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# Taylor Series Based Differential Formulas

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**Abstract:** The paper is a part of student cooperation in AKTION project (Austria-Czech) and concentrates on numerical computations using high-order forward, backward and symmetrical formulas. As an example, the hyperbolic PDE is analyzed, together with multiple integral computations. A multiple integral of a continuous function of $n$ variables can be computed by $n$-ary integration of the function fixing the remaining variables.

*Keywords:* Differential equations, Multiple integrals, Taylor series method, Partial Differential Equations

## 1. INTRODUCTION

The aim of the paper is to show an easy way of very high order difference formulas construction using Taylor series terms. For the point $u_1$ using forward formula (with respect to Fig. 1) we have following equations.

$$u_2 = u_1 + hu_1' + \frac{h^2}{2!}u_1'' + \frac{h^3}{3!}u_1''' + \frac{h^4}{4!}u_1'''' \tag{1}$$

$$u_3 = u_1 + 2hu_1' + \frac{(2h)^2}{2!}u_1'' + \frac{(2h)^3}{3!}u_1''' + \frac{(2h)^4}{4!}u_1'''' \tag{2}$$

$$u_4 = u_1 + 3hu_1' + \frac{(3h)^2}{2!}u_1'' + \frac{(3h)^3}{3!}u_1''' + \frac{(3h)^4}{4!}u_1'''' \tag{3}$$

$$u_5 = u_1 + 4hu_1' + \frac{(4h)^2}{2!}u_1'' + \frac{(4h)^3}{3!}u_1''' + \frac{(4h)^4}{4!}u_1'''' \tag{4}$$

Above mentioned forward formula considers only five points.

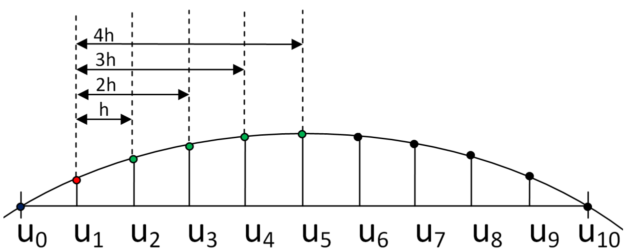


Fig. 1. Forward method

After expressing

$$u_2 - u_1 = DU1 + DU2 + DU3 + DU4 \tag{5}$$

$$u_3 - u_1 = 2DU1 + 2^2DU2 + 2^3DU3 + 2^4DU4 \tag{6}$$

$$u_4 - u_1 = 3DU1 + 3^2DU2 + 3^3DU3 + 3^4DU4 \tag{7}$$

$$u_5 - u_1 = 4DU1 + 4^2DU2 + 4^3DU3 + 4^4DU4 \tag{8}$$

the Taylor series terms $DU1$, $DU2$, $DU3$, $DU4$ can be calculated, where $DUi = \frac{u_1{}^{(i)}}{i!}h^i$.

Similarly, the Taylor series terms can be calculated for the backward and symmetrical differential formulas.

## 2. HYPERBOLIC PDE

One of the most common hyperbolic PDE is the wave equation Burden and Faires (2010); Collins (2006); Grasselli and Pelinovsky (2008); Cheney and Kincaid (2013); Kunovský (1995).

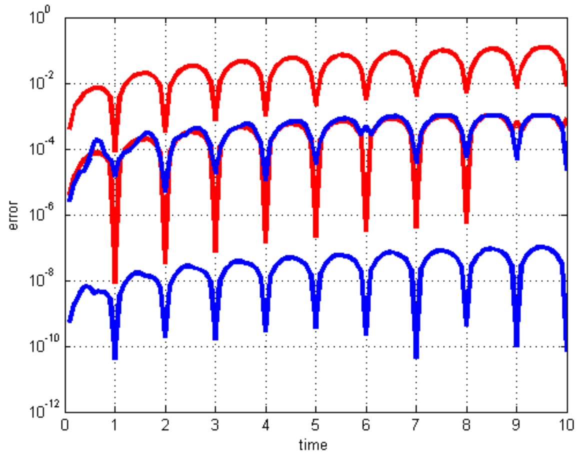


Fig. 2. "Error" functions

The wave equation may describe the oscillations of an ideal string of unit length. The "error" function of hyperbolic PDE solution using a three-point approximation is plotted in Fig. 2 – the upper red function for 10 segments and the lower red function for 100 segments. The "error" can be more effectively decreased by an increase in the order of the difference formula. The upper blue function of Fig. 2

plots the "error" function for 12 segments and also for 100 segments (shown in the blue function down) supposing that a five-point approximation has been used.

## 3. MULTIPLE INTEGRAL COMPUTATION

Here we transform the solution of integral

$$F(x) = \int_a^b f(x)\,\mathrm{d}x$$

into the ordinary differential equation

$$F'(x) = f(x) \tag{9}$$

with initial condition $F(a) = 0$, see Hirayama (2008).

Taylor series method can be used in solution of (9) in form

$$\begin{aligned} F(x_1) &= F(x_0) + \frac{h}{1!}F^{(1)}(x_0) + \frac{h^2}{2!}F^{(2)}(x_0) + \cdots \\ F(x_2) &= F(x_1) + \frac{h}{1!}F^{(1)}(x_1) + \frac{h^2}{2!}F^{(2)}(x_1) + \cdots \\ &\vdots \\ F(x_n) &= F(x_{n-1}) + \frac{h}{1!}F^{(1)}(x_{n-1}) + \frac{h^2}{2!}F^{(2)}(x_{n-1}) + \cdots \end{aligned} \tag{10}$$

where the sequence $x_0, x_1, ..., x_n$ represents a discretization of the interval $\langle a, b\rangle$.

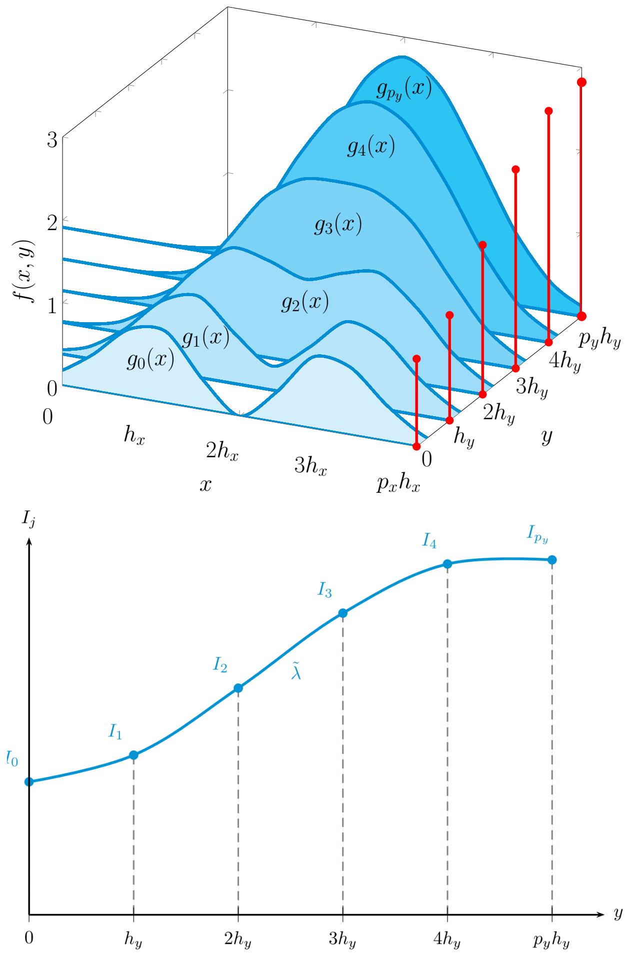


Fig. 3. Sampled integrals as a new function $I_j = \Psi(jh_y)$

### 3.1 Double integral

Without any loss of generality we can suppose all double integrals are in form 11.

$$\int_0^{b_2}\int_0^{b_1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{11}$$

Numerical solution starts by sampling both the $x$ and $y$ axis see Fig. 3.

After sampling we get a new function $\psi$ with values defined at multiples of $h_y$ (see Fig. 3). Using $\psi$, we can approximate the double integral by the following formula.

$$\int_0^{b_2}\int_0^{b_1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \approx \int_0^{b_2} \psi(y)\,\mathrm{d}y \tag{12}$$

Using differential formulas, Taylor series terms can be calculated and Taylor series method presented in (10) can be used for double integral computation.

## 4. CONCLUSIONS

The process of multiple integral can be generalized for $n$-ary integrals by repeating the process above for all subintegrals. Triple integral is computed from a sampled double integral as a function of one variable. The number of integrals grows exponentially with $n$. Integrated function is sampled, continuously integrated by above mentioned process, integrals are composed into a function. The functions then decrease the multiplicity of integration. Once we get a double integral, the last iteration of the process is executed.

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