# Stiffness in Technical Initial Problems

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**Abstract.** Technical initial problems are defined as initial problems where the right-hand side functions of the system are those occurring in the technical practice, that is functions generated by adding, multiplying and superposing elementary functions. Such systems can be expanded into systems with only rationals operations on the right-hand sides of the equations. In such a case the Taylor series terms can easily be calculated [1]. Test examples are presented in the paper. Stiffness in technical initial problems can be eliminated by the TKSL software and solved by the direct use of the explicit and implicit Taylor series methods.

**Keywords:** Differential equations, Taylor series method, TKSL, Stiff systems
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## INTRODUCTION

The "Modern Taylor Series Method" (MTSM) is used for numerical solution of differential equations. The MTSM is based on a recurrent calculation of the Taylor series terms for each time interval. Thus the complicated calculation of higher order derivatives (much criticized in the literature) need not be performed but rather the value of each Taylor series term is numerically calculated. Solving the convolution operations is another typical algorithm used.

An important part of the MTSM is an automatic integration order setting, i.e. using as many Taylor series terms as the defined accuracy requires. Thus it is usual that the computation uses different numbers of Taylor series terms for different steps of constant length.

The MTSM has been implemented in TKSL software [2].

## MODERN TAYLOR SERIES METHOD

The best-known and most accurate method of calculating a new value of a numerical solution of ordinary differential equation $y' = f(t,y),\ \ y(0) = y_0$ is to construct the Taylor series [3, 4].

Methods of different orders can be used in a computation. For instance the 1st order method ($ORD = 1$) means that when computing the new value $y_{i+1}$ only the first Taylor series term is taken into account

$$y_{i+1} = y_i + h \cdot f(t_i, y_i)\,, \text{ resp.} \tag{1}$$
$$y_{i+1} = y_i + DY1_i\,, \tag{2}$$

where $h$ is the integration step.

The 2nd order method ($ORD = 2$) uses Taylor series terms up to the second power of the step $h$

$$y_{i+1} = y_i + h \cdot f(t_i, y_i) + \frac{h^2}{2!} \cdot f^{[1]}(t_i, y_i)\,, \text{ resp.} \tag{3}$$
$$y_{i+1} = y_i + DY1_i + DY2_i\,, \tag{4}$$

and finally the $n-$th order method ($ORD = n$) uses $n$ Taylor series terms in the form

$$y_{i+1} = y_i + h \cdot f(t_i, y_i) + \frac{h^2}{2!} \cdot f^{[1]}(t_i, y_i) + \cdots + \frac{h^n}{n!} \cdot f^{[n-1]}(t_i, y_i)\,, \quad ORD = n\,, \tag{5}$$
$$y_{i+1} = y_i + DY1_i + DY2_i + \cdots + DYN_i\,. \tag{6}$$

It is quite typical of the TKSL to display function *ORD* in the course of computation.

Similarly implicit Taylor series method of order $n$ in the form

$$y_{i+1} = y_i + h \cdot f(t_{i+1}, y_{i+1}) - \frac{h^2}{2!} \cdot f'(t_{i+1}, y_{i+1}) - \cdots - \frac{(-h)^n}{n!} \cdot f^{(n-1)}(t_{i+1}, y_{i+1}), \quad ORD = n, \tag{7}$$

is analyzed.

## TECHNICAL INITIAL PROBLEMS

Technical initial problems are defined as initial problems where the right-hand side functions of the system are those occurring in the technical practice. To test the possibility of the Taylor Series Method the following elementary test problems are analyzed.

### Dahlquist problem

Typical TKSL results of the absolute values of the Taylor series terms $DY1, DY2, \ldots, DYN$ of the well-known Dahlquist equation [5, 6]

$$y' = \lambda y, \quad y(t_0) = y_0, \quad \lambda < 0, \tag{8}$$

are in Fig. 1 ($|h\lambda| = 10$ left, $|h\lambda| = 10^{-4}$ right). Taylor series terms in Fig. 1 right have rapidly decaying trend. Explicit formula (6) has been used.

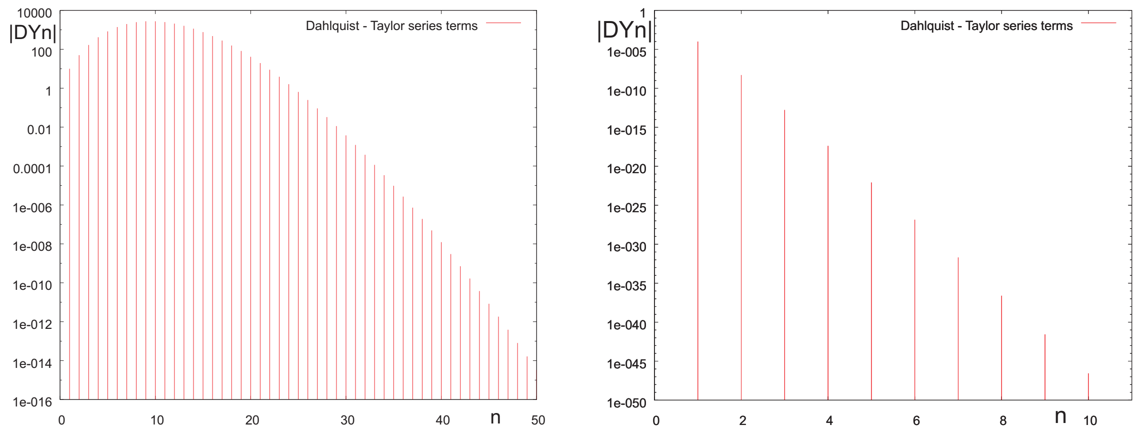


**FIGURE 1.** Taylor series terms - $|h\lambda| = 10$ (left), $|h\lambda| = 10^{-4}$ (right)

Absolute error of implicit Taylor series method of the Dahlquist equation (8) is compared to that of trapezoidal rule and implicit Euler method in Tab. 1.

Absolute error of numerical solution is defined as difference between numerical $y_i$ and analytical $y(t_i)$ solution

$$|\text{Error}(y)| = |y_i - y(t_i)|, \tag{9}$$

where $t_i = h \cdot i$.

Implicit numerical methods provide the best solution of (8) for $\lambda \ll 0$. Examples of solution of implicit numerical methods are shown in Fig. 2 left.

### Semi-analytic computations

Let us consider the initial value problem [7, 8]

$$y' = L(y - \sin(t)) + \cos(t), \quad y(0) = 0, \quad L \ll 0. \tag{10}$$

**TABLE 1.** Implicit numerical methods, absolute error $|\text{Error}(y)|$ in the first step

| $\|h\lambda\|$ | **Trapezoidal rule** | **Implicit Euler method** | **Implicit Taylor method** ($ORD = 9$) |
|---|---|---|---|
| 10 | 0.666712 | $9.08637 \times 10^{-2}$ | $3.24677 \times 10^{-5}$ |
| 100 | 0.960784 | $9.90099 \times 10^{-3}$ | $3.26986 \times 10^{-14}$ |
| 1000 | 0.996008 | $9.99001 \times 10^{-4}$ | $3.59255 \times 10^{-24}$ |
| 5000 | 0.9992 | $1.9996 \times 10^{-4}$ | $3.70846 \times 10^{-31}$ |
| 10000 | 0.9996 | $9.999 \times 10^{-5}$ | $3.62517 \times 10^{-34}$ |

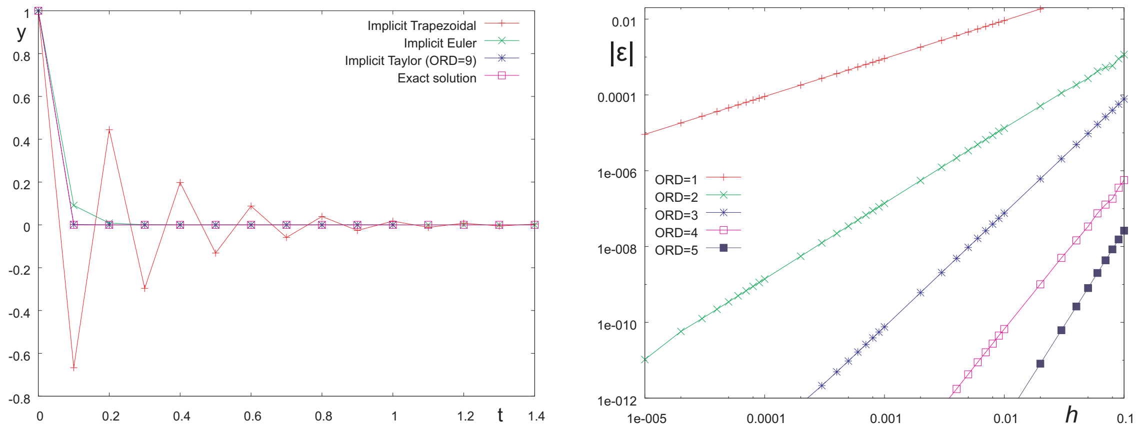


**FIGURE 2.** Solution of (8) using implicit numerical methods ($\lambda = -100, h = 0.1$) - left, Absolute error in $t = 2$, $L = -10000$ explicit Taylor method - right

the exact solution of which is

$$y = \sin(t)\,. \tag{11}$$

If constant $|L|$ increases as presented in [7] the system (10) becomes "stiff" - explicit numerical methods require smaller integration step size for preserving the stability of computation. It is better to use implicit numerical methods for bigger constant $|L|$ .

*Note:* It is an advantage of the explicit Taylor series method using recurrent calculation of the Taylor series terms (implemented in TKSL software) to transform automatically initial value problem (10) into a new system which is independent on constant $L$. The new system is non-stiff and absolute error in $t = 2$ is shown in Fig. 2 right.

## Stiffness in electrical serial RC circuit

Voltage $u_C$ on capacitance $C$ in serial electric RC circuit connected to a voltage $u$ is described by the following differential equation

$$u_C' + a u_C = a u, \quad u_C(0) = 0, \tag{12}$$

where $a = \frac{1}{RC}$.

If the constant $a = \frac{1}{RC}$ is very large (if we use capacitance $C = 5 \cdot 10^{-5}$F and resistance $R = 10\Omega$ then $a = 2000$) the differential equation (12) becomes "stiff".

Numerical solution of differential equation (12) for $a = 2000$ (well known "Stability problem" [5])

$$u_C' = -2000(u_C - \cos(t)), \quad u_C(0) = 0, \quad t \in \langle 0; 1,5 \rangle\,. \tag{13}$$

using implicit numerical methods (Trapezoidal rule, implicit Euler method and implicit Taylor method) can be seen in Fig. 3. The implicit Taylor series method has got the best approximation and the largest integration step $h$.

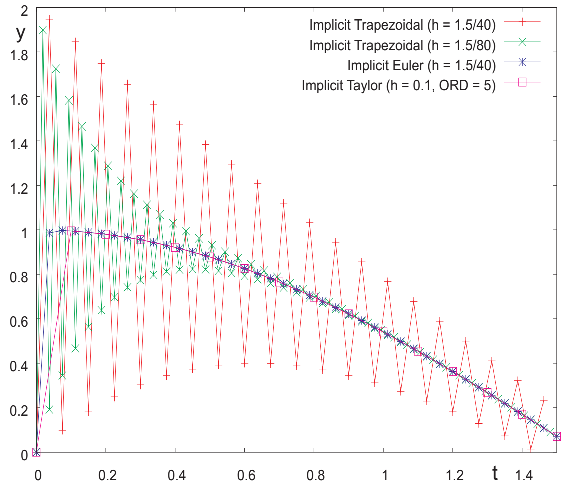


**FIGURE 3.** Solution of (13) using implicit numerical methods

*Note:* For automatic recurrent calculation of higher implicit Taylor series terms the Newton iteration method is used.

## CONCLUSION

Some problems of stiffness of technical initial problems can be eliminated by the TKSL software and by the direct use of the explicit and implicit Taylor series methods as presented in the paper. Positive properties of the Taylor series method are also shown.

Detailed information will be given during the ICNAAM 2012 conference.

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