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Terse walk sets in graphs and induced closure operators

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ABSTRACT

Given a graph G, for every ordinal $\alpha > 1$, we introduce and study closure operators on G induced by sets of α-indexed walks. For such sets, we define a property called terseness and investigate how it affects the induced closure operators. We show, among others, that the induction, if regarded as a map, is one-to-one for terse walk sets. We also determine a poset of closure operators (on a given graph) that is a direct limit of a direct system of sets of terse α-indexed walks ordered by set inclusion for certain ordinals $\alpha > 1$. Possible applications of the closure operators studied in digital topology are indicated.

1. Introduction

It is always worthwhile to deal with nontrivial relationships between different mathematical theories because such relationships bear witness to the interconnectedness of mathematics enabling us to use tools of one theory for studying another. In the present paper, we discuss relationships between graph theory, representing discrete mathematics, and topology, representing continuous mathematics. Particularly, we will introduce and study closure operators on graphs induced by sets of walks of identical (possibly infinite) lengths. While the sets of walks of length 1 induce completely additive closure operators with connectedness coinciding with the usual graph connectedness, the sets of walks of identical lengths greater than 1 induce more sophisticated closure operators. The idea of studying topological properties, connectedness in particular, of a graph with respect to walk sets was used in [16] where special walk sets, called path partitions, were employed to obtain convenient geometric properties of the connectedness. In [18], the concept of

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closure operators on graphs induced by sets of walks of the same finite lengths was introduced and studied. In particular, connectedness with respect to these closure operators was discussed there. A certain type of walk sets was also determined in [18] having the property that the induction of closure operators, if regarded as a map, is one-to-one. The present paper may be seen as a continuation of [18] working with generalized, α-indexed walks for arbitrary ordinals $\alpha > 1$ instead. We will define a certain type of walk sets, more general than the one found in [18], for which the induction of closure operators by walk sets, if regarded as a map, is one-to-one. The walk sets of this type will be called terse and we will determine a set of closure operators on a given graph which is a direct limit of the family of sets of terse α-indexed walks for certain ordinals $\alpha > 1$. Possible applications of the closure operators induced on graphs by terse walk sets (which are shown to contain the well-known Khalimsky and Marcus-Wyse topologies) in digital topology are indicated.

For the graph-theoretic terminology, we refer to [8]. By a *graph* $G = (V, E)$ we understand an (undirected simple) graph (without loops) with $V \neq \emptyset$ the *vertex* set and $E \subseteq \{\{x, y\};\ x, y \in V,\ x \neq y\}$ the set of *edges*. We will say that G is a graph *on* V. As usual, two vertices $x, y \in V$ are said to be *adjacent* (to each other) if $\{x, y\} \in E$. A key role will be played by the concept of a walk. Unlike the usual walks, in the present paper, the walks are allowed to be transfinite. More precisely, given an ordinal $\alpha > 1$, by an *α-walk* (a *walk* for short) in G we understand a sequence $(x_i|\ i < \alpha)$ of vertices of V such that x_i is adjacent to x_{i+1} whenever $i + 1 < \alpha$. If $\alpha > 1$ is a finite ordinal, then $\alpha - 1$ is called the *length* of the walk $(x_i|\ i < \alpha)$. An α-walk is called an *α-path* (a *path* for short) if its members are pairwise different.

By a *closure operator* u on a set X, we mean a topology in Čech's sense [3], i.e., a map u: $\exp X \to \exp X$ (where $\exp X$ denotes the power set of X) which is

(i) grounded (i.e., $u\emptyset = \emptyset$),
(ii) extensive (i.e., $A \subseteq X \Rightarrow A \subseteq uA$), and
(iii) monotone (i.e., $A \subseteq B \subseteq X \Rightarrow uA \subseteq uB$).

The pair (X, u) is then called a *closure space*. Thus, the usual topologies [2] (i.e., Kuratowski closure operators) are the closure operators u on X that are

(iv) additive (i.e., $u(A \cup B) = uA \cup uB$ whenever $A, B \subseteq X$) and
(v) idempotent (i.e., $uuA = uA$ whenever $A \subseteq X$).

Given a cardinal $m > 1$, a closure operator u on a set X and the closure space (X, u) are called an *S_m-closure operator* and an *S_m-closure space* (an *S_m-space* for short), respectively, if the following condition is satisfied:

$$A \subseteq X \Rightarrow uA = \bigcup\{uB; B \subseteq A, \operatorname{card} B < m\}.$$

In [4], S_2-closure operators and S_2-spaces are called *quasi-discrete*. S_2-topologies (S_2-topological spaces) are usually called *Alexandroff topologies* (*Alexandroff spaces*) – cf. [1]. Clearly, every S_2-closure operator is additive and every S_m-closure operator is an S_n-closure operator whenever $m \leq n$. If $m \leq \aleph_0$, then every additive S_m-closure operator is an S_2-closure operator. Of course, if $\operatorname{card} X = n$, then every closure operator on X is an S_{n+1}-closure operator. It is therefore useful to know, for a given closure operator u on X, the minimal cardinal m for which u is an S_m-closure operator. Such a minimal cardinal is an important invariant of the closure space (X, u) as mentioned in [3].

Given an ordinal α, we denote by $\langle\alpha\rangle$ the least cardinal m with $\alpha \leq m$. The predecessor of an isolated ordinal $\alpha > 1$ is denoted by $\alpha - 1$ and, if $\alpha - 1$ is isolated, too, then the predecessor of $\alpha - 1$ is denoted by $\alpha - 2$ and α is said to be *double isolated*.

We will use some basic topological concepts (see e.g. [5]) naturally extended to closure spaces. Given closure operators u, v on a set X, we put $u \le v$ if $uA \subseteq vA$ for every subset $A \subseteq X$. Clearly, $\le$ is a partial order on the set of all closure operators on X. If $u \le v$, then u is said to be *finer* than v and v is said to be *coarser* than u. Note that, for topologies given by open sets, just the converse partial order is usually used.

We will also employ one categorical concept, the one of a direct limit [9] restricted to posets (i.e., partially ordered sets) and isotone maps, hence avoiding the categorical terminology. More precisely, a *direct system* of posets consists of a collection $\{P_\alpha;\ \alpha \in I\}$ of posets indexed by a directed poset I and a collection of isotone maps $\varphi_{\alpha,\beta} : P_\alpha \to P_\beta$ defined for all $\alpha, \beta \in I$, $\alpha \le \beta$, such that

(i) $\varphi_{\alpha,\alpha} = id_{P_\alpha}$ whenever $\alpha \in I$ and
(ii) $\varphi_{\beta,\gamma} \circ \varphi_{\alpha,\beta} = \varphi_{\alpha,\gamma}$ whenever $\alpha, \beta, \gamma \in I$, $\alpha \le \beta \le \gamma$.

We denote such a direct system by $(\{P_\alpha\}, \{\varphi_{\alpha,\beta}\}, I)$. A *direct limit* of the direct system $(\{P_\alpha\}, \{\varphi_{\alpha,\beta}\}, I)$ is a poset P with the property that there exists an isotone map $\varphi_\alpha : P_\alpha \to P$ for every $\alpha \in I$ such that

(1) $\varphi_\beta \circ \varphi_{\alpha,\beta} = \varphi_\alpha$ whenever $\alpha, \beta \in I$, $\alpha \le \beta$;
(2) If G is a poset and $\psi_\alpha : P_\alpha \to G$ is an isotone map for every $\alpha \in I$ such that $\psi_\beta \circ \varphi_{\alpha,\beta} = \psi_\alpha$ for all $\alpha, \beta \in I$, $\alpha \le \beta$, then there exists a unique isotone map $\chi : P \to G$ such that $\chi \circ \varphi_\alpha = \psi_\alpha$ for all $\alpha \in I$.

2. Closure operators on graphs induced by walk sets

Throughout this section, we assumes that there is given an ordinal $\alpha > 1$. We denote by $\mathcal{W}_\alpha(G)$ the set of all α-walks in G. Every subset $\mathcal{B} \subseteq \mathcal{W}_\alpha(G)$ will be called an α-*walk set*, or a *walk set* for short, in G. If every element of $\mathcal{B}$ is even a path, then $\mathcal{B}$ will be called an α-*path set*, or a *path set* for short, in G.

Let $G = (V, E)$ be a graph. Given a subset $\mathcal{B} \subseteq \mathcal{W}_\alpha(G)$, we put $u_\mathcal{B}X = X \cup \{x \in V;$ there exist $(x_i| i < \alpha) \in \mathcal{B}$ and an ordinal $i_0,\ 0 < i_0 < \alpha,$ such that $\{x_i;\ i < i_0\} \subseteq X$ and $x_{i_0} = x\}$ for every $X \subseteq V$.

It may easily be seen that $u_\mathcal{B}$ is an $S_{\langle\alpha\rangle}$-closure operator on G. It will be said to be *induced* by $\mathcal{B}$. It is evident that every walk belonging to $\mathcal{B}$ is a connected subset of the closure space $(V, u_\mathcal{B})$. We clearly have $\mathcal{B} \subseteq \mathcal{D} \Rightarrow u_\mathcal{B} \subseteq u_\mathcal{D}$ whenever $\mathcal{B}, \mathcal{D} \subseteq \mathcal{W}_\alpha(G)$.

Remark 2.1. Let $G = (V, E)$ be a graph and u a closure operator on G. Then it is obvious that there exists an α-walk set $\mathcal{B}$ in G such that $u = u_\mathcal{B}$ if and only if the following condition is satisfied:

If $X \subseteq V$ and $x \in uX - X$, then there exist $(x_i|i < \alpha) \in \mathcal{W}_\alpha(G)$ and an ordinal β, $0 < \beta < \alpha$, such that $\{x_i;\ i < \beta\} \subseteq X$, $x_j \in u\{x_i;\ i < j\}$ for each j, $0 < j < \beta$, and $x \in u\{x_i;\ i < \beta\}$.

In general, $u_\mathcal{B}$ is clearly neither additive nor idempotent. On the other hand, the following assertion is proved in [18] (for sets of walks of finite lengths only but, for the walk sets in our setting, the proof is analogous):

Proposition 2.2. *Let $G = (V, E)$ be a graph and $\mathcal{B} \subseteq \mathcal{W}_\alpha(G)$ a walk set. Then*

(1) The union of a system of closed subsets of $(V, u_\mathcal{B})$ is a closed subset of $(V, u_\mathcal{B})$.
(2) The closure operator $u_\mathcal{B}$ is idempotent if and only if $(V, u_\mathcal{B})$ is an Alexandroff space.

The following Definition and Theorem are also taken from [18] (Definition 3.7 and Theorem 3.8) where they are formulated for a finite ordinal $\alpha > 1$. For an arbitrary ordinal $\alpha > 1$, the Theorem may be proved analogously to that in [18].

Definition 2.3. Let G be a graph. A walk set $\mathcal{B} \subseteq \mathcal{W}_\alpha(G)$ is said to be *strongly terse* if the following condition is true:

If $(x_i|\ i < \alpha), (y_i|\ i < \alpha) \in \mathcal{B}$ are walks with $\{x_0, x_1\} = \{y_{i_0}, y_{i_1}\}$ for some $i_0, i_1 < \alpha$, then $(x_i|\ i < \alpha) = (y_i|\ i < \alpha)$.

Theorem 2.4. *For strongly terse α-walk sets $\mathcal{B}$ in a graph G, the correspondence $\mathcal{B} \mapsto u_\mathcal{B}$ is one-to-one.*

In [18], also some other properties of the closure operators on graphs induced by strongly terse walk sets are discussed. In the present paper, we focus on a property of α-walk sets $\mathcal{B}$ (in a graph) weaker than strong terseness but still sufficient for the correspondence $\mathcal{B} \mapsto u_\mathcal{B}$ to be one-to-one.

Definition 2.5. Let G be a graph. A walk set $\mathcal{B} \subseteq \mathcal{W}_\alpha(G)$ is said to be *terse* if the following condition is true:

If $(x_i|\ i < \alpha) \in \mathcal{W}_\alpha(G)$ has the property that, for every ordinal i_0 with $0 < i_0 < \alpha$, $x_{i_0} \in \{x_i;\ i < i_0\}$ or there exist $(y_j|\ j < \alpha) \in \mathcal{B}$ and j_0, $0 < j_0 < \alpha$, such that $x_{i_0} = y_{j_0}$ and $\{y_j; j < j_0\} \subseteq \{x_i; i < i_0\}$, then $(x_i|\ i < \alpha) \in \mathcal{B}$.

It is evident that every strongly terse walk set is terse but not vice versa.

Example 2.6. Note that every 2-walk set in a graph is terse. A 3-walk set $\mathcal{B}$ in a graph is terse if and only if each of the following four conditions implies $(x, y, z) \in \mathcal{B}$:

(1) $(x, y, t) \in \mathcal{B}$, $(x, z, u) \in \mathcal{B}$,
(2) $(x, y, t) \in \mathcal{B}$, $(y, z, u) \in \mathcal{B}$,
(3) $(x, y, t) \in \mathcal{B}$, $(y, x, z) \in \mathcal{B}$,
(4) $(x, y, t) \in \mathcal{B}$, $x = z$.

Theorem 2.7. *Let $\mathcal{B}, \mathcal{D}$ be terse α-walk sets in a graph. Then $\mathcal{B} \subseteq \mathcal{D}$ if and only if $u_\mathcal{B} \leq u_\mathcal{D}$.*

Proof. The implication $\mathcal{B} \subseteq \mathcal{D} \Rightarrow u_\mathcal{B} \leq u_\mathcal{D}$ is obvious. To prove the converse implication, let $u_\mathcal{B} \leq u_\mathcal{D}$ and let $(x_i|i < \alpha) \in \mathcal{B}$. Then, $x_{i_0} \in u_\mathcal{B}\{x_i;\ i < i_0\} = x_{i_0} \in u_\mathcal{D}\{x_i;\ i < i_0\}$ for each i_0, $0 < i_0 < \alpha$. Hence, for every i_0 with $0 < i_0 < \alpha$, $x_{i_0} \in \{x_i;\ i, i_0\}$ or there exist $(y_j|\ j < \alpha) \in \mathcal{D}$ and j_0, $0 < j_0 < \alpha$, such that $x_{i_0} = y_{j_0}$ and $\{y_j;\ j < j_0\} \subseteq \{x_i;\ i < i_0\}$. Therefore, $(x_i|\ i < \alpha) \in \mathcal{D}$ and we have shown that $\mathcal{B} \subseteq \mathcal{D}$. □

Corollary 2.8. *For terse α-walk sets $\mathcal{B}$ in a graph, the correspondence $\mathcal{B} \mapsto u_\mathcal{B}$ is one-to-one.*

Theorem 2.9. *Let G be a graph and $\mathcal{B} \subseteq \mathcal{W}_\alpha(G)$ a terse walk set. Then, for every $(x_i|i < \alpha) \in \mathcal{W}_\alpha(G)$, $(x_i|i < \alpha) \in \mathcal{B}$ if and only if $x_{i_0} \in u_\mathcal{B}\{x_i;\ i < i_0\}$ for every ordinal i_0, $0 < i_0 < \alpha$.*

Proof. If $(x_i|i < \alpha) \in \mathcal{B}$, then it is obvious that $x_{i_0} \in u_\mathcal{B}\{x_i;\ i < i_0\}$ for every ordinal i_0, $0 < i_0 < \alpha$. Conversely, let $x_{i_0} \in u_\mathcal{B}\{x_i;\ i < i_0\}$ for every ordinal i_0, $0 < i_0 < \alpha$. Then, for every ordinal i_0 with $0 < i_0 < \alpha$, $x_{i_0} \in \{x_i;\ i < i_0\}$ or there exist $(y_j|\ j < \alpha) \in \mathcal{B}$ and j_0, $0 < j_0 < \alpha$, such that $x_{i_0} = y_{j_0}$ and $\{y_j; j < j_0\} \subseteq \{x_i; i < i_0\}$. Since $\mathcal{B}$ is terse, we have $(x_i|\ i < \alpha) \in \mathcal{B}$. □

Let $\alpha, \beta > 1$ be ordinals and G be a graph. For every terse walk set $\mathcal{B} \subseteq \mathcal{W}_\alpha(G)$ in G, we put $\varphi_{\alpha,\beta}(\mathcal{B}) = \{(y_j|\ j < \beta) \in \mathcal{W}_\beta(G)$; for every j_0 with $0 < j_0 < \beta$, $y_{j_0} \in \{y_j;\ j < j_0\}$ or there exist $(x_i|\ i < \alpha) \in \mathcal{B}$ and i_0, $0 < i_0 < \alpha$, such that $y_{j_0} = x_{i_0}$ and $\{x_i;\ i < i_0\} \subseteq \{y_j;\ j < j_0\}$. Clearly, for any terse walk sets $\mathcal{B}, \mathcal{D} \subseteq \mathcal{W}_\alpha(G)$, $\varphi_{\alpha,\alpha}(\mathcal{B}) = \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{D}$ implies $\varphi_{\alpha,\beta}(\mathcal{B}) \subseteq \varphi_{\alpha,\beta}(\mathcal{D})$.

Remark 2.10. Let G be a graph. We clearly have

(1) $\varphi_{\alpha,\alpha}(\mathcal{B}) = \mathcal{B}$ for every walk set $\mathcal{B} \subseteq \mathcal{W}_\alpha(G)$,
(2) $\mathcal{B} \subseteq \mathcal{D}$ implies $\varphi_{\alpha,\beta}(\mathcal{B}) \subseteq \varphi_{\alpha,\beta}(\mathcal{D})$ whenever $\mathcal{B}, \mathcal{D} \subseteq \mathcal{W}_\alpha(G)$ are walk sets,
(3) $(y_j|\ j < \beta) \in \varphi_{\alpha,\beta}(\mathcal{B})$ implies $(y_j|\ j < \alpha) \in \mathcal{B}$ for every walk set $\mathcal{B} \subseteq \mathcal{W}_\alpha(G)$.

Lemma 2.11. *Let $\alpha, \beta > 1$ be ordinals, α double isolated and $\alpha \le \beta$. Let G be a graph and $\mathcal{B}, \mathcal{D}$ terse α-walk sets in G. Then $\mathcal{B} \subseteq \mathcal{D}$ if and only if $\varphi_{\alpha,\beta}(\mathcal{B}) \subseteq \varphi_{\alpha,\beta}(\mathcal{D})$.*

Proof. The implication $\mathcal{B} \subseteq \mathcal{D} \Rightarrow \varphi_{\alpha,\beta}(\mathcal{B}) \subseteq \varphi_{\alpha,\beta}(\mathcal{D})$ is obvious. To prove the converse implication, suppose that $\varphi_{\alpha,\beta}(\mathcal{B}) \subseteq \varphi_{\alpha,\beta}(\mathcal{D})$ and let $(x_i|\ i < \alpha) \in \mathcal{B}$. Let $(y_j|\ j < \beta) \in \mathcal{W}_\beta(G)$ be the walk given by $y_j = x_j$ for every j with $1 < j < \alpha$ and

$$y_j = \begin{cases} x_{\alpha-2} & \text{if } k = \delta + n \text{ where } \delta = \alpha \text{ or } \delta \text{ is a limit ordinal and } n \text{ is an even (finite) ordinal,} \\ x_{\alpha-1} & \text{if } k = \delta + n \text{ where } \delta = \alpha \text{ or } \delta \text{ is a limit ordinal and } n \text{ is an odd (finite) ordinal} \end{cases}$$

for every j with $\alpha \le j < \beta$. Then, for all ordinals j_0 with $0 < j_0 < \alpha$ (including those satisfying $y_{j_0} \in \{y_j;\ j < j_0\}$), we have $y_{j_0} = x_{j_0}$ and $\{x_i;\ i < j_0\} = \{y_j;\ j < j_0\}$. Next, for every ordinal j_0 with $\alpha \le j_0 < \beta$, we have $y_{j_0} \in \{y_j;\ j < j_0\}$. Consequently, $(y_j|\ j < \beta) \in \varphi_{\alpha,\beta}(\mathcal{B})$, so that $(y_j|\ j < \beta) \in \varphi_{\alpha,\beta}(\mathcal{D})$. Therefore, $(y_j|\ j < \alpha) \in \mathcal{D}$ (see Remark 2.10(3)). Since $(y_j|\ j < \alpha) = (x_i|\ i < \alpha)$, we have shown that $\mathcal{B} \subseteq \mathcal{D}$ and the proof is completed. □

Lemma 2.12. *Let $\alpha, \beta > 1$ be ordinals, G a graph, and $\mathcal{B}$ a terse α-walk set in G. Then, $\varphi_{\alpha,\beta}(\mathcal{B})$ is a terse β-walk set in G such that $\varphi_{\gamma,\beta}(\varphi_{\alpha,\gamma}(\mathcal{B})) \subseteq \varphi_{\alpha,\beta}(\mathcal{B})$ for every ordinal γ with $\alpha \le \gamma \le \beta$.*

Proof. Let $(y_j|\ j < \beta) \in \mathcal{W}_\beta(G)$ be a walk such that, for every ordinal j_0 with $0 < j_0 < \beta$, $y_{j_0} \in \{y_j;\ j < j_0\}$ or there exist $(z_k|\ k < \beta) \in \varphi_{\alpha,\beta}(\mathcal{B})$ and k_0, $0 < k_0 < \beta$, such that $y_{j_0} = z_{k_0}$ and $\{z_k;\ k < k_0\} \subseteq \{y_j;\ j < j_0\}$. Suppose that $y_{j_0} \notin \{y_j;\ j < j_0\}$. Then, $z_{k_0} \in \{z_k;\ k < k_0\}$, so that there exist $(x_i|\ i < \alpha) \in \mathcal{B}$ and i_0, $0 < i_0 < \alpha$, such that $z_{k_0} = x_{i_0}$ and $\{x_i;\ i < i_0\} \subseteq \{z_k;\ k < k_0\}$. Thus, we have $y_{j_0} = x_{i_0}$ and $\{x_i;\ i < i_0\} \subseteq \{y_j;\ j < j_0\}$. Therefore, $\{y_j;\ j < j_0\} \in \varphi_{\alpha,\beta}(\mathcal{B})$, so that $\varphi_{\alpha,\beta}(\mathcal{B})$ is terse.

To prove the second part of the statement, let $(z_k|\ k < \beta) \in \varphi_{\gamma,\beta}(\varphi_{\alpha,\gamma}(\mathcal{B}))$. Then, for every ordinal k_0 with $0 < k_0 < \beta$, $z_{k_0} \in \{z_k;\ k < k_0\}$ or there exist $(y_j|\ j < \gamma) \in \varphi_{\alpha,\beta}(\mathcal{B})$ and j_0, $0 < j_0 < \gamma$, such that $z_{k_0} = y_{j_0}$ and $\{y_j;\ j < j_0\} \subseteq \{z_k;\ k < k_0\}$. Suppose that $z_{k_0} \notin \{z_k;\ k < k_0\}$. Then $y_{j_0} \notin \{y_j;\ j < j_0\}$, so that there are $(x_i|\ i < \alpha) \in \mathcal{B}$ and i_0, $0 < i_0 < \alpha$, such that $y_{j_0} = x_{i_0}$ and $\{x_i;\ i < i_0\} \subseteq \{y_j;\ j < j_0\}$. Thus, $z_{k_0} = x_{i_0}$ and $\{x_i;\ i < i_0\} \subseteq \{z_k;\ k < k_0\}$. Consequently, $(z_k|\ k < \beta) \in \varphi_{\alpha,\beta}(\mathcal{B})$, so that $\varphi_{\gamma,\beta}(\varphi_{\alpha,\gamma}(\mathcal{B})) \subseteq \varphi_{\alpha,\beta}(\mathcal{B})$. □

Proposition 2.13. *Let $\alpha, \beta, \gamma > 1$ be ordinals, α double isolated and $\alpha \le \gamma \le \beta$. Let G be a graph and $\mathcal{B}$ a terse α-walk set in G. Then $\varphi_{\gamma,\beta}(\varphi_{\alpha,\gamma}(\mathcal{B})) = \varphi_{\alpha,\beta}(\mathcal{B})$.*

Proof. Let $(y_j|\ j < \beta) \in \varphi_{\alpha,\beta}(\mathcal{B})$. Then, for every ordinal j_0 with $0 < j_0 < \beta$, $y_{j_0} \in \{y_j;\ j < j_0\}$ or there exist $(x_i^{j_0}|\ i < \alpha) \in \mathcal{B}$ and i_0, $0 < i_0 < \alpha$, such that $y_{j_0} = x_{i_0}^{j_0}$ and $\{x_i^{j_0};\ i < i_0\} \subseteq \{y_j;\ j < j_0\}$. For every j_0 such that $0 < j_0 < \beta$ and $y_{j_0} \notin \{y_j;\ j < j_0\}$, let $(z_k^{j_0}|\ k < \gamma) \in \mathcal{W}_\gamma(G)$ be the path with $z_k^{j_0} = x_k^{j_0}$ for every $k < \alpha$ and

$$z_k^{j_0} = \begin{cases} x_{\alpha-2}^{j_0} & \text{if } k = \delta + n \text{ where } \delta = \alpha \text{ or } \delta \text{ is a limit ordinal and } n \text{ is an even (finite) ordinal,} \\ x_{\alpha-1}^{j_0} & \text{if } k = \delta + n \text{ where } \delta = \alpha \text{ or } \delta \text{ is a limit ordinal and } n \text{ is an odd (finite) ordinal} \end{cases}$$

for every ordinal k with $\alpha \le k < \gamma$. Let j_0 be an ordinal such that $0 < j_0 < \beta$ and $y_{j_0} \notin \{y_j;\ j < j_0\}$. For all ordinals k_0, $0 < k_0 < \alpha$ (including those with $z_{k_0} \notin \{z_k;\ k < k_0\}$) we have $z_{k_0}^{j_0} = x_{k_0}^{j_0}$ and

$\{x_i^{j_0};\ i < k_0\} = \{z_k^{j_0};\ k < k_0\}$. Next, for every ordinal k_0, $\alpha \le k_0 < \gamma$, we have $z_{k_0} \in \{z_k;\ k < k_0\}$. Thus, for every ordinal j_0 such that $0 < j_0 < \beta$ and $y_{j_0} \notin \{y_j;\ j < j_0\}$, we have $(z_k^{j_0} |\ k < \gamma) \in \varphi_{\alpha,\beta}(\mathcal{B})$ and also $y_{j_0} = x_{i_0}^{j_0} = z_{i_0}^{j_0}$ and $\{z_k^{j_0};\ k < i_0\} = \{x_i^{j_0};\ i < i_0\} \subseteq \{y_j;\ j < j_0\}$. Thus, $(y_j |\ j < \beta) \in \varphi_{\gamma,\beta}(\varphi_{\alpha,\gamma}(\mathcal{B}))$. We have shown that $\varphi_{\alpha,\beta}(\mathcal{B}) \subseteq \varphi_{\gamma,\beta}(\varphi_{\alpha,\gamma}(\mathcal{B}))$. Since the converse inclusion is true by Lemma 2.12, the proof is completed. □

Given a graph G, we denote by $\mathcal{T}_\alpha(G)$ the set of all terse α-walk sets in G. The sets $\mathcal{T}_\alpha(G)$ ($\alpha > 1$ an ordinal) are understood to be partially ordered by set inclusion. The following statement follows from Remark 2.10(2) and Lemma 2.11:

Proposition 2.14. *Let $\alpha, \beta > 1$ be ordinals and $G = (V, E)$ be a graph. Then, $\varphi_{\alpha,\beta} : \mathcal{T}_\alpha(G) \to \mathcal{T}_\beta(G)$ is an isotone map and, if α is double isolated and $\alpha \le \beta$, then it is an embedding.*

We will need the following observation:

Lemma 2.15. *Let $\alpha, \beta > 1$ be ordinals, α double isolated and $\alpha \le \beta$. Let G a graph and $\mathcal{B}$ a terse α-walk sets in G. Then, $u_{\varphi_{\alpha,\beta}(\mathcal{B})} = u_{\mathcal{B}}$.*

Proof. Let $X \subseteq V$ and $x \in u_{\varphi_{\alpha,\beta}(\mathcal{B})}X$. If $x \in X$, then $x \in u_{\mathcal{B}}X$. Suppose that $x \notin X$. Then there are $(y_j |\ j < \beta) \in \varphi_{\alpha,\beta}(\mathcal{B})$ and an ordinal j_0, $0 < j_0 < \beta$, such that $\{y_j;\ j < j_0\} \subseteq X$ and $y_{j_0} = x$. Let j_1, $0 < j_1 \le j_0$, be the smallest ordinal with $y_{j_1} = x$. Then, there is a walk $(x_i |\ i < \alpha) \in \mathcal{B}$ and an ordinal i_0, $0 < i_0 < \alpha$, such that $x = y_{j_1} = x_{i_0}$ and $\{x_i;\ i < i_0\} \subseteq \{y_j;\ j < j_1\}$. Thus, $\{x_i;\ i < i_0\} \subseteq X$, so that $x \in u_{\mathcal{B}}X$. The inclusion $u_{\varphi_{\alpha,\beta}(\mathcal{B})} \subseteq u_{\mathcal{B}}$ is proved.

Conversely, let $x \in u_{\mathcal{B}}X$. If $x \in X$, then $x \in u_{\varphi_{\alpha,\beta}(\mathcal{B})}X$. Suppose that $x \notin X$. Then, there are $(x_i |\ i < \alpha) \in \mathcal{B}$ and an ordinal i_0, $0 < i_0 < \alpha$, such that $\{x_i;\ i < i_0\} \subseteq X$ and $x_{i_0} = x$. Let $(y_j |\ j < \beta) \in \mathcal{W}_\beta(G)$ be the walk given by $y_j = x_j$ for every j with $1 < j < \alpha$ and

$$y_j = \begin{cases} x_{\alpha-2} & \text{if } k = \delta + n \text{ where } \delta = \alpha \text{ or } \delta \text{ is a limit ordinal and } n \text{ is an even (finite) ordinal,} \\ x_{\alpha-1} & \text{if } k = \delta + n \text{ where } \delta = \alpha \text{ or } \delta \text{ is a limit ordinal and } n \text{ is an odd (finite) ordinal} \end{cases}$$

for every j with $\alpha \le j < \beta$. Then, for all ordinals j_0 with $0 < j_0 < \alpha$ (including those satisfying $y_{j_0} \in \{y_j;\ j < j_0\}$), we have $y_{j_0} = x_{j_0}$ and $\{x_i;\ i < j_0\} = \{y_j;\ j < j_0\}$. Next, for every ordinal j_0 with $\alpha \le j_0 < \beta$, we have $y_{j_0} \in \{y_j;\ j < j_0\}$. Consequently, $(y_j |\ j < \beta) \in \varphi_{\alpha,\beta}(\mathcal{B})$ and we have $\{y_j;\ j < j_0\} \subseteq X$ and $y_{i_0} = x$. Thus, $x \in u_{\varphi_{\alpha,\beta}(\mathcal{B})}X$. The inclusion $u_{\mathcal{B}} \subseteq u_{\varphi_{\alpha,\beta}(\mathcal{B})}$ is proved. Therefore, $u_{\varphi_{\alpha,\beta}(\mathcal{B})} = u_{\mathcal{B}}$ and the proof is completed. □

Given an ordinal γ, we denote by I_γ the set of all ordinals α with $\gamma + 1 < \alpha < \gamma + \omega$ where ω is the least infinite ordinal. Note that $\alpha \in I_\gamma$ means that α is double isolated. Let $G = (V, E)$ be a graph and γ an ordinal. Put $\mathcal{U}^\gamma(G) = \{u;\ u$ is a closure operator on G such that, for every subset $X \subseteq V$ and every point $x \in uX - X$, there exist an ordinal $\alpha \in I_\gamma$, a walk $(x_i |\ i < \alpha) \in \mathcal{W}_\alpha(G)$, and an ordinal β, $0 < \beta < \alpha$, such that $\{x_i;\ i < \beta\} \subseteq X$, $x_j \in u\{x_i;\ i < j\}$ for each ordinal j, $0 < j < \beta$, and $x \in u\{x_i;\ i < \beta\}\}$. The set $\mathcal{U}^\gamma(G)$ is considered to be partially ordered by $\le$ (i.e., "finer than").

Remark 2.10(1) and Proposition 2.13 immediately result in:

Corollary 2.16. *$(\{\mathcal{T}_\alpha(G)\}, \{\varphi_{\alpha,\beta}\}, I_\gamma)$ is a direct system.*

Theorem 2.17. *Let G be a graph and γ an ordinal. Then, the set $\mathcal{U}^\gamma(G)$ is a direct limit of the direct system $(\{\mathcal{T}_\alpha(G)\}, \{\varphi_{\alpha,\beta}\}, I_\gamma)$.*

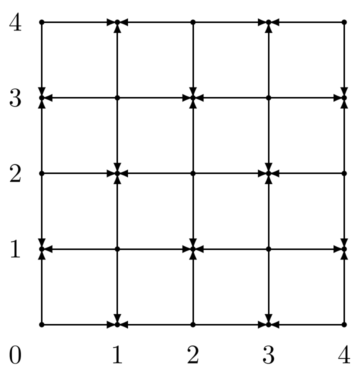
Proof. For every $\alpha \in I_\gamma$ and every $\mathcal{B} \in \mathcal{T}_\alpha(G)$, put $\varphi_\alpha(\mathcal{B}) = u_\mathcal{B}$. By Remark 1, $\varphi_\alpha : \mathcal{T}_\alpha(G) \to \mathcal{U}^\gamma(G)$ is a map such that, for every $u \in \mathcal{U}^\gamma(G)$, there are $\alpha \in I_\gamma$ and $\mathcal{B} \in \mathcal{T}_\alpha(G)$ with $\varphi_\alpha(\mathcal{B}) = u$. By Theorem 2.7, φ_α is an embedding for every $\alpha \in I_\gamma$. We have $\varphi_\beta \circ \varphi_{\alpha,\beta} = \varphi_\alpha$ (so that $\varphi_\alpha(\mathcal{T}_\alpha(G)) \subseteq \varphi_\beta(\mathcal{T}_\beta(G))$) whenever $\alpha, \beta \in I_\gamma$, $\alpha \leq \beta$, by Lemma 2.15. Let P be a poset (with a partial order $\leq$) and, for every $\alpha \in I_\gamma$, let $\psi_\alpha : \mathcal{T}_\alpha(G) \to P$ be an isotone map such that $\psi \circ \varphi_{\alpha,\beta} = \psi_\alpha$ whenever $\alpha, \beta \in I_\gamma$, $\alpha \leq \beta$. For every $u \in \mathcal{U}^\gamma(G)$, put $\chi(u) = \psi_\alpha(\mathcal{B})$ where $\alpha \in I_\gamma$ is the least ordinal with $u \in \varphi_\alpha(\mathcal{T}_\alpha(G))$ and $\mathcal{B} \in \mathcal{T}_\alpha(G)$ is the walk set with $u_\mathcal{B} = u$. Then, $\chi \circ \varphi_\alpha = \psi_\alpha$ for every $\alpha \in I_\gamma$ and $\chi : \mathcal{U}^\gamma(G) \to P$ is clearly a unique map with this property. Let $u, v \in \mathcal{U}^\gamma(G)$, $u \leq v$, and let $\alpha \in I_\gamma$ be the least ordinal such that $u \in \varphi_\alpha(\mathcal{T}_\alpha(G))$ and $\beta \in I_\gamma$ be the least ordinal such that $v \in \varphi_\beta(\mathcal{T}_\beta(G))$. Without loss of generality, we may assume that $\alpha \leq \beta$. Then $u = u_\mathcal{B}$ and $v = u_\mathcal{D}$ for some $\mathcal{B}, \mathcal{D} \in \mathcal{T}_\beta(G)$ and, by Theorem 2.7, $\mathcal{B} \subseteq \mathcal{D}$. Since ψ_β is an isotone map, we have $\psi_\beta(\mathcal{B}) \leq \psi_\beta(\mathcal{D}) = \chi(v)$. Next, we have $u = u_\mathcal{A}$ for some $\mathcal{A} \in \mathcal{T}_\alpha(G)$, so that $\varphi_{\alpha,\beta}(\mathcal{A}) = \mathcal{B}$. Consequently, $\chi(u) = \chi(u_\mathcal{A}) = \psi_\alpha(\mathcal{A}) = \psi_\beta(\varphi_{\alpha,\beta}(\mathcal{A})) = \psi_\beta(\mathcal{B})$. Thus, $\chi(u) \subseteq \chi(v)$ and we have shown that the map χ is isotone. The proof is completed. □

In particular, $\mathcal{U}^0$ is the direct limit of the direct system $(\{\mathcal{T}_\alpha(G)\}, \{\varphi_{\alpha,\beta}\}, \alpha > 1$ finite ordinals).

3. Closure operators on 4- and 8-adjacency graphs on $\mathbb{Z}^2$ induced by terse path sets

Recall that digital topology (cf. [12]) is a theory developed for the study of geometric and topological properties of digital images. Though the classical approach to digital topology is based on using graph theory rather than topology, the topological approach, which was founded in [10] and developed in [17], uses purely topological methods for the purposes of digital topology. It was shown in [15] and [19] that, instead of topologies, closure operators may be used to advantage as basic tools in digital topology. In the following examples, we will propose using closure operators on the digital plane $\mathbb{Z}^2$ that are induced by terse path sets in certain graphs on $\mathbb{Z}^2$. Employing such closure operators is a combination of the classical approach to digital topology and the topological one, thus taking advantages of both of them. We will show that the well-known *Marcus-Wyse topology* [13] and *Khalimsky topology* [11] on $\mathbb{Z}^2$, which often occur in digital topology (cf. [6,7]) may be obtained as closure operators on graphs on $\mathbb{Z}^2$ induced by certain terse path sets.

Example 3.1. Let $G_4 = (\mathbb{Z}^2, E)$ where $E = \{\{(x, y), (z, t)\};\ (x, y), (z, t) \in \mathbb{Z}^2, |x - z| + |y - t| = 1\}$. Then G_4 is called the 4-adjacency graph on $\mathbb{Z}^2$. Put $\mathcal{B}_2 = \{((x_i, y_i)|\ i < 2);\ (x_i, y_i) \in \mathbb{Z}^2$ for every $i < 2$, $|x_0 - x_1| + |y_0 - y_1| = 1$ and $x_0 + y_0$ is even$\}$. Then, $\mathcal{B}_2$ is a terse 2-path set in G_4. A portion of $\mathcal{B}_2$ is shown in the following figure where the paths from $\mathcal{B}_2$ are represented by arrows directed from first to last vertices.



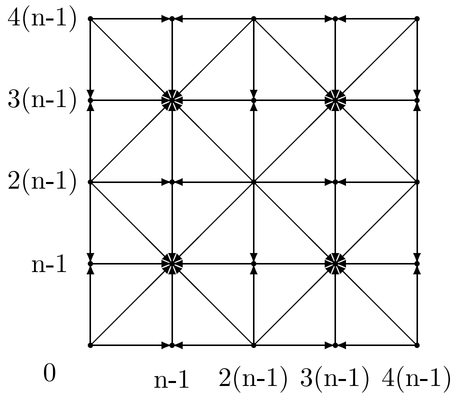
Clearly, $(\mathbb{Z}^2, u_{\mathcal{B}_2})$ is a connected Alexandroff topological space in which the points $(x, y) \in \mathbb{Z}^2$ with $x+y$ even are open while those with $x + y$ odd are closed. The closure operator $u_{\mathcal{B}_2}$ coincides with the Marcus-Wyse topology.

Example 3.2. For an arbitrary finite ordinal $n > 1$, let $G_8 = (\mathbb{Z}^2, E)$ be the graph where $E = \{\{(x, y), (z, t)\};\ (x, y), (z, t) \in \mathbb{Z}^2, |x - z| + |y - t| > 0,\ |x - z| \le 1,\ |y - t| \le 1\}$. The graph G_8 is called 8-adjacency graph on $\mathbb{Z}^2$.

For an arbitrary finite ordinal $n > 1$, let $\mathcal{D}_n$ be the set of all sequences $((x_i, y_i)|\ i < n)$ such that $(x_i, y_i) \in \mathbb{Z}^2$ for every $i < n$ and one of the following eight conditions is satisfied:

(1) $x_0 = x_1 = ... = x_{n-1}$ and there is $k \in \mathbb{Z}$ such that $y_i = 2k(n - 1) + i$ for all $i < n$,
(2) $x_0 = x_1 = ... = x_{n-1}$ and there is $k \in \mathbb{Z}$ such that $y_i = 2k(n - 1) - i$ for all $i < n$,
(3) $y_0 = y_1 = ... = y_{n-1}$ and there is $k \in \mathbb{Z}$ such that $x_i = 2k(n - 1) + i$ for all $i < n$,
(4) $y_0 = y_1 = ... = y_{n-1}$ and there is $k \in \mathbb{Z}$ such that $x_i = 2k(n - 1) - i$ for all $i < n$,
(5) there is $k \in \mathbb{Z}$ such that $x_i = 2k(n - 1) + i$ for all $i < n$ and there is $l \in \mathbb{Z}$ such that $y_i = 2l(n - 1) + i$ for all $i < n$,
(6) there is $k \in \mathbb{Z}$ such that $x_i = 2k(n - 1) + i$ for all $i < n$ and there is $l \in \mathbb{Z}$ such that $y_i = 2l(n - 1) - i$ for all $i < n$,
(7) there is $k \in \mathbb{Z}$ such that $x_i = 2k(n - 1) - i$ for all $i < n$ and there is $l \in \mathbb{Z}$ such that $y_i = 2l(n - 1) + i$ for all $i < n$,
(8) there is $k \in \mathbb{Z}$ such that $x_i = 2k(n - 1) - i$ for all $i < n$ and there is $l \in \mathbb{Z}$ such that $y_i = 2l(n - 1) - i$ for all $i < n$.

It may easily be seen that $\mathcal{D}_n$ is a terse n-path set in G_8. A portion of $\mathcal{D}_n$ is demonstrated in the following figure. The paths belonging to $\mathcal{D}_n$ are represented by arrows directed from first to last terms. Between any pair of neighboring parallel horizontal or vertical arrows (having the same direction), there are $n - 2$ more parallel arrows with the same direction that are not displayed.



Clearly, $(\mathbb{Z}^2, u_{\mathcal{D}_n})$ is a connected S_n-space. The closure operators $u_{\mathcal{D}_n}$, $n > 1$ a finite ordinal, coincide with the closure operators on $\mathbb{Z}^2$ studied in [14]. In particular, $u_{\mathcal{D}_2}$ is an Alexandroff topology, which coincides with the Khalimsky topology on $\mathbb{Z}^2$. Note that, for $n > 2$, the connectedness in $(\mathbb{Z}^2, u_{\mathcal{D}_n})$ cannot be obtained as the usual graph connectedness.

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