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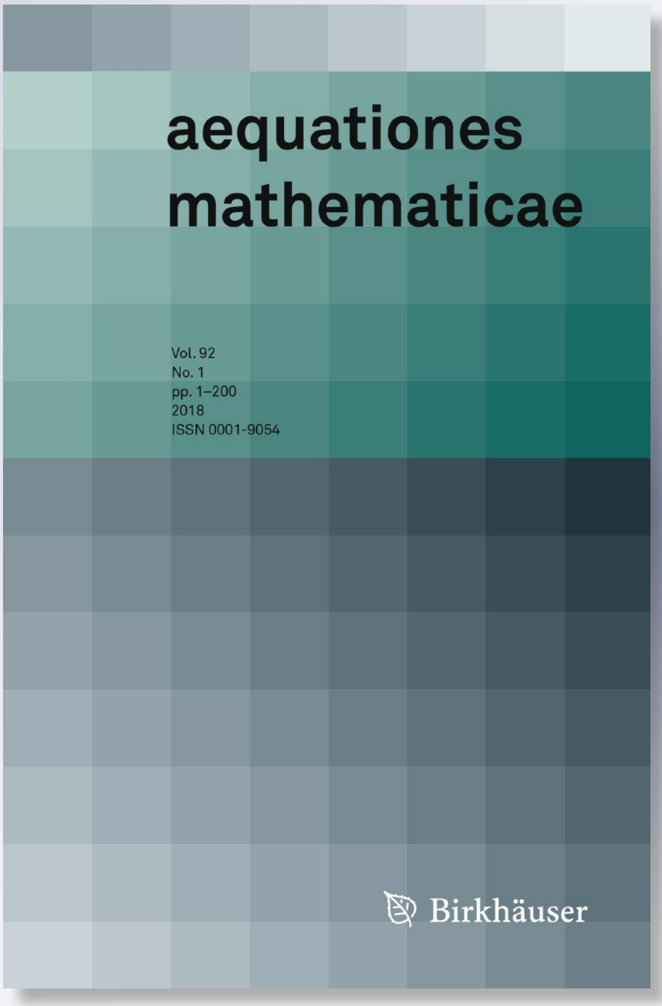

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Abstract. We introduce and discuss a connectedness induced by n-ary relations ($n > 1$ an integer) on their underlying sets. In particular, we focus on certain n-ary relations with the induced connectedness allowing for a definition of digital Jordan curves. For every integer $n > 1$, we introduce one such n-ary relation on the digital plane $\mathbb{Z}^2$ and prove a digital analogue of the Jordan curve theorem for the induced connectedness. It follows that these n-ary relations may be used as convenient structures on the digital plane for the study of geometric properties of digital images. For $n = 2$, such a structure coincides with the (specialization order of the) Khalimsky topology and, for $n > 2$, it allows for a variety of Jordan curves richer than that provided by the Khalimsky topology.

Mathematics Subject Classification. Primary 52C99; Secondary 68R05.

Keywords. n-Ary relation, Connectedness, Digital plane, Khalimsky topology, Jordan curve theorem.

1. Introduction

One of the basic tasks of computer imagery is to provide the digital plane $\mathbb{Z}^2$ with a convenient structure that would enable us to study and process digital images (cf. [9,10]). An important criterion of such a convenience is the validity of a digital analogue of the Jordan curve theorem (recall that the classical Jordan curve theorem states that a simple closed curve in the Euclidean plane separates this plane into exactly two connected components). The classical, graph theoretic, approach to the problem is based on using 4-adjacency and 8-adjacency graphs for structuring $\mathbb{Z}^2$ (see [14,15]). Unfortunately, neither the 4-adjacency nor the 8-adjacency graph alone allows for an analogue of the Jordan curve theorem (cf. [8]) so that a combination of the two adjacency graphs has to be used. Despite this drawback, the classical approach to digital topology, a specific branch of discrete geometry, has been used to solve numerous problems of digital image processing (see, e.g., [1]) and to create a great deal of useful graphic software.

To eliminate the above drawback of the classical approach to digital topology, a new, purely topological approach was proposed in [5] which utilizes a convenient topology for structuring the digital plane, namely the Khalimsky topology. The convenience of the Khalimsky topology for structuring the digital plane was shown in [5] by proving an analogue of the Jordan curve theorem for the topology. The topological approach was then developed by many authors—see, e.g., [3,6,7,11–13,17,18].

Since the Khalimsky topology is an Alexandroff T_0-topology, it is uniquely determined by a partial order on $\mathbb{Z}^2$, the so-called specialization order of the topology. The connectedness in the Khalimsky space then coincides with the connectedness in the underlying (simple) graph of the specialization order. Thus, when studying the connectedness of digital images with respect to the Khalimsky topology, this graph, rather than the Khalimsky topology itself, may be used for structuring the digital plane. A disadvantage of this approach is that Jordan curves in the (specialization order of the) Khalimsky topology can never turn at the acute angle $\frac{\pi}{4}$. It would, therefore, be useful to find some new, more convenient structures on $\mathbb{Z}^2$ that would allow Jordan curves to turn, at some points, at the acute angle $\frac{\pi}{4}$. In the present note, to obtain such a convenient structure, we generalize the specialization order of the Khalimsky topology, hence a binary relation on $\mathbb{Z}^2$, by considering certain n-ary relations on $\mathbb{Z}^2$ ($n > 1$ an integer). We will define a connectedness induced by these relations and prove a digital Jordan curve theorem for this connectedness. Thus, the n-ary relations provide convenient structures on the digital plane for the study of geometric properties of digital images, especially those that are related to boundaries because boundaries of objects in digital images are represented by digital Jordan curves.

2. Preliminaries

Throughout the paper, non-negative integers are considered to be finite ordinals and they are called, as usual, natural numbers. Thus, given a natural number $n > 0$, $(x_i|\ i < n)$ will denote the finite sequence $(x_0, x_1, \ldots, x_{n-1})$ and $(x_i|\ i \leq n)$ the finite sequence $(x_0, x_1, \ldots, x_n)$. Sometimes, such finite sequences will be considered to be just sets, namely the sets $\{x_i;\ i < n\}$ and $\{x_i;\ i \leq n\}$, respectively.

We will work with some basic graph-theoretic concepts only - we refer to [2] for them. By a *graph* $G = (V, E)$, we understand an undirected simple graph without loops where $V \neq \emptyset$ is the *vertex* set of G and $E \subseteq \{\{x, y\};\ x, y \in V,\ x \neq y\}$ is the set of *edges* in G. We will say that G is a graph *on* V. Two vertices $x, y \in V$ are said to be *adjacent* (to each other) if $\{x, y\} \in E$.

Given graphs $G_1 = (V_1, E_1,)$ and $G_2 = (V_2, E_2)$, we say that G_1 is a *subgraph* of G_2 if $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$. If, moreover, $V_1 = V_2$, then G_1 is called a *factor* of G_2 and, if $E_1 = E_2 \cap \{\{x, y\};\ x, y \in V_1\}$, then G_1 is called the *induced subgraph* of G_2.

Recall that a *walk* in G is a (finite) sequence of vertices (i.e., elements of V) such that every pair of consecutive vertices is adjacent. A walk with pairwise different members is called a *path*. A sequence $(x_i|\ i < n)$ of vertices of G with $n > 2$ is called a *circle* in G if $(x_i|i < n)$ is a path in G and $x_0 = x_n$. A subset $A \subseteq V$ is *connected* in G if any two points $x, y \in A$ may be joined by a path contained in A (i.e., there is a path $(x_i|\ i \leq n)$ with $x_0 = x$, $x_n = y$ and $\{x_i|\ i \leq n\} \subseteq A$). A subset $A \subseteq V$ is said to be a *component* of G if it is a maximal (with respect to set inclusion) connected subset of V. A circle C in a graph G is said to be a *simple closed curve* if, for every vertex $z \in C$, C contains precisely two vertices adjacent to z. A simple closed curve J in a graph with vertex set V is called a *Jordan curve* if it separates the set V into precisely two components, i.e., if the induced subgraph $V - J$ has exactly two components.

Given a directed graph (i.e., a set with a binary relation) $D = (X, \rho)$, we define the *underlying graph* of D to be the (undirected simple) graph obtained from D by just ignoring the loops and edge directions, i.e., the graph (X, E) where $E = \{\{x, y\};\ x \neq y \text{ and } (x, y) \in \rho \cup \rho^{-1}\}$.

For every point $(x, y) \in \mathbb{Z}^2$, we denote by $A_4(x, y)$ and $A_8(x, y)$ the sets of all points that are 4-adjacent and 8-adjacent to (x, y), respectively. Thus, $A_4(x, y) = \{(x + i, y + j);\ i, j \in \{-1, 0, 1\},\ ij = 0,\ i + j \neq 0\}$ and $A_8(x, y) = A_4(x, y) \cup \{(x+i, y+j);\ i, j \in \{-1, 1\}\}$. The graphs $(\mathbb{Z}^2, A_4)$ and $(\mathbb{Z}^2, A_8)$ are called the *4-adjacency graph* and *8-adjacency graph*, respectively.

In digital image processing, 4-adjacency and 8-adjacency graphs are the most frequently used structures on the digital plane. But, since the late 1980's, another structure on $\mathbb{Z}^2$ has been used too, namely the Khalimsky topology [5]. For the basic topological concepts used see [4]. The *Khalimsky topology* on $\mathbb{Z}$ is the topology given by the subbase $\{\{2k - 1, 2k, 2k + 1\};\ k \in \mathbb{Z}\}$. (There is another Khalimsky topology on $\mathbb{Z}$, namely the one given by the subbase $\{\{2k, 2k + 1, 2k + 2\};\ k \in \mathbb{Z}\}$, which will not be employed in this note.) The Khalimsky topology on $\mathbb{Z}^m$, $m > 0$ a natural number, is then obtained as the topological product of m copies of the Khalimsky topology on $\mathbb{Z}$. Recall that, given a topology $\mathcal{T}$ on a set X, the *specialization preorder* of $\mathcal{T}$ is the preorder $\leq$ on X defined by $x \leq y \Leftrightarrow x \in \overline{\{y\}}$ for all $x, y \in X$ (where $\overline{}$ denotes the closure operator with respect to $\mathcal{T}$). Since, for every $m > 0$, the Khalimsky topology on $\mathbb{Z}^m$ is T_0 (i.e., for all $t, z \in \mathbb{Z}^m$, $t \in \overline{\{z\}}$ and $z \in \overline{\{t\}}$ imply $t = z$), its specialization preorder is a (partial) order on $\mathbb{Z}^m$. And, since the Khalimsky topology (on $\mathbb{Z}^m$) is an Alexandroff topology (i.e., for all $A \subseteq \mathbb{Z}^m$, $\overline{A} = \bigcup_{z \in A} \overline{\{z\}}$), it is uniquely determined by its specialization order.

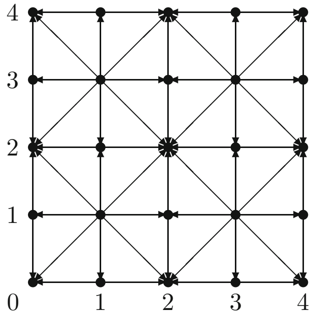

Figure 1. A portion of the specialization order of the Khalimsky topology

The specialization order of the Khalimsky topology on $\mathbb{Z}^2$ coincides with the binary relation $\leq$ on $\mathbb{Z}^2$ given as follows:

For any $(x, y), (z, t) \in \mathbb{Z}^2$, $(x, y) \leq (z, t)$ if and only if one of the following four conditions is satisfied:

(1) $(x, y) = (z, t)$,
(2) x, y are even and $(z, t) \in A_8(x, y)$,
(3) x is even, y is odd, $z = x + i$ where $i \in \{-1, 1\}$, and $t = y$,
(4) x is odd, y is even, $z = x$, and $t = y + i$ where $i \in \{-1, 1\}$.

A portion of the specialization order $\leq$ of the Khalimsky topology is demonstrated in Fig. 1 by a directed graph with vertex set $\mathbb{Z}^2$ where an oriented edge from a point p to a point q means that $q \leq p$.

The underlying graph of the specialization order of the Khalimsky topology coincides with the *connectedness graph* of the topology, i.e., the graph with vertex set $\mathbb{Z}^2$ in which two points are adjacent if and only if they are different and constitute a connected subset of the Khalimsky space. It may be easily seen that connectedness in the Khalimsky space coincides with the connectedness in the connectedness graph of the Khalimsky topology, i.e., in the underlying graph of the specialization order of the topology.

The significant Jordan curve theorem proved for the Khalimsky topology in [5] may be formulated as follows:

Theorem 2.1. *In the underlying graph of the specialization order of the Khalimsky topology, every simple closed curve with at least four points is a Jordan curve.*

It is readily verified that a simple closed curve (and thus also a Jordan curve) in the underlying graph of the specialization order of the Khalimsky topology may never turn at the acute angle $\frac{\pi}{4}$. It could therefore be useful to replace the specialization order of the Khalimsky topology with some more convenient structure (relation on $\mathbb{Z}^2$) that would allow Jordan curves to turn at the acute angle $\frac{\pi}{4}$ at some points. And this is what we will do in the next section.

3. Connectedness induced by n-ary relations

Recall that, given a natural number $n > 0$ and a set X, an *n-ary relation* on X is a subset $\mathcal{R} \subseteq X^n$. Thus, the elements of $\mathcal{R}$ are sequences (ordered n-tuples) $(x_i|\ i < n)$ consisting of elements of X (for the basic properties of n-ary relations see, e.g., [16]). In the sequel, to eliminate the trivial case $n = 1$, we will restrict our considerations to $n > 1$.

Given an n-ary relation $\mathcal{R}$ on a set X, we put
$\mathcal{R}^* = \{(x_i|\ i \leq m);\ (x_i|\ i \leq m) \in X^{m+1},\ 0 < m < n,$ and there exists $(y_i|\ i < n) \in \mathcal{R}$ such that $x_i = y_i$ for every $i \leq m$ or $x_i = y_{m-i}$ for every $i \leq m\}$.
The elements of $\mathcal{R}^*$ will be called $\mathcal{R}$-*initial segments.*

Definition 3.1. Let $\mathcal{R}_j$ be an n-ary relation on a set X_j for every $j = 1, 2, \ldots, m$ ($m > 0$ a natural number). Then we define the *strong product* of the relations $\mathcal{R}_j$, $j = 1, 2, \ldots, m$, to be the n-ary relation $\prod_{j=1}^{m} \mathcal{R}_j$ on the cartesian product $\prod_{j=1}^{m} X_j$ given by $\prod_{j=1}^{m} \mathcal{R}_j = \{((x_i^1, x_i^2, \ldots, x_i^m)|\ i < n);$ there is a nonempty subset $J \subseteq \{1, 2, \ldots, m\}$ such that $(x_i^j|\ i < n) \in \mathcal{R}_j$ for every $j \in J$ and $(x_i^j|\ i < n)$ is a constant sequence for every $j \in \{1, 2, \ldots, m\} - J\}$.

Remark 3.2. Note that the strong product $\prod_{j=1}^{m} \mathcal{R}_j$ in general differs from the usual (cartesian) product $\mathcal{R}$ of the n-ary relations $\mathcal{R}_j$ on X_j, $j = 1, 2, \ldots, m$, which is defined to be the n-ary relation on $\prod_{j=1}^{m} X_j$ given by $\mathcal{R} = \{((x_i^1, x_i^2, \ldots, x_i^m)|\ i < n);\ (x_i^j|\ i < n) \in \mathcal{R}_j$ for every $j \in J\}$ (cf. [16]). Clearly, we always have $\mathcal{R} \subseteq \prod_{j=1}^{m} \mathcal{R}_j$.

Definition 3.3. Let $\mathcal{R}$ be an n-ary relation on a set X. A sequence $C = (x_i|\ i \leq r)$, $r > 0$ a natural number, of elements of X is called an $\mathcal{R}$-*walk* if there is an increasing sequence $(i_k|\ k \leq p)$ of natural numbers with $i_0 = 0$ and $i_p = r$ such that $i_k - i_{k-1} < n$ and $(x_i|\ i_{k-1} \leq i \leq i_k) \in \mathcal{R}^*$ for every k with $0 < k \leq p$. The sequence $(i_k|\ k \leq p)$ is said to be a *binding sequence* of C. An $\mathcal{R}$-walk C is called an $\mathcal{R}$-*path* if its members are pairwise different and it is called an $\mathcal{R}$-*circle* if, for every pair i_0, i_1 of different natural numbers with $i_0, i_1 \leq r$, $x_{i_0} = x_{i_1}$ is equivalent to $\{i_0, i_1\} = \{0, r\}$.

Observe that, if $(x_0, x_1, \ldots, x_r)$ is an $\mathcal{R}$-walk, then $(x_r, x_{r-1}, \ldots, x_0)$ is an $\mathcal{R}$-walk, too ($\mathcal{R}$-walks are closed under reversion). Further, if $C_1 = (x_i|\ i \leq r)$ and $C_2 = (y_i|\ i \leq s)$ are $\mathcal{R}$-walks such that $x_r = y_0$, then, putting $z_i = x_i$ for all $i \leq r$ and $z_i = y_{i-r}$ for all i with $r < i \leq r + s$, we get an $\mathcal{R}$-walk $(z_i|\ i \leq r + s)$ ($\mathcal{R}$-walks are closed under composition). We denote the $\mathcal{R}$-walk $(z_i|\ i \leq r + s)$ by $C_1 \oplus C_2$.

Definition 3.4. Let $\mathcal{R}$ be an n-ary relation on a set X. A set $A \subseteq X$ is said to be $\mathcal{R}$-*connected* if any two different elements $x, y \in A$ can be joined by an $\mathcal{R}$-*walk* contained in A (i.e., there is an $\mathcal{R}$-walk $(x_i|\ i \leq r)$ with $\{x_i|\ i \leq r\} \subseteq A$ such that $x_0 = x$ and $x_r = y$). A maximal $\mathcal{R}$-connected set is called an $\mathcal{R}$-*component*.

If $\mathcal{R}$ is an n-ary relation on a set X and $Y \subseteq X$ is a subset, then there is an n-ary relation on Y induced by $\mathcal{R}$, namely $\mathcal{R} \cap Y^n$. If Y is $\mathcal{R} \cap Y^n$-connected, then we will briefly say that it is $\mathcal{R}$-connected. A similar terminology applies to $\mathcal{R} \cap Y^n$-components.

Note that, given an n-ary relation $\mathcal{R}$ on a set X, every $\mathcal{R}$-initial segment $(x_i|\ i \leq m)$ is $\mathcal{R}$-connected. Indeed, if $x_{i_0}, x_{i_1} \in \{x_i;\ i \leq m\}$ is a pair of different elements and $(y_i|\ i < n) \in \mathcal{R}$ is an n-tuple with $x_i = y_i$ for every $i \leq m$ or $x_i = y_{m-i}$ for every $i \leq m$, then $(x_{i_0}, x_{i_0-1}, x_{i_0-2}, \dots, x_0, x_1, x_2, \dots, x_{i_1})$ or $(x_{i_0}, x_{i_0+1}, x_{i_0+2}, \dots, x_m, x_{m-1}, x_{m-2}, \dots x_{i_1})$, respectively, is an $\mathcal{R}$-walk connecting x_{i_0} and x_{i_1} which is contained in $\{x_i;\ i \leq m\}$. Of course, the union of a finite sequence of nonempty $\mathcal{R}$-connected sets is $\mathcal{R}$-connected if the intersection of every consecutive pair of the sets is nonempty (because $\mathcal{R}$-walks are closed under composition). In particular, every $\mathcal{R}$-walk is $\mathcal{R}$-connected.

Theorem 3.5. *Let $\mathcal{R}_j$ be an n-ary relation on a set X_j and $Y_j \subseteq X_j$ be a subset for every $j = 1, 2, \dots, m$ ($m > 0$ a natural number). If Y_j is $\mathcal{R}_j$-connected for every $i = 1, 2, \dots, n$, then $\prod_{j=1}^m Y_j$ is $\prod_{j=1}^m \mathcal{R}_j$-connected.*

Proof. If $m = 1$, then the statement is trivial. Therefore, we will suppose that $m > 1$.

First, we will show that the statement is true if $Y_j = (y_i^j|\ i \leq p_j)$ is an $\mathcal{R}_j$-initial segment for every $j = 1, 2, \dots, m$. For each $j = 1, 2, \dots, m$, there is a sequence $(x_i^j|\ i < n) \in \mathcal{R}$ such that $y_i^j = x_i^j$ for all $i \leq p_j$ or $y_i^j = x_{p_j-i}^j$ for all $i \leq p_j$ (because $(y_i^j\,|\ i \leq p_j)$ is an $\mathcal{R}_j$-initial segment). Let $y \in \prod_{j=1}^m \{y_i^j;\ i \leq p_j\}$ be an arbitrary element. Then, for each $j = 1, 2, \dots, m$, there is a natural number q_j, $q_j < p_j$, such that $y = (y_{q_1}^1, y_{q_2}^2, \dots, y_{q_m}^m)$. It follows that either $(y_{q_1-i}^1|\ i \leq q_1)$ or $(y_i^1|\ q_1 \leq i \leq p_1)$ is an $\mathcal{R}_1$-initial segment with the first member being $y_{q_1}^1$ and the last one x_0^1. Denote this $\mathcal{R}_1$-initial segment by $(z_i^1|\ i \leq r_1)$ and put $C_1 = ((z_i^1, y_{q_2}^2, y_{q_3}^3, \dots, y_{q_m}^m)|\ i \leq r_1)$. Clearly, C_1 is an $\prod_{j=1}^m \mathcal{R}_j$-initial segment with all members belonging to $\prod_{j=1}^m \{y_i^j;\ i \leq p_j\}$, with the first member y, and with $z_{r_1}^1 = x_0^1$. It follows that either $(y_{q_2-i}^2|\ i \leq q_2)$ or $(y_i^2|\ q_2 \leq i \leq p_2)$ is an $\mathcal{R}_2$-initial segment with the first member being $y_{q_2}^2$ and the last one x_0^2. Denote this $\mathcal{R}_2$-initial segment by $(z_i^2|\ i \leq r_2)$ and put $C_2 = ((x_0^1, z_i^2, y_{q_3}^3, y_{q_4}^4, \dots, y_{q_m}^m)|\ i \leq r_2)$. Clearly, C_2 is a $\prod_{j=1}^m \mathcal{R}_j$-initial segment with all members belonging to $\prod_{j=1}^m \{y_i^j;\ i \leq p_j\}$ such that $z_0^2 = y_{q_2}^2$ and $z_{r_2}^2 = x_0^2$. Thus, $C_1 \oplus C_2$ is a $\prod_{j=1}^m \mathcal{R}_j$-walk with all members belonging to $\prod_{j=1}^m \{y_i^j;\ i \leq p_j\}$, with the first member y, and with the last one $(x_0^1, x_0^2, y_{q_3}^3, y_{q_4}^4, \ldots, y_{q_m}^m)$. Repeating this construction m-times, we get $\prod_{j=1}^m \mathcal{R}_j$-initial segments $C_1, C_2, \ldots, C_m$ with the members of each of them belonging to $\prod_{j=1}^m \{y_i^j;\ i \le p_j\}$ so that $C_1 \oplus C_2 \oplus \cdots \oplus C_m$ is a $\prod_{j=1}^m \mathcal{R}_j$-walk with the first member being y and the last one $(x_0^1, x_0^2, \ldots, x_0^m)$. We have shown that any point of $\prod_{j=1}^m \{y_i^j;\ i \le p_j\}$ can be connected with the point $(x_0^1, x_0^2, \ldots, x_0^m)$ by a $\prod_{j=1}^m \mathcal{R}_j$-walk contained in $\prod_{j=1}^m \{y_i^j;\ i \le p_j\}$.

Second, we will show that the statement is true if $Y_j = (x_i^j|\ i \le p_j)$ is an $\mathcal{R}_j$-walk for every $j = 1, 2, \ldots, m$. If $m = 1$, then the statement is trivial. Let $m > 1$. For each $j = 1, 2, \ldots, m$, let $(i_k^j|\ k \le q_j)$ be a binding sequence of $(x_i^j|\ i \le p_j)$, i.e., a sequence of natural numbers with $i_0^j = 0$ and $i_{q_j}^j = p_j$ such that $(x_i^j|\ i_k^j \le i \le i_{k+1}^j)$ is an $\mathcal{R}_j$-initial segment whenever $k < q_j$. For every $j = 1, 2, \ldots, m$, putting $C_k^j = \{x_i^j;\ i_k^j \le i \le i_{k+1}^j\}$, we get $\{x_i^j;\ i \le p_j\} = \bigcup_{k<q_j} C_k^j$. Therefore, $\prod_{j=1}^m \{x_i^j;\ i \le p_j\} = \bigcup_{k_1<q_1} \bigcup_{k_2<q_2} \cdots \bigcup_{k_m<q_m} \prod_{j=1}^m C_{k_j}^j$ where $\prod_{j=1}^m C_{k_j}^j$ is $\prod_{j=1}^m R_j$-connected whenever $k_j < q_j$, $j = 1, 2, \ldots, m$, by the previous part of the proof. Thus, for any $k_j < q_j$, $j = 1, 2, \ldots, m-1$, $(\prod_{j=1}^m C_{k_j}^j|\ k_m < q_m)$ is a finite sequence of $\prod_{j=1}^m R_j$-connected sets with nonempty intersection for every consecutive pair of them. Hence, the set $\bigcup_{k_m<q_m} \prod_{j=1}^m C_{k_j}^j$ is $\prod_{j=1}^m R_j$-connected. Consequently, for every k_j with $k_j < q_j$, $j = 1, 2, \ldots, m-2$, $(\bigcup_{k_m<q_m} \prod_{j=1}^m C_{k_j}^j|\ k_{m-1} < q_{m-1})$ is a finite sequence of $\prod_{j=1}^m R_j$-connected sets with nonempty intersection for any consecutive pair of them. Thus, the set $\bigcup_{k_{m-1}<q_{m-1}} \bigcup_{k_m<q_m} \prod_{j=1}^m C_{k_j}^j$ is $\prod_{j=1}^m R_j$-connected. After repeating this argument m-times, we get the conclusion that $\bigcup_{k_1<q_1} \bigcup_{k_2<q_2} \cdots \bigcup_{k_m<q_m} \prod_{j=1}^m C_{k_j}^j = \prod_{j=1}^m \{y_i^j;\ i \le p_j\}$ is $\prod_{j=1}^m \mathcal{R}_j$-connected.

Finally, let Y_j be an $\mathcal{R}_j$-connected set for every $j \in \{1, 2, \ldots, m\}$ and let $(x_1, x_2, \ldots, x_m), (y_1, y_2, \ldots, y_m) \in \prod_{j=1}^m X_j$ be arbitrary points. Then, for every $j \in \{1, 2, \ldots, m\}$, there is an $\mathcal{R}_j$-walk $(z_i^j|\ i \le p_j)$ joining the points x_j and y_j which is contained in Y_j. Hence, the set $\prod_{j=1}^m \{z_i^j|\ i \le p_j\}$ contains the points $(x_1, x_2, \ldots, x_m)$ and $(y_1, y_2, \ldots, y_m)$ and is $\prod_{j=1}^m R_j$-connected by the previous part of the proof. Thus, there is a $\prod_{j=1}^m R_j$-walk C joining the points $(x_1, x_2, \ldots, x_m)$ and $(y_1, y_2, \ldots, y_m)$ which is contained in $\prod_{j=1}^m \{z_i^j|\ i \le p_j\}$. Since $\prod_{j=1}^m \{z_i^j|\ i \le p_j\} \subseteq \prod_{j=1}^m Y_j$, C is contained in $\prod_{j=1}^m Y_j$. Therefore, $\prod_{j=1}^m Y_j$ is a $\prod_{j=1}^m \mathcal{R}_j$-connected set. The proof is complete. □

Definition 3.6. An n-ary relation $\mathcal{R}$ on a set X is said to be *plain* if the following two conditions are satisfied:

(a) for every $(x_i|\ i < n) \in \mathcal{R}$, the elements x_i, $i < n$, are pairwise different;
(b) for every $g, h \in \mathcal{R}$, $g \ne h$ implies $\mathrm{card}(g \cap h) \le 1$.

Definition 3.7. Let $\mathcal{R}$ be a plain n-ary relation on a set X. A nonempty, finite and $\mathcal{R}$-connected subset J of X is said to be an $\mathcal{R}$-*simple closed curve* if every point $z \in J$ fulfills one of the following two conditions:

(1) There is a pair $(x_i|\ i \leq p)$, $(y_i|\ i \leq q)$ of $\mathcal{R}$-initial segments such that $\{x_i;\ i \leq p\} \subseteq J$, $\{y_i;\ i \leq q\} \subseteq J$, and $\{z\} = \{x_p\} = \{y_0\} = \{x_i;\ i \leq p\} \cap \{y_i;\ i \leq q\}$ while, for all the other $\mathcal{R}$-initial segments $(z_i|\ i \leq s)$ with $\{z_i;\ i \leq s\} \subseteq J$, we have $z \notin \{z_i;\ i \leq s\}$.
(2) There is an $\mathcal{R}$-initial segment $(x_i|\ i \leq p)$ such that $\{x_i;\ i \leq p\} \subseteq J$ and $z \in \{x_i; 0 < i < p\}$ while, for all the other $\mathcal{R}$-initial segments $(y_i|\ i \leq q)$ with $\{y_i;\ i \leq q\} \subseteq J$, we have $z \notin \{y_i;\ i \leq q\}$.

Proposition 3.8. *Let $\mathcal{R}$ be a plain n-ary relation on a set X. Then every $\mathcal{R}$-simple closed curve J is an $\mathcal{R}$-circle.*

Proof. Since J is nonempty and $\mathcal{R}$-connected, it contains an element $z \in J$ satisfying the condition (1) in Definition 3.7. Let $I_1 = (x_i^1|\ i \leq p_1)$ and $I_2 = (x_i^2|\ i \leq p_2)$ be a pair of $\mathcal{R}$-initial segments such that $I_1 \subseteq J$, $I_2 \subseteq J$, and $\{z\} = \{x_{p_1}^1\} = \{x_0^2\} = I_1 \cap I_2$. As $x_{p_2}^2 \in J$, there is an $\mathcal{R}$-initial segment $I_3 = (x_i^3|\ i \leq p_3)$ such that $I_3 \subseteq J$ and $\{x_{p_2}^2\} = \{x_0^3\} = I_2 \cap I_3$. We may repeat this argument and, since J is finite, after a finite number of steps we will get an $\mathcal{R}$-initial segment $I_k = (x_i^k|\ i \leq p_k)$, $k > 2$ a natural number, with $x_{p_n}^n = x_0^1$. It is evident that the sequence $(x_0^1, x_1^1, \ldots, x_{p_1}^1 = x_1^2, x_2^2, \ldots, x_{p_2}^2 = x_0^3, x_1^3, \ldots, x_{p_{n-1}}^{n-1} = x_0^n, x_1^n, \ldots x_{p_n}^n) = I_1 \oplus I_2 \oplus \cdots \oplus I_k$ is an $\mathcal{R}$-circle. □

Definition 3.9. Let $\mathcal{R}$ be a plain n-ary relation on a set X. An $\mathcal{R}$-simple closed curve J is called an $\mathcal{R}$-*Jordan curve* if the subset $X - J \subseteq X$ consists (i.e., is the union) of precisely two $\mathcal{R}$-components.

4. Plain n-ary relations on the digital plane

From now on, for every natural number $n > 1$, $\mathcal{R}_n$ will denote the plain n-ary relation on $\mathbb{Z}$ given as follows:
$\mathcal{R}_n = \{(x_i|\ i < n) \in \mathbb{Z}^n;$ there is an odd number $l \in \mathbb{Z}$ such that $x_i = l(n-1) + i$ for all $i < n$ or $x_i = l(n-1) - i$ for all $i < n\}$.
Thus, the n-tuples belonging to $\mathcal{R}_n$ are the arithmetic sequences $(x_i|\ i < n)$ of integers with the difference being 1 or -1 and with $x_0 = l(n-1)$ where $l \in \mathbb{Z}$ is an odd number.

The relation $\mathcal{R}_2$ coincides with the specialization order of the Khalimsky topology on $\mathbb{Z}$ and, therefore, $\mathcal{R}_2$-connectedness coincides with connectedness with respect to the Khalimsky topology on $\mathbb{Z}$.

Theorem 4.1. *$\mathbb{Z}$ is $\mathcal{R}_n$-connected (for every natural number $n > 1$).*

Proof. Put $D_l = \{l(n-1)+i;\ i<n\}$ for each $l \in \mathbb{Z}$. Of course, D_l is $\mathcal{R}_n$-connected for every $l \in \mathbb{Z}$ (because $(l(n-1)+i;\ i<n)$ is an $\mathcal{R}_n$-initial segment). Let ω denote the least infinite ordinal and let $(B_i|\ i<\omega)$ be the sequence given by $B_i = D_{\frac{i}{2}}$ whenever i is even and $B_i = D_{-\frac{i+1}{2}}$ whenever i is odd, i.e., $(B_i|\ i<\omega) = (D_0, D_{-1}, D_1, D_{-2}, D_2, \ldots)$. For each $l \in \mathbb{Z}$, there holds $D_l \cap D_{l+1} = \{(l+1)(n-1)\} \neq \emptyset$. Thus, we have $B_0 \cap B_1 \neq \emptyset$. Let i_0 be a natural number with $i_0 > 1$. Then $B_{i_0} \cap B_{i_0-2} \neq \emptyset$ because $B_{i_0} = D_{\frac{i_0}{2}}$ and $B_{i_0-2} = D_{\frac{i_0}{2}-1}$ whenever i_0 is even, while $B_{i_0} = D_{-\frac{i_0+1}{2}}$ and $B_{i_0-2} = D_{-\frac{i_0+1}{2}+1}$ whenever i_0 is odd. Hence, $(\bigcup_{i<i_0} B_i) \cap B_{i_0} \neq \emptyset$ for each i_0, $0 < i_0 < \omega$. Therefore, $\bigcup_{i<\omega} B_i$ is $\mathcal{R}_n$-connected. But $\bigcup_{i<\omega} B_i = \bigcup_{l\in\mathbb{Z}} D_l = \mathbb{Z}$, which proves the statement. □

For every point $x \in \mathbb{Z}$, we put $L(x) = \{y \in \mathbb{Z};\ y < x\}$ and $U(x) = \{y \in \mathbb{Z};\ y > x\}$.

Proposition 4.2. *Let $n > 1$ be a natural number and $z \in \mathbb{Z}$ a point. Then there are points $z_1, z_2 \in \mathbb{Z}$ such that $L(z_1)$ and $U(z_2)$ are $\mathcal{R}_n$-components of the subset $\mathbb{Z} - \{z\}$ of $\mathbb{Z}$ and all the other components of $\mathbb{Z} - \{z\}$ are singletons. If $z = l(n-1)+i$ where $l, i \in \mathbb{Z}$, l is even and $|i| \leq 1$, then $z_1 = z_2 = z$ (so that $\mathbb{Z} - \{z\}$ has no singleton components).*

Proof. There are $l_0, i_0 \in \mathbb{Z}$, l_0 odd and $|i_0| < n$, such that $z = l_0(n-1)+i_0$. Suppose that $|i_0| = n-1$. Then $z = (l_0+1)(n-1)$ or $z = (l_0-1)(n-1)$, so that there is an even number $m \in \mathbb{Z}$ with $z = m(n-1)$. For every $l \in \mathbb{Z}$, put

$$I_l = \begin{cases} (l(n-1)+i|\ i<n) & \text{if } l \text{ is odd,} \\ ((l+1)(n-1)-i|\ i<n) & \text{if } l \text{ is even.} \end{cases}$$

Then $\mathcal{R}_n = \bigcup_{l\in\mathbb{Z}} I_l$. We clearly have $L(z) = \bigcup\{I_l;\ l \leq (m-2)\} \cup \{i;\ (m-1)(n-1) \leq i < z\}$. $L(z)$ is $\mathcal{R}_n$-connected because $(I_l|\ l \leq (m-2))$ is a sequence of n-tuples belonging to $\mathcal{R}_n$ with every pair of consecutive members of the sequence having a point in common, $(i|\ (m-1)(n-1) \leq i < z) \in \mathcal{R}_n^*$ and $(m-1)(n-1) \in I_{m-2} \cap \{i;\ (m-1)(n-1) \leq i < z\}$. Similarly, $U(z) = \bigcup\{I_l;\ l \geq (m+1)\} \cup \{i;\ z < i \leq (m+1)(n-1)\}$ is $\mathcal{R}_n$-connected. Since $L(z)$ and $U(z)$ are maximal $\mathcal{R}_n$-connected subsets which are disjoint and satisfy $L(z) \cup U(z) = \mathbb{Z} - \{z\}$, they are $\mathcal{R}_n$-components

Suppose that $|i_0| < n-1$ where $i_0 \geq 0$. Then $L(l_0(n-1)+1)$ is $\mathcal{R}_n$-connected because it is the union of a sequence of n-tuples belonging to $\mathcal{R}_n$, namely the sequence $(I_l|\ l < l_0)$ in which every pair of consecutive members has a point in common. Further, since $(l_0(n-1)+i|\ i < l_0(n-1)+i_0) \in \mathcal{R}_n^*$ and $L(l_0(n-1)+1) \cap \{l_0(n-1)+i;\ i < l_0(n-1)+i_0\} \neq \emptyset$, the set $L(z) = L(l_0(n-1)+1) \cup \{l_0(n-1)+i;\ i < l_0(n-1)+i_0\}$ is $\mathcal{R}_n$-connected. It is also evident that $L(z)$ is a maximal $\mathcal{R}_n$-connected subset of $\mathbb{Z} - \{z\}$. Further, $U((l_0+1)(n-1)-1)$ is $\mathcal{R}_n$-connected because it is the union of a sequence of n-tuples belonging to $\mathcal{R}_n$, namely the sequence $(I_l|\ l \geq l_0+1)$ in which every pair of consecutive members has a point in common. It is also evident that $U(z)$ is a maximal $\mathcal{R}_n$-connected subset of $\mathbb{Z} - \{z\}$. Clearly, we have $\mathbb{Z} - \{z\} = L(z) \cup \{i;\ z < i < (l_0+1)(n-1)\} \cup U((l_0+1)(n-1)-1)$ where the sets $L(z)$, $\{i;\ z < i < (l_0+1)(n-1)\}$, and $U((l_0+1)(n-1)-1)$ are pairwise disjoint. The singleton subsets of $\{i;\ z < i < (l_0+1)(n-1)\}$ are maximal $\mathcal{R}_n$-connected subsets of $\mathbb{Z} - \{z\}$ because, for every i with $z < i < (l_0+1)(n-1)$, there is no element $j \in \mathbb{Z} - \{z\}$ different from i having the property that both i and j belong to the same $\mathcal{R}_n$-initial segment contained in $\mathbb{Z} - \{z\}$. We have shown that $L(z)$, $U((l_0+1)(n-1)-1)$, and the singletons $\{i\}$, $z < i < (l_0+1)(n-1)$, are maximal $\mathcal{R}_n$-connected subsets of $\mathbb{Z} - \{z\}$. We may show in an analogous way that $U(z)$, $L((l_0-1)(n-1)+1)$, and the singletons $\{i\}$, $(l_0-1)(n-1) < i < z$, are the $\mathcal{R}_n$-components of $\mathbb{Z} - \{z\}$ if $i_0 \leq 0$.

To prove the second part of the statement, suppose that $z = l(n-1)+i$ where $l, i \in \mathbb{Z}$, l is even and $|i| \leq 1$. It was shown in the first part of the proof that $L(z)$ and $U(z)$ are the (only) $\mathcal{R}_n$-components of $\mathbb{Z} - \{z\}$ if $i = 0$ (because then $z = l_0(n-1)+i_0$ where $l_0 = l-1$ is odd and $i_0 = (n-1)$). Suppose that $i = 1$. Then $z = l_0(n-1)+i_0$ where $l_0 = l+1$ and $i_0 = 1-(n-1)$. We have $(l_0-1)(n-1)+1 = l(n-1)+1 = z$, so that $L((l_0-1)(n-1)+1) = L(z)$. Since $\{i;\ (l_0-1)(n-1) < i < z\} \neq \emptyset$, $L(z)$ and $U(z)$ are the (only) $\mathcal{R}_n$-components of $\mathbb{Z} - \{z\}$ according to the previous part of the proof. Using analogous arguments, we may show that $L(z)$ and $U(z)$ are the (only) $\mathcal{R}_n$-components of $\mathbb{Z} - \{z\}$ if $i = -1$. The proof is complete. □

Remark 4.3. Recall [5] that a connected ordered topological space or, briefly, COTS, is a connected topological space X such that, for any three-point subset $Y \subseteq X$, there is a point $x \in Y$ such that Y meets two components of the subspace $X - \{x\}$. It was shown in [5] that a connected topological space X is a COTS if and only if there is a total order on X such that, for every $x \in X$, the sets $L(x)$ and $U(x)$ are components of the subspace $X - \{x\}$. Thus, Proposition 4.2 results in the known fact that $\mathbb{Z}$ with the Khalimsky topology is a COTS [5]. Hence, the relations $\mathcal{R}_n$ may be considered to be generalizations of the Khalimsky topology on $\mathbb{Z}$.

In the sequel, m will denote (similarly to n) a natural number with $m > 0$. Using results of the previous section, we may propose new structures on digital spaces convenient for the study of digital images. Such a structure on $\mathbb{Z}^m$ is obtained as the strong product of m copies of $\mathcal{R}_n$. More formally, we may consider the relation $\mathcal{R}_n^m = \prod_{j=1}^m \mathcal{R}_j$ where $\mathcal{R}_j = \mathcal{R}_n$ for every $j \in \{1, 2, \ldots, m\}$. Since it is evident that the strong product of a family of plain n-ary relations is a plain n-ary relation, $\mathcal{R}_n^m$ is plain. As an immediate consequence of Theorems 3.5 and 4.1 we get:

Theorem 4.4. *$\mathbb{Z}^m$ is $\mathcal{R}_n^m$-connected.*

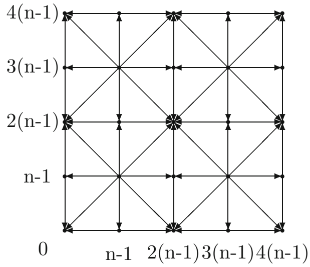

FIGURE 2. A portion of $\mathcal{R}_n^2$

We will restrict our considerations to $m = 2$ because this case is the most important one with respect to possible applications in digital image processing. Thus, we will focus on the n-ary relations $\mathcal{R}_n^2$ ($n > 1$ a natural number).

A portion of $\mathcal{R}_n^2$ is demonstrated in Fig. 2. The ordered n-tuples belonging to $\mathcal{R}_n^2$ are represented by arrows oriented from first to last terms. Between any pair of neighboring parallel horizontal or vertical arrows (having the same orientation), there are $n-2$ more parallel arrows with the same orientation that are not displayed in order to make the Figure transparent. It may easily be seen that $\mathcal{R}_2^2$ coincides with the specialization order of the Khalimsky topology on $\mathbb{Z}^2$ (cf. Fig. 1). Thus, Theorem 2.1 is a Jordan curve theorem for $\mathcal{R}_2^2$. We will prove a Jordan curve theorem for every $\mathcal{R}_n^2$ with $n > 2$.

We denote by G_n the factor of the 8-adjacency graph (with the vertex set $\mathbb{Z}^2$) whose edges are those $\{(x_1, y_1), (x_2, y_2)\} \in A_8$ that satisfy one of the following four conditions for some $k \in \mathbb{Z}$:

$$x_1 - y_1 = x_2 - y_2 = 2k(n-1),$$
$$x_1 + y_1 = x_2 + y_2 = 2k(n-1),$$
$$x_1 = x_2 = 2k(n-1),$$
$$y_1 = y_2 = 2k(n-1).$$

A section of the graph G_n is demonstrated in Fig. 3 where only the vertices $(2k(n-1), 2l(n-1))$, $k, l \in \mathbb{Z}$, are marked out (by bold dots) and thus, on every edge drawn between two such vertices, there are $2n-3$ more (non-displayed) vertices, so that the edge represents $2n-2$ edges in the graph G_n. Clearly, every circle C in G_n is an $\mathcal{R}_n^2$-connected set because it is an $\mathcal{R}_n^2$-circle. Indeed, C consists (i.e., is the union) of a finite sequence of elements of $\mathcal{R}_n^2$, hence $\mathcal{R}_n^2$-initial segments, such that every two consecutive elements have a point in common.

Definition 4.5. A circle J in the graph G_n is said to be *fundamental* if, whenever $((2k+1)(n-1), (2l+1)(n-1)) \in J$ for some $k, l \in \mathbb{Z}$, one of the following two conditions is true:

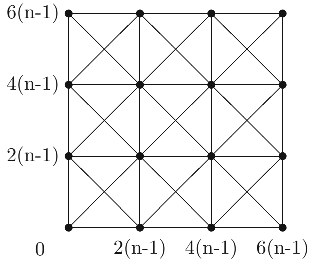

FIGURE 3. $\mathcal{R}_n$-Jordan curves

(1) $\{((2k+1)(n-1)-1,(2l+1)(n-1)-1),(2k+1)(n-1)+1,(2l+1)(n-1)+1))\} \subseteq J$,
(2) $\{((2k+1)(n-1)-1,(2l+1)(n-1)+1),(2k+1)(n-1)+1,(2l+1)(n-1)-1))\} \subseteq J$.

The fundamental circles in G_n are just the circles in Fig. 3 that turn only at (some of) the vertices $(2k(n-1), 2l(n-1))$, $k, l \in \mathbb{Z}$, i.e., the vertices marked out by the bold dots. The following statement is evident:

Proposition 4.6. *Every fundamental circle in G_n^2 is an $\mathcal{R}_n^2$-simple closed curve.*

Theorem 4.7. *If $n > 2$, then every fundamental circle J in G_n is an $\mathcal{R}_n^2$-Jordan curve and the union of any of the two $\mathcal{R}_n^2$-components of $\mathbb{Z}^2 - J$ with J is $\mathcal{R}_n^2$-connected .*

Proof. For every point $z = ((2k+1)(n-1), (2l+1)(n-1))$, $k, l \in \mathbb{Z}$, each of the following four subsets of $\mathbb{Z}^2$ will be called a *fundamental triangle* in G_n (given by z):

(I) $\{(r,s) \in \mathbb{Z}^2;\ 2k(n-1) \leq r \leq (2k+2)(n-1),\ 2l(n-1) \leq s \leq (2l+2)(n-1),\ s \leq r + 2l(n-1) - 2k(n-1)\}$,
(II) $\{(r,s) \in \mathbb{Z}^2;\ 2k(n-1) \leq r \leq (2k+2)(n-1),\ 2l(n-1) \leq s \leq (2l+2)(n-1),\ s \geq 2l(n-1) + (2k+2)(n-1) - r\}$,
(III) $\{(r,s) \in \mathbb{Z}^2;\ 2k(n-1) \leq r \leq (2k+2)(n-1),\ 2l(n-1) \leq s \leq (2l+2)(n-1),\ s \geq r + 2l(n-1) - 2k(n-1)\}$,
(IV) $\{(r,s) \in \mathbb{Z}^2;\ 2k(n-1) \leq r \leq (2k+2)(n-1),\ 2l(n-1) \leq s \leq (2l+2)(n-1),\ s \leq 2l(n-1) + (2k+2)(n-1) - r\}$.

Every fundamental triangle in G_n consists of $2n^2 - n$ points and forms a segment having the shape of a (digital) “rectangular” triangle. The fundamental triangles in G_n given by z are just the triangles in Fig. 3 obtained by dividing the square (segment) with the middle point z and the edge length $2(n-1)$ by one of the two diagonals. Each of the diagonals is the hypotenuse of the two fundamental triangles obtained by dividing the square by the diagonal and z is the middle point of the hypotenuse. Every line segment constituting an edge of a fundamental triangle consists of precisely $2n-1$ points. Clearly, the edges of any fundamental triangle form an $\mathcal{R}_n^2$-simple closed curve. We will show that every fundamental triangle is $\mathcal{R}_n^2$-connected and so is every set obtained from a fundamental triangle by subtracting some of its edges. Let $z = ((2k+1)(n-1), (2l+1)(n-1))$, $k, l \in \mathbb{Z}$, be a point and consider the fundamental triangle $T = \{(r,s) \in \mathbb{Z}^2;\ 2k(n-1) \le r \le (2k+2)(n-1),\ 2l(n-1) \le s \le (2l+2)(n-1),\ s \le r+2l(n-1)-2k(n-1)\}$. Then T is the (digital) triangle ABC with the vertices $A = (2k(n-1), 2l(n-1))$, $B = ((2k+2)(n-1), 2l(n-1))$, $C = ((2k+2)(n-1), (2l+2)(n-1))$. For every $u \in \mathbb{Z}$, $(2k+1)(n-1) \le u \le (2k+2)(n-1)$, the sequence $D_u = ((u,y)|\ 2l(n-1) \le y \le u+2(l-k)(n-1))$ is an $\mathcal{R}_n^2$-walk (contained in T because D_u, if viewed as a set, equals $\{(x,y) \in T;\ x = u\}$), so that D_u is an $\mathcal{R}_n^2$-connected set. Similarly, for every $v \in \mathbb{Z}$, $2l(n-1) \le v \le (2l+1)(n-1)$, the sequence $H_v = ((x,v)|\ v+2(k-l)(n-1) \le x \le (2k+2)(n-1))$ is an $\mathcal{R}_n^2$-walk (contained in T because H_v, if viewed as a set, equals $\{(x,y) \in T;\ y = v\}$), so that H_v is an $\mathcal{R}_n^2$-connected set. We clearly have $T = \bigcup\{D_u;\ (2k+1)(n-1) \le u \le (2k+2)(n-1)\} \cup \bigcup\{H_v;\ 2l(n-1) \le v \le (2l+1)(n-1)\}$. It may be easily seen that $D_u \cap H_v \ne \emptyset$ whenever $(2k+1)(n-1) \le u \le (2k+2)(n-1)$ and $2l(n-1) \le v \le (2l+1)(n-1)$. For every natural number $i < 2n$, we put

$$S_i = \begin{cases} D_{(2k+1)(n-1)+\frac{i}{2}} & \text{if } i \text{ is even,} \\ H_{2l(n-1)+\frac{i-1}{2}} & \text{if } i \text{ is odd.} \end{cases}$$

Then $(S_i|\ i < 2n)$ is a sequence with the property that its members with even indices form the sequence $(D_u|\ (2k+1)(n-1) \le u \le (2k+2)(n-1))$ and those with odd indices form the sequence $(H_v|\ 2l(n-1) \le v \le (2l+1)(n-1))$. Hence, $\bigcup\{S_i|\ i < 2n\} = \bigcup\{D_u;\ (2k+1)(n-1) \le u \le (2k+2)(n-1)\} \cup \bigcup\{H_v;\ 2l(n-1) \le v \le (2l+1)(n-1)\}$ and every pair of consecutive members of $(S_i|\ i < 2n)$ has a non-empty intersection. Thus, since $T = \bigcup\{S_i|\ i < 2n\}$, T is $\mathcal{R}_n^2$-connected. For each of the other three fundamental triangles given by z, the proof is analogous, and the same is true also for every set obtained from a fundamental triangle (given by z) by subtracting some of its edges.

We will say that a (finite or infinite) sequence S of fundamental triangles is a tiling sequence if the members of S are pairwise different and every member of S, excluding the first one, has an edge in common with at least one of its predecessors. Given a tiling sequence S of fundamental triangles, we denote by S' the sequence obtained from S by subtracting, from every member of the sequence, all its edges that are not shared with any other member of the sequence. By the first part of the proof, for every tiling sequence S of fundamental triangles, the set $\bigcup\{T;\ T \in S\}$ is $\mathcal{R}_n^2$-connected and the same is true for the set $\bigcup\{T;\ T \in S'\}$.

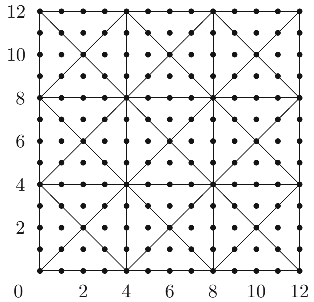

Figure 4. $\mathcal{R}_3$-Jordan curves

Let J be an $\mathcal{R}_n^2$-simple closed curve. Then J constitutes the border of a polygon $S_F \subseteq \mathbb{Z}^2$ consisting of fundamental triangles. More precisely, S_F is the union of some fundamental triangles such that any pair of them is disjoint or meets in just one edge in common. Let U be a tiling sequence of the fundamental triangles contained in S_F. Since S_F is finite, U is finite, too, and we have $S_F = \bigcup\{T;\ T \in U\}$. As every fundamental triangle $T \in U$ is $\mathcal{R}_n^2$-connected, so is S_F. Similarly, U' is a finite sequence with $S_F - J = \bigcup\{T;\ T \in U'\}$ and, since every member of U' is $\mathcal{R}_n^2$-connected (by the first part of the proof), $S_F - J$ is connected, too.
Further, let V be a tiling sequence of fundamental triangles which are not contained in S_F. Since the complement of S_F in $\mathbb{Z}^2$ is infinite, V is infinite, too. Put $S_I = \bigcup\{T;\ T \in V\}$. As every fundamental triangle $T \in V$ is $\mathcal{R}_n^2$-connected, so is also S_I. Similarly, V' is an infinite sequence with $S_I - J = \bigcup\{T;\ T \in V'\}$ and, since every member of V' is connected (by the first part of the proof), $S_I - J$ is connected, too.
It may be easily seen that every $\mathcal{R}_n^2$-walk $C = (z_i|\ i \leq k)$, $k > 0$ a natural number, connecting a point of $S_F - J$ with a point of $S_I - J$ meets J (i.e., meets an edge of a fundamental triangle which is contained in J). Therefore, the set $\mathbb{Z}^2 - J = (S_F - J) \cup (S_I - J)$ is not $\mathcal{R}_n^2$-connected. We have shown that $S_F - J$ and $S_I - J$ are $\mathcal{R}_n^2$-components of $\mathbb{Z}^2 - J$, $S_F - J$ is finite and $S_I - J$ is infinite, with S_F and S_I $\mathcal{R}_n^2$-connected.

As the unions of the $\mathcal{R}_n^2$-components $S_F - J$ and $S_I - J$ with J are the $\mathcal{R}_n^2$-connected sets S_F and S_I, respectively, the proof is complete. □

The fundamental circles in the graph G_n ($n > 2$ a natural number) provide a rich variety of circles to be used for representing borders of objects in digital images. The advantage of circles over Jordan curves in the Khalimsky topology is that they may turn at the acute angle $\frac{\pi}{4}$ at some points.

Example. Every circle in the graph demonstrated in Fig. 4 that does not turn at any point $(4k+2, 4l+2)$, $k, l \in \mathbb{Z}$, is an $\mathcal{R}_3^2$-Jordan curve by Theorem 4.7.

Thus, for example, the triangle with vertices $(0,0),(8,0),(4,4)$ is an $\mathcal{R}_3^2$-Jordan curve but not an $\mathcal{R}_2^2$-Jordan curve. For this triangle to become an $\mathcal{R}_2^2$-Jordan curve, we have to delete the points $(0,0),(1,0),(7,0),(8,0)$. But this will cause a considerable deformation of the triangle.

5. Conclusions

We have shown that every n-ary relation induces connectedness on its underlying set. For certain n-ary relations, which are called plain, the induced connectedness may be used to define the concepts of simple closed curves and Jordan curves in the underlying sets of the relations. For every natural number $n > 1$, we introduced a particular plain n-ary relation $\mathcal{R}_n$ on the digital line $\mathbb{Z}$ and discussed the (plain) relation $\mathcal{R}_n^2$ on the digital plane $\mathbb{Z}^2$ obtained as the strong product of two copies of $\mathcal{R}_n$. We proved that the connectedness on $\mathbb{Z}^2$ induced by $\mathcal{R}_n^2$ allows for digital analogues of the Jordan curve theorem. Thus, we have shown that the n-ary relations $\mathcal{R}_n^2$, $n > 1$ a natural number, provide convenient structures on the digital plane for the study of digital images. While, for $n = 2$, this structure coincides with the Khalimsky topology, for $n > 2$, the structures have the advantage over the Khalimsky topology that they allow Jordan curves to turn at the acute angle $\frac{\pi}{4}$ at some points. Since Jordan curves represent borders of objects in digital images, the structures on $\mathbb{Z}^2$ provided by the n-ary relations $\mathcal{R}_n^2$ may be used in digital image processing for solving problems related to boundaries, such as pattern recognition, boundary detection, contour filling, data compression, etc.

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