

Figure 12: Variant 4: Dependence of the normed amplitude of current density on x for a constant y (expressed in mm) in the conductors $\mathscr{A}_1$ and $\mathscr{A}_2$; 1: 0.05, 2: 1.05, 3: 1.95, 4: 2.95, 5: 3.95, 6: 4.95, 7: 7.95, 8: 0.85, 9: 1.75, 10: 2.65, 11: 3,55.

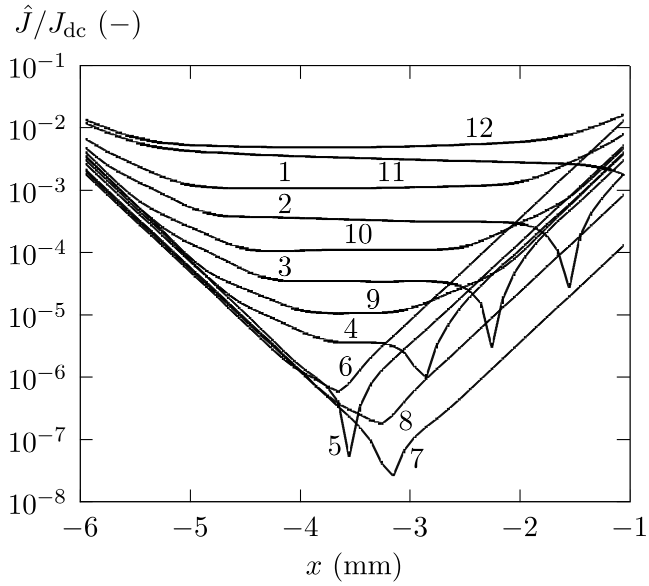

Figure 13: Variant 4: Dependence of the normed amplitude of current density on x for a constant y (expressed in mm) in the conductor $\mathscr{A}_3$; 1: 0.05, 2: 0.65, 3: 1.25, 4: 1.85, 5: 2.65, 6: 7.95, 7: 18.95, 8: 19.95, 9: 23.35, 10: 23.95, 11: 24.55, 12: 24.95.

6 Comparison of newly obtained results with published results

The proposed calculation method is a generalization of the method for calculating current density in one passive conductor occurring in the field $\boldsymbol{B}^{\mathrm{ex}}(x, y, t)$ [5]. This method forms part of the proposed method. In [5], the solution is given to eight examples of the calculation of current density in one passive conductor in both the steady and the transient sinusoidal state.

Maxwell [1, 8, 9], Art. 689 and 690, proposed a method for the calculation of current density in a solitary long cylindrical conductor supplied with variable current. In this case the density is the solution of Bessels differential equation. Since the days of J. C. Maxwell, many papers have been written on the current density in conductors connected to a source of AC voltage as this problem is closely related to the calculation of inductance of various arrangements of solid conductors. Unfortunately, nobody has addressed the possibility of implementing a solitary conductor, except for [7, 10]. Maxwells method can only be applied to the solitary conductor connected to the ideal current source, which cannot be implemented. But even this case is included in the method proposed in the present paper for a conductor of arbitrary cross section, thus not only for a cylindrical conductor.

The methods for the calculation of current density which are, to a certain extent, similar to the method proposed in this paper and form its part were published and applied in [2–4, 11, 12, 18]. They are methods for calculating current density in a pair of long parallel conductors $\mathscr{A}_1$ and $\mathscr{A}_2$, which are active, their cross sections are similar and resistivities symmetrical. Similarity in a plane is the mapping P, which results from combining homothety and identical mapping. A consequence of the similarity of cross sections and the symmetry of resistivity is the symmetry of current density. This means that the current density in one conductor is unambiguously determined by the current density in the other conductor. This allows reducing the number of equations the solution of which is current density.

What is common to the published method and the method proposed in this paper is the replacement of the examined conductors by partial conductors and the application of Kirchhoffs voltage law to independent partial loops. What is different is the choice of independent loops, where the similarity of conductors is made use of in the published methods. The published methods can be applied if for each line of the vector $\boldsymbol{J}$ in the conductor $\mathscr{A}_1$ there is an a priori known line that is its continuation in the conductor $\mathscr{A}_2$. Resistivity symmetry means that for the resistivities ϱ_1 and ϱ_2 in the conductors $\mathscr{A}_1$ and $\mathscr{A}_2$, respectively, it holds $\varrho_2(x', y') = \kappa\, \varrho_1(x, y)$, where $(x', y') = P(x, y)$ and κ has a constant value $\forall (x, y) \in \mathscr{A}_1$.

The conductors examined in [2, 11] are two coaxial tubular conductors (one conductor can be cylindrical). Their common axis goes through the point $(0, 0)$ of the coordinate system xy. The current density and resistivity are assumed to solely depend on $r, r = \sqrt{x^2 + y^2}$. The cross section of the inner conductor is the annular ring $r_i \in [r_{i0}, r_{in}]$, where $0 \le r_{i0} < r_{in}$. The cross section of the outer conductor is the annular ring $r_o \in [r_{on}, r_{o0}]$, where $r_{in} \le r_{on} < r_{o0}$].

In [3, 4, 12, 18] two parallel conductors are examined which are of identical cross sections, symmetrical with respect to the straight line and with symmetrical resistivity. The calculation method in [3,4,12,18] is formulated for cross sections of arbitrary shape, with specific results given for rectangular and circular cross sections.

The method for the calculation of current density as proposed in the present paper is a completely general method and was tested on examples given in [2–

4, 11, 12, 18]. The test results show full agreement with the results published in [2–4, 11, 12, 18]. The calculation using the published methods is quicker because the number of equations for the calculation of current density is reduced due to the symmetry of current density. In the formulation of equations (13), (14) and (15) it is, of course, also possible to take advantage of potential current density symmetry, reduce the number of equations and thus accelerate the calculation.

In [19–21], two methods were proposed and compared for the calculation of the real component $\Re$ of impedance and inductance L of multiconductor transmission lines of long copper strip conductors whose conductivity in the conductor cross section is constant. Forming part of [19, 20] is the method proposed for the calculation of current density in an array of $c, c > 1$, infinitely long cylindrical conductors of arbitrary cross sections. The method was proposed on the same assumptions as given in the Introduction. According to [20], the current density component $J_z(x, y)$ is the solution of an integral equation that is replaced by a system of equations for constant current densities J_i in partial conductors, $i = 1, 2, \ldots, n$,

$$-\mathrm{j}\omega\frac{\mu_0}{2\pi}\sum_{i=1}^{n} J_i \int_{S_i} \log(r)\,\mathrm{d}x'\,\mathrm{d}y' + \frac{1}{\sigma_j}J_j = E_{izj}, \quad j = 1, 2, \ldots, n, \tag{19}$$

where $r = \sqrt{(x - x')^2 + (y - y')^2}$, S_i is the cross section of the ith conductor, σ_j is the conductivity of the jth partial conductor, and E_{izj} is the zth component of an impressed (known) axial electric field in the jth partial conductor. In addition to the system (19) the following equation must hold

$$\sum_{i=1}^{n} J_i S_i = 0. \tag{20}$$

Replacing the last equation in (19) by equation (20) will yield a system of n equations for the current densities $J_i, i = 1, 2, \ldots, n$. What the just described method [20] and my method proposed in this paper have in common is the replacement of the examined conductors by partial conductors. The methods differ in the potential number c of examined conductors; in [20] it holds $c > 1$, in this paper $c > 0$. The majority of the coefficients in equations for the calculation of current density are magnetic fluxes while in [20] they are the integral of logarithmic potential. The method in [20] only solves the steady state. The validity of conditions (14) and (15) gives validity to condition (20) but it does not hold vice versa. In [20] it is thus assumed (and it is not given) that part of the conductors under examination are connected to one terminal of one source while the remaining conductors are connected to the other terminal of this source. Unfortunately, no specific values of current density are given in [19–21]. Therefore it is not possible to compare my specific results with the results obtained by the method from [20]. It is strange that in [19–21] the current density calculated by the method from [20] is not used in the calculation of the quantities $\Re$ and L.

The source of electromagnetic field is the electric charge. In the paper, this field is a quasi-stationary magnetic field produced by a moving electric charge, which is determined by the current density. The result of calculating by the proposed method is the current density. The current density directly determines the magnetic field [11]. The proposed method is thus a direct method in contrast to the published methods, in which the artificially introduced quantity vector potential is usually used. The vector potential is the solution of a second-order differential equation [4, 22]. To solve it numerically, the finite elements method or other methods are used [23, 24].

The current density or the magnetic field is the result of numerical differentiation of the vector potential and thus it is loaded with a non-negligible error [25]. The vector potential is not all-redeeming [22], p. 208, its application leads to the numerical solution of partial differential equations of the second order. However, to establish the current density it is sufficient to solve differential equations of the first order (13). Partial differential equations of the second order cannot be solved without the knowledge of boundary conditions, which constitutes quite a problem. Unnecessary differentiation results in a loss of information, as proved in [26]. The application of classical electromagnetism using partial differential equations for potentials need not always be the optimum method, it is a view of electromagnetism theory from the end.

7 Conclusion

An original method for the calculation of current density in long parallel solid conductors of arbitrary cross section is proposed. Four types of conductor are distinguished

- conductor not connected to the source,
- conductor connected to the ideal current source,
- two or more conductors that form a loop in which there is neither the voltage source nor the current source,
- two or more conductors that form a loop with the voltage source.

The conductors can lie in the external magnetic field. Current in the conductors is assumed to be quasi-stationary, the permeability of the conductor material is constant and equal to the vacuum permeability μ_0, and the displacement current is neglected. The proposed method is a generalization of the method proposed in [5].

The essence of the proposed method consists in the conductors being replaced by partial conductors of rectangular cross section and constant current density. The derivation of the method is in the first place based on Faradays law of electromagnetic induction and is an application of the Biot and Savart law, loop current method, Kirchhoffs voltage law, Ohms law, the Jordan measure theory, and graph theory. These starting points cannot be called in question, the same as the result that follows from them. Current density is generally the solution of a system of equations formed by algebraic equations and ordinary differential equations of the first order. In the steady sinusoidal state the differential equations are replaced by linear algebraic equations for phasors.

The application of the proposed method is demonstrated via solving one example with three conductors of different cross sections, two of which are connected to the source of sinusoidal voltage. The proposed method is compared with methods published up to now.

Acknowledgement

This research work has been carried out in the Centre for Research and Utilization of Renewable Energy (CVVOZE). Authors gratefully acknowledge financial support from the Ministry of Education, Youth and Sports of the Czech Republic under

OP VVV Programme (project No. CZ.02.1.01/0.0/0.0/16_013/0001638 CVVOZE Power Laboratories-Modernization of Research Infrastructure).

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