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Stokes system with local Coulomb's slip boundary conditions: Analysis of discretized models and implementation

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ABSTRACT

The theoretical part of the paper analyzes discretized Stokes systems with local Coulomb's slip boundary conditions. Solutions to discrete models are defined by means of fixed-points of an appropriate mapping. We prove the existence of a fixed-point, establish conditions guaranteeing its uniqueness and examine how they depend on the discretization parameter h and the slip coefficient κ. The second part of the paper is devoted to computational aspects. Numerical experiments are presented.

1. Introduction

The no-slip condition, i.e. the velocity of a fluid vanishes on the boundary of a computational domain, is the standard boundary condition in fluid flow models. It characterizes the adhesion of a fluid on the solid wall. However in many real problems a slip of a fluid is observed (water flow along hydrophobic surfaces, e.g.). The simplest slip model is the Navier slip condition

$$\sigma_\tau = -ku_\tau, \tag{1}$$

where σ_τ is the shear stress, u_τ the tangential component of the velocity vector $\boldsymbol{u}$, and $k > 0$ is an adhesive coefficient. From (1) we see that a slip is instantaneous whenever $\sigma_\tau \neq 0$. Hence (1) is not able to model situations when a slip may occur only if σ_τ attains certain threshold bound g which is at the same time the upper bound of $|\sigma_\tau|$. Such type of the constitutive law between u_τ and σ_τ can be written in the form of an inclusion for an appropriate multi-valued mapping, namely the subgradient of a convex functional. The resulting mathematical formulation involving a fluid model and threshold slip boundary conditions leads to an inequality problem whose complexity depends on the particular choice of the threshold bound g. In the simplest case, g is given a-priori. This corresponds to the Tresca model of friction well-known in solid mechanics. For the mathematical analysis of the Stokes and Navier–Stokes system with the Tresca slip model we refer to [1]. Results have been extended to time dependent problems in [2], while regularity of solutions has been studied in [3]. The case of the slip bound g depending on the value of the tangential component u_τ leads to an implicit type inequality which is more involved [4,5]. Recently, convergence analysis of a finite element approximation of the Stokes system with this type of a solution-dependent g including computational aspects, and numerical experiments are done in [6].

This paper is focused on the Stokes system with the slip model of local Coulomb's type which corresponds to the slip bound $g = \kappa|\sigma_\nu|$, where $\kappa > 0$ is a slip coefficient and σ_ν is the normal stress (see [7]). The slip bound g is again solution-dependent,

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but it depends on σ_ν this time. This fact makes the resulting mathematical problem much more involved compared with the previous one due to the fact that σ_ν is only a functional over a trace space and its norm appears in the definition of g. To our knowledge the existence of a solution is still open in this case. A possible way how to overcome the low regularity of σ_ν is to use a non-local version of Coulomb's law [8,9] or to impose a-priori bounds on σ_ν as in [10]. However, the weak formulation of this problem can be derived provided that σ_ν is sufficiently smooth. This helps to understand the definition of the respective discrete problem. The main attention of the paper is paid to the analysis of discrete models of the Stokes system with local Coulomb's slip and their properties depending on the discretization parameter h and the slip coefficient κ. In particular, we will study the existence of solutions and establish sufficient conditions under which the solution is unique. With these results at hand, convergence analysis can be done as soon as continuous setting of the problem will be settled. It is worth mentioning that the conditions guaranteeing uniqueness of the solution are mesh dependent for given κ unlike the case when g depends on $|u_\tau|$ as shown in [6].

The paper is organized as follows: In Section 2 we present the weak formulation of the problem and its fixed-point variant assuming that σ_ν is sufficiently regular. To define the respective fixed-point mapping Ψ, three field formulation of the auxiliary Stokes system with the Tresca slip model is used, i.e. the formulation in terms of the velocity field $\boldsymbol{u}$, pressure p and the Lagrange multiplier λ releasing the impermeability condition prescribed on the slip part S of the boundary. Section 3 is devoted to the definition of the discrete problem and the analysis of its properties. To this end the mixed finite element formulation of the Stokes system with Tresca slip by P_1+bubble/P_1 elements is used. As far as the discretization of λ is concerned, two variants of elements on S are considered: piecewise linear on the partition of S which is generated by the triangulation of the computational domain and piecewise constant on a coarse partition of S. Using the discrete Lagrange multiplier λ_h, the discrete form Ψ_h of the fixed-point mapping Ψ is introduced. Solutions to the discrete problem are defined by fixed-points of Ψ_h. The existence and possible uniqueness of fixed-points of Ψ_h (and so of solutions to the discrete problem) are studied. The algebraic formulation of the problem is given in Section 4. Finally, results of numerical experiments are presented in Section 5.

2. Setting of the problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with the Lipschitz boundary $\partial\Omega$ which is split into two non-empty, non-overlapping portions Γ and S open in $\partial\Omega$. In Ω we shall consider the Stokes system with the no-slip condition on Γ, impermeability and the Coulomb type slip boundary conditions on S, respectively. The *classical* formulation of this problem reads as follows: find the velocity $\boldsymbol{u} = (u_1, u_2)$ and the pressure p satisfying:

$$\left.\begin{aligned}
-\operatorname{div}(2\mu\mathbb{D}\boldsymbol{u}) + \nabla p &= \boldsymbol{f} && \text{in } \Omega\\
\operatorname{div}\boldsymbol{u} &= 0 && \text{in } \Omega\\
\boldsymbol{u} &= \boldsymbol{0} && \text{on } \Gamma\\
u_\nu &= 0 && \text{on } S\\
|\sigma_\tau(\boldsymbol{u})| &\le \kappa|\sigma_\nu(\boldsymbol{u},p)| && \text{on } S\\
\kappa|\sigma_\nu(\boldsymbol{u},p)|u_\tau &= -|u_\tau|\sigma_\tau(\boldsymbol{u}) && \text{on } S
\end{aligned}\right\} \tag{2}$$

where $\boldsymbol{f}$ represents volume forces acting on the fluid, $\mu > 0$ is the (constant) dynamic viscosity, $\kappa > 0$ is the slip coefficient, $u_\nu = \boldsymbol{u}\cdot\boldsymbol{\nu}$, $u_\tau = \boldsymbol{u}\cdot\boldsymbol{\tau}$ is the normal, and tangential components of $\boldsymbol{u}$ on S, respectively. Further, $\sigma_\tau(\boldsymbol{u}) = 2\mu\mathbb{D}\boldsymbol{u}\boldsymbol{\nu}\cdot\boldsymbol{\tau}$, $\sigma_\nu(\boldsymbol{u},p) = 2\mu\mathbb{D}\boldsymbol{u}\boldsymbol{\nu}\cdot\boldsymbol{\nu} - p$ is the shear stress, and normal stress corresponding to $(\boldsymbol{u},p)$, respectively and $\mathbb{D}\boldsymbol{u} = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathrm{T}})$ is the symmetric part of the gradient of $\boldsymbol{u}$. To simplify presentation we shall assume that $2\mu = 1$ in Sections 2 and 3.

The last condition in (2) says that the slip at a point $x \in S$ may occur only when $|\sigma_\tau(\boldsymbol{u}(x))| = \kappa(x)|\sigma_\nu(\boldsymbol{u}(x), p(x))|$ and $\sigma_\tau(\boldsymbol{u})$ and u_τ have the opposite signs.

To give the *weak* formulation of (2) we shall *assume* that the velocity $\boldsymbol{u}$ and the pressure p are *regular* in such a way that $\sigma_\nu(\boldsymbol{u},p) \in L^2(S)$.

We introduce the following function spaces:

$$\begin{aligned}
&\boldsymbol{W}(\Omega) = \{\boldsymbol{v} \in (H^1(\Omega))^2 |\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma\},\\
&\boldsymbol{V}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}(\Omega) |\ v_\nu = 0 \text{ on } S\},\\
&Q(\Omega) = \{q \in L^2(\Omega) |\ \int_\Omega q\,dx = 0\}.
\end{aligned}$$

If $\sigma_\nu(\boldsymbol{u},p) \in L^2(S)$, the weak formulation of (2) leads to the following inequality type problem:

$$\left.\begin{aligned}
&\text{Find}\quad (\boldsymbol{u},p) \in \boldsymbol{V}(\Omega)\times Q(\Omega)\quad \text{s. t.}\\
&a(\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) - b(\boldsymbol{v}-\boldsymbol{u},p) + j(|\sigma_\nu(\boldsymbol{u},p)|, v_\tau) - j(|\sigma_\nu(\boldsymbol{u},p)|, u_\tau) \quad\ge\quad (\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u})_{0,\Omega}\\
&b(\boldsymbol{u},q) = 0 \qquad \forall(\boldsymbol{v},q) \in \boldsymbol{V}(\Omega)\times Q(\Omega),
\end{aligned}\right\} (\mathscr{P})$$

where $a : \boldsymbol{W}(\Omega) \times \boldsymbol{W}(\Omega) \to \mathbb{R}$, $b : \boldsymbol{W}(\Omega) \times Q(\Omega) \to \mathbb{R}$ are the bilinear forms defined by

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{v} dx, \qquad b(\boldsymbol{v}, q) = \int_\Omega \operatorname{div} \boldsymbol{v} q dx,$$

and $j : L^2_+(S) \times \boldsymbol{W}(\Omega) \to \mathbb{R}$ is the slip term

$$j(g, v_\tau) = \kappa \int_S g|v_\tau| ds\,.$$ [1]

The Green formula applied to the bilinear forms a, b together with an appropriate choice of test functions $\boldsymbol{v}$ in $(\mathscr{P})$ leads to (2) provided that the solution $(\boldsymbol{u}, p)$ is smooth enough.

Problem $(\mathscr{P})$ can be also formulated by means of the fixed-point approach. To this end, the unknown argument $|\sigma_\nu(\boldsymbol{u}, p)|$ appearing in j will be replaced by a function $g \in L^2_+(S)$ given a-priori. Instead of $(\mathscr{P})$ we consider much simpler problem $(\mathscr{P}^g)$ which is parametrized by g:

$$\left.\begin{aligned} &\text{Find} \quad (\boldsymbol{u}^g, p^g) \in \boldsymbol{V}(\Omega) \times Q(\Omega) \quad \text{s. t.} \\ &a(\boldsymbol{u}^g, \boldsymbol{v} - \boldsymbol{u}^g) - b(\boldsymbol{v} - \boldsymbol{u}^g, p^g) + j(g, v_\tau) - j(g, u^g_\tau) \ \geq \ (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}^g)_{0,\Omega} \\ &b(\boldsymbol{u}^g, q) = 0 \qquad \forall (\boldsymbol{v}, q) \in \boldsymbol{V}(\Omega) \times Q(\Omega)\,. \end{aligned}\right\} (\mathscr{P}^g)$$

It is well-known that $(\mathscr{P}^g)$ has a unique solution for any $g \in L^2_+(S)$ (see [1,2]).

Under the assumption that $\sigma_\nu(\boldsymbol{u}^g, p^g) \in L^2(S)$ for any $g \in L^2_+(S)$ one can define the mapping $\Psi : L^2_+(S) \to L^2_+(S)$ by

$$\Psi(g) = |\sigma_\nu(\boldsymbol{u}^g, p^g)| \text{ on } S \quad \forall g \in L^2_+(S)\,. \tag{3}$$

Comparing the definition of $(\mathscr{P})$ and $(\mathscr{P}^g)$ we see that a pair $(\boldsymbol{u}, p)$ solves $(\mathscr{P})$ if and only if it solves $(\mathscr{P}^g)$ with $g = |\sigma_\nu(\boldsymbol{u}, p)|$, i.e. $|\sigma_\nu(\boldsymbol{u}, p)|$ is a fixed-point of Ψ.

Now we present an alternative form of the fixed-point mapping Ψ which will be discretized and used in the next section. This form is based on the dualization of the impermeability condition $v_\nu = 0$ on S.

Denote

$$X(S) = \{\varphi \in L^2(S) \,|\, \exists \boldsymbol{v} \in \boldsymbol{W}(\Omega) : \varphi = v_\nu \text{ on } S\}$$

and let $X'(S)$ be its dual with the duality pairing $\langle\ ,\ \rangle$.

For any $g \in L^2_+(S)$ we consider the problem:

$$\left.\begin{aligned} &\text{Find} \quad (\boldsymbol{u}^g, p^g, \lambda^g) \in \boldsymbol{W}(\Omega) \times Q(\Omega) \times X'(S) \quad \text{s. t.} \\ &a(\boldsymbol{u}^g, \boldsymbol{v} - \boldsymbol{u}^g) - b(\boldsymbol{v} - \boldsymbol{u}^g, p^g) - \langle \lambda^g, v_\nu - u^g_\nu \rangle + j(g, v_\tau) - j(g, u^g_\tau) \ \geq \ (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}^g)_{0,\Omega} \\ &b(\boldsymbol{u}^g, q) + \langle \psi, u^g_\nu \rangle = 0 \qquad \forall (\boldsymbol{v}, q, \psi) \in \boldsymbol{W}(\Omega) \times Q(\Omega) \times X'(S)\,. \end{aligned}\right\} (\mathscr{M}^g)$$

It is easy to verify that $(\mathscr{M}^g)$ has a unique solution $(\boldsymbol{u}^g, p^g, \lambda^g)$ for any $g \in L^2_+(S)$. In addition, its first two components $(\boldsymbol{u}^g, p^g)$ solve $(\mathscr{P}^g)$ and $\lambda^g = \sigma_\nu(\boldsymbol{u}^g, p^g)$. Thus, the mapping Ψ from (3) can be also expressed by

$$\Psi(g) = |\lambda^g| \text{ on } S\,. \tag{4}$$

Remark 1. Problem $(\mathscr{P}^g)$ is the weak formulation of the Stokes system with the so-called Tresca type slip condition in which the slip bound function g is given a priori.

Remark 2. The weak formulation $(\mathscr{P})$ is written in a formal way. It is correct provided that $\sigma_\nu(\boldsymbol{u}, p) \in L^2(S)$ otherwise a non-local form of Coulomb's slip has to be used as in [8,9].

3. Discretization of $(\mathscr{P})$

This section deals with a discretization of problem $(\mathscr{P})$. To this end we introduce the discrete form of the fixed-point mapping Ψ defined by (4). It will be based on a finite element discretization of $(\mathscr{M}^g)$. In what follows we suppose that Ω is a *polygonal* domain and to simplify our presentation that the slip part S is represented by *one straight segment parallel to* the x_1-axis.

[1] For the sake of simplicity of our presentation we consider $\kappa > 0$ to be constant on S.

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Let $\{\mathscr{T}_h\}$ be a family of regular triangulations of $\overline{\Omega}$, where $h > 0$ stands for the norm of $\mathscr{T}_h$. On any $\mathscr{T}_h \in \{\mathscr{T}_h\}$ we construct the following finite element spaces:

$$\begin{aligned}
\mathscr{S}_h(\Omega) &= \{\boldsymbol{v}_h \in (C(\overline{\Omega}))^2 | \quad \boldsymbol{v}_h|_{\mathbb{T}} = \bar{\boldsymbol{v}}_h|_{\mathbb{T}} + \boldsymbol{b}_{\mathbb{T}}, \quad \bar{\boldsymbol{v}}_h|_{\mathbb{T}} \in (P_1(\mathbb{T}))^2 \\
&\qquad \boldsymbol{b}_{\mathbb{T}} \in (P_3(\mathbb{T}))^2 \text{ is the bubble function, } \forall \mathbb{T} \in \mathscr{T}_h\}\,, \\
\mathscr{S}_h^0(\Omega) &= \mathscr{S}_h(\Omega) \cap (H_0^1(\Omega))^2\,, \\
\boldsymbol{W}_h(\Omega) &= \mathscr{S}_h(\Omega) \cap \boldsymbol{W}(\Omega)\,, \\
Q_h(\Omega) &= \{q_h \in C(\overline{\Omega}) | \quad q_h|_{\mathbb{T}} \in P_1(\mathbb{T}) \ \forall \mathbb{T} \in \mathscr{T}_h, \ \int_\Omega q_h dx = 0\}\,, \\
\Lambda_h(S) &= \{\psi_h \in C(\bar{S}) | \quad \exists \boldsymbol{v}_h \in \boldsymbol{W}_h(\Omega) : \ \psi_h = v_{h\nu} \text{ on } S\}\,, \\
\Lambda_{h+}(S) &= \{\psi_h \in \Lambda_h(S) | \quad \psi_h \geq 0 \text{ on } S\}\,.
\end{aligned}$$

Remark 3. Owing to the assumption on the shape of S it holds that

$$\Lambda_h(S) = \{\psi_h \in C(\bar{S}) | \quad \psi_h|_{\Delta} \in P_1(\Delta), \ \ \forall \Delta \in \mathscr{D}_h, \psi_h(a) = 0, \ \ \forall a \in \bar{\Gamma} \cap \bar{S}\}\,,$$

where $\mathscr{D}_h = \mathscr{T}_{h|S}$ is the partition of $\bar{S}$ generated by $\mathscr{T}_h$.

It is well-known that the pairs $\{\mathscr{S}_h^0(\Omega), Q_h(\Omega)\}$, $\{\boldsymbol{W}_h(\Omega), \Lambda_h(S)\}$ satisfy the following LBB-conditions:

$$\exists \beta_1 > 0 : \quad \sup_{\boldsymbol{v}_h \in \mathscr{S}_h^0(\Omega)} \frac{b_1(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{1,\Omega}} \geq \beta_1 \|q_h\|_{0,\Omega}\,, \quad \forall q_h \in Q_h(\Omega) \text{ and } \forall h > 0\,, \tag{5}$$

and

$$\exists \beta_2 > 0 : \quad \sup_{\boldsymbol{v}_h \in \boldsymbol{W}_h(\Omega)} \frac{b_2(v_{h\nu}, \psi_h)}{\|\boldsymbol{v}_h\|_{1,\Omega}} \geq \beta_2 \|\psi_h\|_{-1/2,S}\,, \quad \forall \psi_h \in \Lambda_h(S) \text{ and } \forall h > 0\,, \tag{6}$$

respectively, where

$$\begin{aligned}
b_1(\boldsymbol{v}_h, q_h) &= \int_\Omega \operatorname{div} \boldsymbol{v}_h q_h dx\,, \quad (\boldsymbol{v}_h, q_h) \in \boldsymbol{W}_h(\Omega) \times Q_h(\Omega)\,, \\
b_2(v_{h\nu}, \psi_h) &= \int_S v_{h\nu} \psi_h ds\,, \quad (\boldsymbol{v}_h, \psi_h) \in \boldsymbol{W}_h(\Omega) \times \Lambda_h(S)\,, \\
\|\psi_h\|_{-1/2,S} &= \sup_{\boldsymbol{v} \in \boldsymbol{W}(\Omega)} \frac{b_2(v_\nu, \psi_h)}{\|\boldsymbol{v}\|_{1,\Omega}}\,.
\end{aligned}$$

For (5), (6) we refer to [11], and [12], respectively. From (5), (6) and Theorem 3.1 in [13] it follows that the sum $b_1 + b_2$ satisfies the LBB-condition, as well:

$$\exists \beta_3 := \beta_3(\beta_1, \beta_2) > 0 : \quad \sup_{\boldsymbol{v}_h \in \boldsymbol{W}_h(\Omega)} \frac{b_1(\boldsymbol{v}_h, q_h) + b_2(v_{h\nu}, \psi_h)}{\|\boldsymbol{v}_h\|_{1,\Omega}} \geq \beta_3(\|q_h\|_{0,\Omega} + \|\psi_h\|_{-1/2,S})\,, \tag{7}$$

holds for any $(q_h, \psi_h) \in Q_h(\Omega) \times \Lambda_h(S)$ and any $h > 0$.

For any $g_h \in \Lambda_{h+}(S)$ we define the discretization of $(\mathscr{M}^{g_h})$ by:

$$\left.\begin{aligned}
&\text{Find} \quad (\boldsymbol{u}_h^{g_h}, p_h^{g_h}, \lambda_h^{g_h}) \in \boldsymbol{W}_h(\Omega) \times Q_h(\Omega) \times \Lambda_h(S) \quad \text{s. t.} \\
&a(\boldsymbol{u}_h^{g_h}, \boldsymbol{v}_h - \boldsymbol{u}_h^{g_h}) - b_1(\boldsymbol{v}_h - \boldsymbol{u}_h^{g_h}, p_h^{g_h}) - b_2(v_{h\nu} - u_{h\nu}^{g_h}, \lambda_h^{g_h}) + j(g_h, v_{h\tau}) - j(g_h, u_{h\tau}^{g_h}) \geq (\boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u}_h^{g_h})_{0,\Omega} \\
&b_1(\boldsymbol{u}_h^{g_h}, q_h) + b_2(u_{h\nu}^{g_h}, \psi_h) = 0 \qquad \forall (\boldsymbol{v}_h, q_h, \psi_h) \in \boldsymbol{W}_h(\Omega) \times Q_h(\Omega) \times \Lambda_h(S)\,.
\end{aligned}\right\} (\mathscr{M}^{g_h})_h$$

Theorem 1. *There exists a unique solution to $(\mathscr{M}^{g_h})_h$ for any $g_h \in \Lambda_{h+}(S)$. In addition, there exists a constant*[2] $c := c(\|\boldsymbol{f}\|_{0,\Omega}, \beta_1, \beta_2)$ *which does not depend on $g_h \in \Lambda_{h+}(S)$, κ and h such that*

$$\|\boldsymbol{u}_h^{g_h}\|_{1,\Omega} + \|p_h^{g_h}\|_{0,\Omega} + \|\lambda_h^{g_h}\|_{-1/2,S} \leq c\,. \tag{8}$$

Proof. Korn's inequality and (7) ensure the existence and uniqueness of the solution as well as the estimate (8) for some constant $c > 0$. Only what it remains to show is that c does not depend on the above mentioned quantities. Inserting $\boldsymbol{v}_h = \boldsymbol{0}$ and $2\boldsymbol{u}_h^{g_h}$ into $(\mathscr{M}^{g_h})_h$ we see that

$$a(\boldsymbol{u}_h^{g_h}, \boldsymbol{u}_h^{g_h}) \leq a(\boldsymbol{u}_h^{g_h}, \boldsymbol{u}_h^{g_h}) + j(g_h, u_{h\tau}^{g_h}) = (\boldsymbol{f}, \boldsymbol{u}_h^{g_h})_{0,\Omega}\,, \tag{9}$$

[2] In the sequel the symbol c denotes a generic positive constant. If one needs to point out that c depends on specific parameters then the ones appear as the arguments of c.

and

$$a(\boldsymbol{u}_h^{g_h}, \boldsymbol{v}_h) - b_1(\boldsymbol{v}_h, p_h^{g_h}) - b_2(v_{h\nu}, \lambda_h^{g_h}) + j(g_h, v_{h\tau}) \geq (\boldsymbol{f}, \boldsymbol{v}_h)_{0,\Omega} \tag{10}$$

holds for any $\boldsymbol{v}_h \in \boldsymbol{W}_h(\Omega)$. The bound of $\|\boldsymbol{u}_h^{g_h}\|_{1,\Omega}$ by a constant $c := c(\|\boldsymbol{f}\|_{0,\Omega})$ is a consequence of (9). If we restrict ourselves to test functions $\boldsymbol{v}_h \in \mathscr{S}_h^0(\Omega)$ then (10) becomes

$$a(\boldsymbol{u}_h^{g_h}, \boldsymbol{v}_h) - b_1(\boldsymbol{v}_h, p_h^{g_h}) = (\boldsymbol{f}, \boldsymbol{v}_h)_{0,\Omega} \quad \forall \boldsymbol{v}_h \in \mathscr{S}_h^0(\Omega). \tag{11}$$

From this and (5) it follows that $\|p_h^{g_h}\|_{0,\Omega} \leq c := c(\|\boldsymbol{f}\|_{0,\Omega}, \beta_1)$.

To show uniform boundedness of $\|\lambda_h^{g_h}\|_{-1/2,S}$ with respect to $g_h \in \Lambda_{h+}(S)$, κ and h, we introduce the space

$$\boldsymbol{W}_h^0(\Omega) = \{\boldsymbol{v}_h \in \boldsymbol{W}_h(\Omega) |\ \boldsymbol{v}_h = (0, v_{h2})\}.$$

Inserting $\boldsymbol{v}_h \in \boldsymbol{W}_h^0(\Omega)$ into (10) we obtain:

$$a(\boldsymbol{u}_h^{g_h}, \boldsymbol{v}_h) - b_1(\boldsymbol{v}_h, p_h^{g_h}) - b_2(v_{h2}, \lambda_h^{g_h}) = (\boldsymbol{f}, \boldsymbol{v}_h)_{0,\Omega} \quad \forall \boldsymbol{v}_h \in \boldsymbol{W}_h^0(S), \tag{12}$$

using that $v_{h\tau} = 0$ and $v_{h\nu} = \pm v_{h2}$ on S for any $\boldsymbol{v}_h \in \boldsymbol{W}_h^0(\Omega)$. Since

$$\sup_{\boldsymbol{v}_h \in \boldsymbol{W}_h(\Omega)} \frac{b_2(v_{h2}, \psi_h)}{\|\boldsymbol{v}_h\|_{1,\Omega}} = \sup_{\boldsymbol{v}_h \in \boldsymbol{W}_h^0(\Omega)} \frac{b_2(v_{h2}, \psi_h)}{\|\boldsymbol{v}_h\|_{1,\Omega}}, \tag{13}$$

there exists a constant $c := c(\|\boldsymbol{f}\|_{0,\Omega}, \beta_1, \beta_2)$ such that $\|\lambda_h^{g_h}\|_{-1/2,S} \leq c$ making use of (6) and (12). □

Remark 4. The fact that the constant c in (8) does not depend on h is a consequence of (5) and (6).

Remark 5. Let $(\boldsymbol{u}_h^{g_h}, p_h^{g_h}, \lambda_h^{g_h})$ solve $(\mathscr{M}^{g_h})_h$. Then $\boldsymbol{u}_h^{g_h}$ is the discrete velocity field which is the solution to the following discrete velocity formulation:

$$\left.\begin{aligned} \boldsymbol{u}_h^{g_h} \in \boldsymbol{V}_{h,\mathrm{div}}(\Omega) \quad : \quad & a(\boldsymbol{u}_h^{g_h}, \boldsymbol{v}_h - \boldsymbol{u}_h^{g_h}) + j(g_h, v_{h\tau}) - j(g_h, u_{h\tau}^{g_h}) \\ & \geq (\boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u}_h^{g_h})_{0,\Omega} \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_{h,\mathrm{div}}(\Omega), \end{aligned}\right\} (\mathscr{P}^{g_h})_h$$

where

$$\begin{aligned} \boldsymbol{V}_{h,\mathrm{div}}(\Omega) &= \{\boldsymbol{v}_h \in \boldsymbol{W}_h(\Omega) |\ b_1(\boldsymbol{v}_h, q_h) = 0 \quad \forall q_h \in Q_h(\Omega), \\ &\qquad b_2(v_{h\nu}, \psi_h) = 0 \quad \forall \psi_h \in \Lambda_h(S)\} \\ &= \{\boldsymbol{v}_h \in \boldsymbol{W}_h(\Omega) |\ b_1(\boldsymbol{v}_h, q_h) = 0 \quad \forall q_h \in Q_h(\Omega),\ v_{h\nu} = 0 \text{ on } S\}. \end{aligned} \tag{14}$$

Now we shall investigate how the solution $(\boldsymbol{u}_h^{g_h}, p_h^{g_h}, \lambda_h^{g_h})$ to $(\mathscr{M}^{g_h})_h$ depends on $g_h \in \Lambda_{h+}(S)$. Let $\Phi_h : \Lambda_{h+}(S) \to \boldsymbol{W}_h(\Omega) \times Q_h(\Omega) \times \Lambda_h(S)$ be the mapping defined by

$$\Phi_h(g_h) = (\boldsymbol{u}_h^{g_h}, p_h^{g_h}, \lambda_h^{g_h}) \quad \forall g_h \in \Lambda_{h+}(S), \tag{15}$$

and denote

$$|||\Phi_h(g_h)||| := \|\boldsymbol{u}_h^{g_h}\|_{1,\Omega} + \|p_h^{g_h}\|_{0,\Omega} + \|\lambda_h^{g_h}\|_{-1/2,S}.$$

Theorem 2. *The mapping Φ_h is Lipschitz continuous in $\Lambda_{h+}(S)$:*

$$\exists c := c(\beta_1, \beta_2) \text{ s. t. } |||\Phi_h(g_h) - \Phi_h(\overline{g}_h)||| \leq c\kappa \|g_h - \overline{g}_h\|_{0,S} \quad \forall g_h, \overline{g}_h \in \Lambda_{h+}(S) \tag{16}$$

and c does not depend on $\boldsymbol{f}$, g_h, $\overline{g}_h$, κ and h.

Proof. Let $g_h, \overline{g}_h \in \Lambda_{h+}(S)$ be given and $(\boldsymbol{u}_h, p_h, \lambda_h) := (\boldsymbol{u}_h^{g_h}, p_h^{g_h}, \lambda_h^{g_h})$, $(\overline{\boldsymbol{u}}_h, \overline{p}_h, \overline{\lambda}_h) := (\boldsymbol{u}_h^{\overline{g}_h}, p_h^{\overline{g}_h}, \lambda_h^{\overline{g}_h})$ be the solution to $(\mathscr{M}^{g_h})_h$, and $(\mathscr{M}^{\overline{g}_h})_h$, respectively. Then $\boldsymbol{u}_h, \overline{\boldsymbol{u}}_h$ solve problem $(\mathscr{P}^{g_h})_h$, and $(\mathscr{P}^{\overline{g}_h})_h$, respectively, see Remark 5. Inserting $\boldsymbol{v}_h = \overline{\boldsymbol{u}}_h$ into $(\mathscr{P}^{g_h})_h$ and $\boldsymbol{v}_h = \boldsymbol{u}_h$ into $(\mathscr{P}^{\overline{g}_h})_h$, adding both inequalities, we get:

$$a(\boldsymbol{u}_h - \overline{\boldsymbol{u}}_h, \boldsymbol{u}_h - \overline{\boldsymbol{u}}_h) \leq j(g_h - \overline{g}_h, \overline{u}_{h\tau}) - j(g_h - \overline{g}_h, u_{h\tau}) \leq \kappa \|g_h - \overline{g}_h\|_{0,S} \|u_{h\tau} - \overline{u}_{h\tau}\|_{0,S}.$$

Consequently

$$\|\boldsymbol{u}_h - \overline{\boldsymbol{u}}_h\|_{1,\Omega} \leq c\kappa \|g_h - \overline{g}_h\|_{0,S}, \tag{17}$$

where c depends on the norm of the respective trace mapping and on the constant of Korn's inequality.

Subtracting Eqs. (11) satisfied by $(\boldsymbol{u}_h, p_h)$ and $(\overline{\boldsymbol{u}}_h, \overline{p}_h)$ we have:

$$a(\boldsymbol{u}_h - \overline{\boldsymbol{u}}_h, \boldsymbol{v}_h) - b_1(\boldsymbol{v}_h, p_h - \overline{p}_h) = 0 \quad \forall \boldsymbol{v}_h \in \mathscr{S}_h^0(\Omega).$$

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This, (5) and (17) entail:

$$\exists c := c(\beta_1): \quad \|p_h - \overline{p}_h\|_{0,\Omega} \le c\kappa \|g_h - \overline{g}_h\|_{0,S}\,. \tag{18}$$

Finally, subtracting Eqs. (12) corresponding to $(\mathscr{M}^{g_h})_h$ and $(\mathscr{M}^{\overline{g}_h})_h$ we obtain:

$$a(\boldsymbol{u}_h - \overline{\boldsymbol{u}}_h, \boldsymbol{v}_h) - b_1(\boldsymbol{v}_h, p_h - \overline{p}_h) - b_2(v_{h\nu}, \lambda_h - \overline{\lambda}_h) = 0 \quad \forall \boldsymbol{v}_h \in \boldsymbol{W}_h^0(\Omega)\,.$$

From this, (6), (13), (17) and (18) we may conclude that

$$\exists c := c(\beta_1, \beta_2): \ \|\lambda_h - \overline{\lambda}_h\|_{-1/2,S} \le c\kappa \|g_h - \overline{g}_h\|_{0,S}\,. \quad \square \tag{19}$$

Now we proceed to the discretization of the fixed-point mapping Ψ defined by (4). The main difficulty we face in the discrete case is the fact that if $\psi_h \in \Lambda_h(S)$ then $|\psi_h|$ belongs to $\Lambda_{h+}(S)$ if it does not change the sign on S or its sign changes only at the nodes of $\mathscr{D}_h$. To overcome this difficulty we define the mapping $\Psi_h : \Lambda_{h+}(S) \to \Lambda_{h+}(S)$ representing the discretization of Ψ by

$$\Psi_h(g_h) = r_h|\lambda_h^{g_h}|\,, \tag{20}$$

where r_h is the piecewise linear Lagrange interpolation operator on $\mathscr{D}_h$ and $\lambda_h^{g_h}$ is the discrete normal stress given by the third component of the solution to $(\mathscr{M}^{g_h})_h$.

Definition 1. By the solution to the discrete Stokes system with the local Coulomb type slip condition we call any triplet $(\boldsymbol{u}_h^{g_h}, p_h^{g_h}, \lambda_h^{g_h})$ solving $(\mathscr{M}^{g_h})_h$, where g_h is a fixed-point of Ψ_h.

In the next part of this section the existence of a fixed-point of Ψ_h and its possible uniqueness is investigated. Since we shall use the global inverse inequalities for the space $\Lambda_h(S)$, we shall suppose that any partition $\mathscr{D}_h \in \{\mathscr{D}_h\}$ of S satisfies the following condition:

$$\exists \gamma > 0: \ \gamma h \le |\Delta| \le h \tag{21}$$

holds for any $\Delta \in \mathscr{D}_h$ and $h > 0$.

Theorem 3. *Let (21) be satisfied. Then the mapping Ψ_h has at least one fixed-point for any h, and κ. All fixed-points of Ψ_h belong to the set*

$$\mathscr{B}_h = \{\psi_h \in \Lambda_{h+}(S) |\ \|\psi_h\|_{0,S} \le ch^{-1/2}\}\,,$$

where the constant $c > 0$ does not depend on h, and κ.

Proof. We use Brouwer's fixed-point theorem. Clearly, Ψ_h is continuous in $\Lambda_{h+}(S)$. From the approximation properties and the inverse inequalities valid for functions from $\Lambda_h(S)$ it follows that there exists a constant c which does not depend on h and such that

$$\begin{aligned}\|r_h|\lambda_h^{g_h}|\|_{0,S} &\le \|r_h|\lambda_h^{g_h}| - |\lambda_h^{g_h}|\|_{0,S} + \|\lambda_h^{g_h}\|_{0,S} \le ch\|\lambda_h^{g_h}\|_{1,S} + \|\lambda_h^{g_h}\|_{0,S} \\ &\le c\|\lambda_h^{g_h}\|_{0,S} \le ch^{-1/2}\|\lambda_h^{g_h}\|_{-1/2,S} \overset{(8)}{\le} ch^{-1/2}\end{aligned} \tag{22}$$

holds for any $g_h \in \Lambda_{h+}(S)$, h and also any κ in view of (8). Hence $\Psi_h(\mathscr{B}_h) \subseteq \mathscr{B}_h$. $\square$

Theorem 4. *Let (21) be satisfied. Then the mapping Ψ_h is Lipschitz continuous in $\Lambda_{h+}(S)$:*

$$\exists c: \ \|\Psi_h(g_h) - \Psi_h(\overline{g}_h)\|_{0,S} \le c\kappa h^{-1/2}\|g_h - \overline{g}_h\|_{0,S} \quad \forall g_h, \overline{g}_h \in \Lambda_{h+}(S)\,,$$

where the constant c does not depend on $\boldsymbol{f}$, g_h, $\overline{g}_h$, κ, and h.

Proof. Let $g_h, \overline{g}_h \in \Lambda_{h+}(S)$. Then

$$\|\Psi_h(g_h) - \Psi_h(\overline{g}_h)\|_{0,S} = \|r_h(|\lambda_h^{g_h}| - |\lambda_h^{\overline{g}_h}|)\|_{0,S} \le \|r_h(\big||\lambda_h^{g_h}| - |\lambda_h^{\overline{g}_h}|\big|)\|_{0,S}$$

$$\le \|r_h|\lambda_h^{g_h} - \lambda_h^{\overline{g}_h}|\|_{0,S} \overset{(22)}{\le} ch^{-1/2}\|\lambda_h^{g_h} - \lambda_h^{\overline{g}_h}\|_{-1/2,S} \overset{(19)}{\le} c\kappa h^{-1/2}\|g_h - \overline{g}_h\|_{0,S}$$

using the monotonicity of the piecewise linear Lagrange interpolation operator r_h.

Consequence 1. *If $\kappa < ch^{1/2}$ for an appropriate constant c, which does not depend on h, the mapping Ψ_h is contractive. Consequently, the fixed-point of Ψ_h is unique and the method of successive approximations*

$$\left.\begin{array}{l} g_0 \in \Lambda_{h+}(S) \ \textit{arbitrary;} \\ g_{k+1} = \Psi_h(g_k), \ k = 0, 1, \ldots \end{array}\right\} \tag{23}$$

is convergent. Notice that the condition on κ guaranteeing contractivity of Ψ_h is mesh dependent.

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Each iterative step in (23) leads to the problem of finding the solution $(\boldsymbol{u}_h^{g_k}, p_h^{g_k}, \lambda_h^{g_k})$ of $(\mathscr{M}^{g_k})_h$. Then $g_{k+1} = r_h|\lambda_h^{g_k}|$.

Below we present an alternative definition of the Lagrange multiplier space $\Lambda_h(S)$. Instead of piecewise linear functions we use piecewise constant functions. As we shall see this choice simplifies the definition of the discrete fixed-point mapping on the one hand but the impermeability condition on S is satisfied only in an integral mean value sense on the other hand.

Let $\{\mathscr{D}_H\}$, $H \to 0_+$ be a system of partitions of S into segments $\Delta \in \mathscr{D}_H, |\Delta| \le H\ \forall \Delta \in \mathscr{D}_H$. Notice that $\mathscr{D}_H$ is independent of $\mathscr{T}_h|_S$. With any $\mathscr{D}_H$ we associate the space of piecewise constant functions on $\mathscr{D}_H$:

$$\Lambda_H(S) = \{\psi_H \in L^2(S) \mid \psi_H|_\Delta \in P_0(\Delta) \quad \forall \Delta \in \mathscr{D}_H\}$$

and denote

$$\Lambda_{H+}(S) = \{\psi_H \in \Lambda_H(S) \mid \psi_H \ge 0 \text{ on } S\}\,.$$

The bilinear form b_2 will be now defined on $\boldsymbol{W}_h(\Omega) \times \Lambda_H(S)$:

$$b_2(v_{h\nu}, \psi_H) = \int_S v_{h\nu}\psi_H\, ds \qquad (\boldsymbol{v}_h, \psi_H) \in \boldsymbol{W}_h(\Omega) \times \Lambda_H(S)\,.$$

Next, we shall *suppose* that there exist constants $\bar{\gamma}, \bar{\bar{\gamma}} > 0$ which do not depend on $h, H > 0$ such that

$$\bar{\gamma} \le H/h \le \bar{\bar{\gamma}} \tag{24}$$

and $\bar{\gamma}$ is sufficiently large. Then the couple $\{\boldsymbol{W}_h(\Omega), \Lambda_H(S)\}$ satisfies the following LBB-condition:

$$\exists \bar{\beta}_2 > 0 : \sup_{\boldsymbol{v}_h \in \boldsymbol{W}_h(\Omega)} \frac{b_2(v_{h\nu}, \psi_H)}{\|\boldsymbol{v}_h\|_{1,\Omega}} \ge \bar{\beta}_2 \|\psi_H\|_{-1/2,S} \tag{25}$$

holds for any $\psi_H \in \Lambda_H(S)$ and any h, H satisfying (24) [14,15]. In view of (5), (25) the same holds for the sum $b_1 + b_2$.

For the discretization of the Stokes problem with Coulomb's slip condition we use the same approach as above but with the space $\Lambda_H(S)$ instead of $\Lambda_h(S)$. For any $g_H \in \Lambda_{H+}(S)$ we consider the problem:

$$\left.\begin{aligned}
&\text{Find} \quad (\boldsymbol{u}_h^{g_H}, p_h^{g_H}, \lambda_H^{g_H}) \in \boldsymbol{W}_h(\Omega) \times Q_h(\Omega) \times \Lambda_H(S) \quad \text{s. t.}\\
&a(\boldsymbol{u}_h^{g_H}, \boldsymbol{v}_h - \boldsymbol{u}_h^{g_H}) - b_1(\boldsymbol{v}_h - \boldsymbol{u}_h^{g_H}, p_h^{g_H}) - b_2(v_{h\nu} - u_{h\nu}^{g_H}, \lambda_H^{g_H}) + j(g_H, v_{h\tau}) - j(g_H, u_{h\tau}^{g_H}) \ge (\boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u}_h^{g_H})_{0,\Omega}\\
&b_1(\boldsymbol{u}_h^{g_H}, q_h) + b_2(u_{h\nu}^{g_H}, \psi_H) = 0 \qquad \forall(\boldsymbol{v}_h, q_h, \psi_H) \in \boldsymbol{W}_h(\Omega) \times Q_h(\Omega) \times \Lambda_H(S)\,.
\end{aligned}\right\} (\mathscr{M}^{g_H})_{h,H}$$

Problem $(\mathscr{M}^{g_H})_{h,H}$ has a unique solution. Let us observe that this time the discrete velocity $\boldsymbol{u}_h^{g_H}$ satisfies the impermeability condition on S only in an integral sense:

$$\int_\Delta u_{h\nu}^{g_H}\, ds = 0 \quad \forall \Delta \in \mathscr{D}_H\,.$$

Let $\Phi_{hH} : \Lambda_{H+}(S) \to \boldsymbol{W}_h(\Omega) \times Q_h(\Omega) \times \Lambda_H(S)$ be the mapping defined by

$$\Phi_{hH}(g_H) = (\boldsymbol{u}_h^{g_H}, p_h^{g_H}, \lambda_H^{g_H})\,.$$

It is easy to show that Theorems 1 and 2 remain valid with the appropriate modifications, namely

$$\exists c := c(\|f\|_{0,\Omega}, \beta_1, \bar{\beta}_2) : \|\boldsymbol{u}_h^{g_H}\|_{1,\Omega} + \|p_h^{g_H}\|_{0,\Omega} + \|\lambda_H^{g_H}\|_{-1/2,S} \le c\,, \tag{26}$$

where c does not depend on $g_H \in \Lambda_{H+}(S)$, κ, H, h, and

$$\exists c := c(\beta_1, \bar{\beta}_2) : |\!|\!|\Phi_{hH}(g_H) - \Phi_{hH}(\bar{g}_H)|\!|\!| \le c\kappa \|g_H - \bar{g}_H\|_{0,S} \quad \forall g_H, \bar{g}_H \in \Lambda_{H+}\,, \tag{27}$$

where c does not depend on $\boldsymbol{f}$, g_H, $\bar{g}_H$ κ, h, and H, respectively.

Since $\psi_H \in \Lambda_H(S)$ if and only if $|\psi_H| \in \Lambda_{H+}(S)$ one can define the approximation $\Psi_{hH} : \Lambda_{H+}(S) \to \Lambda_{H+}(S)$ of the fixed-point mapping Ψ simply by

$$\Psi_{hH}(g_H) = |\lambda_H^{g_H}| \quad \forall g_H \in \Lambda_{H+}(S)\,.$$

To prove the existence and possible uniqueness of a fixed-point of Ψ_{hH} in $\Lambda_{H+}(S)$ we shall *suppose* that in addition to (24)

$$\exists \gamma > 0 : \qquad |\Delta'| \le \gamma|\Delta| \quad \forall \Delta, \Delta' \in \mathscr{D}_H \text{ and } \forall H > 0\,. \tag{28}$$

Then also Theorems 3 and 4 remain true with appropriate modifications.

Theorem 5. *Let (24) and (28) be satisfied. Then the mapping Ψ_{hH} has at least one fixed-point for any κ, h, and H. All fixed-points belong to the set*

$$\mathscr{B}_H = \{\psi_H \in \Lambda_{H+}(S) \mid \|\psi_H\|_{0,S} \le cH^{-1/2}\}\,,$$

where c does not depend on κ, h, and H.

Proof. Let $g_H \in \Lambda_{H+}(S)$. Then

$$\| |\lambda_H^{g_H}| \|_{0,S} \le cH^{-1/2} \|\lambda_H^{g_H}\|_{-1/2,S} \overset{(26)}{\le} cH^{-1/2}$$

making use of the inverse inequality. Hence $\Psi_{hH}(\mathscr{B}_H) \subseteq \mathscr{B}_H$. □

Theorem 6. *Let (24) and (28) be satisfied. Then the mapping Ψ_{hH} is Lipschitz continuous in $\Lambda_{H+}(S)$:*

$$\exists c > 0 : \ \|\Psi_{hH}(g_H) - \Psi_{hH}(\bar{g}_H)\|_{0,S} \le c\kappa H^{-1/2} \|g_H - \bar{g}_H\|_{0,S} \quad \forall g_H, \bar{g}_H \in \Lambda_{H+}(S),$$

where $c > 0$ does not depend on $\boldsymbol{f}$, g_H, $\bar{g}_H$, κ, h, and H.

Proof. Let $g_H, \bar{g}_H \in \Lambda_{H+}(S)$. Then

$$\begin{aligned} \|\Psi_{hH}(g_H) - \Psi_{hH}(\bar{g}_H)\|_{0,S} &= \| |\lambda_H^{g_H}| - |\lambda_H^{\bar{g}_H}| \|_{0,S} \le \|\lambda_H^{g_H} - \lambda_H^{\bar{g}_H}\|_{0,S} \\ &\le cH^{-1/2} \|\lambda_H^{g_H} - \lambda_H^{\bar{g}_H}\|_{-1/2,S} \overset{(27)}{\le} cH^{-1/2}\kappa \|g_H - \bar{g}_H\|_{0,S}. \end{aligned}$$ □

Thus, Consequence 1 can be modified in a straightforward way.

4. Algebraic formulation and solvers

The mixed finite element approximation of $(\mathscr{P}^g)$ uses the P1+bubble/P1 finite element pair [16] satisfying the LBB-stability condition. The matrices are assembled by vectorized codes [17]. To release the impermeability constraint and to regularize the non-smooth term j, we introduce the Lagrange multipliers $\boldsymbol{\lambda}_\nu$, and $\boldsymbol{\lambda}_\tau$, respectively, in the resulting algebraic formulation. The corresponding KKT (Karush–Kuhn–Tucker) system reads as follows: find $(\mathbf{u}, \boldsymbol{\lambda}_\tau, \boldsymbol{\lambda}_\nu, \mathbf{p}) \in \mathbb{R}^{n_u} \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_p}$ such that

$$\mathbf{A}\mathbf{u} - \mathbf{b} + \mathbf{T}^T\boldsymbol{\lambda}_\tau + \mathbf{N}^T\boldsymbol{\lambda}_\nu + \mathbf{B}^T\mathbf{p} = \mathbf{0}, \tag{29}$$

$$\mathbf{B}\mathbf{u} - \mathbf{E}\mathbf{p} - \mathbf{c} = \mathbf{0}, \tag{30}$$

$$\mathbf{N}\mathbf{u} = \mathbf{0}, \tag{31}$$

$$\left.\begin{array}{l} (\mathbf{T}\mathbf{u})_i = 0 \Rightarrow |\lambda_{\tau i}| \le \kappa g_i, \\ (\mathbf{T}\mathbf{u})_i > 0 \Rightarrow \lambda_{\tau i} = \kappa g_i, \\ (\mathbf{T}\mathbf{u})_i < 0 \Rightarrow \lambda_{\tau i} = -\kappa g_i, \end{array}\right\} \ i \in \mathscr{N} := \{1, \dots, n_c\}, \tag{32}$$

where $\mathbf{A} \in \mathbb{R}^{n_u \times n_u}$ is the symmetric, positive definite stiffness matrix arising from the discretization of the 2^{nd} order elliptic operator in $(2)_1$, $\mathbf{b} \in \mathbb{R}^{n_u}$ is the discrete source term, and $\mathbf{B} \in \mathbb{R}^{n_p \times n_u}$ is the full row rank matrix representing the divergence operator in $(2)_2$. The symmetric, positive semidefinite matrix $\mathbf{E} \in \mathbb{R}^{n_p \times n_p}$ and $\mathbf{c} \in \mathbb{R}^{n_p}$ arise from the elimination of the bubble components (on the element level). Here, n_u, n_p stand for the dimension of the solution components $\mathbf{u}$, and $\mathbf{p}$, respectively. Further, the number of the nodes belonging to $\bar{S} \setminus \bar{\Gamma}$ is denoted by n_c. The normal, tangential vectors to S at these nodes define the full row rank matrices $\mathbf{T}$, and $\mathbf{N} \in \mathbb{R}^{n_c \times n_u}$, respectively. The non-smooth term $j(g, v_\tau)$ is approximated by the composite trapezoidal formula, in which the nodal values of g multiplied by the length of segments belonging to the partition $\mathscr{D}_h$ appear. These products are denoted again as g_i and $\mathbf{g} = (g_1, \dots, g_{n_c})^T$. Let us mention that $\boldsymbol{\lambda}_\nu$, $\boldsymbol{\lambda}_\tau$ represent the discretizations of $-\sigma_\nu$ and $-\sigma_\tau$ on S, respectively.

Using simplified notation

$$\boldsymbol{\mu} = \left(\boldsymbol{\mu}_\tau^T, \boldsymbol{\mu}_\nu^T, \mathbf{q}^T\right)^T \in \mathbb{R}^{2n_c+n_p}, \quad \mathbf{C} = \left(\mathbf{T}^T, \mathbf{N}^T, \mathbf{B}^T\right)^T \in \mathbb{R}^{(2n_c+n_p)\times n_u}, \tag{33}$$

one can compute $\mathbf{u}$ from (29): $\mathbf{u} = \mathbf{A}^{-1}(\mathbf{b} - \mathbf{C}^T\boldsymbol{\lambda})$. Eliminating $\mathbf{u}$, we arrive at the dual minimization problem in terms of the Lagrange multipliers:

$$\boldsymbol{\lambda} = \arg \min_{\boldsymbol{\mu} \in \Lambda(\kappa \mathbf{g})} q(\boldsymbol{\mu}), \tag{34}$$

where $\Lambda(\kappa\mathbf{g}) = \{\boldsymbol{\mu} \in \mathbb{R}^{2n_c+n_p} : \ |\boldsymbol{\mu}_\tau| \le \kappa\mathbf{g}\}$, the inequality "$\le$" and the absolute value $|\cdot|$ are understood component wisely, $q(\boldsymbol{\mu}) = \frac{1}{2}\boldsymbol{\mu}^T\mathbf{F}\boldsymbol{\mu} - \boldsymbol{\mu}^T\mathbf{d}$ with $\mathbf{F} = \mathbf{C}\mathbf{A}^{-1}\mathbf{C}^T + \operatorname{diag}(\mathbf{0}, \mathbf{0}, \mathbf{E})$, and $\mathbf{d} = \mathbf{C}\mathbf{A}^{-1}\mathbf{b} - (\mathbf{0}^T, \mathbf{0}^T, \mathbf{c}^T)^T$. Note that $\mathbf{F}$ is symmetric, positive definite.

The quadratic programming problem (34) will be solved in Section 5 by a variant of the path-following interior point method (IPM) [18,19] and by the semismooth Newton method (SSNM) [20,21]. The method of successive approximations (23) is implemented as follows.

Algorithm MSA Given $\boldsymbol{\lambda}^0 \in \mathbb{R}^{2n_c+n_p}$, $\varepsilon > 0$, and set $k := 0$.

(i) Solve $\boldsymbol{\lambda}^{k+1} = \arg\min q(\boldsymbol{\mu})$ subject to $\boldsymbol{\mu} \in \Lambda(\kappa|\boldsymbol{\lambda}_\nu^k|)$.
(ii) Compute $err^k := \|\boldsymbol{\lambda}^{k+1} - \boldsymbol{\lambda}^k\|/\|\boldsymbol{\lambda}^k\|$.
(iii) If $err^k \le \varepsilon$, stop, else set $k := k+1$ and go to Step *(i)*.
(iv) Return $\bar{\boldsymbol{\lambda}} = \boldsymbol{\lambda}^{k+1}$ and $\bar{\mathbf{u}} = \mathbf{A}^{-1}(\mathbf{b} - \mathbf{C}^T\bar{\boldsymbol{\lambda}})$.

Here, $\boldsymbol{\lambda}_\nu^k$ denotes the second component of $\boldsymbol{\lambda}^k$.

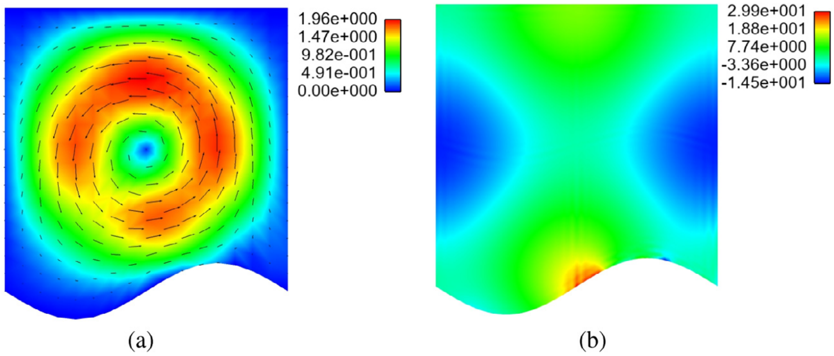

Fig. 1. Velocity field (a) and pressure distribution (b) in Ω.

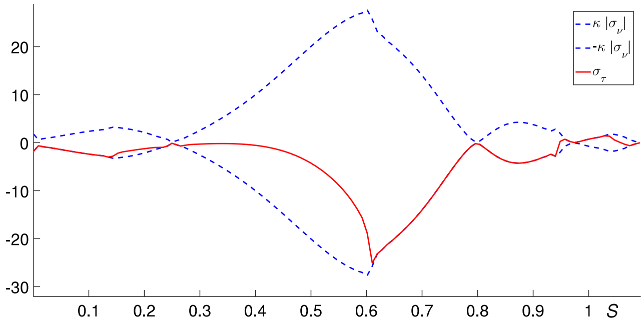

Fig. 2. Distribution of σ_τ and $\pm\kappa|\sigma_\nu|$ along S.

5. Numerical experiments

This section presents numerical results of two model examples. Since the pressure p is determined up to an arbitrary additive constant, one more boundary condition will be used to guarantee its uniqueness. In both examples, the boundary of Ω will be split into *three* non-empty, non-overlapping parts. Besides the no-slip, slip conditions on Γ, and S, respectively, a value of the stress vector $\boldsymbol{\sigma} = 2\mu\mathbb{D}\boldsymbol{u}\boldsymbol{\nu} - p\boldsymbol{\nu}$ will be prescribed on Γ_N. The numerical implementation uses the Algorithm MSA presented in Section 4. Each iterative step *(i)*, corresponding to the Stokes system with the Tresca type slip boundary condition is solved by both, the interior point (IPM) and the semismooth Newton (SSNM) method.

5.1. Example 1

The computational domain Ω is seen in Fig. 1. It is a square like domain with the curved bottom $S = \{(x, -0.1\sin(2\pi x)), x \in (0, 1)\}$ representing the slip part of $\partial\Omega$. The zero velocity is prescribed on the top Γ of Ω. The rest of $\partial\Omega$ represents the part Γ_N. The source term $\boldsymbol{f} = -\mu\,\mathrm{div}(2\mathbb{D}\boldsymbol{u}_{exp}) + \nabla p_{exp}$ with $\mu = 1$, $\boldsymbol{u}_{exp}(x, y) = (-\cos(2\pi x)\sin(2\pi y) + \sin(2\pi y), \sin(2\pi x)\cos(2\pi y) - \sin(2\pi x))$, and $p_{exp}(x, y) = 2\pi(\cos(2\pi y) - \cos(2\pi x))$. Further, $\boldsymbol{\sigma} = \boldsymbol{\sigma}_{exp|\Gamma_N}$ on Γ_N, where $\boldsymbol{\sigma}_{exp}$ is the stress vector resulting from $\boldsymbol{u}_{exp}$ and p_{exp}. Finally, the slip coefficient $\kappa = 1$. Data are chosen in such a way that both, stick and slip zones appear on S.

The orientation and magnitude of the velocity vector $\boldsymbol{u}$ in Ω are depicted in Fig. 1(a). Pressure distribution is shown in Fig. 1(b). The distributions of the shear stress σ_τ, the threshold slip bound $\kappa|\sigma_\nu|$, the pressure p and the tangential velocity u_τ along straightened S are plotted in Figs. 2 and 3. One can see that S is split into the stick and slip parts accordingly to the mutual relation between σ_τ and $\kappa|\sigma_\nu|$. In Table 1 we compare the number of the fixed-point iterations *iter* and the number of the matrix–vector multiplications N_F for IPM and SSNM on different meshes, characterized by n_u, n_p, and n_c. The relative error in the Algorithm MSA is set $\varepsilon = 10^{-7}$. One can see that the number *iter* is the same for IPM and SSNM and practically it does not depend on the size of this problem. The number N_F which characterizes the computational efficiency strongly depends on the norm of the partition of Ω. It is lower for SSNM on coarser and slightly higher on finer partitions compared with IPM.

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Table 1
Comparison of IPM and SSNM.

$n_u/n_p/n_c$	IPM		SSNM	
	iter	N_F	*iter*	N_F
544 / 289 / 17	16	4639	16	**1808**
2112 / 1089 / 33	19	6532	19	**3986**
8320 / 4225 / 65	17	9727	17	**5909**
18624 / 9409 / 97	17	**8789**	17	10455
33024 / 16641 / 129	16	**10123**	17	12729

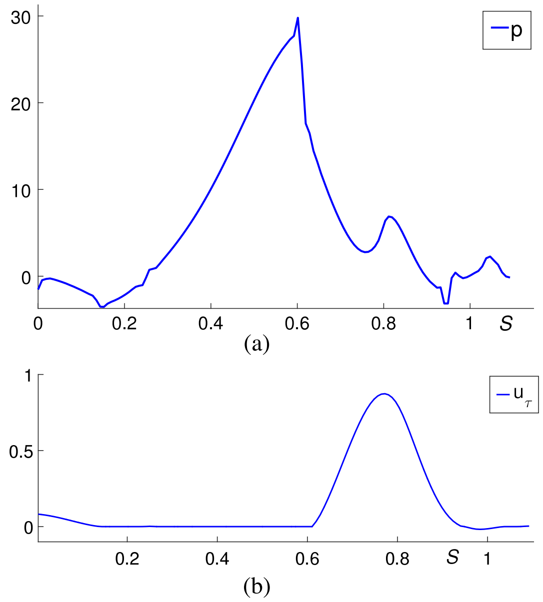

Fig. 3. Distribution of p (a) and u_τ (b) along S.

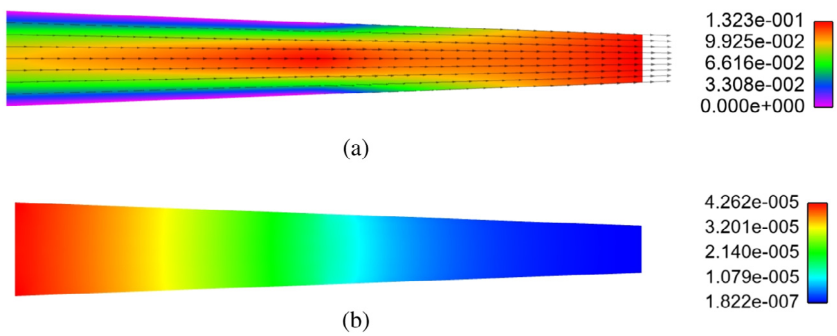

Fig. 4. Velocity field (a) and pressure distribution (b) in Ω, $\kappa = 0.3$.

5.2. Example 2

The nozzle is represented by the trapezoidal domain Ω depicted in Fig. 4. The Dirichlet condition $\boldsymbol{u} = (-160y^2/9 + 1/10, 0)$ is prescribed on the left vertical side $\Gamma = \{0\} \times (-0.075, 0.075)$, while the stress vector $\boldsymbol{\sigma}$ vanishes on the right vertical side $\Gamma_N = \{1\} \times (-0.0375, 0.0375)$. The rest of $\partial\Omega$ represents the slip part S. Further $\boldsymbol{f} = 0$ [N $\cdot$ m^{-3}] and $\mu = 0.001003$ [Pa $\cdot$ s].

In computations we use two values of slip coefficients, namely $\kappa = 0.3$ and 0.6. Again, the data are chosen in such a way that both, the stick, and slip zones appear on S. This time the relative error in the ALGORITHM MSA is set $\varepsilon = 10^{-8}$. The same physical quantities as in the previous example are plotted in Figs. 4–9 for the slip coefficient $\kappa = 0.3$ and 0.6. As expected, the slip zone for $\kappa = 0.3$ is larger than the one for $\kappa = 0.6$. From Table 2 we see that the number of the fixed-point iterations

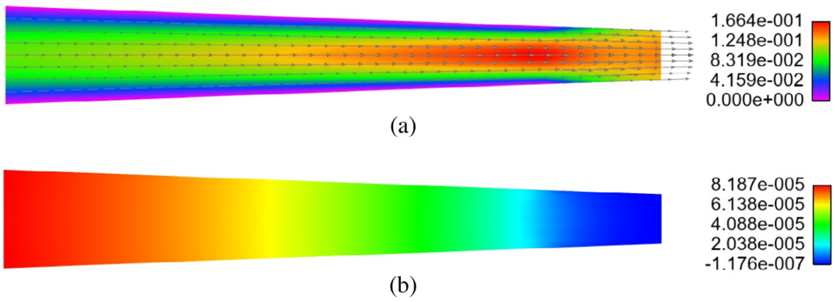

Fig. 5. Velocity field (a) and pressure distribution (b) in Ω, $\kappa = 0.6$.

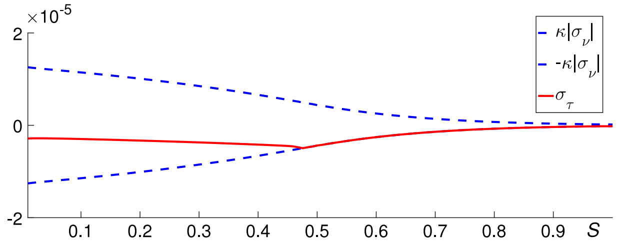

Fig. 6. Distribution of σ_τ and $\pm\kappa|\sigma_\nu|$ along S, $\kappa = 0.3$.

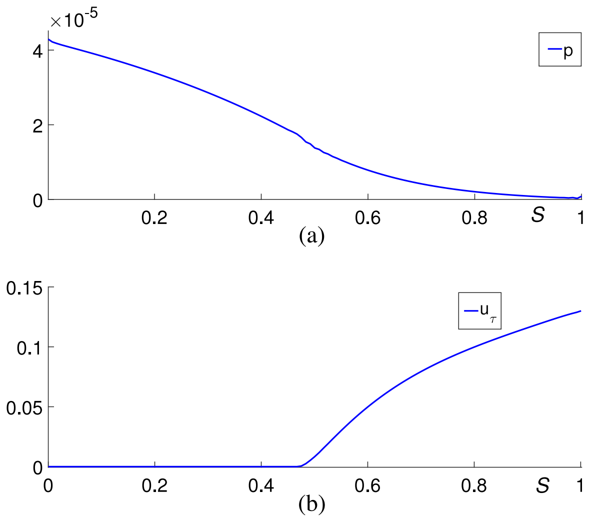

Fig. 7. Distribution of p (a) and u_τ (b) along S, $\kappa = 0.3$.

iter again does not depend on the size of the problem. On the other hand SSNM needs at least seven times less matrix–vector multiplications than IPM.

In Examples 1 and 2 two numerical methods for solving individual iterative steps of MSA, namely IPM and SSNM, are compared. Our computational experiments (not only the ones presented in this paper) demonstrate that SSNM is superior over IPM as far as the number of the matrix–vector multiplications is concerned. Another possibility how to solve our problem and to avoid the method of successive approximations is to implement SSNM directly to the original fixed-point formulation which is given by (29)–(31) and the following slight modification of (32): instead of g_i we use the ith component of the vector $|\boldsymbol{\lambda}_\nu|$. Unfortunately this approach turned out not to be very robust, meaning that SSNM either converged and in that case convergence was fast or the method totally failed changing for example the shape of the computational domain

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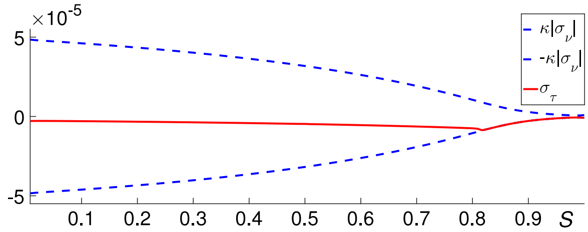

Fig. 8. Distribution of σ_τ and $\pm\kappa|\sigma_\nu|$ along S, $\kappa = 0.6$.

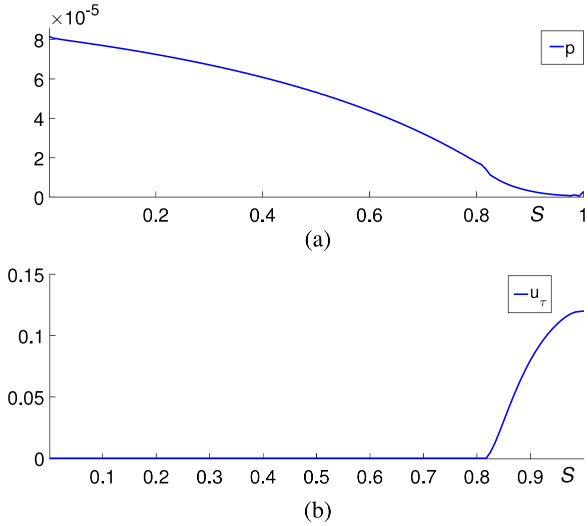

Fig. 9. Distribution of p (a) and u_τ (b) along S, $\kappa = 0.6$.

Table 2
Comparison of IPM and SSNM.

$n_u/n_p/n_c$	$\kappa = 0.3$				$\kappa = 0.6$			
	IPM		SSNM		IPM		SSNM	
	iter	N_F	*iter*	N_F	*iter*	N_F	*iter*	N_F
720 / 369 / 80	11	25201	11	**3629**	18	46835	18	**6514**
2720 / 1377 / 160	10	32243	10	**4885**	15	56377	15	**7888**
6000 / 3025 / 240	14	58835	10	**5807**	19	92069	15	**11553**
10560 / 5313 / 320	14	82050	10	**7262**	19	109864	15	**14627**
16400 / 8241 / 400	14	89426	10	**9341**	19	121433	15	**17015**
23520 / 11809 / 480	14	88700	10	**10126**	19	155552	16	**20052**

(among others). One of possible explanations of this behaviour is that the matrices of linear algebraic systems involved in SSNM are no longer symmetric. On that account we preferred the implementation via MSA.

6. Conclusions

This paper is devoted to the theoretical analysis of the discretized Stokes system with Coulomb's slip boundary conditions and its numerical implementation. In the theoretical part we proved the existence of at least one solution using the fixed-point method. Its uniqueness was established by the Banach fixed-point theorem. It must be quoted that the condition guaranteeing the uniqueness result is only *sufficient* under which the respective fixed-point mapping is contractive. The second part focuses on computational aspects. Computations are based on the method of successive approximations using the dual formulation of each iterative step. Such formulation is derived from the KKT-system by eliminating the velocity vector $\boldsymbol{u}$. The resulting quadratic programming problem is solved by both, the semismooth Newton and interior-point methods. The numerical experiments demonstrate the applicability of the proposed approach.

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