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Adaptive Solution of Laplace Equation

Václav Valenta*, Václav Šátek†, Jiří Kunovský* and Patricia Humenná*

*University of Technology, Faculty of Information Technology,
Božetěchova 2, 612 66 Brno, Czech Republic [1]
†IT4Innovations, VŠB Technical University of Ostrava,
17. listopadu 15/2172, 708 33 Ostrava-Poruba, Czech Republic

Abstract. The paper is a part of student cooperation in AKTION project (Austria-Czech). Method of aposteriori error estimation based on weighted averaging to improve initial triangulation to get better solution of the planar elliptic boundary-value problem of second order and numerical illustrations of the method are presented in the paper.

Keywords: Laplace Equation, Gradient, Partial Derivatives, Weighted Averaging
PACS: 02,89

INTRODUCTION

We call a set of elements T triangulation of an open polygon Ω in Cartesian coordinate system in the plane when

- all elements of T are fully covering Ω including boundary, $\bigcup_{t\in T} t = \overline{\Omega}$
- any two elements have disjoint interiors
- any side of an element $t_1 \in T$ is a side of another element $t_2 \in T$.

We are working with *nonobtuse regular triangulation* consisting only of triangles in this article. This comes from [1]. We call a triangulation T covering a polygon Ω nonobtuse regular when all inner angles in the triangle $t \in T$ are less than or equal to 90 degrees and we call T *shape-regular* if $\sigma > 0$ exists such that $\rho_t/h_t > \sigma$ for $\forall t \in T$, where ρ_t is a diameter of a circle inscribed to t and h_t is the longest side in triangle t.

We call a vertex a of a triangle $t \in T$ *inner vertex* when $a \in \Omega$ and *boundary vertex* when $a \in \partial\Omega$. When vertex a is an inner vertex, then we call $N(a) = \{b|\overline{ab}$ is a side of a triangle $t \in T\}$ a set of neighbours of a.

PARTIAL DERIVATIVES

It is important to be able to calculate the gradient vector at a vertex $a = [x,y]$ of a second order function u. Let's call $\Pi_t[u]$ linear interpolant of function u on triangle t. This function approximates the function u with a plane on the triangle t.

Let's have vertices a, b, c of a triangle t and a_x, a_y be the x, y coordinates of a vertex a, then

$$D(a,b,c) = \begin{vmatrix} a_x - c_x & b_x - c_x \\ a_y - c_y & b_y - c_y \end{vmatrix} \tag{1}$$

Let us define: now three basis functions for a triangle $t = \overline{abc}$ and a vertex $p = [x,y]$.

$$N_t^a(p) = \frac{D(p,b,c)}{D(a,b,c)} \quad N_t^b(p) = \frac{D(p,a,c)}{D(a,b,c)} \quad N_t^c(p) = \frac{D(p,a,b)}{D(a,b,c)}$$

The important feature of basis function $N_t^a(p)$ is that its value $N_t^a(a)$ is one and for other points b and c, zero.

With these basis functions, we can define the linear interpolant $\Pi_t[u]$ of a triangle $t = \overline{abc}$.

[1] kunovsky@fit.vutbr.cz

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$$\Pi_t[u](p) = u(a)N_t^a(p) + u(b)N_t^b(p) + u(c)N_t^c(p) \tag{2}$$

The gradient of linear interpolant $\nabla\Pi_t[u]$ can be calculated in the following way

$$\frac{\partial \Pi_t[u]}{\partial x} = -\frac{u(a)(c_x - b_x) + u(b)(a_x - c_x) + u(c)(b_x - a_x)}{D(a,b,c)} \tag{3}$$

$$\frac{\partial \Pi_t[u]}{\partial y} = \frac{u(a)(c_y - b_y) + u(b)(a_y - c_y) + u(c)(b_y - a_y)}{D(a,b,c)} \tag{4}$$

The partial derivative of a function u can be calculated in the following way

$$\frac{\partial u}{\partial x}(a) = \frac{\partial \Pi_t[u]}{\partial x}(a) + O(h_t) \tag{5}$$

where the h_t is the longest side of the triangle t.

THE AVERAGING METHOD

Consider an inner vertex $a \in \Omega$. This vertex can be a boundary vertex for multiple triangles and the value of gradient can be different for every triangle with the error $O(h_t)$. We used a method which calculates more precise gradient of the function u using a weighted average of all gradients of elements around a vertex a.

Let's define $b = (b^1, b^2, \dots, b^n)$ is a *ring* around a vertex a in the triangulation. Now it is important to define two possible situations.

- a **is an inner vertex**. Then sum of all inner angles of all trinangles around vertex a is 2π

$$\angle b^n a b^1 + \angle b^1 a b^2 + \ldots + \angle b^{n-1} a b^n = 2\pi \tag{6}$$

- a **is a boundary vertex**. There exists an inner vertex b^j and $(a, b^{j+1}, \dots, b^n, b^1, \dots, b^{j-1})$ is a ring around internal vertex.

When $r = (b^1, \dots, b^n)$ is a ring around vertex a, then this ring defines also list of triangles $tr = (t_1 = \overline{b^1 a b^n}, t_2 = \overline{b^2 a b^1} \ldots, t_n = \overline{b^n})$ and finally a weight vector $w = (w_1, w_2, \dots, w_n)$.

for each triangle $t \in tr$, we define a linear interpolant of function $u(x,y)$, so we have $\Pi_{t_1}[u](x,y), \dots, \Pi_{t_n}[u](x,y)$.

We can simply create derivatives from these interpolants, all interpolants are linear functions of two variables. Gradient of an interpolant of a triangle $t = \overline{abc}$ is

$$\frac{\partial \Pi_t[u](x,y)}{\partial x} = \frac{1}{D(a,b,c)} \Big(\big(u(b_x, b_y) - u(a_x, a_y) \big)(c_x - a_x) - \big(u(c_x, c_y) - u(a_x, a_y) \big)(b_x - a_x) \Big) \tag{7}$$

$$\frac{\partial \Pi_t[u](x,y)}{\partial y} = \frac{1}{D(a,b,c)} \Big(\big(u(b_x, b_y) - u(a_x, a_y) \big)(c_y - a_y) - \big(u(c_x, c_y) - u(a_x, a_y) \big)(b_y - a_y) \Big) \tag{8}$$

When we have a vertex a and list of linear interpolants for each triangle in tr, we have a set W of weight vectors w which satisfy the following condition, where k is x or y (weight vectors differs for variable x and y).

$$w_1 \frac{\Pi_{t_1}[u](x,y)}{\partial k} + w_2 \frac{\Pi_{t_2}[u](x,y)}{\partial k} + \ldots + w_n \frac{\Pi_{t_n}[u](x,y)}{\partial k} = \frac{\partial u(x,y)}{\partial k} \tag{9}$$

For all $u(x,y)$ of second order polynomial functions, it is correct to use the basis functions $1, x, y, x^2, xy, y^2$, which are equivalent to a system of six equations which are related to every basis. It is possible to convert every triangle to local coordinates (ϕ, ζ) with origin at point a. Then when calculating a derivative by x, we can simplify the derivative of basis functions y and y^2 which are constant functions and they behave like basis function 1, thus we solve a system of four equations in n unknowns.

The error of this weighted gradient is $O(h_t^2)$. More details and proof can be found in [1], [2], [3] .

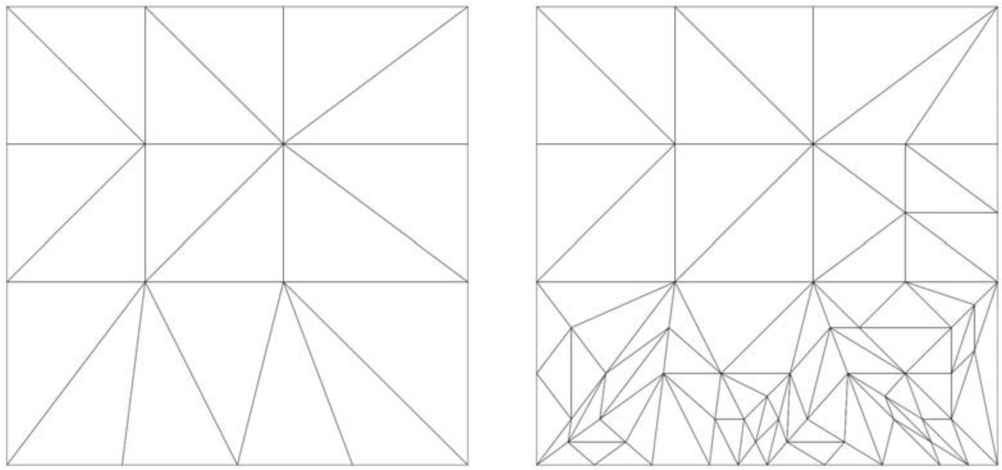

FIGURE 1. Initial triangulation (left), Final triangulation after 10 iterations (Right)

APOSTERIORI ERROR ESTIMATION

Now we will focus on a Dirichlet boundary value problem in a form $-\Delta u(x,y) = 0$ on a polygonal domain Ω with a nonobtuse regular triangulation T. We wanted to achive a more precise solution of boundary value problem using aposteriori error estimation. There are many types of error in the solution such as truncation error or discretization error which is the most important in most cases. When trying to reduce the discretization error, we need to identify triangles with big error. We use for this identification aposteriori error estimation which does not need to have a precise solution. For each triangle $t \in T$, we define a linear interpolant $\Pi_t[u_n]$ where u_n is the numerical solution of the boundary value problem. We also calculate a linear derivative functions $u_x(x,y)$ and $u_y(x,y)$ using weighted averaging method with error of $O(h_t^2)$. The aposteriori error is defined in the following way, where $|t|$ is the area of the triangle t

$$\eta = \frac{1}{|t|}\left(\int_t (\frac{\Pi_t[u_n](x,y)}{\partial x} - u_x(x,y))^2 + (\frac{\Pi_t[u_n](x,y)}{\partial y} - u_y(x,y))^2 \mathrm{d}x\mathrm{d}y\right) \tag{10}$$

In practice, this formula can be simplified to a calculation of a volume of a prism. We calculate the value of gradient of the linear interpolant and two gradients using weighted averaging (one for x and one for y) in all vertices a, b and c of t.

When the aposteriori error estimation is calculated for every triangle $t \in T$, we choose the biggest error and update the triangulation.

ADAPTIVE SOLUTION OF A MODEL PROBLEM

Let's consider the following boundary value problem with dirichlet boundary conditions.

$$-\Delta u(x,y) = 0 \text{ in } \Omega = \langle 0,1\rangle \times \langle 0,1\rangle \tag{11}$$

$$u(x,1) = u(0,y) = u(1,y) = 0, \quad u(x,0) = \sin \pi x \tag{12}$$

The following algorithm has been used:

1. Create initial very rough triangulation consisting of 19 triangles covering the domain Ω.
2. Solve boundary value problem using finite elements method
3. Calculate aposteriori error estimation for each triangle in the triangulation
4. If the biggest error is greater than threshold, update the triangulation and continue with point 2.

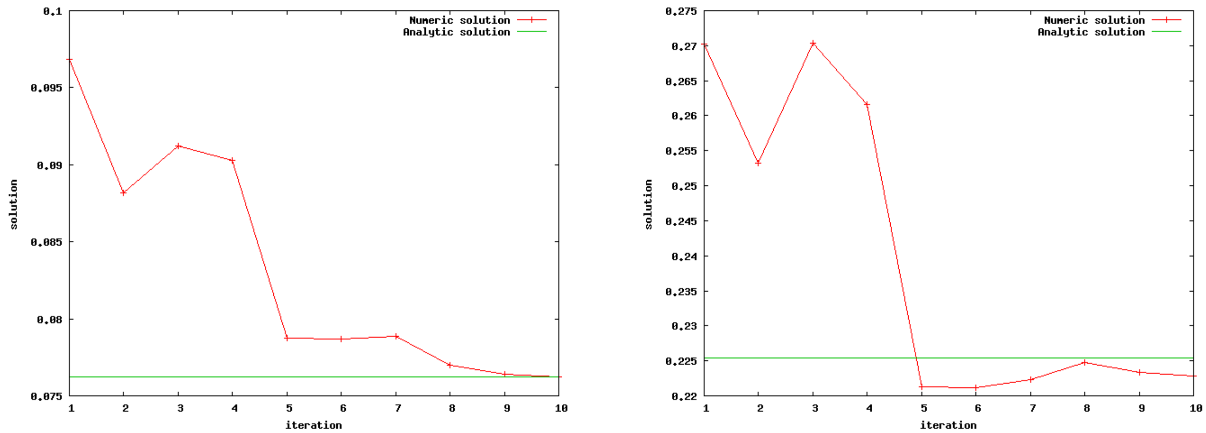

FIGURE 2. Solution at points $[0.3, 0.3]$ (left) and $[0.3, 0.6]$ (right)

5. We have found the solution

The update of triangulation is performed in the following way. We consider reducing a triangle t with neighbors t_{ab}, t_{bc} and t_{ac} which share side with triangle t.

1. Create centers of all sides of triangle t which are c_{ab}, c_{bc} and c_{ac}.
2. Replace triangle t by four new triangles $\overline{ac_{ab}c_{ac}}$, $\overline{c_{ab}bc_{bc}}$, $\overline{c_{bc}cc_{ac}}$ and $\overline{c_{ab}c_{bc}c_{ac}}$
3. Replace each triangle of t_{ab}, t_{bc} and t_{ac} by two triangles

CONCLUSIONS

We have got more precise solution of Laplace equation using iterative approach based on apposteriori error estimation. The triangulation was updated in a very simple way and there is a possibility to provide a smarter algorithm. This implementation can easily degrade triangle angles very close to zero and this causes aposteriori error estimation to provide worse solution from equilateral triangle.

Detailed information will be given during the ICNAAM 2013 conference.

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